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The clique operator on circular-arc graphs

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ABSTRACT

A *circular-arc graph* G is the intersection graph of a collection of arcs on the circle and such a collection is called a *model* of G. Say that the model is *proper* when no arc of the collection contains another one, it is *Helly* when the arcs satisfy the Helly Property, while the model is *proper Helly* when it is simultaneously proper and Helly. A graph admitting a Helly (resp. proper Helly) model is called a *Helly* (resp. *proper Helly*) circular-arc graph. The *clique graph* $K(G)$ of a graph G is the intersection graph of its cliques. The *iterated clique graph* $K^i(G)$ of G is defined by $K^0(G) = G$ and $K^{i+1}(G) = K(K^i(G))$. In this paper, we consider two problems on clique graphs of circular-arc graphs. The first is to characterize clique graphs of Helly circular-arc graphs and proper Helly circular-arc graphs. The second is to characterize the graph to which a general circular-arc graph K-converges, if it is K-convergent. We propose complete solutions to both problems, extending the partial results known so far. The methods lead to linear time recognition algorithms, for both problems.

1. Introduction

We consider two problems on clique graphs of circular-arc graphs. The first is to characterize clique graphs of Helly circular-arc graphs and proper Helly circular-arc graphs. The second is to characterize the graph to which a circular-arc graph K-converges, if it is K-convergent. We propose complete solutions to both problems, extending the partial results known so far. The methods lead to linear time-recognition algorithms, for both problems. First, we describe the notation employed and related works.

If $G = (V(G), E(G))$ is a graph, we denote by n and m the values of $|V(G)|$ and $|E(G)|$. Denote by C_n the induced cycle with n vertices. A *trivial* graph is a graph with only one vertex. A *complete set* is a subset of pairwise adjacent vertices, while a *clique* is a maximal complete set. The *clique graph* $K(G)$ of G is the intersection graph of its cliques. A graph is a *clique graph* if it is isomorphic to $K(G)$ for some graph G [1,2]. If $\mathcal{C}$ is a class of graphs, we denote by $K(\mathcal{C})$ the class of graphs whose members are clique graphs of the members of $\mathcal{C}$. One of the common questions on clique graphs is to characterize and recognize clique graphs of classes of graphs. In fact, clique graphs of several classes have been characterized and several algorithms are known for testing if a graph is a clique graph of some class [3]. For many of these classes there are also polynomial-time recognition algorithms. However, recently the complexity of the recognition of clique graphs of arbitrary graphs was proved to be NP-Hard [4]. A class of graphs $\mathcal{C}$ is K-*fixed* whenever $K(\mathcal{C}) = \mathcal{C}$ and it is K-*closed* when $K(\mathcal{C}) \subset \mathcal{C}$. In [3] it is remarked that a large number of the classes whose clique graphs have been characterized so far are K-fixed or K-closed. For example, Hedman [5] proved that the class of interval graphs is K-closed, while that of proper interval graphs is K-fixed. These results yield linear time algorithms for the recognition of clique graphs of interval and proper interval graphs.

The *iterated clique graph* is defined by $K^0(G) = G$ and $K^{i+1}(G) = K(K^i(G))$. The analysis of the K-*behavior* of a clique graph is one of the main topics about iterated clique graphs. A graph G is K-*null* if $K^i(G)$ is the trivial graph, for some i. Say

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that G is K-*periodic* with period i if $K^i(G) = G$ for some $i > 0$. When the period is 1 the K-periodic graph is called *self-clique*. A graph is K-*convergent* when it is K-null or $K^i(G)$ is K-periodic for some $i \geq 0$. If G is not K-convergent, then $|V(K^i(G))|$ is unbounded when $i \to \infty$; in this case G is K-*divergent*. To determine the K-behavior of a graph G means to decide if G is K-null, K-convergent or K-divergent. Other questions related to iterated graphs include determining the speed of convergence, its period or the speed of divergence (see [3]). For the general case, the problem of determining the K-behavior of a graph is not known even to be computable. Nevertheless, polynomial-time algorithms to decide the K-behavior of a few classes are known. This is the case for cographs [6], P_4-tidy graphs [7] and complete multipartite graphs [8]. Clique-Helly graphs K-converge to graphs with period either 1 or 2 [9], interval graphs are K-null and octahedra of dimension at least 3 K-diverge. The K-behavior of circular-arc graphs can be determined in linear time. We show how to combine the results of [10–12] in order to obtain the desired algorithm.

A graph is co-bipartite if its vertex set can be partitioned into two complete sets. The *k-th power* of a graph G is the graph G^k that has the same vertices as G and two vertices are adjacent whenever their distance is at most k. The *neighborhood* $N_G(v)$ of a vertex v is the set of its adjacent vertices, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. When there is no ambiguity, we may simply write $N(v)$ or $N[v]$. A vertex v is *universal* when $N[v] = V(G)$. Vertex v *dominates* vertex w when $N[w] \subseteq N[v]$, and they are *twins* when w also dominates v. A *dismantling* of a graph G is the subgraph obtained from G by iteratively removing a vertex v, which is dominated by some vertex $w \neq v$, in the subgraph so far obtained. A set of vertices H of G is a *dismantling set* if the vertices of H are precisely those that are not removed when computing some dismantling. It is not hard to see that the dismantling is unique up to isomorphism. In [10] it is proved that the K-behavior is the same for a graph and its dismantling, i.e., they are both K-null, or they are both K-convergent and not K-null, or they are both K-divergent. For general graphs, the dismantling of a graph can be computed in polynomial-time.

