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The rank associated to a projective curve

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A B S T R A C T

We define the C-rank associated to a projective curve and describe the strata of points having constant rank.

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1. Introduction

Let $Y \subset \mathbb{P}^n$ be a nondegenerate variety. Let $x \in \mathbb{P}^n$ be a point, we define the Y-rank of x as the smallest number r such that x is in the linear span of r points in Y. The definition of Y-rank is based on the definition of rank of matrices, since the rank of a matrix $A \in \mathbb{C}^{m\times n}$ is the smallest number r such that A can be written as a sum of r rank one matrices. If we let $Y \subset \mathbb{P}(\mathbb{C}^{m\times n})$ be the variety of rank one matrices, the rank of a matrix $A \in \mathbb{C}^{m\times n}$ is the Y-rank of $[A] \in \mathbb{P}(\mathbb{C}^{m\times n})$.

If $\mathbb{P}^n = \mathbb{P}(V_1 \otimes \cdots \otimes V_r)$ is the projective space associated to the tensor product of r vector spaces and Y is the variety of decomposable tensors, the Y-rank is called the tensor rank. If $\mathbb{P}^n = \mathbb{P}(S_{m,d})$ is the projective space associated to the vector space of degree d homogeneous forms in m variables, and Y is the locus of forms that are the dth power of a linear form the Y-rank of a form is simply called the rank of the form. We refer to the survey [10] and the references in it (for example [3,1]) for these and other examples of Y-rank.

In this article we will consider as the variety Y a nonsingular and nondegenerate curve $C \subset \mathbb{P}^n$, and the C-rank will be denoted rank.

The rank induced by a rational normal curve of degree d is related to the rank of binary forms. This interpretation comes from the fact that we can consider the rational normal curve in $\mathbb{P}(S_{2,d})$ as the locus of binary forms that are the dth power of a linear form. The rank of a degree n binary form P is the smallest number r such that P can be written as

$$P = L_1^d + \cdots + L_r^d,$$

with L_i linear forms. In [4] we give a description of all the strata of binary forms having constant rank and give a simple algorithm to calculate the rank of a given form.

It is clear from the definitions that a point in a secant line to C (and not in C) has rank two. But a point in a tangent line will have in general greater rank, although it is a limit of rank two points. This behaviour is already present in the case of the rational normal curve. For example the form $P = X^{d-1}Y$ is a limit of rank two forms, since it belongs to the tangent line of the rational normal curve in the point X^d. But the rank of P is d. In fact, the only forms having rank d are those on tangent lines to the rational normal curve, that is, forms that can be written as $P = L_1^{d-1}L_2$ with L_1 and L_2 different (up to scalar) linear forms. In this article we will show that this behaviour occurs also for certain immersions of curves in projective space. For example, if C is immersed in $\mathbb{P}^n$ by the complete linear system associated to a degree d divisor ($d \geq 4g+1$), then a point in a tangent line to C (different from the tangency point) has rank $n-g$, although it is a limit of rank 2 points (Theorem 2). If

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$d \geq 10g + 1$ then the only points having rank $n - g$ are those on tangent lines and not in C (Theorem 1). So if $d \geq 10g + 1$ we have that the closure of rank $n - g$ points also includes all rank one points, and the closure of rank two points also includes all rank $n - g$ points (and therefore all rank one points).

We also define the border rank of a point $x \in \mathbb{P}^n$ as the least integer r such that x is in the linear space spanned by r points in C or a limit of those. The set of points having border rank less than r is the r-secant variety to C, $\mathrm{Sec}^r(C)$, that is, the closure of the union of all $\mathbb{P}^{r-1}$'s spanned by r points in C. Therefore the border rank of a point x is the least integer r such that $x \in \mathrm{Sec}^r(C)$. Clearly the border rank of a point is less than or equal to its rank. In order to study the rank function associated to the curve C we will compute the rank of points whose border rank is r but its actual rank is greater.

Let C_r denote the locus of rank r points in $\mathbb{P}^n$. We will prove the following theorems:

Theorem 1. *Let $C \subset \mathbb{P}^n$ be a curve immersed by the complete linear system associated to a degree d line bundle L. Let $r \geq 2$ such that $d \geq 2r + 10g - 3$. Then*

1. $\overline{C}_r \setminus \overline{C}_{r-1} = C_r \cup C_{n-g-r+2}$.
2. $C_r = \overline{C}_r \setminus \overline{C}_{n-g-r+2}$.
3. $C_{n-g-r+2} = \overline{C}_{n-g-r+2} \setminus \overline{C}_{r-1}$.

Theorem 2. *Let $C \subset \mathbb{P}^n$ be a curve immersed by the complete linear system associated to a degree d line bundle L. Let $r \geq 2$ such that $d \geq 2r + 4g - 3$. Then*

1. $\overline{C}_r \setminus \overline{C}_{r-1} \subset C_r \cup C_{n-g-r+2}$.
2. $C_r = \overline{C}_r \setminus \overline{\overline{C}_r \cap C_{n-g-r+2}}$.
3. $\overline{\overline{C}_r \cap C_{n-g-r+2}} \setminus \overline{C}_{r-1} \subset C_{n-g-r+2}$.

Theorem 3. *Let $C \subset \mathbb{P}^n$ be a curve immersed by the complete linear system associated to a degree d line bundle L, with $d \geq 10g - 1$. Then $n - g$ is the least integer r such that every point $\mathbb{P}(H^0(L)^*)$ has rank less than or equal to r.*

The first statement of Theorem 1 shows that the border rank of a point x is r if and only if its rank is r or $n - g - r + 2$. On the other hand, in Theorem 2 we cannot assure that a rank $n - g - r + 2$ point has border rank r. For example let $C \subset \mathbb{P}^4$ be an elliptic normal curve of degree 5, that is, $g = 1$, $n = 4$ and $d = 5$. We will show that every point in $\mathbb{P}^4$ has rank less than or equal to 3. If we have $\overline{C}_2 \setminus \overline{C}_1 = C_2 \cup C_3$ that would mean that $\overline{C}_2 = \overline{C}_1 \cup C_2 \cup C_3$. Since $C = C_1 = \overline{C}_1$, then $\overline{C}_2 = \mathbb{P}^4$. But the closure of rank 2 points is the closure of the union of secant lines to the curve, that is, the secant variety of C, which is a threefold. Therefore there are rank three points whose border rank is not two.

If $r \geq 2$ is such that $2r \leq d - 10g + 3$, combining Theorem 1 with Theorem 3 we get

$$\overline{C}_r = \left(\bigcup_{k \leq r} C_k \right) \cup \left(\bigcup_{k \geq n-g-r+2} C_k \right)$$

$$\overline{C}_{n-g-r+2} = \left(\bigcup_{k \leq r-1} C_k \right) \cup \left(\bigcup_{k \geq n-g-r+2} C_k \right).$$

So we get a nice stratification for the lowest and highest values of the rank function.

Theorem 1 generalizes the results obtained in [4] for a rational normal curve.

In the case of elliptic normal curves, that is, genus one curves immersed by a complete linear system of degree $n + 1$, we improve Theorem 1 in the following way:

Theorem 4. *Let $C \subset \mathbb{P}^n$ be an elliptic normal of degree $n + 1$. Let $r \geq 2$ such that $n + 1 \geq 2r + 4$. Then*

1. $\overline{C}_r \setminus \overline{C}_{r-1} = C_r \cup C_{n-r+1}$.
2. $C_r = \overline{C}_r \setminus \overline{C}_{n-r+1}$.
3. $C_{n-r+1} = \overline{C}_{n-r+1} \setminus \overline{C}_{r-1}$.

This article is structured as follows. In Section 2 we show general bounds for the rank function. In Section 3 we study linear spaces spanned by effective divisors, and the multiplication map induced by a decomposition of a line bundle L as a tensor product $L_1 \otimes L_2$. In Section 4 we prove the main theorems. In Section 5 we state the main result in [4] and give a proof of it. Finally in Section 6 we give a detailed description of the rank function associated to an elliptic normal curve.

2. Bounds for the rank

We will work over the field $\mathbb{C}$ of complex numbers. A curve will be assumed reduced, irreducible and nonsingular.

Definition 5. Let $C \subset \mathbb{P}^n$ be a nondegenerate and nonsingular curve. Let $x \in \mathbb{P}^n$ be a point, we define the C-rank of x (and note it rank x) as the smallest number r such that x is in the linear span of r points in C. We define the C-border rank of x (and note it $\underline{\mathrm{rank}}$ x) as the smallest number r such that x is in the linear span of r points in C or in a limit of those.

Since C is nondegenerate we can find $n+1$ points in C spanning the whole ambient space. Therefore every point $x \in \mathbb{P}^n$ lies in the span of these $n+1$ points, and so rank $x \leq n+1$. The following proposition (which is a well known fact and can be generalized for any nondegenerate variety, see [11, Proposition 5.1]) sharpens the bound.

Proposition 6. *Let $C \subset \mathbb{P}^n$ be a nondegenerate curve. Then* $\operatorname{rank}(x) \leq n$ *for all* $x \in \mathbb{P}^n$.

