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Prime spectra of lattice-ordered abelian groups

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Abstract

We prove that for each ℓ-group G, the topological space $\mathsf{Spec}(G)$ satisfies a condition Idω. Generalising a previous construction of Delzell and Madden we show that for each nondenumerable cardinal there is a completely normal spectral space that is not homeomorphic to $\mathsf{Spec}(G)$ for any ℓ-group G. We show also that a stronger form of property Idω, called Id, suffices to ensure that a completely normal spectral space is homeomorphic to $\mathsf{Spec}(G)$ for some ℓ-group G.

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0. Introduction

By a *generalized spectral space* we understand a topological space X fulfilling the following properties:

(S0) X is T_0, i.e., for each pair of distinct points of X, at least one has an open neighborhood not containing the other.

(S1) The set $\mathbb{D}(X)$ of quasi-compact open subsets of X is a lattice under union and intersection, and constitutes a basis for the topology of X.

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(S2) The space X is *sober*, i.e., if a nonempty closed subset F is not the closure of a singleton $\{x\}$, then there are closed sets F_1, F_2 such that $F=F_1\cup F_2$ and $F_1\neq F$, $F_2\neq F$.

Quasi-compact generalized spectral spaces are known in the literature as *spectral spaces*.

A generalized spectral space is called *completely normal* if whenever points x and y are in the closure of a singleton $\{z\}$, then either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$.

Let G be a lattice-ordered *abelian* group, or ℓ-group for short. It is well known (see, for instance, [2, Ch. 10]), that the set $\mathsf{Spec}(G)$ of prime ℓ-ideals of G equipped with the topology having as a basis the sets

$$\hat{g}=\{P\in \mathsf{Spec}(G)\mid g\notin P\},\quad \text{for } g\in G$$

is a completely normal generalized spectral space. This space is quasi-compact if and only if G has a strong order unit.

Moreover one has [2, Proposition 10.1.3]

(IL) *The lattice of ℓ-ideals of an ℓ-group G is isomorphic to the lattice of open subsets of* $\mathsf{Spec}(G)$.

In this paper we prove (Theorem 2.2) that for each ℓ-group G, $\mathsf{Spec}(G)$ satisfies a topological condition that we call (Idω). Moreover, we show that each nondenumerable set admits a structure of a completely normal spectral space not satisfying Property (Idω). In other words, we show that from each nondenumerable cardinal we can obtain an example of a completely normal spectral space that is not homeomorphic to $\mathsf{Spec}(G)$ for any ℓ-group G. In this way we simplify a previous construction of Delzell and Madden [8].

We also show (Theorem 3.3) that a stronger form of Property (Idω), called (Id), suffices to ensure that a completely normal generalized spectral space is homeomorphic to $\mathsf{Spec}(G)$ for some ℓ-group G. Our proof is based on the properties of the *Priestley power of a totally ordered group on a completely normal generalized spectral space.*

These Priestley powers were introduced in [9] as a common generalization of Boolean [3, Ch. IV, Section 5] and Hahn [5] powers of totally ordered groups. Our proof of Theorem 3.3 shows that each completely normal generalized spectral space X satisfying the (Id) property is homeomorphic to the spectrum of the Priestley power of any nontrivial totally ordered archimedean group over X. On the other hand, in Section 5 we construct an ℓ-group G such that $\mathsf{Spec}(G)$ does not satisfy Property (Id).

In Section 4 we give a new proof of the well-known characterization of those posets that are induced by $\mathsf{Spec}(G)$ for some ℓ-group G.

In the definition of Priestley powers an important role is played by the *patch topology* over a generalized spectral space. For the convenience of the reader, the relations between the spectral and the patch topologies needed in this paper are given in some detail in Section 1.

1. Generalized spectral spaces

Let X be a generalized spectral space. The sets of the form $U \setminus V$, for U, V in $\mathbb{D}(X)$ form a basis for a Hausdorff topology on X, called *the patch topology on* X (see [11, 14]). Whenever we refer to topological properties such as open, quasi-compact, etc., if we do not specify which topology on X we mean, then we shall mean the original (so-called *spectral*) topology, and not the patch topology (unless these topologies happen to be the same).

The next lemma is in the folklore of the theory of spectral spaces. For lack of a precise reference, we give a proof of it.