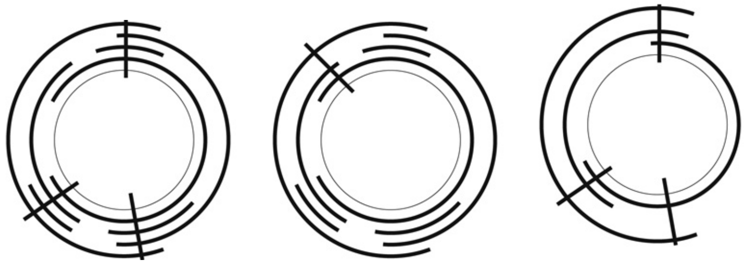
A *circular-arc* (CA) model $\mathcal{M}$ is a pair $(C, \mathcal{A})$, where C is a circle and $\mathcal{A}$ is a collection of arcs of C. When traversing the circle C, we will always choose the clockwise direction. If s, t are points of C, write (s, t) to mean the arc of C defined by traversing the circle from s to t. Call s, t the *extremes* of (s, t), while s is the *beginning point* and t the *ending point* of the arc. For $A \in \mathcal{A}$, write $A = (s(A), t(A))$. The *extremes* of $\mathcal{A}$ are those of all arcs in $\mathcal{A}$. Without loss of generality, we assume that all arcs of $\mathcal{A}$ are open arcs, no two extremes coincide, and no single arc covers C. We will say that $\epsilon > 0$ is *small enough* if ϵ is smaller than the quarter of the minimum arc distance between two consecutive extremes of $\mathcal{A}$.

When no arc of $\mathcal{A}$ contains any other, $\mathcal{M}$ is a *proper* circular-arc (PCA) model. When every set of pairwise intersecting arcs share a common point, $\mathcal{M}$ is called a *Helly* circular-arc (HCA) model. If no two arcs of $\mathcal{A}$ cover C, then the model is called *normal*. A *proper Helly* circular-arc model (PHCA) is one which is both HCA and PCA. Similarly, a *normal Helly* circular-arc model (NHCA) is one which is simultaneously normal and Helly. Finally, an *interval model* (I) is a CA model where $\bigcup_{A \in \mathcal{A}} A \neq C$, while a *proper interval* (PI) model is an interval model which is also proper. A CA (resp. PCA, HCA, PHCA, NHCA, I, PI) graph is the intersection graph of a CA (resp. PCA, HCA, PHCA, NHCA, I, PI) model. For simplicity of notation, we are going to use the same abbreviations to denote the corresponding subclasses of circular-arc graphs but in italic style, for instance *HCA* denotes the class of HCA graphs. Two CA models are *equivalent* when they have the same intersection graph. Circular-arc graphs and their subclasses have been receiving much attention recently [13,14]. For CA, PCA, HCA, I, PI and PHCA graphs, there are several characterizations and linear time recognition algorithms which construct a model (see [15–19,14]).

In this paper, we consider the questions of characterizing and recognizing the clique graphs of HCA graphs and characterizing the graph to which a CA graph K-converges, if it is K-convergent. Partial results concerning these problems have been reported in [20,21], respectively. In relation to characterizing clique graphs of HCA graphs, in [20] it is proved that $K(HCA) \subset PCA \cap HCA$ thus the class is K-closed. The same paper also describes characterizations for the clique graphs of HCA graphs, but these characterizations did not lead to a polynomial time recognition algorithm, and the complexity of recognizing clique graphs of HCA graphs remained so far open. As for the problem of characterizing the graph to which a CA graph K-converges, in [21] it has been proved that an HCA graph G is K-periodic if and only if G is isomorphic to C_n^k with $n > 3k$. Moreover, K-periodic Helly circular-arc graphs are always self-clique.

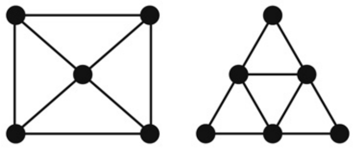
We propose complete solutions to these problems. We characterize the clique graph of an HCA graph G, by proving that $K(G)$ is either a PHCA graph, or $K(G) \setminus U$ is co-bipartite and PHCA, where U is the set of universal vertices of $K(G)$, with $|U| \geq 2$. In addition, we prove that the class of PHCA graphs is fixed. These characterizations lead to linear time recognition algorithms for the classes of clique graphs of HCA graphs and PHCA graphs. As for the K-behavior of a general CA graph G, first we employ the dismantling of G to observe that its K-behavior follows from the results of [11,12]. That is, we conclude that G is K-null if and only if its dismantling is K-null; GK-converges to a graph which is not trivial if and only if its dismantling is C_n^k, with $n > 3k$; and G is K-divergent otherwise. Next, we prove that a K-convergent CA graph always K-converges to its dismantling. Furthermore, we characterize the K-convergent CA graphs, which are not K-null, by showing that the class is exactly the class of NHCA graphs, which are not interval graphs. Finally, we describe how the algorithm of [11] can be adapted to compute the dismantling of a CA graph in $O(n)$ time, given a CA model of it. This algorithm implies that the K-behavior of a CA graph can be decided in linear time.

The characterizations of the clique graphs of HCA and PHCA are described in Section 2, while Section 3 contains the results on the K-behavior of general CA graphs. Some further comments form Section 4.



(a) Clique points. (b) A non-clique intersection point. (c) A clique model.

Fig. 1. Example of the construction of a clique model, given an HCA model $\mathcal{M}$. In (a), the clique points of $\mathcal{M}$ are shown, and they are all intersection points. In (b) there is an intersection point which is properly dominated and in (c) the clique model of $\mathcal{M}$ (w.r.t. the set of clique points) is shown.



(a) 4-wheel. (b) 3-sun.

Fig. 2. Family of minimal PCA graphs that are not PHCA graphs.

2. Characterization of clique graphs of HCA graphs

In this section we characterize the classes $K(HCA)$ and $K(PHCA)$ relating them to the *PHCA* class. In this sense, the characterizations are very similar to those of Hedman for $K(I)$ and $K(PI)$ graphs. Recall that $K(HCA) \subset PCA$ [20]. For the characterizations we need to describe the PCA models of the clique graphs, both for HCA and for PHCA graphs.