Proof. Let $x \in \mathbb{P}^n \setminus C$. Let $V_x = \{H \subset \mathbb{P}^n : x \in H\}$ be the linear system of hyperplanes containing x. The linear system V_x has no base points since the intersection of all members in V_x is the set $\{x\}$. Therefore by Bertini's Theorem the intersection of a general member of V_x and C is nonsingular, that is, the general member of V_x cuts C in d different points (where d is the degree of C). This shows that we can find a hyperplane H containing x and spanned by d points in C. We can choose n of these points that still generate H, and so rank $x \leq n$. □

As a corollary we have:

Corollary 7. *Let $C \subset \mathbb{P}^2$ be a curve that is not a line. Then the rank function associated to C is*

$$r(x) = \begin{cases} 1 & \text{if } x \in C \\ 2 & \text{if } x \notin C. \end{cases}$$

Let $\operatorname{Sec}^r(C)$ be the r-secant variety of C, defined as the closure of the union all $(r-1)$-planes spanned by r points in C,

$$\operatorname{Sec}^r(C) = \overline{\bigcup \langle p_1, \ldots, p_r \rangle}.$$

It is a known fact that $\operatorname{Sec}^r(C)$ is an irreducible variety of dimension $\min\{2r-1, n\}$. Clearly, if $2r-2 \leq n$, the variety $\operatorname{Sec}^r(C)$ contains all points of rank less than or equal to r. Therefore $\lceil \frac{n+1}{2} \rceil$ is an upper bound for the border rank. A point x has border rank less than or equal to r if and only if $x \in \operatorname{Sec}^r(C)$. Therefore the set of points having border rank r is $\operatorname{Sec}^r(C) \setminus \operatorname{Sec}^{r-1}(C)$. We will denote this set $(\operatorname{Sec}^r)^\circ(C)$.

3. Linear spaces spanned by divisor of a curve

We will establish some properties of linear spaces spanned by divisors of a curve $C \subset \mathbb{P}^n$. If D is an effective divisor on C, we will consider the linear span of D, $\langle D \rangle$ defined as as the intersection of all hyperplanes in $\mathbb{P}^n$ whose intersection with H contains D. It is clear that if D consists of r different points, then the linear span of D is the usual linear span of the points. If $C \subset \mathbb{P}^n$ is nonsingular, then for $p \in C$, the linear space $\langle 2p \rangle$ is the tangent line to C at p. If C is immersed in $\mathbb{P}^n$ by the complete linear system associated to a line bundle L, the linear system of hyperplanes in $\mathbb{P}^n = \mathbb{P}(H^0(L)^*)$ is $H^0(L)$ and the linear system of hyperplanes such that its intersection with C contains D can be identified with $H^0(L(-D))$. Notice that a point $x \in \operatorname{Sec}^r(C)$ lies on $\langle D \rangle$ for a degree r divisor D.

The following lemma summarizes the main results concerning the linear span of a divisor.

Lemma 8. *Let $C \subset \mathbb{P}(H^0(L)^*)$ be a nonsingular and nondegenerate curve, where L is a line bundle of degree d, and let D, D_1, D_2 be effective divisors on C. Then*

1. $\dim\langle D \rangle = h^0(L) - h^0(L(-D))$. *In particular, if* $\deg D \leq d+1-2g$, *we have* $\dim\langle D \rangle = \deg D - 1$, *that is,* $\langle D \rangle$ *has the expected dimension.*
2. *The linear span of $\langle D_1 \rangle$ and $\langle D_2 \rangle$ is $\langle D_1 + D_2 - D_1 \cap D_2 \rangle$.*
3. *If $\deg(D_1 + D_2 - D_1 \cap D_2) \leq d+1-2g$, then $\langle D_1 \rangle \cap \langle D_2 \rangle = \langle D_1 \cap D_2 \rangle$.*

Proof. 1. The first statement is obvious from the definition of $\langle D \rangle$. The second statement follows by Riemman–Roch.
2. The intersection of an hyperplane with C contains D_1 and D_2 if and only if it contains the divisor $D_1 + D_2 - D_1 \cap D_2$.
3. Clearly $\langle D_1 \cap D_2 \rangle \subset \langle D_1 \rangle \cap \langle D_2 \rangle$. The other inclusion follows using (i), (ii) and counting dimensions. □

Let $C \subset \mathbb{P}(H^0(L)^*)$ for a degree d line bundle L and let $r \geq 2$ such that $2r \leq d+1-2g$. Let us assume that there are two distinct degree r effective divisors D_1 and D_2 such that $x \in \langle D_1 \rangle \cap \langle D_2 \rangle$, then $x \in \langle D_1 \cap D_2 \rangle$, that is, $x \in \operatorname{Sec}^{r-1}(C)$. Therefore for a point $x \in (\operatorname{Sec}^r)^\circ(C)$ there is a unique degree r divisor D such that $x \in \langle D \rangle$. So if $x \in (\operatorname{Sec}^r)^\circ(C)$, and $x \in \langle D \rangle$ with D a degree r divisor with multiple points, we cannot have rank $x \leq r$. Thus it makes sense to study the set of points belonging to linear spaces spanned by divisors with multiple points.

Definition 9. Let $C \subset \mathbb{P}^n$ be a nonsingular curve. Let $r \geq 2$. We define the variety $\operatorname{Sec}^{r,2}(C)$ as

$$\operatorname{Sec}^{r,2}(C) = \overline{\bigcup \langle T_p(C), p_1, \ldots, p_{r-2} \rangle}$$

where $T_p(C)$ is the tangent line to C at p, and $p, p_1, \ldots, p_{r-2}$ runs over all subset of $r-1$ points in C such that

$$\dim\langle T_p(C), p_1, \ldots, p_{r-2} \rangle = r-1.$$

Notice that a linear space of the form $\langle T_p(C), p_1, \ldots, p_{r-2} \rangle$ can also be written as $\langle 2p + p_1 + \cdots + p_{r-2} \rangle$.

Let us consider the correspondence

$$\Sigma = \overline{\{(x, p, p_1, \dots, p_{r-2}, x) \in \mathbb{P}^n \times C^{r-1} : \ x \in \langle 2p + p_1 + \cdots + p_{r-2} \rangle\}}.$$

For general $p, p_1, \dots, p_{r-2}$ we have that $\langle 2p+p_1+\cdots+p_{r-2}\rangle$ is a linear variety of dimension $r-1$. Therefore we have that the general fiber of the second projection has dimension $r-1$. Thus Σ is an irreducible variety of dimension $2r-2$. Since the image of the first projection is the variety $\mathrm{Sec}^{r,2}(C)$, we must have that $\mathrm{Sec}^{r,2}(C)$ is an irreducible variety of dimension less than or equal to $2r-2$. Since $\mathrm{Sec}^{r-1}(C) \subsetneq \mathrm{Sec}^{r,2}(C) \subsetneq \mathrm{Sec}^{r}(C)$ we must have $\dim \mathrm{Sec}^{r,2}(C) = 2r-2$.

From now on we will assume that C is immersed as a nonsingular curve in $\mathbb{P}^n$ by the complete linear system associated to a line bundle L of degree $d \geq 2g+1$. In particular we have $d = n + g$ and $\mathbb{P}^n = \mathbb{P}(H^0(L)^*)$. We will denote $\mathcal{L}$ the linear system of effective divisors associated to a line bundle L.

Suppose that $L = L_1 \otimes L_2$, with L_1 and L_2 line bundles such that $h^0(L_i) \neq 0$ for $i = 1, 2$. In that case we have a multiplication morphism

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu} H^0(L).$$

If we let $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_s\}$ be a basis for $H^0(L_1)$ and $H^0(L_2)$ respectively, we define the matrix of the multiplication morphism μ as the $r \times s$ matrix with entries $\mu(e_i \otimes f_j)$. Notice that its entries are linear forms in $H^0(L)^*$. If $D_i \in \mathcal{L}_i$ then the divisor $D_1 + D_2 \in \mathcal{L}$ is the divisor of a hyperplane $H \subset \mathbb{P}(H^0(L)^*)$. We will abuse notation and call the corresponding effective divisor H.

Given a point $x \in \mathbb{P}(H^0(L)^*)$ we consider the bilinear form

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu_x} \mathbb{C},$$

defined up to scalar multiple. We will denote $W_1(x) \subset \mathbb{P}(H^0(L_1))$ the left-kernel of μ_x (that no longer depends on the choice of a representant of x). The condition for a divisor $D_1 \in \mathcal{L}_1$ to belong to $W_1(x)$ is that $x \in H$ for every hyperplane H such that $D_1 + D_2 = H$, and $D_2 \in \mathcal{L}_2$. In a similar way we define $W_2(x) \subset \mathbb{P}(H^0(L_2))$.

In order to study the C-rank we introduce the sets

$$\{x \in \mathbb{P}(H^0(L)^*) : \mathrm{rank}\ \mu_x \leq r\}.$$

Notice that the entries of the matrix of μ restricted to C are rational functions of the form e_if_j. Since the ring of rational functions on C is commutative, all two by two minors of this matrix vanish. This shows that $C \subset \{x \in \mathbb{P}(H^0(L)^*) : \mathrm{rank}\ \mu_x \leq 1\}$. In [5] it is shown that under certain circumstances the ideal of C is generated by the two by two minors of this matrix. On the other hand in [12] it is shown that

Theorem 10 (*Ravi*). *If* $\deg L_1, \deg L_2 \geq 2g + r$ *and* $\deg L \geq 4g + 2r + 1$ *then*

$$\mathrm{Sec}^r(C) = \{x \in \mathbb{P}(H^0(L)^*) : \ \mathrm{rank}(\mu_x) \leq r\}.$$

Therefore at least set theoretically the secant variety is determinantal.

The sets of points such the rank of μ_x is not the maximum are characterized by the following lemma.

