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Smoothness of coalgebras and higher degrees of *Hoch* ☆

Marco A. Farinati[a,b,*], Andrea Solotar[a,b,1]

[a] *Dto. de Matemática Facultad de Cs., Exactas y Naturales Universidad de Buenos Aires, Ciudad Universitaria Pab I, 1428 Buenos Aires, Argentina*

[b] *Equipe de Topologie et Dynamique, Université Paris Sud, 91405 Orsay, France*

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Abstract

We prove that if C is a cocommutative k-coalgebra such that $dim_{\bar{k}}(\bar{k}e \wedge \bar{k}e) < N$ for all group-like elements $e \in C \otimes \bar{k}$, then smoothness of C is equivalent to the condition $Hoch^*(C) = 0$ for all $* \geqslant N$. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

This paper is devoted to the study of the relationship between the functor $Hoch^*$ defined by Doi [5] and the notion of smoothness for coalgebras introduced in [8]. In [8] we proved, in characteristic zero, that if C is a cocommutative smooth locally finite k-coalgebra, where locally finite means that the dimension of the space of primitive elements of any irreducible component of the $\bar{k}$-coalgebra $C \otimes \bar{k}$ is finite, then $Hoch^*(C)$ can be computed in terms of Ω^1_C, and consequently, $Hoch^*(C)$ vanishes for $* \gg 0$. The C-comodule Ω^1_C is a universal object for coderivations with cosymmetric bicomodules as source, it can be constructed as the cosymmetric part of the C-bicomodule $(C \otimes C)/\Delta(C)$ and it is also naturally isomorphic to $Hoch^1(C)$ (see [8]).

This characterization of $Hoch^*(C)$ when C is a locally finite smooth coalgebra can be seen as the coalgebra version of the Hochschild–Kostant–Rosenberg Theorem that

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* Corresponding author.

E-mail addresses: mfarinat@dm.uba.ar (M.A. Farinati), asolotar@dm.uba.ar (A. Solotar).

[1] Research member of CONICET (Argentina).

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says precisely when one can compute Hochschild homology as the exterior algebra on Kähler differentials.

It is known that the condition $HH_*(A)=0$ for $*\gg 0$ is sufficient in order to assure that a commutative algebra A (essentially of finite type over k) is k-smooth (see for example [2] or [3]). This reciprocal statement to the Hochschild–Kostant–Rosenberg theorem for coalgebras (or one should say "Rodicio's conjecture for coalgebras") is true, and this is our main theorem (see Theorem 3.5 and Corollary 3.6). In the coalgebra case, this Hochschild cohomological condition for smoothness has the advantage that it also implies a structure theorem for the original coalgebra. It is well known that if $(A,\mathscr{M})$ is a commutative k-algebra which is the completion of a local and regular ring with $dim_k(\mathscr{M}/\mathscr{M}^2)=n<\infty$, then it is (Lipman–Nagata–Zariski Theorem for $char(k)=0$ and a corollary of Cohen's Theorem for arbitrary characteristic) a ring of formal power series on n-variables. In the coalgebra case the situation is simpler, we keep the hypotheses of C being smooth, irreducible (i.e. local), with finite dimensional space of primitives (analogous to $dim_k(\mathscr{M}/\mathscr{M}^2)<\infty$) but there are no hypotheses concerning "completeness", and we still have that C must be the graded dual of a polynomial algebra (see [8] for an explicit map in characteristic zero and Theorem 1.7 here for arbitrary characteristic).

The main results of this work can be summarized in the following theorem.

Theorem A. *If k be an algebraically closed field of arbitrary characteristic and C is a cocommutative k-coalgebra such that, for all $e\in G(C)=\{c\in C\mid \Delta(c)=c\otimes c\}$, $dim_k(k.e\wedge k.e)=d_e<N+1$ for a fixed integer N, then the following assertions are equivalent*:
1. *C is smooth.*
2. $C\cong\bigoplus_{e\in G(C)}B((k.e\wedge k.e)/k.e)$.
3. *$Hoch^*(C,C)=0$ for $*>N$.*
4. *$Hoch^*(M,C)=0$ for $*>N$ and all C-bicomodules M.*

Here we use Sweedler's notation $B(V)$ for the biggest cocommutative subcoalgebra of the cofree coalgebra on the vector space V. If V is finite dimensional, $B(V)$ is the graded dual of the symmetric algebra on V^*.

The contents of this work are organized as follows:

In Section 1 we recall the notion of smoothness for coalgebras introduced in [8], we also recall some of the fundamental properties and we prove a generalization to arbitrary characteristic of a structure theorem for smooth coalgebras.

In Section 2 we use a Künneth formula in order to compute easily $Hoch^*(M,C)$, for C a smooth irreducible coalgebra, and M an arbitrary bicomodule.

In Section 3 we prove, following Tate, that given any irreducible cocommutative coalgebra there exists a differential graded cofree coalgebra quasi-isomorphic to it. This is the key point of the proof of Theorem 3.5. Before the proof of Theorem 3.5, we make the computation for a small dimensional example illustrating the methods to be used in the proof of this theorem.

Unless stated otherwise, k will be an algebraically closed field of arbitrary characteristic, all coalgebras will be cocommutative (or eventually graded-cocommutative) k-coalgebras.