Fix an HCA model $\mathcal{M} = (C, \mathcal{A})$ of a graph G and denote by $\mathcal{A}(p)$ the collection of arcs that contain the point $p \in C$. Clearly, the vertices of G corresponding to $\mathcal{A}(p)$ induce a complete set. If this set is a clique, then p is called a *clique point*. For points $p \neq p'$ on the circle, p *(properly) dominates* p' if $\mathcal{A}(p')$ is (properly) contained in $\mathcal{A}(p)$. When $\mathcal{A}(p) = \mathcal{A}(p')$ then p, p' are *equivalent*. In $\mathcal{M}$, every non properly dominated point is a clique point and vice versa, therefore, there is a one-to-one correspondence between cliques of G and non-equivalent clique points (see Fig. 1 (a)). An *intersection segment* (s, t) is a pair of consecutive extremes where s is a beginning point and t is an ending point. Points inside intersection segments are called *intersection points*. Every clique point of $\mathcal{M}$ must be an intersection point, but the converse is not necessarily true [22], because there can be multiple intersection segments that are contained in exactly the same set of arcs (see Fig. 1 (b)). However, when $\mathcal{M}$ is an NHCA model, then every intersection point is also a clique point because (s, t) is exactly the intersection between the arc whose beginning point is s and the arc whose ending point is t.

The *arc reduction* of a clique point p is the arc $(s, t(A_k))$ where s is the beginning point of the intersection segment that contains p, $A_k \in \mathcal{A}(p)$ and $t(A_k)$ is the ending point farthest from p when traversing C. Observe that if $s = s(A_k)$ then the arc reduction of p is precisely A_k. In such cases, we say that p and its arc reduction A_k are *strong*. Non-strong clique points as well as non-strong arc reductions are referred to as *weak*. If $\mathcal{M}$ is a PHCA model then every clique point is strong, i.e., all the arc reductions of $\mathcal{M}$ are arcs of $\mathcal{M}$. A *clique point representation* of $\mathcal{M}$ is a maximal set of non-equivalent clique points. Define the *clique model* (w.r.t. a clique point representation Q) as the model formed by the arc reductions of Q (see Fig. 1 (c)). In particular, any clique model of an HCA (resp. NHCA, PHCA) graph G is a PCA (resp. PHCA) model of $K(G)$ [22]. We sum up this discussion with two results for future reference.

Theorem 1 *([22]). Let $\mathcal{M}$ be an HCA (resp. NHCA, PHCA) model of a graph G. Then, every clique model of $\mathcal{M}$ is a PCA (resp. PHCA) model of $K(G)$.*

Proposition 2. *The clique model of every PHCA model $\mathcal{M}$ is the submodel of $\mathcal{M}$ induced by its strong arcs.*

The construction of the clique model can be done in $O(n)$ time when the input is an HCA model of G. For this, first compute every clique point in $O(n)$ time as in [23] and then compute every arc reduction as in [22]. PHCA graphs are characterized by a family of two forbidden PCA graphs, 4-wheels and 3-suns (see Fig. 2). We employ this characterization in the proof that *PHCA* is K-fixed.

Theorem 3 *([17]). Let G be a PCA graph. Then G is a PHCA graph if and only if G contains neither* 4*-wheels nor* 3*-suns as induced subgraphs.*

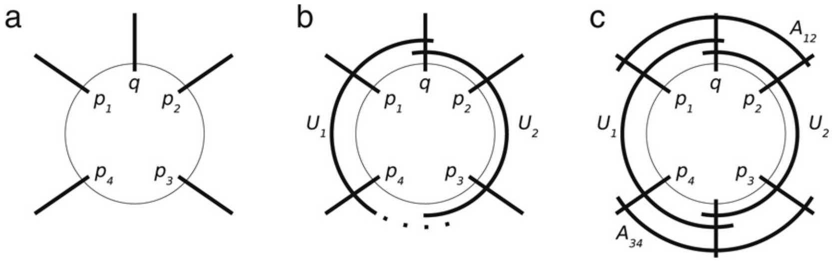


Fig. 3. Proof of Lemma 5.

A graph G is *clique-Helly* if every family of pairwise intersecting cliques of G has a non-empty intersection. Every clique-Helly graph that contains an induced 3-sun H must contain an induced $K_{1,3}$ [24,25]. In [20] it is not only proved that $K(HCA) \subset PCA$, but also it is proved that every graph in $K(HCA)$ is clique-Helly. Therefore, graphs in $K(HCA)$ contain no induced 3-sun, because $K_{1,3}$ is not a PCA graph. We remark this fact for future reference, and analyze how does the *center* of a 4-wheel look like in clique graphs of HCA graphs. The *center* of a 4-wheel is the vertex of degree four.

Lemma 4. *Graphs in $K(HCA)$ contain no 3-sun as an induced subgraph.*

Lemma 5. *Let G be an HCA graph and $\mathcal{M} = (C, \mathcal{A})$ be any HCA model of G. If $K(G)$ contains a 4-wheel as an induced subgraph then*

(i) *$\mathcal{M}$ has two arcs covering the circle,*
(ii) *$K(G)$ has at least two universal vertices, and*
(iii) *the center of every induced 4-wheel of $K(G)$ is universal.*

Proof. Let v_1, v_2, v_3, v_4, u be vertices of $H = K(G)$ that induce a 4-wheel, where v_1, v_2, v_3, v_4 is the cycle in that order and u is universal to the vertices of the cycle. Since G is HCA, then in $\mathcal{M}$ every v_i $(1 \le i \le 4)$ is represented by a point p_i and u is represented by a point q. In every circular-arc model, p_1, p_2, p_3 and p_4 should be in that circular order, or in the reverse one. Without loss of generality, assume the former and suppose that q lies between p_1 and p_2 (see Fig. 3 (a)). Since u is adjacent to v_4 in H, then there exists some arc $U_1 \in \mathcal{A}$ crossing both q and p_4. But v_4 is not adjacent to v_2, so U_1 does not cross p_2. Then U_1 crosses p_1 and, since v_1 is not adjacent to v_3, it follows that U_1 does not cross p_3. The same argument can be used to show that there is an arc $U_2 \in \mathcal{A}$ crossing q and p_3 but not p_1 and p_4 (see Fig. 3 (b)). Since v_3 is adjacent to v_4 in H then there is an arc $A_{34} \in \mathcal{A}$ crossing p_3 and p_4 and, since v_3 is not adjacent to v_1 and v_4 is not adjacent to v_2, it follows that A_{34} crosses neither p_1 nor p_2. Analogously, there is an arc $A_{12} \in \mathcal{A}$ crossing p_1 and p_2 that crosses neither p_3 nor p_4. As a consequence of these facts, U_1, U_2, A_{12} and A_{34} are all different (see Fig. 3 (c)). Now, since $\mathcal{M}$ is Helly and U_1, U_2, A_{34} cover the circle, then U_1, U_2 must share a common point inside A_{34}, i.e., U_1, U_2 cover the circle. Also, A_{12} and A_{34} do not intersect since otherwise A_{12}, A_{34}, U_1 and U_2 correspond to a complete set with no common point, which is impossible. Then there are at least two different cliques, one containing U_1, U_2, A_{12} and the other containing U_1, U_2, A_{34} that are both universal, because every clique point is crossed by either U_1 or U_2. In other words, H has two universal vertices. Finally, point q is crossed by both U_1 and U_2, so u is universal in H. □