Lemma 11. *Let $C \subset \mathbb{P}(H^0(L)^*)$ be a nonsingular and nondegenerate curve, where L is a line bundle of degree $d \geq 2g+1$. Let D_1, D_2 be effective divisors on C, such that $D_1 + D_2 \in \mathcal{L}$, and let L_1 and L_2 be the corresponding line bundles.*

1. *If $W_1(x) \neq 0$, and $D \in \mathcal{L}_1$ corresponds to a section $s \in W_1(x)$, then $x \in \langle D \rangle$.*
2. *If $h^0(L_1) \leq h^0(L_2)$ we get the following description:*

$$\{x \in \mathbb{P}(H^0(L)^*) : \ \mathrm{rank}(\mu_x) < h^0(L_1)\} = \bigcup_{D \in \mathcal{L}_1} \langle D \rangle.$$

Proof. A point $x \in \mathbb{P}(H^0(L)^*)$ corresponds to the linear system $V_x \subset H^0(L)$ of hyperplanes that contain x. If $s \in W_1(x)$, then $st \in V_x$ for all $t \in H^0(L_2)$. This means that for all $D_2 \in \mathcal{L}_2$, the divisor $D + D_2$ corresponds to a hyperplane containing x. But as D_2 runs through $\mathcal{L}_2$, $D + D_2$ runs through all the hyperplanes containing D. So $H^0(L(-D)) \subset V_x$, that is, $x \in \langle D \rangle$. This proves the first statement and also proves that

$$\{x \in \mathbb{P}(H^0(L)^*) : \ \mathrm{rank}(\mu_x) < h^0(L_1)\} \subset \bigcup_{D \in \mathcal{L}_1} \langle D \rangle.$$

Now given $x \in \langle D \rangle$ for $D \in \mathcal{L}_1$, we know that $H^0(L(-D)) \subset V_x$. Therefore for any $D_2 \in \mathcal{L}_2$, $D + D_2$ is a hyperplane containing x. If $s \in H^0(L_1)$ is the section corresponding to D, the previous statement means that $s \in W_1(x)$ and so rank $\mu_x < h^0(L_1)$. □

Given a decomposition $L = L_1 \otimes L_2$ and $x \in C$ such that $W_2(x)$ has base points, we can consider $W_2(x)$ as a base point free subspace of $H^0(L_2(-E))$, where E is the base locus of $W_2(x)$. Then we have

Lemma 12. *If the restriction of the multiplication morphism*

$$H^0(L_1) \otimes H^0(L_2(-E)) \xrightarrow{\mu} H^0(L(-E))$$

to $H^0(L_1) \otimes W_2(x)$ is surjective, then $x \subset \langle E \rangle$.

Proof. We identify $H^0(L(-E))$ with the subspace of $H^0(L)$ of hyperplane sections containing $\langle E\rangle$. We are stating that every section in $H^0(L(-E))$ can be written as a sum

$$s_1t_1 + \cdots + s_kt_k,$$

where $s_1, \ldots, s_k \in H^0(L_1)$ and $t_1, \ldots, t_k \in W_2(x)$. If $D_1, \ldots, D_k$ are the effective divisors associated to $t_1, \ldots, t_k$ respectively, then $\langle E\rangle \subset \langle D_i\rangle$ for each $1 \le i \le k$. Therefore the intersection of hyperplanes containing $\langle E\rangle$ contains x, that is, $x \in \langle E\rangle$. □

We will use Lemma 12 together with the following lemmas:

Lemma 13 ([12]). *Let $W \subset H^0(L_2)$ be a base point free subspace of codimension l. Then the multiplication morphism*

$$\mu : H^0(L_1) \otimes H^0(L_2) \to H^0(L)$$

restricted to $H^0(L_1) \otimes W$ is surjective if $\deg L_2 \ge 2g + 1$ and $\deg L_1 \ge 2g + l + 1$.

Lemma 14 ([5]). *Assume L_2 is a base point free line bundle and $\deg L_2 \ge 2g$. If $\deg L_1 + \deg L_2 \ge 4g+1$ or $\deg L_1 + \deg L_2 = 4g$ and $L_1^{-1} \otimes L_2 \ne \mathcal{O}_C, \omega_C$, or, if C is hyperelliptic, any other multiple of the g^1_2, then the multiplication morphism*

$$\mu : H^0(L_1) \otimes H^0(L_2) \to H^0(L)$$

is surjective.

Lemma 15 ([7]). *Let $W \subset H^0(L_2)$ be a base point free linear system. If*

$$h^1(L_1 \otimes L_2^{-1}) \le \dim W - 2.$$

then the multiplication morphism

$$H^0(L_1) \otimes W \to H^0(L)$$

is surjective.

Finally we recall the definition of the the discriminant variety $\Delta_L \subset H^0(L)$ for a line bundle L in C.

Definition 16. Let L be a line bundle in a curve C. We define the discriminant variety $\Delta_L \subset H^0(L)$ as the set of sections s such that its associated effective divisor D has multiple points. If L is a very ample bundle then Δ_L is an irreducible variety that does not contain hyperplanes and is called dual variety (the corresponding projective variety $\mathbb{P}(\Delta_L) \subset \mathbb{P}(H^0(L))$ is the dual variety of the immersion of C in $\mathbb{P}(H^0(L)^*)$).

4. Proofs of the main theorems

We know that if $2r-1 \le n$ then points having rank less than or equal to r belong to $\mathrm{Sec}^r(C)$. In order to give a description of the sets of points having constant rank we will calculate the rank of the points in $\mathrm{Sec}^r(C)$ having rank greater than r.

First we give a necessary and sufficient condition for a point x to have rank less than or equal to r.

Proposition 17. *Let $x \in \mathbb{P}(H^0(L)^*)$. Then $\mathrm{rank}(x) \le r$ if and only if there exists a decomposition $L = L_1 \otimes L_2$ with $\deg L_1 = r$, such that $W_1(x) \not\subset \Delta_{L_1}$.*

Proof. Let $x \in \mathbb{P}(H^0(L)^*)$ such that $\mathrm{rank}(x) \le r$. Then there are r points in C, $p_1, \ldots, p_r$ such that $x \in \langle p_1, \ldots, p_r\rangle$. Let D be the divisor $D = \sum_{i=1}^r p_i$, L_1 the line bundle associated to D and $L_2 = L_1^{-1} \otimes L$. Let $s \in H^0(L_1)$ be the section associated to D. Clearly we have $\mu_x(s \times H^0(L_2)) = 0$ and so $s \in W_1(x)$. On the other hand, as the p_i's are distinct, we have $s \notin \Delta_{L_1}$.

Now let L_1, L_2 be line bundles with $\deg L_1 = r$, $L_1 \otimes L_2 = L$ and $W_1(x) \not\subset \Delta_{L_1}$. Let $s \in W_1(x) \setminus \Delta_{L_1}$. Then the divisor associated to s has the form $p_1 + \cdots + p_r$, $p_i \ne p_j$ and $r = \deg(L_1)$. By Lemma 11 we have $x \in \langle p_1 + \cdots + p_r\rangle$, that is, $\mathrm{rank}(x) \le r$. □

In the following proposition we estimate the increase of the rank in a limit position.

Proposition 18. *Let $r \ge 2$ such that $\deg L \ge 2g + r$ and such that $2r - 2 \le n = \deg L - g$. Let $x \in \mathbb{P}(H^0(L)^*)$ be a point such that $x \in (\mathrm{Sec}^r)^\circ(C)$. Then $\mathrm{rank}(x) = r$ or $\mathrm{rank}(x) \ge n - g - r + 2$.*

Proof. First notice that as we have $2r - 2 \le n$, we must have $(\mathrm{Sec}^r)^\circ(C) \ne \emptyset$. Now, as $x \in \mathrm{Sec}^r(C)$, there is a divisor D of degree r such that $x \in \langle D\rangle$ and such that $\dim\langle D\rangle = r - 1$.

Suppose that $\mathrm{rank}(x) \ne r$. Then we must have $\mathrm{rank}(x) > r$, because all points having rank less than or equal to $r - 1$ belong to $\mathrm{Sec}^{r-1}(C)$. Therefore we must have multiple points in the support of D, and at most $r - 1$ distinct points.

Let $\mathrm{rank}(x) = k$ and let $E = q_1 + \cdots + q_k$ be an effective divisor such that $x \in \langle E\rangle$ with $\dim\langle E\rangle = k - 1$. We must have $\deg(D \cap E) \le r - 1$. As $\mathrm{rank}(x) > r$ we cannot have $x \in \langle D \cap E\rangle$. But as $x \in \langle D\rangle \cap \langle E\rangle$, this intersection cannot be spanned by the divisor $D \cap E$. Therefore by Lemma 8 we have

$$\deg D + \deg E - \deg(D \cap E) \ge \deg L + 2 - 2g = n + 2 - g,$$

that is,

$$r + k - \deg(D \cap E) \ge n + 2 - g.$$

Finally,

$$\operatorname{rank}(x) = k \geq n + 2 - g - r + \deg(D \cap E) \geq n - g - r + 2. \quad \square$$

Now we give an upper bound to the rank of points in $(\mathrm{Sec}^r)^\circ(C)$.

Proposition 19. *Let* $r \geq 2$ *such that* $\deg L \geq 2r + 4g - 3$ *(or* $\deg L \geq 2r - 2$ *if* $g = 0$*) and let* $x \in \mathbb{P}(H^0(L)^*)$ *be a point such that* $x \in (\mathrm{Sec}^r)^\circ(C)$. *Then* $\operatorname{rank}(x) \leq n - g - r + 2$.

Proof. Notice that if $g \geq 1$, we have $n \geq 2r$ and so $(\mathrm{Sec}^r)^\circ(C) \neq \emptyset$. On the other hand if $g = 0$, $n = \deg L \geq 2r - 2$ and we algo get $(\mathrm{Sec}^r)^\circ(C) \neq \emptyset$. As in the previous proposition let D be a degree r divisor such that $x \in \langle D \rangle$ and such that $\dim\langle D \rangle = r - 1$. Let F be the divisor $F = D + K$ where K is the canonical divisor, and let L_1 be the line bundle associated to F. Let $L_2 = L_1^{-1} \otimes L$.