Lemma 1.1. *Let X be a topological space such that the quasi-compact open subsets are closed under finite intersection and form a basis for the open sets. Then the following are equivalent*:

(i) *X is sober.*

(ii) *Each quasi-compact open subset of X is compact in the patch topology.*

(iii) *If $\mathscr{C}$ is a collection of quasi-compact open subsets of X and F is a closed subset of X such that the collection $\mathscr{C} \cup \{F\}$ has the finite intersection property, then $F \cap \bigcap \mathscr{C} \neq \emptyset$.*

Proof. Suppose (i) holds true and let U be a quasi-compact open subset of X. The intersections of U with the quasi-compact open and with the closed subsets of X form a subbasis for the closed sets of U as a subspace of X with the patch topology. Therefore, the patch-compactness of U will follow if every collection $\mathscr{C}$ of closed subsets and quasi-compact open subsets of X such that $U \in \mathscr{C}$ and $\mathscr{C}$ is maximal with respect to having the finite-intersection property, has a nonempty intersection (cf. [11, Theorem 1]). Let $\mathscr{C}$ be such a collection, and let $\mathscr{C}_0$ denote the subcollection of $\mathscr{C}$ formed by the elements that are closed in X. Note that $\mathscr{C}_0 \neq \emptyset$, because $X \in \mathscr{C}_0$. Let $F_0 = \bigcap \mathscr{C}_0$. For each quasi-compact open $V \in \mathscr{C}$, $\{U \cap V \cap F \mid F \in \mathscr{C}_0\}$ is a collection of closed subsets of the quasi-compact space $U \cap V$ with the finite intersection property. Therefore $U \cap V \cap F_0 \neq \emptyset$ for each quasi-compact open $V \in \mathscr{C}$. In particular, $F_0 \neq \emptyset$, and the maximality of $\mathscr{C}$ implies that F_0 cannot be the union of two proper closed subsets. Hence, by (i), there is a $z \in X$ such that $F_0 = \overline{\{z\}}$, and it is easy to check that $z \in \bigcap \mathscr{C}$. This shows that (i) implies (ii). To prove that (ii) implies (iii), suppose that (ii) holds, and let $\mathscr{C}$ be a collection of quasi-compact open subsets of X and F be a closed subset of X such that the collection $\mathscr{C} \cup \{F\}$ has the finite intersection property. We may assume $\mathscr{C} \neq \emptyset$. Take $U_0 \in \mathscr{C}$. Then $\mathscr{C}_0 := \{U_0 \cap F \cap U \mid U \in \mathscr{C}\}$ is a collection of patch-closed subsets of the patch-compact U_0, whence $\emptyset \neq \bigcap \mathscr{C}_0 = F \cap \bigcap \mathscr{C}$. Finally, to prove that (iii) implies (i) assume that (iii) holds, and let F be a nonempty closed subset of X. Suppose that F is not the closure of a singleton $\{x\}$. Then for each $x \in F$ we can find a quasi-compact open U_x such that $x \notin U_x$ and $U_x \cap F \neq \emptyset$. Since $F \cap \bigcap_{x \in F} U_x = \emptyset$, by (iii) there are a finite number of points in F, say $x_1, \ldots, x_n$ such that $F = (F \cap (X \backslash U_{x_1})) \cup \cdots \cup (F \cap (X \backslash U_{x_n}))$, and F is a finite union of proper closed subsets. □

Generalized spectral spaces were introduced by Stone in 1937 [15] with the aim of extending to distributive lattices his celebrated theorem on the representation of Boolean algebras. To be precise, he considered topological spaces satisfying properties (S0), (S1) and (iii) in Lemma 1.1.

The next proposition establishes the connections between general spectral spaces and distributive lattices with smallest element in the way most relevant to the purposes of this paper.

For each topological space X, the set of open subsets of X is a distributive lattice under union and intersection, which we denote by $\mathcal{O}(X)$.

Proposition 1.2. *The generalized spectral spaces X and Y are homeomorphic if and only if the lattices $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic.*

Proof. The "only if" part is trivial. To prove the "if" part, observe that the lattice $\mathcal{O}(X)$ uniquely determines the sublattice $\mathbb{D}(X)$ of quasi-compact open sets. Hence, if the lattices $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic, then the lattices $\mathbb{D}(X)$ and $\mathbb{D}(Y)$ are isomorphic, and by a classical result of Stone [15] (see also [1, Ch. IV] or [10, Section 11]), we obtain that the spaces X and Y are homeomorphic. □

The following result is an immediate consequence of the above proposition and Property (IL).