1. Smooth coalgebras

We begin by recalling the definition of smooth coalgebra given in [8] and some of their properties proved in the same article.

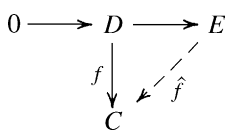
Definition 1.1. Given a cocommutative k-coalgebra C, a *square zero extension* of C is a cocommutative coalgebra D together with a monomorphism of coalgebras $C \to D$, such that the quotient D/C is a C-bicomodule.

In other words, a square zero extension of C is a cocommutative coalgebra D containing C such that the wedge of C with itself viewed as a subspace of D is the whole space D, we write $C \wedge_D C = D$. We recall from [12] that if V and W are two subspaces of a coalgebra D, the subspace $V \wedge W$ of D is defined by $V \wedge W = \{d \in D$ such that $\Delta(d) \in V \otimes D + D \otimes W\} = Ker((p_V \otimes p_W) \circ \Delta)$, where p_V (respectively p_W) is the canonical projection $D \to D/V$.

Remark. Of course this definition can be made in the context of general coalgebras omitting the word 'cocommutative', but in this work we will use only cocommutative extensions and mapping extensions properties with respect to cocommutative extensions. Mapping extension property with respect to general extensions would correspond to the notion of 'quasi-free' instead of smooth.

The definition of smoothness is then given in terms of an extension property with respect to square zero extensions.

Definition 1.2. Let C be a cocommutative coalgebra, we call C *smooth* if for any square zero extension $D \to E$ and any morphism of cocommutative coalgebras $f : D \to C$, then there exists a morphism of coalgebras $\hat{f} : E \to C$ extending f.



This definition can also be written in cohomological terms. We know after [5] that the class of coalgebra extensions

$$0 \to C \to D \to M \to 0$$

(where $C \to D$ is a coalgebra map and M is a C-bicomodule, the projection $D \to M$ being D-bicolinear) is in 1–1 bijection with $H^2(M,C)$. If we are interested in square zero extensions of cocommutative coalgebras, then D must be cocommutative and consequently M must be cosymmetric. Let us call as in [8], $H^2(M,C)^{sym}$ or $H^2_{Har}(M,C)$ the subgroup of $H^2(M,C)$ consisting on symmetric cocycles, then it is proved easily in [8] that C is smooth if and only if $H^2_{Har}(M,C)=0$ for all cosymmetric C-comodules M. If $\frac{1}{2}\in k$, then $H^2_{Har}(M,C)$ is a direct summand of $H^2(M,C)$.

Proposition 1.3. *Let C and D be cocommutative coalgebras*:

1. *If C is smooth, then Ω^1_C is an injective C-comodule* (*Proposition* 3.4 *of* [8]).
2. *If C is smooth, then $C_{[S]}$ is smooth for all multiplicative subsets $S \subset C^*$* (*Proposition* 1.6 *of* [8]).
3. *$C \otimes D$ is smooth if and only if C and D are smooth* (*part* 2 *of Lemma* 2.8 *and Lemma* 2.9 *of* [8] *for the "if" part and the "only if" part, respectively*).
4. *If $C=\bigoplus_{i\in I} C_i$ and C is smooth, then C_i is smooth for all $i\in I$* (*Proposition* 3.4.3 *of* [6], *also a consequence of* 2).
5. *If K is any subcoalgebra of C, then there is an exact sequence*
$$0 \to \Omega^1_K \to \Omega^1_C \square_C K \to \frac{K\wedge_C K}{K}$$
and if one assumes K smooth, then the last morphism of the above sequence is a split epimorphism (*Proposition* 3.5 *and* 3.6 *of* [8]).

It is evident that Property 4 is only a half of what we wanted, as a good definition of smoothness should be checked locally. We want to prove that if C is a coalgebra such that every localization at maximal ideals of C^* is smooth, then C is smooth.

Proposition 1.4. *If $C=\bigoplus_i C_i$ is a k-coalgebra with every C_i smooth, then C is smooth.*

Proof. It is enough to prove that given a square zero extension
$$0 \to C \to E \to M \to 0$$
there exists $\pi : E \to C$ a coalgebra morphism splitting the inclusion $C \hookrightarrow E$.

Considering such an extension, and denoting by $G(-)$ the set of group-like elements of a given coalgebra, it is clear that $G(C)=\coprod_i G(C_i)$ and that $G(C)\subseteq G(E)$. We claim that $G(C)=G(E)$, because if not, considering $e\in G(E)-G(C)$, $\Delta(e)=e\otimes e\notin C\otimes E+E\otimes C$ and so $C\wedge_E C\neq E$.

Since $E=\bigwedge_E^\infty kG(E)=\bigoplus_i\bigwedge_E^\infty kG(C_i)$, we have, denoting $E_i:=\bigwedge_E^\infty kG(C_i)$, that $C_i\hookrightarrow E_i$ and $C_i\wedge_E C_i=C_i\wedge_{E_i} C_i=E_i$. So the extension $C\hookrightarrow E$ is a direct sum of square zero extensions $\bigoplus C_i\hookrightarrow\bigoplus E_i$; each C_i is smooth, so the inclusions $C_i\hookrightarrow E_i$ split by $\pi_i : E_i\to C_i$. If we consider the map $\pi=\bigoplus_i \pi_i : \oplus E_i\to\oplus C_i$, it has the desired properties. □

We will recall an immediate consequence of Lemma 12.1.1 from [12] that will be used in the proof of the Structure Theorem 1.7 and also when constructing models for coalgebras and in the proof of our main results.