First assume $g \geq 1$. Let $s \in H^0(L_D)$ be the section associated to D and let $\{t_1, \dots, t_g\}$ be a basis for $H^0(L_K)$. Then the subset $\{s \otimes t_1, \dots, s \otimes t_g\} \subset H^0(L_1)$ is linearly independent and is included in $W_1(x)$. As $h^0(L_1) = g + r - 1$, the codimension of $W_1(x)$ is less than or equal to $r - 1$. In particular, the rank of the bilinear form μ_x is also less than or equal to $r - 1$. If $g = 0$, then the divisor $D + K$ is a degree $r - 2$ divisor and $h^0(L_1) = r - 1$, so we also get rank $\mu_x \leq r - 1$.

First we assume that $r = 2$. Then the rank of μ_x is one, and therefore $W_2 = W_2(x)$ is a hyperplane in $H^0(L_2)$. We know that Δ_{L_2} does not contain hyperplanes, so $W_2 \not\subset \Delta_{L_2}$. This means that the rank of x is less than or equal to $\deg L_2 = n - g$.

Now assume that $r \geq 3$. If the rank of μ_x is less than $r - 1$, we can use Theorem 10 to show that $x \in \mathrm{Sec}^{r-2}(C)$.

So we must have rank $\mu_x = r - 1$. Suppose that $\operatorname{rank}(x) > n - g - r + 2$. Then, as the degree of L_2 is $n - g - r + 2$, we must have $W_2 \subset \Delta_{L_2}$ by the Proposition 17. By Bertini's Theorem W_2 must have base points and one of them must be a multiple point of all the elements of W_2. Let E be the base locus of W_2. We can consider W_2 as a base point free subspace of $H^0(L_2(-E))$. Therefore

$$\dim W_2 \leq h^0(L_2(-E)) \leq \deg(L_2) - \deg(E) + 1 - g.$$

This means that

$$\deg E \leq \deg L_2 + 1 - g - \dim W_2 = r - 1.$$

Then we have

$$2 \leq \deg E \leq r - 1.$$

We will show that the restriction of the multiplication map

$$H^0(L_1) \otimes H^0(L_2(-E)) \to H^0(L(-E))$$

to $H^0(L_1) \otimes W_2$ is surjective, and so, by Lemma 12, $x \in \langle E \rangle$. But as $\deg E \leq r - 1$ that would mean $x \in \mathrm{Sec}^{r-1}(C)$ which is a contradiction. Then we must have $\operatorname{rank}(x) \leq n - g - r + 2$.

In order to prove that

$$H^0(L_1) \otimes W_2 \to H^0(L(-E))$$

is surjective we first assume that $\deg E < r-1$. Let l be the codimension of W_2 in $H^0(L_2(-E))$. We have $1 \leq l \leq r-1-\deg E$, so we are in the hypothesis of Lemma 13 and therefore the restriction of μ is surjective.

If $\deg E = r - 1$, we have $W_2 = H^0(L_2(-E))$. Since $\deg L_1 + \deg L_2(-E) = \deg L - \deg E = 2r + 4g - 3 - (r - 1) = 4g + r - 2 \geq 4g + 1$, by Lemma 14, the multiplication map is surjective (here we use $r \geq 3$). $\square$

Now we are ready to prove Theorem 2, but first we state and prove an alternate version of it using secant varieties.

Theorem 20. *Let* $C \subset \mathbb{P}^n$ *be a curve, such that the immersion is induced by the complete linear system associated to a line bundle* L. *Let* $r \geq 2$ *such that* $\deg L \geq 2r + 4g - 3$. *Then*

1. $(\mathrm{Sec}^r)^\circ(C) \subset C_r \cup C_{n-g-r+2}$.
2. $C_r = \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r,2}(C)$.
3. $\mathrm{Sec}^{r,2}(C) \setminus \mathrm{Sec}^{r-1}(C) \subset C_{n-g-r+2}$.

Proof. The first statement is a direct consequence of Propositions 18 and 19.

Let $x \in C_r$, that is, $\operatorname{rank}(x) = r$. Then there is a degree r divisor D with no multiple points such that $x \in \langle D \rangle$. Therefore $x \in \mathrm{Sec}^r(C)$. We want to show that $x \notin \mathrm{Sec}^{r,2}(C)$. If $x \in \mathrm{Sec}^{r,2}(C)$, then there would be a degree r divisor E with multiple points such that $x \in \langle E \rangle$. Since $\deg D + \deg E \leq \deg L + 1 - 2g$, by Lemma 8 we must have $\langle D \rangle \cap \langle E \rangle = \langle D \cap E \rangle$, and so $x \in \langle D \cap E \rangle$. But as E has multiple points, the divisor $D \cap E$ would consist of at most $r - 1$ points, and so $\operatorname{rank}(x) \leq r - 1$ which is a contradiction.

Now let $x \in \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r,2}(C)$. Then there is a degree r divisor D such that $x \in \langle D \rangle$. As $x \notin \mathrm{Sec}^{r,2}(C)$, the divisor D has no multiple points. Therefore $\operatorname{rank}(x) \leq r$. As $\mathrm{Sec}^{r-1}(C) \subset \mathrm{Sec}^{r,2}(C)$ we cannot have $\operatorname{rank}(x) < r$.

For the third statement let x be a point in $\mathrm{Sec}^{r,2}(C) \setminus \mathrm{Sec}^{r-1}(C)$. As $x \in (\mathrm{Sec}^r)^\circ(C)$, we must have $\operatorname{rank}(x) = r$ or $\operatorname{rank}(x) = n - g - r + 2$. But we just proved that if $\operatorname{rank}(x) = r$, then $x \notin \mathrm{Sec}^{r,2}(C)$. Therefore we have $\operatorname{rank}(x) = n - g - r + 2$. $\square$

Now we prove Theorem 2 using Theorem 20:

Proof of Theorem 2. One has to notice that C_r is open in $\mathrm{Sec}^r(C)$, and therefore $\overline{C}_r = \mathrm{Sec}^r(C)$. Similarly $C_{n-g-r+2} \cap \mathrm{Sec}^r(C)$ is open in $\mathrm{Sec}^{r,2}(C)$, and thus $\overline{C_{n-g-r+2} \cap \overline{C}_r} = \mathrm{Sec}^{r,2}(C)$.

Form now on we assume that $g \geq 1$. See Section 5 for an improvement of Theorem 1 for a rational normal curve.

Next we show that points having large rank belong to small secant varieties. This proposition will let us prove Theorem 1 for curves of positive genus.

Proposition 21. *Let* $r \geq 1$ *such that* $\deg L \geq 2r + 10g - 3$ *and consider*

$$C^+_{n-g-r+2} = \{x \in \mathbb{P}(H^0(L)^*) : \text{ rank } x \geq n - g - r + 2\}.$$

Then

1. $C^+_{n-g+1} = \emptyset$*, that is, there are no points* $x \in \mathbb{P}(H^0(L)^*)$ *having rank greater than* $n - g$*.*
2. *If* $r \geq 2$*, then* $C^+_{n-g-r+2} \subset \mathrm{Sec}^r(C)$*.*

Proof. Let x be a point such that rank $x \geq n - g - r + 2$. We will show by a case by case analysis that $x \in \mathrm{Sec}^l(C)$ for l in a finite set of values.

Let L_1 and L_2 be line bundles of degree $2g+r-1$ and $\deg L - 2g - r + 1 = n - g - r + 1$ respectively such that $L = L_1 \otimes L_2$. Let μ_x be the bilinear form induced by the multiplication map. We have $1 \leq \text{rank } \mu_x \leq g + r = h^0(L_1)$.

1. If rank $\mu_x < g + r$, then there is a section $s \in H^0(L_1)$ such that $s \in W_1(x)$. Therefore by Lemma 11 we have that $x \in \langle D\rangle$ where D is the divisor associated to s. As D has degree $2g + r - 1$, we have that $x \in \mathrm{Sec}^{2g+r-1}(C)$. Then $2g + r - 1$ is the first value of l.
2. Now assume that rank $\mu_x = g + r$. Since rank $x > n - g - r + 1 = \deg L_2$, we have $W_2(x) \subset \Delta_{L_2}$. Therefore $W_2(x)$ has base points. Let E be the base locus of $W_2(x)$. For E we have $2 \leq \deg E \leq g + r$. We can consider $W_2(x)$ as a base point free subspace of $H^0(L_2(-E))$, of codimension $g + r - \deg(E)$.
   (a) If $\deg E = g + r$, then $W_2(x) = H^0(L_2(-E))$. In this case we are in the hypothesis of Lemma 14 and therefore
   $$H^0(L_1) \otimes W_2(x) \to H^0(L(-E))$$
   is surjective. By Lemma 12, $x \in \langle E\rangle$, and since $\deg E = g + r$, $x \in \mathrm{Sec}^{g+r}(C)$. We have that $l = g + r$ is the second value of l.
   (b) If $k = \deg E < g + r$ we consider the following cases
      i. If $k \geq g + 2$, then the codimension of $W_2(x)$ in $H^0(L_2(-E))$ is $r - 2$ (in this case we cannot have $r = 1$ nor $r = 2$). Therefore using Lemma 13 we have that the restriction of μ to $H^0(L_1) \otimes W_2$ is surjective. Then $x \in \langle E\rangle$, and since $\deg E \leq g + r - 1$, $x \in \mathrm{Sec}^{g+r-1}(C)$. We have that $l = g + r - 1$ is the third value of l.
      ii. Finally we consider the case $2 \leq k \leq g + 1$. Let us consider the projection morphism
      $$\pi_E : \mathbb{P}(H^0(L)^*) \setminus \langle E\rangle \to \mathbb{P}(H^0(L(-E))^*).$$
      The restriction of π_E to $C \setminus \langle E\rangle$ extends to all of C. Since the codimension of $W_2(x)$ in $H^0(L_2(-E))$ is $g+r-k < g+r$, the rank of
      $$\mu_x : H^0(L_1) \otimes H^0(L_2(-E)) \to \mathbb{C}$$
      is not maximal. This implies that there is divisor $F \in \mathcal{L}_1$ such that $\pi_E(x)$ lies in the linear span of F in $\mathbb{P}(H^0(L(-E))^*)$. Therefore x lies in the linear span of $F + E$ in $\mathbb{P}(H^0(L)^*)$. Since $\deg F + E = 2g + r - 1 + k \leq 3g + r$, we have that $x \in \mathrm{Sec}^{3g+r}(C)$, and $3g + r$ is the last value of l.