Corollary 1.3. *Let G be an ℓ-group and X be a generalized spectral space. Then X is homeomorphic to* $\mathsf{Spec}(G)$ *if and only if $\mathcal{O}(X)$ is isomorphic to the lattice of ℓ-ideals of G.*

Given a subset A of a partially ordered set $\langle X, \leq \rangle$, the initial (final) section of A is the set $A\!\downarrow := \{x \in X \mid x \leq a \text{ for some } a \in A\}$ ($A\!\uparrow := \{x \in X \mid a \leq x \text{ for some } a \in A\}$). A subset of X is called *decreasing* (*increasing*) provided it coincides with its initial (final) section.

A *root system* is a partially ordered set such that the final section of each of its elements is totally ordered.

If X is now a generalized spectral space, then X is T_0, so that the relation defined by $x \leq y$ if and only if y belongs to the closure of the singleton $\{x\}$ ($x, y \in X$), is a partial order relation, called the *specialization* (*partial*) *order*.

Note that a generalized spectral space X is completely normal if and only if X, endowed with the specialization order, is a root system.

It is well known and easy to check that for each ℓ-group G, the specialization order of $\mathsf{Spec}(G)$ coincides with the set theoretical inclusion of prime ℓ-ideals of G.

In what follows, we shall consider all generalized spectral spaces as being equipped with the specialization order. Then for each subset A of a generalized spectral space

X, $A\downarrow$ ($A\uparrow$) will always mean the initial (final) section of A in the specialization order of X. Analogously, we shall say that A is decreasing (increasing) if it is decreasing (increasing) in the specialization order of X.

Note that the open subsets of a generalized spectral space are decreasing and the closed subsets are increasing.

Spectral spaces equipped with the patch topology and the specialization order are called *Priestley spaces* (see [7, 14]).

Lemma 1.4. *Let X be a generalized spectral space. An open (closed) set in the patch topology of X is open (closed) in the spectral topology if and only if it is decreasing (increasing).*

Proof. We already noted that the open sets in the spectral topology are decreasing. Suppose that $Y\subseteq X$ is open in the patch topology and decreasing. We want to show that for each $x\in Y$, there is $W\in\mathbb{D}(X)$ such that $x\in W\subseteq Y$. Since Y is open in the patch topology, there are U, V in $\mathbb{D}(X)$ such that $x\in U\backslash V\subseteq Y$. If $U\subseteq Y$, we can take $W=U$. Otherwise $Z:=U\cap(X\backslash Y)$ is nonempty and compact in the patch topology. Let $t\in Z$. Since Y is decreasing, $t\not\leq x$, and then there is $W_t\in\mathbb{D}(X)$ such that $x\in W_t$ and $t\notin W_t$. By the patch-compactness of Z we conclude that there is a finite number of elements in Y, say $t_1,\ldots,t_n$ such that $Z\subseteq(X\backslash W_{t_1})\cup\cdots\cup(X\backslash W_{t_n})$, and we can take $W=U\cap W_{t_1}\cap\cdots\cap W_{t_n}$. To complete the proof note that A is closed in the patch topology and increasing if and only if $X\backslash A$ is open in the patch topology and decreasing. □

The following property can be easily proved (see [13, p. 509]):

(IFC) Let X be a generalized spectral space. If $A\subseteq X$ is compact in the patch topology, then the initial and final sections of A are both closed in the patch topology.

We shall consider generalized spectral spaces X satisfying the following property:

(Id) For U, V in $\mathbb{D}(X)$, the set $(U\backslash V)\downarrow$ is quasi-compact open.

Property (Id) – which stands for "interior-decreasing" – has a nice lattice theoretical interpretation.

A *generalized op-Heyting algebra* is a lattice L such that, for each $a,b\in L$, there exists the element

$$a*b:=\min\{c\mid a\vee c\geq b\}$$

By dualizing an argument on p. 174 of [1], one obtains that such a lattice L is necessarily distributive, and has a smallest element $0:=a*a$, for any $a\in L$.

An *op-Heyting algebra* is a generalized op-Heyting algebra having a greatest element.

Observe that the order duals of these lattices are, respectively, the relatively pseudocomplemented lattices and the Heyting algebras (see [1, Ch. IX]).