Given a local k-algebra A essentially of finite type it can be always written as a quotient R/I where R is a local regular algebra with maximal ideal $\mathscr{M}$ and I is an ideal of R verifying $I \subseteq \mathscr{M}^2$ (see for example [3, Lemma 1.1.2]). On the coalgebra side, the local coalgebras are the irreducible ones and there is an analogous description.

Lemma 1.5. *If C is a pointed cocommutative irreducible k-coalgebra and e is its unique group-like element, then there is a monomorphism of coalgebras denoted by $i: C \hookrightarrow B(P(C))$ where $P(C) = \{x \in C \mid \Delta(x) = x \otimes e + e \otimes x\}$.*

Remark. The restriction of i to $P(C)$ is the identity.

Corollary 1.6. *With the above notations, $ke \wedge_{B(P(C))} ke \subseteq C$.*

Remark. As we said before the lemma, in the algebra case we write $A = R/I$ where R is local and regular and $I \subseteq \mathscr{M}^2$. For coalgebras, C is embedded in the smooth coalgebra $B(P(C))$. Dualizing the exact sequence

$$0 \to C \to B(P(C)) \to Coker \to 0$$

we obtain

$$\begin{array}{ccccccccc} 0 & \longleftarrow & A & \longleftarrow & B(P(C))^* & \longleftarrow & I & \longleftarrow & 0 \\ & & \| & & & & \| & & \\ & & C^* & & & & C^{\perp} & & \end{array}$$

Then $I \subseteq \mathscr{M}^2 = (k.e^{\perp})^2 = (ke \wedge ke)^{\perp} \;\Leftrightarrow ke \wedge ke \subseteq I^{\perp} = C$.

Next, we shall prove a structure theorem for smooth coalgebras in arbitrary characteristic. The corresponding result for algebras is a corollary of Cohen's Theorem (see for example [11, Corollary 28.J]).

Theorem 1.7. *If C is an irreducible smooth k-coalgebra such that $dim_k(P(C)) < \infty$, then $C \cong B(P(C))$.*

Proof. By the previous lemma there is a monomorphism of coalgebras $i: C \to B(P(C))$. Next, identifying C with $i(C)$, consider the short exact sequence (see Proposition 1.3) for the inclusion $C \to B(P(C))$:

$$0 \to \Omega^1_C \to C \square_{B(P(C))} \Omega^1_{B(P(C))} \to \frac{C \wedge_{B(P(C))} C}{C} \to 0.$$

C is smooth, so Ω^1_C is an injective C-comodule, and, since C is irreducible, it is free (see for example [13], Appendix 2). In order to compute its rank, we shall use the same short exact sequence for the inclusion $k.e \hookrightarrow C$ (remark that $k.e$ is also a smooth coalgebra), then we have

$$0 \to \Omega^1_{k.e} \to k.e \Box_C \Omega^1_C \to \frac{k.e \wedge_C k.e}{k.e} \to 0.$$

It is clear that $\Omega^1_{k.e} = 0$, so if Ω^1_C is a C-comodule of rank n, then $n = dim_k(k.e \Box_C \Omega^1_C) = dim_k(k.e \wedge_C k.e/k.e) = dim_k(P(C))$.

Note that $dim_k(P(C)) = dim_k(P(B(P(C))))$. Using again the first short exact sequence we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_C & \longrightarrow & C \Box_{B(PC))} \Omega^1_{B(P(C))} & \longrightarrow & \frac{C \wedge_{B(P(C))} C}{C} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & C^n & \stackrel{j}{\longrightarrow} & C \Box_{B(P(C))} B(P(C))^n & \longrightarrow & \frac{C \wedge_{B(P(C))} C}{C} & \longrightarrow & 0. \end{array}$$

This sequence splits, then there is a split epimorphism $p: C^n \cong C \Box_{B(P(C))} B(P(C))^n \to C^n$ such that $p \circ j = id_{C^n}$. Since both p and j restrict to the socle $soc(C^n) \cong k^n$, the identity $p|_{soc} \circ j|_{soc} = id_{soc}$ implies that $p|_{soc}$ is an isomorphism of vector spaces, in particular $p|_{soc}$ is injective. This means that $Ker(p) \cap soc(C^n) = soc(Ker(p)) = Ker(p|_{soc}) = 0$, and so $Ker(p) = 0$.

On the other hand, $Ker(p)$ identifies with $Coker(j)$, as a consequence, $(C \wedge_{B(P(C))} C)/C = 0$, or equivalently $C \wedge_{B(P(C))} C = C$. Inductively, it follows that $C = (\bigwedge^n_{B(P(C))} C) \wedge_{B(P(C))} C = \bigwedge^{n+1}_{B(P(C))} C$ for all $n \in \mathbb{N}$. However,

$$B(P(C)) = \bigwedge^{\infty}_{B(P(C))} k.e = \bigcup_n \bigwedge^{n}_{B(P(C))} k.e \subseteq \bigcup_n \bigwedge^{n}_{B(P(C))} C = C$$

from where we obtain $C = B(P(C))$. □

2. Hochschild cohomology of $C = B(V)$

The cohomology groups $Hoch^*$ are easily computable for cocommutative cofree coalgebras. We recall that, for a k-coalgebra C and a bicomodule M, the groups $Hoch^*(M, C)$ can be computed by means of a standard complex, but they are also the values of the right derived functors of a cotensor product (see [5]):

$$Hoch^*(M, C) = Cotor^*_{C^e}(M, C) = R(-\Box_{C^e} C)^*(M)$$

where $C^e = C \otimes C^{\mathrm{op}}$, and we identify the category of right C^e-comodules with the category of (k-symmetric) C-bicomodules. Fixing a coalgebra C over a field, the category of C-comodules has enough injectives and the cotensor product is defined as a kernel, so it is right exact, and the definition of right derived functor is the classical one.