Let us prove the first statement of the proposition, so let us assume that $r = 1$. In this case the possibility 2 (b) i is not possible, so l can take the values $2g$, $g + 1$, and $3g + 1$. Then we can assume that $x \in \mathrm{Sec}^{3g+1}(C)$, and using Theorem 20 we can conclude that $1 \leq \text{rank } x \leq 3g + 1$ or $n - 4g + 1 \leq \text{rank } x \leq n - g$. Since we are assuming that rank $x > n - g$ this is a contradiction, and therefore there are no points having rank greater than $n - g$.

Now we will prove the second statement. If the value of l comes from 2 (b) i, then $r \geq 3$. If this is the case, and if $g = 1$, then $l = r$ and so $x \in \mathrm{Sec}^r(C)$ as we wanted. In any other case we must have $r < l$. Let us assume that $x \notin \mathrm{Sec}^r(C)$ and let j be the least integer such that $x \in \mathrm{Sec}^j(C)$. We have $r < j \leq l \leq 3g + r$. Since $x \in (\mathrm{Sec}^j)^\circ(C)$, by the Theorem 20 we have $\mathrm{rank}(x) = j$ or $\mathrm{rank}(x) = n - g - j + 2$. In the first case, since $\deg L > 2r - 2 + 5g$, we have

$$\text{rank } (x) = j \leq 3g + r < n - g - r + 2.$$

On the other hand, if $\mathrm{rank}(x) = n - g - j + 2$, since $r < j$, we have

$$\mathrm{rank}(x) = n - g - j + 2 < n - g - r + 2.$$

In either case we get a contradiction. Therefore $x \in \mathrm{Sec}^r(C)$. $\square$

We can now prove Theorem 3:

Proof of Theorem 3. The case $r = 1$ of the previous proposition shows that $n - g$ is an upper bound for the rank. To show that $n - g$ is the minimum bound we have to show that there are points having rank $n - g$. But a point lying on a tangent line to C has rank $n - g$ by Theorem 20.

Finally we give the following alternate form of Theorem 1:

Theorem 22. *Let $C \subset \mathbb{P}^n$ be a curve, such that the immersion is induced by the complete linear system associated to a line bundle L. Let $r \geq 2$ such that $\deg L \geq 2r + 10g - 3$. Then*

1. $(\mathrm{Sec}^r)^\circ(C) = C_r \cup C_{n-g-r+2}$.
2. $C_r = \mathrm{Sec}^r(C) \setminus \mathrm{Sec}^{r,2}(C)$.
3. $C_{n-g-r+2} = \mathrm{Sec}^{r,2}(C) \setminus \mathrm{Sec}^{r-1}(C)$.

Proof of Theorem 1. As we have done to prove Theorem 2, we only have to notice that C_r is an open set in $\mathrm{Sec}^r(C)$ and that $C_{n-g-r+2}$ is an open set in $\mathrm{Sec}^{r,2}(C)$.

Proof of Theorem 22. We only have to show that if rank $(x) = n-g-r+2$, then $x \in \mathrm{Sec}^{r,2}(C) \setminus \mathrm{Sec}^{r-1}(C)$. By Theorem 20 we have that $x \notin \mathrm{Sec}^{r-1}(C)$. On the other hand Proposition 21 shows that $x \in \mathrm{Sec}^r(C)$. Since $\mathrm{rank}(x) \neq r$, we must have $x \in \mathrm{Sec}^{r,2}(C)$.

5. Rational normal curves

In this section we state the main result of [4], and give a proof using the results of the previous section. We consider the degree d rational normal curve in $\mathbb{P}^d$, that is, the immersion of the projective line in $\mathbb{P}^d$ by the complete linear system of degree d divisors on $\mathbb{P}^1$. This application is the Veronese map, that in homogeneous coordinates is given by

$$\begin{array}{ccc} \mathbb{P}^1 & \to & \mathbb{P}^d \\ [t,u] & \mapsto & \left[t^d, t^{d-1}u, \dots, tu^{d-1}, u^d\right] \end{array}$$

As it was mentioned in the introduction, the rational normal curve can be defined as the locus, in the space of degree d binary forms, of forms that are the dth power of a linear form. So we can consider instead the rational normal curve as the image of the map

$$\begin{array}{ccc} \mathbb{P}(S_1) & \to & \mathbb{P}(S_d) \\ [L] & \mapsto & [L^d]. \end{array}$$

The main result in [4] is the following:

Theorem 23. *Let $\mathbb{P}(S_d)$ be the projective space of binary forms of degree d, $C \subset \mathbb{P}(S_d)$ the Veronese curve of dth powers of linear forms and for each $1 \leq r \leq d$, let $S_{d,r} \subset \mathbb{P}(S_d)$ be the projectivization of the set of degree d forms having rank r.*

1. *For each $r \geq 2$ such that $d \geq 2r - 1$ we have*

$$\begin{aligned} \overline{S}_{d,r} \setminus \overline{S}_{d,r-1} &= S_{d,r} \cup S_{d,d-r+2} \\ S_{d,r} &= \overline{S}_{d,r} \setminus \overline{S}_{d,d-r+2} \\ S_{d,d-r+2} &= \overline{S}_{d,d-r+2} \setminus \overline{S}_{d,r-1} \end{aligned}$$

2. *If $d = 2k - 2 = d$, that is, $k = d - k + 2$, we have*

$$S_{d,k} = \overline{S}_{d,k} \setminus \overline{S}_{d,k-1}$$

Proof. Let $r \geq 2$ and $2r - 2 \leq d$. By Proposition 18, a form having border rank r has either rank r or rank greater than or equal to $d-r+2$. On the other hand, by Proposition 19, a form having border rank r has rank less than or equal to $d-r+2$. This shows that for each $r \geq 2$ such that $2r - 2 \leq d$ we get

$$\overline{S}_{d,r} \setminus \overline{S}_{d,r-1} \subset S_{d,r} \cup S_{d,d-r+2}.$$

Therefore if $2r - 2 \leq d$ a rank r form has border rank r. Now let us consider $d = 2k - 1$. As r goes from 2 to k, $d - r + 2$ goes from d to $k + 1$. This shows that the inclusion must be an equality. If $d = 2k - 2$, as r goes from 2 to $k - 1$, $d - r + 2$ goes from d to $k + 1$. This shows that the inclusion must be an equality and also shows that $\overline{S}_{d,k} \setminus \overline{S}_{d,k-1} = S_{d,k}$. □

6. Elliptic normal curves

In this section we expand the results to immersions of elliptic curves. We will consider an elliptic curve C, a fixed point $p_0 \in C$ and the immersion of C in projective space $\mathbb{P}^n$ by the degree $d = n+1$ line bundle L associated to the divisor $(n+1)p_0$. We will call such an immersion an elliptic normal curve of degree $n + 1$.

We will prove the following theorem.

Theorem 4. *Let $C \subset \mathbb{P}^n$ be an elliptic normal of degree $n + 1$. Let $r \geq 2$ such that $n + 1 \geq 2r + 4$. Then*

1. $\overline{C}_r \setminus \overline{C}_{r-1} = C_r \cup C_{n-r+1}$.
2. $C_r = \overline{C}_r \setminus \overline{C}_{n-r+1}$.
3. $C_{n-r+1} = \overline{C}_{n-r+1} \setminus \overline{C}_{r-1}$.

We will prove the theorem separately for odd and even values of n. For the lower values of n we will do a case by case analysis.

For an elliptic normal curve C of degree $n+1$ there are two generalizations of Ravi's Theorem regarding the ideal of the secant variety $\mathrm{Sec}^r(C)$ due to Fisher [6] that we will use.

Theorem 24 (*Fisher*). *Let $C \subset \mathbb{P}^n$ be an elliptic normal curve of degree $n+1$. Let $r \geq 1$ such that $n \geq 2r$ and let L_1 and L_2 be two line bundles such that $L_1 \otimes L_2 = L$. Then the ideal of $\mathrm{Sec}^r(C)$ is generated by the $(r+1) \times (r+1)$ minors of the matrix associated to the multiplication morphism*

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu} H^0(L)$$

if and only if

1. $\deg L_1, \deg L_2 \geq r+2$ *and*
2. *if* $\deg L_1 = \deg L_2 = r+2$ *then* $L_1 \not\sim L_2$.