We are now ready to give both characterizations of $K(PHCA)$ and $K(HCA)$. As a corollary we will also obtain that $K(NHCA) = K(PHCA)$ which resembles the fact that $K(I) = K(PI)$. A preliminary version of the proof for the PHCA case appeared in [23].

Theorem 6. *Let H be a graph. Then $H = K(G)$ for some PHCA graph G if and only if H is a PHCA graph.*

Proof. Fix a PHCA graph G and its PHCA model $\mathcal{M}$. Graph $H = K(G)$ is PCA [20] and it does not contain 3-suns as induced subgraphs by Lemma 4. If two arcs of $\mathcal{M}$ cover the circle then, since $\mathcal{M}$ is proper, these two are universal arcs, thus H is a clique which is clearly a PHCA graph. If not, then by Lemma 5, H has no 4-wheels as induced subgraphs. Hence, by Theorem 3, H is PHCA.

For the converse, let H be a PHCA graph. If H is a proper interval graph, then there exists a proper interval graph G such that $K(G) = H$ [5], and so the result follows. Otherwise, let $\mathcal{M} = (C, \mathcal{A})$ be a PHCA model of H. We may assume without loss of generality that $\mathcal{M}$ is normal [17], i.e., there are no two arcs that together cover the circle. By Theorem 1, it suffices to find a PHCA supermodel of $\mathcal{M}$ whose clique model is $\mathcal{M}$. Let $\mathcal{Q}$ be the set of arc reductions of $\mathcal{A}$ and $\mathcal{N} = \mathcal{A} \setminus \mathcal{Q}$. By Proposition 2, $\mathcal{Q}$ is a subset of arcs of $\mathcal{A}$. Note that since H is not an interval graph then every arc of $\mathcal{A}$ contains at least one ending point of some other arc.

Fix a small enough ϵ. For $A_i \in \mathcal{A}$, call $NEXT_t(A_i)$ to the arc whose ending point appears first when traversing C from $s(A_i)$. For each arc $A_i \in \mathcal{N}$ let B_i be the arc $(s(NEXT_t(A_i)) - \epsilon, s(A_i) + \epsilon)$ (see Fig. 4). If two arcs B_i, B_j share their beginning points, then modify one of them so that none of them is included in the other. We claim that $\mathcal{M}' = (C, \mathcal{A} \cup \{B_i : A_i \in \mathcal{N}\})$ is PHCA and has $\mathcal{M}$ as its clique model.

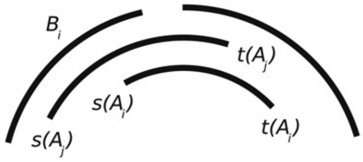


Fig. 4. Example of the arc B_i associated with A_i, where $A_j = NEXT_t(A_i)$.

First we prove that $\mathcal{M}'$ is proper, i.e., no arc of $\mathcal{M}'$ is contained in some other arc. Fix some B_i for $1 \le i \le n$. First observe that every arc A_j containing the beginning point of B_i does not cross $s(A_i)$, because the first arc crossing $s(A_i)$ is $NEXT_t(A_i)$. Thus $B_i \not\subset A_j$ for $1 \le i, j \le n$. Also $A_i \not\subset B_j$ for $1 \le i, j \le n$, because every beginning point that lies in B_j crosses $s(A_j)$ and ϵ is small enough. Finally, if B_i is contained in B_j then $s(B_i)$ appears after $s(NEXT_t(A_j)) = s(B_j) + \epsilon$ because ϵ is small enough. Since $t(NEXT_t(A_j))$ appears after $t(B_j)$ then $NEXT_t(A_j)$ contains B_i which is a contradiction. Thus, $\mathcal{M}'$ is a proper model.

Now, we must see that $\mathcal{M}'$ is Helly and normal, and for this it is enough to show that there are no two nor three arcs covering the circle. If B_i together with a set of arcs cover the circle then $NEXT_t(A_i)$ with this set of arcs also cover the circle, because ϵ is small enough. Model $\mathcal{M}$ is Helly and normal, thus the smallest set of arcs covering the circle has size at least four and the same holds for $\mathcal{M}'$. Therefore, $\mathcal{M}'$ is a PHCA model.

Finally, we have to prove that $\mathcal{M}'$ has $\mathcal{M}$ as its clique model. Every arc $A_i \in \mathcal{N}$ is strong, because $\mathcal{M}'$ is PHCA and the first ending point that appears from $s(A_i)$ is $t(B_i)$. Also, the next ending point that appears from $s(B_i)$ is the beginning point of either $NEXT_t(A_i)$ or some B_j for $1 \le j \le n$, thus B_i is not strong. Last, the extreme that appears after $s(A_i)$ is not changed for $A_i \in \mathcal{Q}$, so it must be an ending point. Consequently, $\mathcal{M}$ is the submodel of $\mathcal{M}'$ induced by the strong arcs and the result follows from Proposition 2. □

Theorem 7. *Let H be a graph and U the set of universal vertices of H. Then $H = K(G)$ for some HCA graph G if and only if:*

(i) *H is a PHCA graph or*
(ii) *$H \setminus U$ is a co-bipartite PHCA graph and $|U| \ge 2$.*