A generalized spectral space X satisfies property (Id) if and only if the lattice $\mathbb{D}(X)$ is a generalized op-Heyting algebra. Indeed, if $U,\ V \in \mathbb{D}(X)$, then it is easy to check that $U * V$ must be defined as $(V \backslash U)\downarrow$.

Since the sets of the form $U\backslash V$, with $U,\ V$ in $\mathbb{D}(X)$ constitute a basis for the open sets in the patch topology, we have the following property:

(PI) A generalized spectral space X satisfies property (Id) if and only if the initial sections of patch-open sets are open.

We are going to close this section with a class of examples of completely normal spectral spaces that will play an important role in the rest of the paper.

Example 1.5. Let Z be an infinite set, and α, β be two distinct elements not belonging to Z. We shall denote by $\mathsf{S}(Z)$ the spectral space obtained by equipping the set $Z \cup \{\alpha, \beta\}$ with the topology having the following open sets:
(1) All subsets of Z,
(2) All sets of the form $Y \cup \{\alpha\}$, where Y is a cofinite subset of Z, and
(3) All sets of the form $Y \cup \{\alpha, \beta\}$, where Y is a cofinite subset of Z.
Since for each $x \in Z \cup \{\beta\}$, the singleton $\{x\}$ is closed, and the closure of $\{\alpha\}$ is $\{\alpha, \beta\}$, $\mathsf{S}(Z)$ is a completely normal spectral space. The quasi-compact open sets are the finite subsets of Z and the sets of the forms (2) and (3). Since $((Z \cup \{\alpha, \beta\}) \backslash (Z \cup \{\alpha\}))\downarrow = \{\alpha, \beta\}$, the space $\mathsf{S}(Z)$ does not satisfy Property (Id).

2. The property (Idω)

In this section we are going to consider a weaker form of Property (Id), which we call Property (Idω).

We are going to use the following notation. For each element g of an ℓ-group G,

$$\check{g} := \mathsf{Spec}(G) \backslash \hat{g} = \{P \in \mathsf{Spec}(G) \mid g \in P\}.$$

Lemma 2.1. *Let G be an abelian ℓ-group. For any $g, h \in G^+$ there exists a sequence $(g_n)_{n \in \omega} \subseteq G^+$ such that*

$$g_0 \geq g_1 \geq \cdots \geq g_n \geq g_{n+1} \geq \cdots$$

and

$$(\hat{g} \cap \check{h})\downarrow = \bigcap_{n \in \omega} \hat{g}_n.$$

Proof. For each $n \in \omega$, define

$$g_n = (g - nh)^+.$$

Since for all $n \in \omega$, $P \in \check{h}$ and $P \notin \hat{g}_n$ imply $0 \leq g \leq g \vee nh = g_n + nh \in P$, one has that $\hat{g} \cap \check{h} \subseteq \hat{g}_n$, and since each $\hat{g}_n$ is a decreasing set, it follows that $(\hat{g} \cap \check{h})\downarrow \subseteq \bigcap_{n \in \omega} \hat{g}_n$.

To prove the other inclusion, suppose that $P \notin (\hat{g} \cap \check{h}) \downarrow$. We claim that g belongs to the ℓ-ideal J generated in G by P and h. Indeed, if $g \notin J$, then there would be a prime ℓ-ideal Q such that $J \subseteq Q$ and $g \notin Q$, and this would imply that $P \subseteq Q \in \hat{g} \cap \check{h}$, a contradiction.

Since $g \in J$, there are $p \in P^+$ and $n_0 \in \omega$ such that $0 \leq g \leq p + n_0 h$, and this implies that $g_{n_0} = (g - n_0 h) \vee 0 \leq p \vee 0 = p \in P$. Therefore $P \notin \hat{g}_{n_0} \supseteq \bigcap_{n \in \omega} \hat{g}_n$. □

Since $U \in \mathbb{D}(\mathsf{Spec}(G))$ if and only if $U = \hat{g}$ for some $g \in G$, one has the following theorem.

Theorem 2.2. *For each ℓ-group G, the completely normal generalized spectral space* $\mathsf{Spec}(G)$ *satisfies the following condition*:

(Idω) *If U and V are quasi-compact open subsets of X, then there is a sequence $\{W_n\}_{n \in \omega}$ of quasi-compact open subsets of X such that*

$$W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n \supseteq W_{n+1} \supseteq \cdots$$

and

$$(U \backslash V) \downarrow = \bigcap_{n \in \omega} W_n.$$

It is easy to check that the completely normal spectral space $\mathsf{S}(Z)$ constructed in Example 1.5 satisfies property (Idω) if and only if Z is a denumerable set. Hence when Z is nondenumerable, the space $\mathsf{S}(Z)$ is an example of a completely normal spectral space that is not homeomorphic to $\mathsf{Spec}(G)$ for any ℓ-group G (cf. the example given by Delzell and Madden [8]).