Lemma 25 (*Fisher*). *Let $C \subset \mathbb{P}^n$ be an elliptic normal curve of degree $n+1$. Let $r \geq 1$ such that $n \geq 2r+1$. Then the ideal of $\mathrm{Sec}^r(C)$ is generated by the $(r+1) \times (r+1)$ minors of the matrices A and B, associated to the multiplication matrix defined by the pairs L_1, L_2 and L_1', L_2' where $L_1 \not\sim L_1'$, $L_1 \not\sim L_2'$, and $\deg L_i, \deg L_i' \geq r+1$.*

Notice that Lemma 25 for $r = 1$ and $n = 3$ is just the well known fact that an elliptic normal curve of degree four in $\mathbb{P}^3$ is the complete intersection of two quadrics.

6.1. $n = 2$

We already shown that for every plane curve the rank function takes the values 1 and 2, so every point in C has rank one, and the rest have rank two.

6.2. $n = 3$

Points in C have rank one. We know that every point in $\mathbb{P}^3 \setminus C$ has rank less than or equal to 3. We also know that every point in $\mathbb{P}^3$ lies on a secant or tangent line to C.

Let us analyze the multiplication matrices in this case. So let D and E be degree two divisors such that $D + E \sim 4p_0$, and let L_1 and L_2 the associated line bundles. If $D \not\sim E$, the multiplication morphism

$$H^0(L_1) \otimes H^0(L_2) \to H^0(L)$$

is surjective (by Lemma 14) and therefore the entries of the matrix are linearly independent. Then the determinant of the matrix is the equation of a nonsingular quadric.

If on the other hand $D \sim E$, we can make the matrix symmetric and therefore its entries span a three dimensional subspace. Then the determinant of the matrix is the equation of a cone.

The curve C is the complete intersection of any two of the quadrics just described [8]. Furthermore any two of the quadrics generate the ideal of C.

Let $x \in \mathbb{P}^3 \setminus C$ and let D be a degree two divisor such that $x \in \langle D \rangle$. By the results that we will prove in Section 6.4 we have the following possibilities:

- $2D \not\sim 4p_0$. In this case there exists a divisor $E \neq D$ such that $x \in \langle E \rangle$, and x lie on exactly two secant lines to C. If one of this lines is a honest secant line we have rank $x = 2$. If the two lines are tangent lines then rank $x = 3$.
- $2D \sim 4p_0$. Here we have the following options:
  - $\mu_x \neq 0$. In this case x lies on exactly one secant line to C, namely $\langle D \rangle$. Therefore if this secant line is a honest one we have rank $x = 2$, and otherwise we have rank $x = 3$.
  - $\mu_x = 0$. In this case x lies on a pencil of secant lines to C, and by Proposition 30 (Section 6.4) a general member is a honest secant line. Therefore rank $x = 2$.

We characterize the rank three points in the following way. Given a tangent line $\langle 2p \rangle$ such that $4p \not\sim 4p_0$, there are four different tangent lines cutting it. More precisely, they are the tangent lines of the form $\langle 2q_i \rangle$ for q_i such that $2p + 2q_i \sim 4p_0$. So we get four rank three points in each tangent line $\langle 2p \rangle$ with $4p \not\sim 4p_0$.

Now we analyze tangent lines of the form $\langle 2p \rangle$ such that $4p \sim 4p_0$. There are 16 of those (one for each point in C such that $4p = 0$ in the law group) and they are divided in four sets of four, each corresponding with one of the four cones containing C. Moreover, the four lines in each set all pass through the center of the cone. Since we know that each center is a rank two point, each point in these lines excepting the center of the cone and the point of tangency is a rank three point. See [8] for other interpretations of these 16 lines.

Therefore the closure of rank three points is the union of the 16 tangent lines to points of order 4 and the set $\{\langle 2p_1 \rangle \cap \langle 2p_2 \rangle : p_1 \neq p_2\}$.

Notice that in the previous two cases there are points having rank n. For $n \geq 4$ we will show that every point has rank less than or equal to $n-1$.

6.3. $n = 4$

As in the previous cases points in C have rank one, points on honest secant lines have rank two. A point lying on a tangent line (and not lying on a secant line) has rank three by Proposition 19. We will show that the remaining points all have rank three.

We first prove the following proposition for an elliptic normal curve $C \subset \mathbb{P}^n$, $n = 2k \geq 4$.

Proposition 26. *Let $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$, $n = 2k \geq 4$. Then the family of secant $\mathbb{P}^{k+1}$'s to C containing x is parametrized by C.*

Proof. For each $p \in C$ let us consider the divisor $kp_0 + p$. It is a well known fact that in this way we go through all the equivalence classes of degree $k + 1$ divisors on C. Let L_1 and L_2 be the line bundles associated to $(k + 1)p_0 - p$ and $kp_0 + p$ respectively and let μ be the multiplication morphism associated to the pair. Since $x \notin \mathrm{Sec}^k(C)$, the rank of μ_x is k and therefore $W_2(x)$ is a one dimensional subspace. Let s be a generator of $W_2(x)$ and let D be the effective divisor associated to s. By Lemma 11 $x \in \langle D\rangle$. This way we find for each $p \in C$ a degree $k + 1$ divisor D_p linearly equivalent to $kp_0 + p$ and such that $x \in \langle D_p\rangle$.

Reciprocally, if D is a degree $k + 1$ divisor such that $x \in \langle D\rangle$, there is a unique $p \in C$ such that $D \sim kp_0 + p$, so D must be D_p. □

We go back for the case $n = 4$. The previous proposition shows that a point x in $\mathbb{P}^4 \setminus \mathrm{Sec}^2(C)$ lies on a one dimensional family of secant $\mathbb{P}^2$'s to C. In order to prove that rank $x = 3$ we have to show that one of these secant $\mathbb{P}^2$'s cuts C in three different points. If this is not the case then every secant $\mathbb{P}^2$ to C passing through x is of the form $\langle D\rangle$, where $D = 2p + q$. We consider the projection from x to $\mathbb{P}^3$. Since $x \notin \mathrm{Sec}^2(C)$, the image of C by this projection is a non singular curve C'. A secant $\mathbb{P}^2$ to C of the form $\langle 2p + q\rangle$ passing through x projects to a tangent line to C' that cuts C' in a third point. Notice that given $D = 2p + q$ and $D' = 2p' + q'$, we cannot have $p = p'$ because in that case the planes $\langle D\rangle$ and $\langle D'\rangle$ would both contain the tangent line $T_p(C)$. That would mean that $\langle D\rangle \cap \langle D'\rangle = T_p(C)$ and since x lies in both planes we would have $x \in T_p(C)$ which is a contradiction. This shows that projecting all planes of the form $\langle D\rangle$ containing x we obtain all the tangent lines to C'. Therefore we are showing that every tangent line to C' is a trisecant tangent line. But since we are working over a characteristic zero field, this is not possible (see [9], Theorem 3.1).

Therefore one of the secant $\mathbb{P}^2$'s to C passing through x must cut C in three different points, showing that x has rank three.

Notice that there are two kinds of points having rank three. The first kind are the ones lying on tangent lines. The closure of this set is the tangential surface. The second kind are those on $\mathbb{P}^4 \setminus \mathrm{Sec}^2(C)$. The closure of this set is the whole projective space $\mathbb{P}^4$.

6.4. Odd n

Let us recapitulate what we know in this case from Theorem 2. So let $n = 2k + 1$. Theorem 20 let us calculate the rank of all points in $\mathrm{Sec}^r(C)$ for $1 \leq r \leq k$. We only miss points on $\mathbb{P}^n \setminus \mathrm{Sec}^k(C)$. We will show that for these points we have two possible values of the rank: $k + 1$ or $k + 2$. Notice that there are points in $(\mathrm{Sec}^k)^\circ(C)$ having rank $k + 2$. Also notice that in this case points in $\mathrm{Sec}^{k+1,2}(C) \setminus \mathrm{Sec}^k(C)$ will not all have the same rank, so we will make a different argument as before.

In order to compute the rank of a point $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ we will study the family of secant $\mathbb{P}^k$'s to C that pass through x. For general x this family will be a finite set, in fact we will show that there will be at least two secant $\mathbb{P}^k$'s through x. If one of these planes is a honest secant plane, then x has rank $k + 1$. If neither of theses planes is a honest secant plane x will have rank $k + 2$.

On the other hand we will show that for some x's the family of secant $\mathbb{P}^k$'s through x is a pencil. In this case we will show that the general member of the pencil must be a honest secant plane, and x has therefore rank $k + 1$.

We start showing what happens if a point $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lies on two different secant $\mathbb{P}^k$'s to C.

Lemma 27. *Let $n = 2k + 1 \geq 3$, $x \notin \mathrm{Sec}^k(C)$ and let D, E be two $k + 1$ degree divisors such that $x \in \langle D\rangle$ and $x \in \langle E\rangle$. Then $\langle D\rangle \cap \langle E\rangle = \{x\}$, $\dim(\langle D\rangle + \langle E\rangle) = 2k$ and $\langle D + E\rangle = \langle D\rangle + \langle E\rangle$.*

Proof. We only have to show that $\dim(\langle D\rangle + \langle E\rangle) = 2k$, because in that case the dimension of $\langle D\rangle \cap \langle E\rangle$ is zero and so $\langle D\rangle \cap \langle E\rangle = \{x\}$.

Suppose that $\dim(\langle D\rangle + \langle E\rangle) < 2k$. Since $\langle D\rangle + \langle E\rangle = \langle D + E - D \cap E\rangle$, the degree of the divisor $D + E - D \cap E$ is less than or equal to $2k + 1$. Therefore, by Lemma 8 we have $\langle D\rangle \cap \langle E\rangle = \langle D \cap E\rangle$. Since $x \in \langle D\rangle \cap \langle E\rangle$, we have $x \in \langle D \cap E\rangle$. Now D and E are different divisors, and therefore $\deg(D \cap E) < k + 1$. This implies that $x \in \mathrm{Sec}^k(C)$ which is a contradiction.