We will now show inductively that if V is a finite dimensional k-vector space, then $Hoch^r(B(V)) \cong B(V)^{\binom{dim_k(V)}{r}}$ for all $r \geqslant 0$. Suppose first that $dim_k(V) = 1$. Then $C = B(k.x) = sh(k.x)$ (see [7]),

$$Hoch^*(C) = \begin{cases} C & \text{if } * = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

If M is an arbitrary C-bicomodule, $Hoch^*(M, C) = 0$ for $* \geqslant 2$ because C admits the following short injective resolution:

$$0 \to C \to C \otimes C \to C \otimes k.x \otimes C \to 0.$$

If $dim_k(V) = n$, then we can compute $Hoch^*(B(V))$ using induction on the dimension, because if C and D are arbitrary coalgebras and $E = C \otimes D$, then $Hoch^*(E) = Hoch^*(C) \otimes Hoch^*(D)$ (see for example [8]). So, if V is a k-vector space with $dim_k(V) = n$, choose a basis $\{x_1, \dots, x_n\}$ of V, then $B(V) = B(kx_1 \oplus \cdots \oplus kx_n) \cong \bigotimes_{i=1}^n B(kx_i)$. It is immediate to compute inductively that $Hoch^r(B(V)) = B(V)^{\binom{n}{r}}$ for all $r \geqslant 0$. In particular, $Hoch^r(B(V))$ vanishes for $r > dim_k(V)$.

More generally, if we are interested in computing $Hoch^*$ with coefficients, we can use induction on the dimension of V in order to find a "co-Koszul" type resolution:

$$\begin{aligned} &0 \to C \to C \otimes C \to C \otimes V \otimes C \to \\ &\quad C \otimes \Lambda^2 V \otimes C \to \cdots \to C \otimes \Lambda^{dim(V)} V \otimes C \to 0. \end{aligned}$$

So we can conclude that $Hoch^*(M, C) = 0$ for all $* > dim_k(V)$ and all C-bicomodules M.

The structure theorem of the previous section tells us that if C is a smooth irreducible coalgebra with $dim_k(P(C)) < \infty$ then C is isomorphic to $B(P(C))$. As a consequence, we have proved the following theorem generalizing Theorem 7.1 of [8] to arbitrary characteristic.

Theorem 2.1. *If C is a smooth irreducible k-coalgebra such that $dim_k(P(C)) < \infty$, then $Hoch^*(M, C) = 0$ for $* > dim_k(P(C))$ and all C-bicomodules M.*

One of the main results of this work is to prove the reciprocal statement that we have conjectured in [8] for k a field of characteristic zero. In order to do it, we shall next construct models for cocommutative coalgebras.

3. Models and cohomology

The theory of Tate models [14] for commutative finitely generated k-algebras is a useful tool for the computation of homology and cohomology. The idea is to replace a commutative algebra A by its minimal model, i.e. a commutative differential graded algebra of type $(\Lambda V, d) = (\tilde{A}, d)$ such that $\tilde{A}_0 / d(\tilde{A}_1) \cong A$, $H_i(\tilde{A}, d) = 0$ for $i > 0$, and $d(V) \subseteq V.\tilde{A}^+$. Even if it seems to be a more complicated object, in practice this method can be used to isolate the difficulties. The algebra $\tilde{A}$ is of type ΛV where V is a graded vector space $V=\bigoplus_{n\geqslant 0} V_n$, $\Lambda V = S(\bigoplus_{n\geqslant 0} V_{2n})\otimes E(\bigoplus_{n\geqslant 0} V_{2n+1})$ and $E(-)$ and $S(-)$ are, respectively, the exterior and the symmetric algebra functors for $char(k)=0$, and the divided power versions in positive characteristic. More concretely:

Lemma 3.1. *If A and B are two commutative noetherian k-algebras and $f: A\to B$ is an algebra epimorphism, then there is a graded k-vector space $V=\bigoplus_{n\geqslant 1} V_n$ such that $dim_k(V_n)<\infty$ for all $n\geqslant 1$, a differential d of degree -1 on $A\otimes \Lambda V$, and a quasi-isomorphism of algebras $\tilde{f}: A\otimes\Lambda V\to B$. Even more, if $(B,\mathscr{M}_B)$ is local, one can always find a local and smooth algebra A and an epimorphism $f: A\to B$ such that the differential graded algebra $(A\otimes\Lambda V, d)$ verifies $d(V_1)\subseteq \mathscr{M}^2$ and $d(V_p)\subseteq \mathscr{M}.V_{p-1}+((\Lambda^+V.\Lambda^+V).A\otimes\Lambda V)_{p-1}$, where Λ^+V is the augmentation ideal of $\Lambda V\to k$.*