3. Priestley powers

Let $\langle X, \leq \rangle$ be a root system and H be a totally ordered abelian group. Given a function $f: X \to H$, define its *support* as

$$\mathsf{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

We shall need to consider also the following subsets involving the maximal elements of $\mathsf{supp}(f)$:

$$\mathsf{ms}(f) := \{x \in \mathsf{supp}(f) \mid \text{ for each } y \in X, \text{ if } y > x, \text{ then } f(y) = 0\},$$

$$\mathsf{ms}_+(f) := \{x \in \mathsf{ms}(f) \mid f(x) > 0\}, \quad \mathsf{ms}_-(f) := \{x \in \mathsf{ms}(f) \mid f(x) < 0\},$$

$$\mathsf{supp}_+(f) := (\mathsf{ms}_+(f) \downarrow) \cap \mathsf{supp}(f) \text{ and } \mathsf{supp}_-(f) := (\mathsf{ms}_-(f) \downarrow) \cap \mathsf{supp}(f).$$

Recall that the *Hahn power of H over X*, denoted by $\mathsf{V}(X, H)$, is the ℓ-group of all functions $f: X \to H$ such that $\mathsf{supp}(f)$ satisfies the ascending chain condition, with addition defined pointwise and with the *lexicographic order*, i.e., $f > 0$ if and only if $\mathsf{ms}_+(f) = \mathsf{ms}(f)$ (see [5; 2, p. 27; 6, Theorem 51.3]). Recall also that for each $f \in \mathsf{V}(X,H)$ one has:

(1) $\mathsf{supp}(f^+) = \mathsf{supp}_+(f)$ and $f^+(x) = f(x)$ for each $x \in \mathsf{supp}(f^+)$
(2) $\mathsf{supp}(f^-) = \mathsf{supp}_-(f)$ and $f^-(x) = -f(x)$ for each $x \in \mathsf{supp}(f^-)$.

Let X be a completely normal generalized spectral space, and let H be a totally ordered abelian group. The set of all continuous functions with quasi-compact support from X, endowed with the patch topology, to H, endowed with the discrete topology, is a group under pointwise addition, which we denote by $\mathsf{Cont}_0(X,H)$. The set X equipped with the specialization order is a root system, and it follows from Lemma 3 (i) of [9] that for each $f \in \mathsf{Cont}_0(X,H)$, $\mathsf{supp}(f) \subseteq \mathsf{ms}(f)\downarrow$. Then one can define the lexicographic order on $\mathsf{Cont}_0(X,H)$, to obtain an ordered group. This ordered group is called the *Priestley power of H over X* (see [9] for details).

Proposition 4 in [9] asserts that Property (Id) implies that $\mathsf{Cont}_0(X,H)$, ordered lexicographically, is an ℓ-group. More precisely, the proof there shows that if Property (Id) holds in X, then for each $f \in \mathsf{Cont}_0(X,H)$, $\mathsf{supp}_+(f)$ is compact open in the patch topology of X. This implies, using Lemma 1 (ii) (the "patchwork property") of [9], that the function $f' : X \to H$ defined by the stipulation

$$f'(x) = \begin{cases} f(x) & \text{if } x \in \mathsf{supp}_+(f), \text{ and } 0 \\ 0 & \text{otherwise,} \end{cases}$$

is the supremum of f and 0 in the lexicographic order of $\mathsf{Cont}_0(X,H)$. From this one concludes that $\mathsf{Cont}_0(X,H)$ is an ℓ-group that satisfies conditions (1) and (2).

Note that since $|f| = f^+ + f^-$, it follows from (1) and (2) that

(3) for each $f \in \mathsf{Cont}_0(X,H), \mathsf{supp}(|f|) = \mathsf{supp}(f)$.