Therefore $\dim(\langle D\rangle + \langle E\rangle) = 2k$ and $x = \langle D\rangle \cap \langle E\rangle$. We also proved that $D \cap E = \emptyset$, so we have $\langle D\rangle + \langle E\rangle = \langle E + D\rangle$. □

Notice that in the lemma we showed that if $x \in \langle D\rangle$ for D a $k + 1$ degree divisor and $x \notin \mathrm{Sec}^k(C)$, then every $k + 1$ divisor E such that $x \in \langle E\rangle$ must verify $D + E \sim (n + 1)p_0$. In other words E must belong to the linear system $H^0(L(-D))$.

The following lemma (which is a refinement of Proposition 5.2 in [2]) gives a condition on x for being on exactly two secant $\mathbb{P}^k$'s to C.

Lemma 28. *Let $n = 2k + 1 \geq 3$, and $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$. Let D be a degree $k + 1$ divisor such that $x \in \langle D\rangle$. If $2D \not\sim (n + 1)p_0$, then there are exactly two secant $\mathbb{P}^k$'s to C containing x.*

Proof. We know that x must lie on a secant $\mathbb{P}^k$ to C of the form $\langle D\rangle$. Any other secant $\mathbb{P}^k$ must be of the form $\langle E\rangle$ for $E \sim (n+1)p_0 - D$.

Let $L_1 = L_D$ be the line bundle associated to D, $L_2 = L \otimes L_1^{-1}$ and μ the multiplication morphism

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu} H^0(L).$$

The section $s \in H^0(L_1)$ associated to D lies in $W_1(x)$, so rank $\mu_x \le k$. If rank $\mu_x < k$, and since $L_1 \not\sim L_2$, we can use the Theorem 24 to show that $x \in \mathrm{Sec}^{k-1}(C)$, which is a contradiction. Therefore rank $\mu_x = k$ and $\dim W_2(x) = 1$ (if $k = 1$ we know that the rank of μ_x is one and no less). Let E be a divisor in $\mathcal{L}_2$ whose section generates $W_2(x)$. By Lemma 11 we have $x \in \langle E\rangle$. Since $D + D \not\sim (n+1)p_0$, $D \ne E$ and so $\langle D\rangle$ and $\langle E\rangle$ are the two different secant $\mathbb{P}^k$ containing x. □

Newt we characterize the points $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lying on an infinite number of secant $\mathbb{P}^k$'s to C.

Lemma 29. *Let $n = 2k+1 \ge 3$, and $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$. Then x lies on an infinite number of $\mathbb{P}^k$'s to C if and only if there is a $k+1$ divisor D such that $2D \sim (n+1)p_0$ and the rank of the multiplication matrix*

$$H^0(L_D) \otimes H^0(L_D) \to H^0(L)$$

at x is $k-1$. Notice that for $k = 1$ this means that the multiplication matrix at x is the zero matrix.

Proof. Let $M = M_x$ be the family of secant $\mathbb{P}^k$'s to C containing x.

First assume that M is an infinite set and let $D \in M$. The previous lemma shows that if $2D \not\sim (n+1)p_0$ then M is a set having two elements. Therefore we must have $2D \sim (n+1)p_0$. By Lemma 27 if $E \in M$, then $\langle E + D\rangle$ is a hyperplane and therefore $D \sim E$. We conclude that M is a linear system contained in $\mathbb{P}(H^0(L_1))$. In fact, $M = \mathbb{P}(W_1(x))$. Since M is infinite we must have rank $\mu_x \le k-1$. And since $x \notin \mathrm{Sec}^k(C)$ by Theorem 24 we must have rank $\mu_x \ge k-1$. This works if $k > 1$.

Now if $k = 1$ we have that x lies in a pencil of lines through x, that is, on a cone with center the point x. Since every point in C lies on a secant line of the form $\langle E\rangle$ for $E \sim D$, we must have that the curve C lies on the cone. On the other hand we know that the curve C lies on the set $\{y \in \mathbb{P}(H^0(L)^*) : \text{rank } \mu_y \le 1\}$. Since $h^0(L_D) = 2$, this set is a quadric, moreover, the equation which defines it is the determinant of the multiplication matrix. In this case the multiplication matrix is symmetric, and therefore its entries do not span the space $H^0(L)$, but a hyperplane. This hyperplane corresponds with the point x so at this point the matrix vanishes.

If we now assume that $2D \sim (n+1)p_0$ and that rank $\mu_x = k-1$, we must have $\dim W_1(x) = 2$ and so $\mathbb{P}(W_1(x)) = M$ is an infinite set. □

Notice that there are four (up to linear equivalence) divisors D such that $2D = (n+1)p_0$. If $n = 3$ this means that C lie on four different cones.

Also notice that if $x \in \mathbb{P}^k \setminus \mathrm{Sec}^k(C)$, $x \in \langle D\rangle$, $2D = (n+1)p_0$ and rank $\mu_x = k$, then $\langle D\rangle$ is the unique secant $\mathbb{P}^k$ to C containing x.

Next we compute the rank of a point $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lying on an infinite number of secant $\mathbb{P}^k$'s to C.

Proposition 30. *Let $n = 2k+1 \ge 3$ and let $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ be a point lying on an infinite number of $\mathbb{P}^k$'s secant to C. Then* rank $x = k+1$.

Proof. We know that there is a pencil of degree $k+1$ divisors containing x. We also know that the intersection of any two members of the pencil is the point x. Therefore the pencil has no base points. Using Bertini's Theorem we know that the general member of the pencil cuts C in a nonsingular set, that is, in a set of $k+1$ different points. Therefore we find a $\mathbb{P}^k$ secant to C spanned by $k+1$ different points in C, and so rank $x = k+1$. □

Finally we compute the rank of a point $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lying on an finite number secant $\mathbb{P}^k$'s to C.

Proposition 31. *Let $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lying on an finite number of secant $\mathbb{P}^k$'s to C.*

1. *If one of the secant $\mathbb{P}^k$'s is a honest secant plane, then* rank $x = k+1$.
2. *If none of the secant $\mathbb{P}^k$'s is a honest secant plane, then* rank $x = k+2$.

Proof. The first statement is obvious.

So assume that none of the secant $\mathbb{P}^k$ to C containing x is a honest secant plane. We know that rank $x \ge k+2$, so we have to show that rank $x \le k+2$.

Let L_1 and L_2 be line bundles of degrees k and $k+2$ respectively and let us consider the multiplication morphism

$$H^0(L_1) \otimes H^0(L_2) \xrightarrow{\mu} H^0(L).$$

Since $x \notin \mathrm{Sec}^k(C)$ the rank of μ_x is k. Suppose that rank $x > k+2$. Then we must have $W_2 = W_2(x) \subset \Delta_{L_2}$ by the Proposition 17. Therefore W_2 has base points and since $\operatorname{codim} W_2 = k$, the base locus of W_2 is a divisor E such that $2 \le \deg(E) \le k$. Once again we consider W_2 as a base point free subspace of $H^0(L_2(-E))$ and once again we show that

$$H^0(L_1) \otimes W_2 \xrightarrow{\mu} H^0(L(-E))$$

is surjective. That would mean that $x \in \langle E\rangle$ which is a contradiction.

So we will prove that

$$H^0(L_1) \otimes W_2 \xrightarrow{\mu} H^0(L(-E))$$

is surjective.

First we show that we can choose L_1 and L_2 in such a way that if $\deg E = 2$, then $L_1 \not\sim L_2(-E)$.

Let D, D' be the degree $k+1$ divisors such that they span linear spaces containing x (if there is only one D, we put $D' = D$). Let Φ be the family of $k-1$ divisors

$$\Phi = \{F : \deg F = k-1 \text{ and } D = 2p + F \text{ for } p \in C\}.$$

The family is not empty because we assume that D has multiple points.

Let us consider the set

$$M = \{q \in \operatorname{supp} D : \ 2q \sim D' - F \text{ with } F \in \Phi\}.$$

Let $r \in C$ be a point such that $r \notin M$.

Let L_2 be the line bundle associated to $D + r$ and L_1 be the line bundle associated to $D' - r$. Clearly $L_1 \otimes L_2 = L$.

Since $x \in \langle D + r\rangle$, we have $D + r \in W_2$. So assume that $\deg E = 2$ and $L_1 \sim L_2(-E)$. Put $E = 2p$. Since $D + r \in W_2$ and since r does not lie on the support of D, p must be one of the double points of D. Since we are assuming that $L_1 \sim L_2(-2p)$, we have that $D' - r \sim D + r - 2p = r + F$ for $F \in \Phi$. But then we would have $D' - F \sim 2r$ which is not possible by the choice of r.

Now we prove the surjectivity of μ. First suppose that $\deg E = k$, that is, $W_2 = H^0(L_2(-E))$. If $k \geq 3$, then $\deg L_1 + \deg L_2(-E)) = k + 2 \geq 5$. If $k = 2$, that is $\deg E = 2$, we can assume that $L_1 \not\sim L_2(-E)$. So $\deg L_1 + \deg L_2(-E) = 4$ and $L_1 \not\sim L_2(-E)$. In either case we can use Lemma 14.

If $\deg E = r < k$, then the codimension of W_2 in $H^0(L_2(-E))$ is $k - r$. If $\deg E \geq 3$ μ is surjective by Lemma 13.