This Lemma is a classical result [14], so we are just going to comment the first step of the proof:

Since A is noetherian, $Ker(f)$ is finitely generated, so take a set $\{a_1,\dots,a_n\}$ spanning (as A-module) the kernel of f. Take $V_1=\bigoplus_{i=1}^n k.e_i$ and define the differential on $A\otimes\Lambda^*(V_1)$ as the unique derivation verifying

$$d_1(a)=0 \quad \forall a\in A; \qquad d_1(e_i)=a_i, \quad i=1,\dots n.$$

The homology of $A\otimes\Lambda^*(V_1)$ is $A/\langle a_1,\dots,a_n\rangle = B$ in degree zero. The homology of $A\otimes\Lambda^*(V_1)$ in higher degrees may be zero or not; in the next steps this procedure is iterated adding V_i with $i\geqslant 2$ in order to kill higher homology classes and extending d_1.

The second part of the lemma comes from the fact that every noetherian local algebra $(B,\mathscr{M}_B)$ may be presented as a quotient A/I with $(A,\mathscr{M}_A)$ noetherian local smooth, and $I\subseteq\mathscr{M}_A^2$ (see for example [10]).

The importance of this method for Hochschild homology computation has been extensively proved. For example, it is the key of the proof of the reciprocal of the Hochschild–Kostant–Rosenberg Theorem for algebras (namely Rodicio's conjecture, see [2] or [3]).

For coalgebras, the situation is as follows:

Proposition 3.2. *If C is an irreducible k-coalgebra with $dim_k(P(C))<\infty$ and group-like element e, then there is a graded k-vector space $W=\bigoplus_{n\geqslant 1} W_n$ with $dim_k(W_n)<\infty$ for all n and a quasi-isomorphism*

$$C\to (B(P(C))\otimes\Lambda W, d),$$

where ΛW is as coalgebra the graded dual of the exterior algebra on $\bigoplus_{n\geqslant 1} W_n^*$. The differential d comes from a differential $d': k[|P(C)^*|]\otimes\Lambda(\bigoplus W_n^*)\to C^*$ which is a derivation satisfying $d'(P(C)^*)\subseteq (k.e^{\perp})^2$ and $d'(W_p^*)\subseteq ke^{\perp}.W_{p-1}^*+((\Lambda^+W^*.\Lambda^+W^*).C^*\otimes\Lambda W^*)_{p-1}$. Here Λ^+W^* denotes the augmentation ideal of $\Lambda W^*\to k$.

Proof. From Lemma 1.5, we have a monomorphism $C \to B(P(C))$ extending the identity of $P(C)$. This map induces an epimorphism $k[|U|] \to C^*$, where $U = P(C)^*$. Since $k[|U|]$ is a noetherian k-algebra (see for example [1]), we are able to use the previous lemma, which says that there is a graded k-vector space $V = \bigoplus_{n\geqslant 1} V_n$ and a quasi-isomorphism

$$\tilde{q} : (k[|U|] \otimes \Lambda V, d') \to C^*.$$

This map induces in turn, a morphism q

$$q : C = C^{*0} \to ((k[|U|] \otimes \Lambda V)^0, (d')^0) = k[|U|]^0 \otimes \Lambda V^* = B(P(C)) \otimes \Lambda V^*,$$

where V^* is the graded dual vector space $\bigoplus_{n\geqslant 1} V_n^* =: W$. The map d' is a derivation, so $d{:=}(d')^0$ is a coderivation. Also, $\tilde{q}$ is a morphism of differential graded algebras, so q is a morphism of differential graded coalgebras, and its transpose is the quasi-isomorphism $\tilde{q}$; as a consequence, q is also a quasi-isomorphism.

We know that d is a degree $+1$ map (d' is of degree -1), and by the remark after Corollary 1.6 we have that $d'(V_1) \subseteq (ke^{\perp})^2$, the rest is a formal consequence of this fact (see [14]). □

Remark. Taking $W_0 = P(C)$ and $W = \bigoplus_{n\geqslant 0} W_n$, there is a quasi-isomorphism between C and $(\Lambda W, d)$.

The above proposition may be used to compute Hochschild cohomology groups of coalgebras thanks to the following fact:

Theorem 3.3 (Proposition 7.7.1 together with Theorem 9.1.4 of [6])**.** *Given two differential positively graded k-coalgebras C and D, if $f : C \to D$ is a quasi-isomorphism then (via f) $Hoch^*(C) \cong Hoch^*(D)$.*

Theorem 9.1.4 of [6] states that if there is a 'derived Morita equivalence' between C and D, then $Hoch^*(C) \cong Hoch^*(D)$, and Proposition 7.7.1 says precisely that if there is a quasi-isomorphism $f : C \to D$, then taking $C_f \in \mathscr{D}(C \otimes D^{\mathrm{op}})$ and ${}_fC \in \mathscr{D}(D \otimes C^{\mathrm{op}})$, there are isomorphisms ${}_fC \square^R_C C_f \cong D$ (isomorphism in $\mathscr{D}(D \otimes D^{\mathrm{op}})$) and $C_f \square^R_D {}_fC \cong C$ (isomorphism in $\mathscr{D}(C \otimes C^{\mathrm{op}})$), and hence a derived Morita equivalence. The notation ${}_fC$ (resp. C_f) means that we view C as right (resp. left) C-comodule and left (resp. right) D-comodule via f.