Suppose that the compact open sets in the patch topology of X satisfy the ascending chain condition. If Property (Id) hold in X, then for each $f \in \mathsf{V}(X,H) \cap \mathsf{Cont}_0(X,H)$ we have that $f \vee_{\mathsf{V}} 0$ and $f \vee_{\mathsf{C}} 0$ are both given by (1), where $\vee_{\mathsf{V}}$ and $\vee_{\mathsf{C}}$ denote, respectively, the join operation in $\mathsf{V}(X,H)$ and in $\mathsf{Cont}_0(X,H)$. Hence we have that when the compact open sets in the patch topology of X satisfy the ascending chain condition, Property (Id) guarantees that the lattice structures of the Priestley power $\mathsf{Cont}_0(X,H)$ and of the Hahn power $\mathsf{V}(X,H)$ are compatible in the following sense:

(C) If f, g are in $\mathsf{V}(X,H) \cap \mathsf{Cont}_0(X,H)$, then $f \vee_{\mathsf{V}} g = f \vee_{\mathsf{C}} g$.

Remark. As a matter of fact, if $H \neq (0)$, then Property (Id) is equivalent to the property that for each $f \in \mathsf{Cont}_0(X,H)$, $\mathsf{supp}_+(f)$ is compact open in the patch topology of X. Indeed, suppose that this property holds, and let $U = V \setminus W$, where V, W are quasi-compact open subsets in the spectral topology of X. Define $f : X \to H$ by the following stipulations, where a denotes a positive element of H:

$$f(x)=\begin{cases} a & \text{if } x\in U, \\ -a & \text{if } x\in V\backslash U, \\ 0 & \text{if } x\notin V. \end{cases}$$

It is plain that $f\in \mathsf{Cont}_0(I,H)$, and since every maximal element of U is a maximal element of V, we have that $\mathsf{ms}_+(f)$ is exactly the set of maximal elements of U. Therefore, $\mathsf{supp}_+(f)=(\mathsf{ms}_+(f)\downarrow)\cap \mathsf{supp}(f)=U\downarrow\cap V=U\downarrow$. Then the hypothesis implies that $U\downarrow$ is compact open in the patch topology of X. Since $U\downarrow$ is decreasing, it is also open in the spectral topology, proving Property (Id).

The next example shows that in the absence of Property (Id), the lexicographic order can define a lattice structure on $\mathsf{Cont}_0(I,H)$ that is not compatible with $\mathsf{V}(I,H)$.

Example 3.1. Let $X=\mathsf{S}(\omega)$ (see Example 1.5). It is obvious that all subsets of X satisfy the ascending chain condition with respect to the specialization order. If f, g are in $\mathsf{Cont}_0(X,\mathbb{Z})$, then $f(x)\geq g(x)$ for all $x\in X$ if and only if $f(n)\geq g(n)$ for all $n\in\omega$ and $f(\beta)\geq g(\beta)$. Therefore, the lexicographic order coincides with the pointwise order of functions, and $\mathsf{Cont}_0(X,\mathbb{Z})$ is an ℓ-subgroup of the power ℓ-group $\mathbb{Z}^X$. Let $f: X\to\mathbb{Z}$ be defined by the prescription

$$f(x)=\begin{cases} 1 & \text{if } x=\beta, \\ -1 & \text{if } x\in\omega\cup\{\alpha\}. \end{cases}$$

One has that $f\in\mathsf{Cont}_0(X,\mathbb{Z})\cap\mathsf{V}(X,\mathbb{Z})$, but $\mathsf{supp}(f\vee_{\mathsf{C}}0)=\{\beta\}$ and $\mathsf{supp}(f\vee_{\mathsf{V}}0)=\{\alpha,\beta\}$.

Given a generalized spectral space X, we denote by $\mathscr{F}(X)$ the lattice formed by the closed subsets of X, ordered by inclusion.

Lemma 3.2. *Let X be a completely normal generalized spectral space satisfying condition (Id), and H be a nontrivial archimedean totally ordered group. If G denotes the Priestley power* $\mathsf{Cont}_0(X,H)$, *then the map*

$$T\mapsto J(T)=\{g\in G\mid g(x)=0 \quad \textit{for each } x\in T\}$$

defines an anti-isomorphism of the lattice $\mathscr{F}(X)$ onto the lattice $\mathscr{I}(G)$ of ℓ-ideals of G. The inverse anti-isomorphism is given by

$$I\mapsto Z(I)=\{x\in X\mid f(x)=0 \quad \textit{for each } f\in I\}.$$

Proof. Throughout this proof, we consider X endowed with the patch topology. Therefore *open* will mean *open in the patch topology*, and similarly for *closed* and *compact*. Then by Lemma 1.4 the elements of $\mathscr{F}(X)$ are the closed increasing subsets of X.