Finally if $\deg E = 2$ and $k > 2$, we assume again that $L_1 \not\sim L_2(-E)$. Let us consider the projection from $\langle E\rangle$, $\pi_E : \mathbb{P}^n \setminus \langle E\rangle \to \mathbb{P}(H^0(L(-E))^*) = \mathbb{P}^{n-2}$. We cannot have $x \in \langle E\rangle$ because $x \notin \operatorname{Sec}^2(C)$. The image of C by π_E is an elliptic normal curve in $\mathbb{P}^{n-2}$ immersed by the complete linear system $(n+1)p_0 - E$.

Let $x' = \pi_E(x)$ and consider the multiplication matrix associated to the pair of line bundles L_1 and $L_2(-E)$. Its rank in x' is $k-2$ since the codimension of W_2 in $H^0(L_2(-E))$ is $k - \deg E = k-2$. Since $L_1 \not\sim L_2(-E)$, by Theorem 24 we must have $x' \in \operatorname{Sec}^{k-2}(C')$. If F is a degree $k-2$ divisor such that $x' \in \langle F\rangle \subset \mathbb{P}(H^0(L(-E))^*)$, then $x \in \langle E + F\rangle \subset \operatorname{Sec}^k(C)$, which is a contradiction. □

Now we prove Theorem 4 for n odd.

Proof of Theorem 4 (*First Part*)**.** Let $n = 2k+1$ and $r \geq 2$ such that $n + 1 \geq 2r + 4$, that is, $k \geq r + 1$. We know that if $x \in \operatorname{Sec}^{k-1}(C)$ then we must have $1 \leq \operatorname{rank} x \leq k-1$ or $n - k + 2 \leq \operatorname{rank} x \leq n-1$. The last inequality in this case becomes $k + 3 \leq \operatorname{rank} x \leq n-1$. If $x \in \mathbb{P}^n \setminus \operatorname{Sec}^{k-1}(C)$, then $k \leq \operatorname{rank} x \leq k+2$. This means that all the inclusions in Theorem 2 are equalities if $n + 1 \geq 2r + 4$.

Notice that the theorem we just proved characterizes the sets C_r for $r \leq k-1$ and $r \geq k+3$. The following proposition describes the sets C_k, C_{k+1} and C_{k+2}.

Proposition 32. *Let $C \subset \mathbb{P}^n$ be an elliptic normal of degree $n + 1 = 2k + 2$. Then*

1. $C_k = \overline{C}_k \setminus \overline{\overline{C}_k \cap C_{n-k+1}}$.
2. $C_{k+1} = (\mathbb{P}^n \setminus \overline{C}_k) \setminus C'_{k+2}$.
3. $C_{k+2} = (\operatorname{Sec}^{k,2}(C) \setminus \operatorname{Sec}^{k-1}(C)) \cup C'_{k+2}$.

Here C'_{k+2} is defined as the set of points $x \in \mathbb{P}^n \setminus \operatorname{Sec}^k(C)$ lying on a finite number of secant $\mathbb{P}^k$'s to C and such those are not honest.

Proof. The first statement is the second statement of Theorem 2 applied to this case.

We also know from Theorem 2 that points on $\overline{C}_k \setminus \overline{C}_{k-1}$ and not having rank k have rank $k+2$. But in Proposition 31 we showed that there are rank $k+2$ points in $\mathbb{P}^n \setminus \overline{C}_k$. They are the points $x \in \mathbb{P}^n \setminus \operatorname{Sec}^k(C)$ lying on a finite number of secant $\mathbb{P}^k$'s to C and as such those are not honest, that is, the points $x \in C'_{k+2}$. Therefore the set C_{k+2} is the union of $\operatorname{Sec}^{k,2}(C) \setminus \operatorname{Sec}^{k-1}(C)$ (this is the set of points in $\overline{C}_k \setminus \overline{C}_{k-1}$ not having rank k) and C'_{k+2}.

Finally from Proposition 31 we get that $C_{k+1} = (\mathbb{P}^n \setminus \overline{C}_k) \setminus C'_{k+2}$. □

6.5. Even n

Let $n = 2k$, $n \geq 6$. In this case Theorem 2 let us compute the rank of all points in $\mathbb{P}^n \setminus \operatorname{Sec}^k(C)$. Since $\dim \operatorname{Sec}^k(C) = 2k-1$, $\operatorname{Sec}^k(C)$ is a hypersurface. We will show that points in $\mathbb{P}^n \setminus \operatorname{Sec}^k(C)$ have rank $k+1$ or $k+2$. We conjecture that in fact all points have rank $k+1$. Notice that in $\operatorname{Sec}^k(C)$ there are already points having rank $k+1$ and $k+2$.

Proposition 33. *Let* $n = 2k \geq 6$ *and let* $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$. *Then* $\mathrm{rank}\, x = k+1$ *or* $\mathrm{rank}\, x = k+2$.

Proof. Let $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$. We know that rank $x > k$ so we assume that rank $x > k+2$ and arrive at a contradiction.

Let L_1 and L_2 be line bundles of degree $k-1$ and $k+2$ respectively and let us consider μ the multiplication morphism associated to the pair. The matrix μ_x must have rank $k-1$ because $x \notin \mathrm{Sec}^{k-1}(C)$. Since rank $x > k+2$, the subspace $W_2 = W_2(x)$ must have base points. Since codim $W_2 = k-1$, the base locus is a divisor E such that $2 \leq \deg E \leq k-1$.

Once again we consider W_2 as a base point free subspace of $H^0(L_2(-E))$, and once again we try to show that the restriction of μ

$$H^0(L_1) \otimes W_2 \xrightarrow{\mu} H^0(L(-E))$$

is surjective.

If $\deg E = k-1$, we have $W_2 = H^0(L_2(-E))$, and since $5 \leq k+2 = \deg L_1 + \deg L_2(-E)$, we know that the restriction of μ is surjective (Lemma 14).

Now assume $\deg E < k-1$. Since C is an elliptic curve, its canonical divisor is $K = 0$. Therefore $h^1(L_1 \otimes L_2(-E)^{-1}) = h^0(L_2(-E) \otimes L_1^{-1})$. The degree of $L_2(-E) \otimes L_1^{-1}$ is $k+2-\deg E - k + 1 = 3 - \deg E$. Therefore $h^1(L_1 \otimes L_2(-E)^{-1})$ is equal to 0 or 1. Since $\dim W_2 = 3$, using Lemma 15 we know that the restriction of μ is surjective. □

Now we are able to prove Theorem 4.

Proof of Theorem 4 (*Second Part*). Let $n = 2k$ and $r \geq 2$ such that $n+1 \geq 2r+4$, that is, $k > r+1$. We know that if $x \in \mathrm{Sec}^{k-2}(C)$ then we must have $1 \leq \mathrm{rank}\, x \leq k-2$ or $n-k+3 \leq \mathrm{rank}\, x \leq n-1$. The last inequality in this case becomes $k+3 \leq \mathrm{rank}\, x \leq n-1$. If $x \in \mathbb{P}^n \setminus \mathrm{Sec}^{k-2}(C)$, then $k-1 \leq \mathrm{rank}\, x \leq k+2$. This means that all the inequalities in Theorem 2 are equalities if $n+1 \geq 2r+4$.

Remember that we proved that $x \in \mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ lies on a curve of secant $\mathbb{P}^k$'s to C. In order to prove that all points in $\mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ have rank $k+1$ we need to prove that one of the members of the curve is a honest secant $\mathbb{P}^k$. In the case $n = 4$ this fact is true due to the fact that for a nonsingular curve immersed in $\mathbb{P}^3$ not all tangent lines are trisecant. For $n \geq 6$ we were not able to prove an equivalent statement, but we conjecture that all points in $\mathbb{P}^n \setminus \mathrm{Sec}^k(C)$ have rank $k+1$.

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References

[1] M.V. Catalisano, A.V. Geramita, A. Gimigliano, Ranks of tensors, secant varieties of segre varieties and fat points, Linear algebra and its applications 355 (2002) 263–285.
[2] L. Chiantini, C. Ciliberto, On the concept of *k*-secant order of a variety, Journal of the London Mathematical Society 2 (73) (2006) 436–454.
[3] C. Ciliberto, Geometric aspects of polynomial interpolation in more variables and of Waring's problem, in: Proceedings of the ECM, Barcelona, 2000.
[4] G. Comas, M. Seiguer, On the rank of a binary form, Foundations of Computational Mathematics (2010), doi:10.1007/s10208-010-9077-x.
[5] D. Eisenbud, J. Koh, M. Stillman, Determinantal equations for curves of high degree, American Journal of Mathematics 110 (3) (1988) 513–539.
[6] T. Fisher, Pfaffian presentations of elliptic normal curves, Transactions of the American Mathematical Society 362 (5) (2010) 2525–2540.
[7] M.L. Green, Koszul cohomology and the geometry of projective varieties, Journal of Differential Geometry (19) (1984) 125–171.
[8] K. Hulek, Projective geometry of elliptic curves, in: Lecture Notes in Mathematics, vol. 997, Springer Verlag, 1983, pp. 228–266.
[9] H. Kaji, On the tangentially degenerate curves, Journal of the London Mathematical Society 2 (33) (1986) 430–440.
[10] J.M. Landsberg, Geometry and the complexity of matrix multiplication, American Mathematical Society. Bulletin. New Series 45 (2) (2008) 247–284.
[11] J.M. Landsberg, Z. Teitler, On the ranks of tensors and symmetric tensors, Foundations of Computational Mathematics 10 (3) (2010) 339.
[12] M.S. Ravi, Determinantal equations for secant varieties of curves, Communications in Algebra 22 (8) (1994) 3103–3106.