Since our purpose is to obtain $Hoch^*((B(P(C) \otimes \Lambda W, d))$, we shall next show that there exists a co-Koszul-type resolution for this coalgebra as comodule over its enveloping coalgebra.

Let us denote $(D, d) = (\Lambda W, d)$ a differential graded coalgebra coming from the construction given in Proposition 3.2. Here $W = \bigoplus_{n\geqslant 0} W_n$ and $W_0 = P(C)$. As usual, $D^e = D \otimes D^{\mathrm{op}}$. The procedure is divided into five steps:

Step 1: Suppose that $W=\bigoplus_{n\geqslant 0} W_{2n}$, $(D,d)=(\Lambda W,0)=(S(W),0)$ and $dim_k(W)<\infty$. Forgetting the grading, $S(W)$ is isomorphic to $B(W)$, and so it has a resolution of type

$$0\to B(W)\to B(W)^e\to B(W)\otimes W\otimes B(W)\to B(W)\otimes E^2W\otimes B(W)\to\cdots.$$

This fact is clear if $dim_k(W)=1$ (since in this case $B(V)$ is the tensor coalgebra on $W=kx$). For $dim_k(W)>1$ and $\{x_1,\dots,x_n\}$ a k-basis of W, by tensoring the resolutions of $B(kx_i)$, $1\leqslant i\leqslant n$ (see example of Section 2), we obtain a complex of $B(W)^e$-comodules which is, using a Künneth formula, a resolution of $B(W)$. This resolution has on degree j free $B(W)^e$-comodules of rank $dim_k(E^jW)$.

Step 2: Suppose now $W=\bigoplus_{n\geqslant 0} W_{2n+1}$, $(D,d)=(\Lambda W,0)=(E(W),0)$ and $dim_k(W)<\infty$. If $dim_k(W)=1$, then $E(W)=(k[x]/x^2)^*$ and it has the following resolution:

$$0\longrightarrow E(W)\longrightarrow E(W)^e\longrightarrow E(W)^e\longrightarrow E(W)^e\longrightarrow\cdots$$

i.e. the dual complex of the periodic complex

$$\cdots\xrightarrow{.(1\otimes x-x\otimes 1)}(k[x]/x^2)^e\xrightarrow{.(1\otimes x+x\otimes 1)}(k[x]/x^2)^e\xrightarrow{.(1\otimes x-x\otimes 1)}(k[x]/x^2)^e\xrightarrow{m}k[x]/x^2\to 0.$$

If $dim(W)>1$, we tensorize the resolutions as we did in the first case.

Step 3: Suppose $(D,d)=(\Lambda(\bigoplus_{n\geqslant 0} W_n),0)$ with $dim_k(W)<\infty$.

In this case, we obtain a resolution by tensoring those of the previous cases since $(\Lambda W,0)=(S(\bigoplus_{n\geqslant 0} W_{2n}),0)\otimes(E(\bigoplus_{n\geqslant 0} W_{2n+1}),0)$.

Step 4: Suppose $(D,d)=(\Lambda W,d)$ and $dim_k(W)<\infty$.

The resolution is now the complex which, as graded vector space, is the same as above. We shall now construct the differential. In order to do so, since $dim_k(W)<\infty$, we shall take the graded dual of the complex, considering then the (local) algebra $A=k[|W_0^*|]\otimes\Lambda W_{\geqslant 1}^*$.

Using Tate's Lemma 3.1, there is a model for $HH_*(A,d')$ [4]. This model may be obtained by tensoring the resolution $(A\otimes\Lambda\bar{V}\otimes A,\delta)$ of A as A^e-module over A^e by A, where $V_n=W_n^*$, $\bar{V}_n\cong V_n$ $(\bar{v}\mapsto v)$ but with different grading, we set $|\bar{v}|=|v|+1$. The differential δ is defined by

$$\begin{aligned}\delta(a\otimes\overline{v_1}\wedge\cdots\wedge\overline{v_n}\otimes b)\\ =(-1)^n\Bigg(&d'(a)\otimes\overline{v_1}\wedge\cdots\wedge\overline{v_n}\otimes b+(-1)^{|a\otimes\overline{v_1}\wedge\cdots\wedge\overline{v_n}|}a\otimes\overline{v_1}\wedge\cdots\wedge\overline{v_n}\otimes d'(b)\\ &+\sum_{i=1}^{n}(-1)^{|\overline{v_1}\wedge\cdots\wedge\overline{v_{i-1}}|.|\overline{v_i}|}ad'(v_i)\otimes\overline{v_1}\wedge\cdots\widehat{\overline{v_i}}\cdots\wedge\overline{v_n}\otimes b\\ &+\sum_{i=1}^{n}(-1)^{|\overline{v_{i+1}}\wedge\cdots\wedge\overline{v_n}|.(|\overline{v_i}|-1)}a\otimes\overline{v_1}\wedge\cdots\widehat{\overline{v_i}}\cdots\wedge\overline{v_n}\otimes d'(v_i)b\Bigg)\end{aligned}$$

$$+\sum_{i=1}^{n}(-1)^{|\overline{v_1}\wedge\cdots\wedge\overline{v_{i-1}}|.|\overline{v_i}|}av_i\otimes\overline{v_1}\wedge\cdots\widehat{\overline{v_i}}\cdots\wedge\overline{v_n}\otimes b$$

$$+\sum_{i=1}^{n}(-1)^{|\overline{v_{i+1}}\wedge\cdots\wedge\overline{v_n}|.(|\overline{v_i}|)}a\otimes\overline{v_1}\wedge\cdots\widehat{\overline{v_i}}\cdots\wedge\overline{v_n}\otimes v_ib.$$