Proof. Fix an HCA graph G and one of its HCA models $\mathcal{M}$. As in Theorem 6, H is PCA and contains no 3-suns as induced subgraphs. By condition (iii) of Lemma 5, the center of every induced 4-wheel of H is universal, thus $H \setminus U$ has no 4-wheels as induced subgraphs and consequently $H \setminus U$ is PHCA, by Theorem 3. Then if $U = \emptyset$, it follows that H is a PHCA graph. Otherwise, by conditions (i) and (ii) of Lemma 5, two arcs U_1, U_2 of $\mathcal{M}$ cover the circle and H has at least two universal vertices. Then, every clique point of $\mathcal{M}$ is crossed by at least one of U_1, U_2, so the cliques of G can be partitioned into families Q_1 and Q_2 such that all the cliques of Q_i contain the vertex of G that corresponds to U_i, for $i \in \{1, 2\}$. In other words, the set of vertices of H corresponding to Q_i is a complete set for $i \in \{1, 2\}$, thus condition (ii) of this theorem holds.

For the converse, if H is PHCA then the result follows from Theorem 6. Suppose then that $H \setminus U$ is a co-bipartite PHCA graph and that $|U| \ge 2$. Let V_1, V_2 be a co-bipartition of $H \setminus U$ and $\mathcal{M}_H$ be a PHCA model of $H \setminus U$. Each V_i is a complete set, therefore $\mathcal{M}_H$ has one point p_i which is crossed by every arc of V_i for $i \in \{1, 2\}$. Also, since $H \setminus U$ has no universal vertices, then no arc crosses both points p_1 and p_2. Let $A_1, \dots, A_n$ be a circular ordering of the arcs in $\mathcal{M}_H$ where, w.l.o.g., p_2 lies in the segment (s_n, s_1) and that p_1 lies in (s_x, s_{x+1}) for some $1 \le x \le n$. Note that $A_1, \dots, A_x$ are the arcs corresponding to V_1 and $A_{x+1}, \dots, A_n$ are the arcs corresponding to V_2.

For each $A_i \in \mathcal{A}$ define the arc $B_i = (s(A_i) - \epsilon, s(A_i) + \epsilon)$ for some small enough ϵ. Let $\mathcal{M}_1$ be the model obtained from $\mathcal{M}_H$ by adding every B_i for $A_i \in \mathcal{A}$. Clearly $\mathcal{M}_1$ is Helly and each of its intersection segments is of the form $(s(A_i), t(B_i))$. Also, every B_i contains only one intersection segment which is $(s(A_i), t(B_i))$. Then, it follows that every intersection point is a clique point, and the arc reduction of $p_i \in (s(A_i), t(B_i))$ is A_i. In other words, $\mathcal{M}_H$ is the clique model of $\mathcal{M}_1$. Therefore, by Theorem 1, the intersection graph G_1 of $\mathcal{M}_1$ is an HCA graph whose clique graph is $H \setminus U$.

Let $U_1 = (s(A_1) - 3\epsilon, s(A_{x+1}) - 2\epsilon)$ and $U_2 = (s(A_{x+1}) - 3\epsilon, s(A_1) - 2\epsilon)$ and define $\mathcal{M}_2$ as the model obtained from $\mathcal{M}_1$ by adding U_1 and U_2. Suppose, to obtain a contradiction, that $\mathcal{M}_2$ is not Helly. By definition, U_i, A_j does not cover the circle for $i \in \{1, 2\}$ and $1 \le j \le n$. Then neither U_i and B_j can cover the circle. The only pair of arcs that cover the circle is formed by U_1 and U_2 and therefore the non-Helly cliques have exactly three arcs. If one of these arcs is B_i then we can exchange it for A_i, for $1 \le i \le n$. Hence, we assume w.l.o.g. that one of the arcs is U_1 and the other two arcs are A_i, A_j, where $s(A_i) \in U_1$ and $t(A_j) \in U_1$. No arc in $A_1, \dots, A_x$ crosses $s(A_1) - 4\epsilon$, no arc in $A_{x+1}, \dots, A_n$ crosses $s(A_{x+1})$ and U_1 crosses neither $s(A_1) - 4\epsilon$ nor $s(A_{x+1})$, therefore $1 \le i \le x$ and $x + 1 \le j \le n$. Since $t(A_j) \in U_1$, then A_j crosses the beginning point of U_1 which is $s(A_1) - 3\epsilon$. Thus, since ϵ is small enough, A_j intersects A_1. But then A_1, A_i, A_j cover the circle, which is a contradiction to the fact that $\mathcal{M}_H$ is PHCA. Therefore, it follows that the intersection graph G_2 of $\mathcal{M}_2$ is HCA.

Let Q_1, Q_2 be two cliques of G_1, corresponding to vertices v_1, v_2 of $H \setminus U$. In $\mathcal{M}_2$, clique points of $\mathcal{M}_1$ are still clique points, because in $\mathcal{M}_2$ each B_i contains only the intersection segment $(s(A_i), t(B_i))$, so Q_1 and Q_2 are also cliques of G_2. Even more, since U_i crosses only clique points corresponding to vertices of V_i ($i \in \{1, 2\}$) then Q_1 and Q_2 intersect if and only if they intersect in G_1. That is, Q_1 and Q_2 intersect in G_2 if and only if v_1 is adjacent to v_2 in $H \setminus U$, thus $H \setminus U$ is an induced subgraph of $K(G_2)$. Now, $\mathcal{M}_2$ has at most two more clique points than $\mathcal{M}_1$, because the inclusion of U_1 and U_2 has only two more intersection segments, $(s(U_1), t(U_2))$ and $(s(U_2), t(U_1))$. On the other hand, there is at least one more clique point in $\mathcal{M}_2$

because $\{U_1, U_2\}$ is contained in one universal clique and G_1 has no universal cliques. Hence, G_2 is isomorphic to $H \setminus U$ plus one or two universal vertices. Adding $|U| - 1$ or $|U| - 2$ pairwise disjoint arcs into $(s(U_1), t(U_2))$ into $\mathcal{M}_2$, an HCA model of a graph G is obtained, where $K(G) = H$. □