This complex is a resolution because of the following argument:

Consider the filtration induced by V, and the associated graded complex $(\mathscr{C}_*,\bar{\bar{\delta}}_*)$. Apply the functor $(-)^0$. The complex obtained is acyclic since the second part of Tate's Lemma implies that the 'internal' differential on the associated graded is zero, so we are in the situation of $(\Lambda W,0)$. As a consequence, the original complex was acyclic, and hence a resolution.

Step 5: Suppose $(D,d)=(\Lambda W,d)$ with W a graded k-vector spaces with arbitrary dimension. We can write $W=\varinjlim W^i$ with W^i finite dimensional, consequently $(D,d)=\varinjlim_i(\Lambda W^i,d_i)$. For each coalgebra $(\Lambda W^i,d_i)$, we have already constructed a resolution, in such a way that they fit together into an inductive system of resolutions. The functor $\varinjlim$ commutes with homology, so the direct limit of the resolution is a resolution of (D,d).

Lemma 3.4. *If C is a cocommutative irreducible k-coalgebra with $dim_k(P(C))<\infty$ and $n\in\mathbb{N}$ such that $HH_n(C^*)\neq 0$, then $Hoch^n(C,C)\neq 0$ as well.*

Proof. Since $dim_k(P(C))<\infty$, C admits a differential graded model of type $(B(P(C))\otimes\Lambda W,d)$ with $dim_k(W_n)<\infty$ for all n and consequently, C^* has $(k[|P(C)^*|]\otimes\Lambda W^*,d')$ as differential graded model.

If we use the Koszul type resolution for the models of C and C^*, the complex computing $HH_*(C^*,C^*)$ is the graded dual of the one computing $Hoch^*(C,C)$. By the universal coefficients theorem, $(Hoch^n(C))^*=HH_n(C^*)$, and so the Lemma is proved. □

In order to describe $Hoch^*(C)$ when C is not a smooth coalgebra, let us first deal with an example. Suppose then that C has a model of type $(B(W_0)\otimes\Lambda W_1,d)$ where $dim_k(W_0)=dim_k(P(C))=2$, $dim_k(W_1)=1$, and k is a characteristic zero field. Next we write the double complex whose total complex is used to compute $HH_*(C^*)$. Keeping the notation $V=W^*$, the bicomplex $\mathscr{C}_{**}=(\Lambda V\otimes\Lambda\bar{V},\partial)$ is obtained after identification $\Lambda V\otimes\Lambda\bar{V}\cong(\Lambda V\otimes\Lambda\bar{V}\otimes\Lambda V)\otimes_{(\Lambda\bar{V})^e}\Lambda\bar{V}$. It has components

$$\mathscr{C}_{pq}=\bigoplus_{s,j,i/i+j=p,s+i+2j=q}k[|V_0|]\otimes E^s(V_1)\otimes E^i(\bar{V}_0)\otimes S^j(\bar{V}_1),$$

$$\mathscr{C}_{**} = \begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & 0 & \leftarrow & k[|V_0|]\otimes V_1\otimes S^3\bar V_1 \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & \leftarrow & k[|V_0|]\otimes V_1\otimes S^2\bar V_1 & \leftarrow & \begin{pmatrix} k[|V_0|]\otimes S^3\bar V_1\oplus \\ k[|V_0|]\otimes V_1\otimes S^2\bar V_1\otimes \bar V_0\end{pmatrix} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \leftarrow & k[|V_0|]\otimes V_1\otimes \bar V_1 & \leftarrow & \begin{pmatrix} k[|V_0|]\otimes S^2\bar V_1\oplus \\ k[|V_0|]\otimes V_1\otimes \bar V_1\otimes \bar V_0\end{pmatrix} & \leftarrow & k[|V_0|]\otimes S^2\bar V_1\otimes \bar V_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k[|V_0|]\otimes V_1 & \leftarrow & \begin{pmatrix} k[|V_0|]\otimes \bar V_1\oplus \\ k[|V_0|]\otimes V_1\otimes \bar V_0\end{pmatrix} & \leftarrow & k[|V_0|]\otimes V_1\otimes E^2\bar V_0 & \leftarrow & k[|V_0|]\otimes \bar V_1\otimes E^2\bar V_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
k[|V_0|] & \leftarrow & k[|V_0|]\otimes \bar V_0 & \leftarrow & k[|V_0|]\otimes E^2\bar V_0 & \leftarrow & 0
\end{array}$$

where $|V_0|=0$, $|V_1|=1$ and $|\bar V_i|=|V_i|+1$. The vertical maps are induced by the differential in $(k[|V_0|]\otimes E(V_1))\otimes E(\bar V_0)\otimes S(\bar V_1)$, i.e. ∂ is a derivation satisfying $\partial(v)=d'(v)$ and $\partial(\bar v)=-\beta(d'(v))$ where β is also a derivation, such that $\beta(v)=\bar v$ and $\beta(\bar v)=0$.

The horizontal maps are zero, because of the cocommutativity of C (the algebra ΛV is graded commutative and the horizontal differential is essentially the graded commutator).

Remarks. (1) The homology of the total complex is the direct sum of the homologies of the columns.

(2) Consider the elements of $k[|V_0|]\otimes\bar V_1\otimes E^2(\bar V_0)$ (in degree 4), they are all cycles. Then look at the image of $k[|V_0|]\otimes S^2\bar V_1\otimes\bar V_0$ by ∂:

$$\begin{aligned}\partial(\bar v_1\bar v_1'\bar v_0) &= \partial(\bar v_1)\bar v_1'\bar v_0+\bar v_1\partial(\bar v_1')\bar v_0+\bar v_1\bar v_1'\partial(\bar v_0)\\ &= -\beta(d'v_1)\bar v_1'\bar v_0-\bar v_1\beta(d'v_1')\bar v_0+0.\end{aligned}$$

Using that $d'(V_1)\subset\langle (V_0)^2\rangle$, we have that $\beta(d'(V_1))\subset k[|V_0|]V_0\otimes\bar V_0$, and so

$$\partial(\bar v_1\bar v_1'\bar v_0)\in k[|V_0|]V_0\otimes\bar V_1\otimes E^2(\bar V_0).$$

This says that the space of cycles is bigger than the image of $k[|V_0|]\otimes S^2\bar V_1\otimes\bar V_0$ by ∂, and hence $HH_4(C^*)\neq 0$.

(3) A similar argument proves that $HH_{4+2k}(C^*)\neq 0$ for all $k\in\mathbb{N}$.

(4) Consequently, $Hoch^{4+2k}(C)\neq 0$ $\forall k\geqslant 0$.

Using arguments similar to those in [2,3], the above example can be generalized in order to obtain the following theorem:

Theorem 3.5. *If C is a cocommutative non smooth k-coalgebra with $dim_k(P(C))<\infty$, then for all $n\in\mathbb{N}$, there exists $m>n$ such that $Hoch^m(C)\neq 0$.*

Proof. Using Lemma 3.4, it is sufficient to prove that for all $n\in\mathbb{N}$, there exists $m>n$ such that $HH_m(C^*)\neq 0$. After Proposition 3.2, C admits a differential graded model of type $(B(P(C))\otimes \Lambda W, d)$ with $W=\bigoplus_{n\geqslant 1} W_n$. Since C is not smooth, C cannot be isomorphic to $B(P(C))$, and this is the same as saying that $W_1\neq 0$ (the homology in degree zero of the model is the Kernel of the first differential $B(P(C))\to B(P(C))\otimes W_1$).

In addition, the primitive elements of C can be identified with the primitive elements of the model. Following Corollary 1.6 and the remark after it, the translation of this property in the model of C^* is that $d(W_1^*)\subset (P(C)^*)^2\subset k[|P(C)^*|]\otimes \Lambda W^*$. This condition is exactly the one needed in order to construct non zero elements on infinitely many degrees in $HH_*(C^*)$ (see [2] or [3]). As a consequence, given n, there exists $m>n$ such that $Hoch^m(C)\neq 0$. □

Corollary 3.6. *Let k be a field (not necessarily algebraically closed) and C a cocommutative k-coalgebra. If* $dim_{\bar{k}}(\bar{k}e\wedge \bar{k}e)<\infty$ *for all group-like element* $e\in C\otimes_k \bar{k}$ *and C is not k-smooth, then for all* $n\in\mathbb{N}$ *there exists* $m>n$ *such that* $Hoch^m(C)\neq 0$.

Proof. One can assume that $k=\bar{k}$ because $Hoch^*(C|k)\otimes\bar{k}=Hoch^*(C\otimes\bar{k}|\bar{k})$. Also we can suppose that C is irreducible, because if not, write $C=\bigoplus_i C_i$ with C_i irreducible. We know (see for example [9]) that $Hoch^*(\bigoplus_i C_i)=\bigoplus_i Hoch^*(C_i)$. If all C_i were smooth, then (Proposition 1.4) C would be smooth too, so there must be at least one C_{i_0} not smooth. Now use the Theorem above and conclude that there exists $m>n$ such that $Hoch^m(C_{i_0})\neq 0$. □

Remarks. (1) The proof of Theorem 3.5 relies on the main results of [2,3]. This says in addition that the nonzero homology groups that one finds have all the same parity. As a consequence, if C is a locally finite coalgebra such that there exists an odd number i and an even number j with $Hoch^i(C)=0=Hoch^j(C)$, then C is smooth.

(2) If C is cocommutative irreducible non smooth, and $dim(P(C))<\infty$, then the algebra C^* is not smooth.

We comment now how the results obtained up to now can be used to give a proof of Theorem A stated in the introduction.

It is clear that 4 implies 3. The decomposition of a cocommutative coalgebra into irreducible components plus part 4 of Proposition 1.3 plus the Structure Theorem 1.7 give 1 implies 2.

$2\Rightarrow 3$ and $2\Rightarrow 4$ are the computations of Section 2.

Finally $3\Rightarrow 1$ is Theorem 3.5 above.

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