Corollary 8. *Let H be a graph. Then $H = K(G)$ for some NHCA graph G if and only if H is a PHCA graph.*

Proof. By Theorem 1, the clique model of any NHCA model of G is a PHCA model of $K(G)$. The converse follows from Theorem 6. □

The check of whether a graph is PHCA takes linear time [17]. Thus, the recognition problem for graphs in $K(PHCA)$ can be solved in linear time. For the recognition of graphs in $K(HCA)$ we need to take into account the universal arcs. Let H be a graph that is candidate to be in $K(HCA)$. If H is not a PCA graph, then it is not in $K(HCA)$ [20]. Otherwise, a PCA model can be found using the algorithms in [26,15] in $O(n + m)$ time and, if needed, it can be normalized in $O(n)$ time as in [27]. In this model, universal vertices correspond to arcs containing exactly $n - 1$ extremes of other arcs. Thus, the removal of the universal arcs can be done in $O(n)$ time, while counting how many universal arcs are there. If there is only one universal arc then H is not in $K(HCA)$, otherwise let $\mathcal{M}$ be the resulting model. Then $H \in K(HCA)$ if and only if $H \setminus U$ is PHCA and either $U = \emptyset$ or $H \setminus U$ co-bipartite. Testing if $H \setminus U$ is co-bipartite can be done in $O(n)$ time from $\mathcal{M}$, e.g. as in [28], and testing if $H \setminus U$ is PHCA is done in $O(n)$ time [17]. To sum up, the recognition of graphs in $K(HCA)$ can also be done in linear time.

Theorems 6 and 7 also give procedures to compute an inverse graph with respect to operator K. That is, given a graph H in $K(HCA)$ ($K(PHCA)$), find a graph G such that $K(G) = H$. We omit the details here but it is not hard to see that both algorithms can be implemented to run in $O(n)$ time when the input is a model of H.

3. *K*-behavior of circular-arc graphs

In this section we develop an algorithm to find out to which graph does a circular-arc graph K-converge, when it does K-converge. Moreover, we describe how to determine the K-behavior of a general circular-arc graph. We employ the two theorems below. The first of them characterizes the dismantling of a circular-arc graph. The second theorem specifies exactly when does the dismantling K-converge.

Theorem 9 (*[11]*)**.** *Let G be a non-complete graph. Then the following are equivalent:*

(i) *G is isomorphic to C_n^k for some pair of values n, k.*
(ii) *G is a PCA graph without dominated vertices.*
(iii) *G has a unique PCA model with arcs $A_1, \ldots, A_n$ where $t(A_i)$ lies immediately after $s(A_{i+k})$.*
(iv) *G has a PCA model where every beginning point is followed by an ending point.*

Theorem 10 (*[29]*)**.** *Graph C_n^k is K-convergent if and only if it is complete or $n > 3k$.*

These two theorems can be used to actually decide the K-behavior of a general CA graph and also lead to a polynomial-time algorithm to determine the K-behavior of a circular-arc graph G. A circular-arc graph is K-null if its dismantling is K-null; it K-converges to a graph which is not K-null if its dismantling is C_n^k with $n > 3k$; or it K-diverges otherwise. However, much more can be said about the graph to which GK-converges when it does, because this graph is self-clique and thus unique. Denote by $G[H]$ the subgraph of G induced by H. We start with two useful propositions that are easy enough to prove.

Proposition 11. *For every graph G there exists a dismantling set that contains no properly dominated vertex of G.*

Proposition 12. *Let H be a dismantling set of a graph G. If G' is an induced subgraph of G that contains every vertex of H and the vertices of $G \setminus G'$ are dominated by vertices of G' then H is a dismantling set of G'.*

Next, we show that every CA graph that is neither K-null nor K-divergent must be HCA. For this we need to show how the dismantling of a circular-arc graph can be computed when a circular-arc model $\mathcal{M}$ is given. The *CA-dismantling algorithm* can be divided in two steps, first remove every arc A_i that is contained in some arc A_j to obtain a PCA submodel $\mathcal{M}'$ of $\mathcal{M}$. Clearly, every removed arc corresponds to a dominated vertex. Second, iteratively remove every arc whose beginning point is immediately followed by some other beginning point. Since $\mathcal{M}'$ is PCA, every such arc is dominated. At the end, every beginning point is followed by an ending point. If $\mathcal{M}'$ has some non-universal arc then the model so obtained has no dominated arcs by Theorem 9. Otherwise, the dismantling of $\mathcal{M}$ is any trivial model.

Theorem 13. *A circular-arc graph G is K-convergent to a non-trivial graph if and only if G is a non-interval NHCA graph.*

Proof. Suppose first that G is K-convergent to a non-trivial graph and let H be some of its dismantling sets. If $|H| = 1$ then G is K-null [30] which is impossible, so $|H| > 1$. Then H is not a complete set and therefore, by Theorem 9, $G[H]$ is isomorphic to C_n^k for some pair of values n, k. If $n \leq 3k$ then G is K-divergent by Theorem 10 which is also impossible, so $n > 3k$. Then G contains an induced cycle of length at least 4, that is, G is not an interval graph. Let $\mathcal{M}$ be a circular-arc model of G and call $\mathcal{M}_H$ to the submodel of $\mathcal{M}$ induced by the arcs corresponding to vertices of H. We can assume, w.l.o.g., that $\mathcal{M}_H$

was computed by the CA-dismantling algorithm. Call $N_1, \ldots, N_s$ to the arcs of G that were removed by the CA-dismantling algorithm, where N_i was removed after N_{i+1} for $1 \leq i \leq s-1$. We prove by induction on i that $\mathcal{M}_i = \mathcal{M}_H \cup \{N_1 \ldots, N_i\}$ is an NHCA graph.

For the base case $i = 0$, H is isomorphic to C_n^k with $n > 3k$ and so, by Theorem 9, $\mathcal{M}_H$ has no two nor three arcs that together cover the circle, i.e., $\mathcal{M}_H$ is normal and HCA. For the inductive case, observe that by construction N_i is either properly contained in some arc $A \in \mathcal{M}_{i-1}$; or $\mathcal{M}_i$ is proper and $s(N_i)$ is followed by the beginning point of some arc $A \in \mathcal{M}_{i-1}$. In either case, if N_i together with a subset of arcs $\mathcal{A}$ in $\mathcal{M}_i$ cover the circle, then also A covers the circle with $\mathcal{A}$. Consequently, by the inductive hypothesis, $|\mathcal{A} \cup \{N_i\}| \geq 4$, i.e., $\mathcal{M}_i$ is NHCA.

For the converse, assume that G is a non-interval NHCA graph. We employ induction to show that in every step of the dismantling process, the subgraph so far obtained contains a hole, that is, an induced cycle of length at least 4. Since G is NHCA and non-interval, G itself contains at least one hole. By the induction hypothesis the subgraph obtained after a certain number of removals of dominated vertices also contains a hole. Let $\mathcal{M}$ be the CA model corresponding to this subgraph, and let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be the set of arcs of some minimum hole, where A_i intersects A_{i-1} and A_{i+1} for $1 \leq i \leq k$. Examine the removal of the next dominated arc. If the removed arc is not one of $\mathcal{A}$ then the hole in $\mathcal{M}$ is preserved in the next step. Otherwise, some arc $A_i \in \mathcal{A}$ is either contained in an arc B_i or $\mathcal{M}$ is proper and $s(A_i)$ is followed by $s(B_i)$. In either case, B_i intersects A_{i-1} and A_{i+1} and since $\mathcal{A}$ induces a minimum hole, then B_i is adjacent to either none or all of the arcs in $\mathcal{A} \setminus \{A_{i-1}, A_{i+1}\}$. In the former case, $(\mathcal{A} \setminus \{A_i\}) \cup \{B_i\}$ induces a hole and the invariant holds. In the latter case, let A_j be the arc that is not contained in B_i, whose beginning point appears first in a traversal from $s(A_i)$. Then B_i, A_j, A_{j+1} must cover the circle which is impossible, because $\mathcal{M}$ is normal and Helly. Consequently, the dismantling of G contains a hole, meaning that it is neither a single vertex nor isomorphic to $C_{n,k}$ for $n \leq 3k$, i.e., G is K-convergent and not K-null. □

Now we proceed to prove that K-convergent circular-arc graphs K-converge to their dismantlings, which is the main theorem of this section.

Theorem 14. *If a circular-arc graph K-converges then it K-converges to its dismantling.*

Proof. If G is K-null then the dismantling of G is trivial [30] and so GK-converges to its dismantling. So, assume that G is not K-null which implies that G is NHCA by Theorem 13. We prove the theorem by induction on k where k is the minimum number such that $K^k(G) = K^{k+1}(G)$. This induction is well defined because K-periodic circular-arc graphs are self-clique [21].

In the base case, $K(G) = G$. By Theorem 1, the clique model $\mathcal{M}_Q$ of $\mathcal{M}$ is a PHCA model of $K(G)$, thus $\mathcal{M} = \mathcal{M}_Q$ is a PHCA model. Then, by Proposition 2, $\mathcal{M}_Q = \mathcal{M}$ has only strong arcs which implies that every beginning point of $\mathcal{M}$ is followed by an ending point. Hence, by Theorem 9, G contains no dominated arcs, i.e., $G[H] = G = K(G)$.

Now we proceed with the inductive case. Let H be a dismantling set of G that it must contain at least two vertices, because G is not K-null. Let $\mathcal{M} = (C, \mathcal{A} \cup \mathcal{N})$ be an NHCA model of G where $\mathcal{A}$ is the set of arcs corresponding to H. Without loss of generality, we may assume that if an arc A contains another arc B then there is some ending point between $s(A)$ and $s(B)$. We refer to this condition of $\mathcal{M}$ as the *s-ordering* condition. Also, by Proposition 11, we may assume that no vertex of H is properly dominated in G, hence if $A \in \mathcal{A}$ is dominated by $N \in \mathcal{N}$ then they must be twins. In the case that A and N are twins, assume that $s(N), s(A), t(N), t(A)$ appear in that order in $\mathcal{M}$.

Claim 1. *Every arc of $\mathcal{A}$ is strong. Let p be the first clique point of $\mathcal{M}$ that appears from $s(A)$ for $A \in \mathcal{A}$ and let B be the arc crossing p whose ending point is farthest from p. By Proposition 11 we have assumed that B does not properly dominate A in $\mathcal{M}$, then B crosses exactly the same clique points as A. If $A \neq B$ then B and A are twins, but we have also assumed that, in this case, $t(A)$ appears farther from p than $t(B)$, a contradiction. Then p is a strong clique point and A is its arc reduction. Therefore, every arc of $\mathcal{A}$ is strong and the claim is proved.*

Claim 2. *Every arc that belongs to $\mathcal{M}$ but not to $\mathcal{M}_Q$ is dominated by some strong arc in $\mathcal{M}$. If B belongs to $\mathcal{M}$ but not to $\mathcal{M}_Q$, it is because B is not an arc reduction. Then, either $s(B)$ is followed by some beginning point $s(B')$ in $\mathcal{M}$ or $(s(B), t)$ is an intersection segment which contains weak clique points, for some ending point t. In the former case, B does not contain B' by the s-ordering condition of $\mathcal{M}$, thus B is dominated by B'. In the latter case, there is some arc B' that crosses $s(B)$ and that reaches farther than $t(B)$, i.e., B is contained B'. If B' is strong, then the proof of the claim is complete. Otherwise, by applying this reasoning iteratively and using the fact that domination is a transitive relation, we obtain that B is dominated by some strong arc of $\mathcal{M}$.*

Claim 3. *Every arc reduction of a weak clique point of $\mathcal{M}$ is dominated by a strong arc of $\mathcal{M}$. Suppose that B is the arc reduction of the weak clique point p and let A be the arc of $\mathcal{M}$ that crosses p and reaches farthest. By definition, $B = (s, t(A))$ where s is the first beginning point that appears from p in a counter-clockwise traversal of C. Since p is weak, then A crosses s and so, by the s-ordering condition of $\mathcal{M}$, there is some ending point between $s(A)$ and s. Hence, A crosses the clique point q that appears first from p in a counter-clockwise traversal of C. By definition, the arc reduction of q has $t(A)$ as its ending point, so the arc reduction of q dominates B. If q is a strong clique point, then the claim is proved. Otherwise, as in Claim 2, we can apply this reasoning and use the fact that domination is transitive to obtain that B is dominated by some strong arc of $\mathcal{M}$.*

By Theorem 1, the clique model $\mathcal{M}_Q$ of $\mathcal{M}$ is a PHCA model of $K(G)$. By definition, the arcs of $\mathcal{M}_Q$ are precisely the arc reductions of $\mathcal{M}$, thus $\mathcal{M}_Q$ contains all the strong arcs of $\mathcal{M}$. By Claim 3, we can compute the dismantling of $\mathcal{M}_Q$ by first removing all the arcs that are weak arc reductions of $\mathcal{M}$ and then applying the dismantling procedure in the resulting model. In terms of vertices and graphs, the dismantling of $K(G)$ is isomorphic to the dismantling of $G[S]$ where $S \subseteq V(G)$ is the set of vertices that correspond to strong arcs of $\mathcal{M}$. Since $\mathcal{M}_Q$ contains all the strong arcs of $\mathcal{M}$ then, by Claim 1, $\mathcal{A}$ is a subset of arcs of $\mathcal{M}_Q$, i.e., H is both a subset of vertices of $K(G)$ and a subset of S. Finally, by Claim 2, the vertices of $G \setminus G[S]$ are all dominated by some vertex of S. Thus, by Proposition 12, H is a dismantling set of $G[S]$ and therefore it is also a dismantling set of $K(G)$. Hence, by the induction hypothesis, $K(G)K$-converges to $G[H]$ which concludes the proof. □

To end this section, we describe an implementation of the CA-dismantling algorithm which runs in $O(n)$ time. This implementation is rather similar to the one in [11] for computing the maximum independent set of a CA graph. The main difference is that our algorithm eliminates all the dominated arcs, while the algorithm of [11] eliminates all the dominating arcs. The input of our algorithm is some circular-arc model $\mathcal{M} = (C, \mathcal{A})$ and the output is an induced submodel of it. Recall that the algorithm is divided in two steps. The first one is the removal of included arcs and the second step is the removal of arcs whose beginning points are followed by another beginning point. For the first step of the algorithm, traverse twice the circle C from some beginning point $s(A)$, while maintaining the position of the farthest ending point t viewed so far. This farthest ending point is initialized to $t(A)$. When a beginning point $s(A_i)$ is reached, check if $t(A_i)$ reaches farther than t. If so, then set $t := t(A_i)$, otherwise, A_i is contained in the arc whose ending point is t, so we can remove it. Since the circle is traversed twice, then every contained arc is eventually removed and we obtain model $\mathcal{M}'$.

Call *s-sequence* to a maximal set of consecutive beginning points. For the second step of the algorithm, first initialize a set $\mathcal{S}$ containing each non-singleton s-sequence. Each s-sequence can be represented by its first and last beginning points. Now, choose some s-sequence $s_1, \dots, s_k$ of $\mathcal{S}$. Beginning point s_1 is followed by the beginning point s_2, thus we need to remove $A = (s_1, t(A))$ from $\mathcal{M}'$, and update (s_1, s_k) in $\mathcal{S}$ to (s_2, s_k). Let e_1 and e_2 be the previous and next extremes of $t(A)$ in $\mathcal{M}'$. If e_1 or e_2 is not a beginning point, then every non-singleton s-sequence is contained in $\mathcal{S}$. Otherwise, we may have to remove the s-sequences S_1 containing e_1 and S_2 containing e_2 from $\mathcal{S}$, and insert the non-singleton s-sequence $S_1 \cup S_2$. These can all be done in $O(1)$ if two references are maintained in the first and last beginning points of each s-sequence. The first reference of each beginning point informs the position of its s-sequence in $\mathcal{S}$. The second reference links these first and last beginning points of the s-sequence. Hence, the whole algorithm can be implemented in $O(n)$ time.

4. Conclusions

We have considered two basic problems on clique graphs, for the class of circular-arc graphs. First, we have characterized clique graphs of Helly circular-arc graphs and proper Helly circular-arc graphs. The clique graphs of Helly circular-arc graphs were shown to be the graphs obtained from proper Helly circular-arc graphs by the addition of 0 or at least 2 universal vertices. On the other hand, the proper Helly circular-arc graphs were shown to form a fixed class, under the clique graph operator. Next, we have described a method for deciding the K-behavior of a general circular-arc graph G, that is, to determine if G is K-null, K-convergent or K-divergent. In addition, we have proved that K-convergent circular-arc graphs always K-converge to their dismantlings. Furthermore, we have proved that a K-convergent circular-arc graph either is K-null or it is a non interval normal Helly circular-arc graph.

Besides the structural results above described, the methods also lead to fast algorithms for the corresponding recognition problems. The characterizations of the clique graphs of Helly circular-arc graphs and proper Helly circular-arc graphs imply linear time algorithms for recognizing the graphs of these classes. The analysis of the K-behavior of a general circular-arc graph also leads to a linear time algorithm for deciding if a given circular-arc graph is K-null, K-convergent or K-divergent. The time complexity of the mentioned algorithms reduce to $O(n)$, if a circular-arc model of the graph is given as input. We have also described an $O(n)$ time algorithm for constructing the dismantling of a general circular-arc graph G, given a circular-arc model of it. This algorithm was employed for finding the graph to which GK-converges, if so. We remark that the algorithms for finding the dismantlings of general graphs require $O(nm)$ time [31,32] or $O(n^3/\log n)$ time [33]. Finally, we mention that the methods also imply a linear time algorithm for recognizing normal Helly circular-arc graphs, because G is NHCA if and only if G is an interval graph or the dismantling of G is isomorphic to C_n^k for $n > 3k$.

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