Since the topology of H is discrete, it follows that $\mathsf{supp}(f)$ is open for each $f \in G$. Then by properties (IFC) and (PI) one has that

(4) $\mathsf{supp}(f)\downarrow$ is closed and open for each $f \in G$.

Choose a strictly positive element $a \in H$. For each compact open set V, f_V will denote the function taking the value $a > 0$ on V and 0 on $X \setminus V$.

Let $S, T \in \mathscr{F}(X)$. It is plain that $J(S)$ is a subgroup of G. Suppose that $0 \leq f \in J(S)$, and that $0 \leq g \notin J(S)$. Then there is $x \in \mathsf{ms}(g) \cap S$, and since S is increasing, $x \nleq y$ for each $y \in \mathsf{supp}(f)$. Hence $x \in \mathsf{ms}(f-g)$, and $f(x) - g(x) = -g(x) < 0$, i.e., $g \nleq f$. Since by (3) $f \in J(S)$ if and only if $|f| \in J(S)$, we have shown that $J(S)$ is absolutely convex. It is obvious that $S \subseteq T$ implies that $J(T) \subseteq J(S)$. Hence we have proved that J is an order-reversing mapping of $\mathscr{F}(X)$ into $\mathscr{I}(G)$.

Let I, $J \in \mathscr{I}(G)$. It is plain that $Z(I)$ is closed in X. Then to prove that $Z(I) \in \mathscr{F}(X)$ it is sufficient to show that $Z(I)$ is increasing. Let $s \in Z(I)$ and suppose that there is $t \in X$ such that $t > s$ and $t \notin Z(I)$. Then there is $g \in I$ such that $g \leq 0$ and $t \in \mathsf{supp}(g)$. By hypothesis, $s \notin \mathsf{supp}(g)$ and, since $\mathsf{supp}(g)$ is quasi-compact and the quasi-compact open sets form a basis of the patch topology of X, we can find a quasi-compact open set U such that $s \in U$ and $U \cap \mathsf{supp}(g) = \emptyset$. Since the patch topology is Hausdorff, all compact sets are closed. Hence by (4), $V := U \cap (\mathsf{supp}(g)\downarrow)$ is a closed and open subset of the compact set U. Therefore V is a compact open set containing s. It is easy to check that $0 \leq f_V \leq g$. But this contradicts the hypothesis that $s \in Z(I)$. Therefore, we have that $Z(I)$ is an increasing subset of X. It is obvious that $I \subseteq J$ implies that $Z(J) \subseteq Z(I)$. Hence we have proved that Z is an order-reversing mapping of $\mathscr{I}(G)$ into $\mathscr{F}(X)$.

To complete the proof we are going to show that $ZJ = id_{\mathscr{F}(X)}$ and $JZ = id_{\mathscr{I}(G)}$.

Let $S \in \mathscr{F}(X)$. It is plain that $S \subseteq Z(J(S))$. To prove the other inclusion, suppose that $z \notin S$. Then there is a compact open set $V \subseteq X$ such that $z \in V$ and $V \cap S = \emptyset$. Hence $f_V \in J(S)$, and since $f_V(z) \neq 0$, we have that $z \notin Z(J(S))$. Therefore $Z(J(S)) = S$.

Let $I \in \mathscr{I}(G)$. Since it is obvious that $I \subseteq J(Z(I))$, we need only to prove the other inclusion. Let $0 \leq g \in J(Z(I))$. Since $\mathsf{supp}(g) \cap Z(I) = \emptyset$, for each $x \in \mathsf{ms}(g)$ there is a function $f_x \in I$ such that $x \in \mathsf{supp}(f_x)$. By (4), the sets $U_x = \mathsf{supp}(f_x)\downarrow$ are open. Since they are obviously decreasing, $\mathsf{supp}(g) \subseteq \bigcup_{x \in \mathsf{ms}(g)} U_x$. Since $\mathsf{supp}(g)$ is compact, there is a finite number of elements $x_1, \ldots, x_n$ in $\mathsf{ms}(g)$ such that $\mathsf{ms}(g) \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. Let $h = |f_{x_1}| \vee \cdots \vee |f_{x_n}| \in I$. Since the functions g and h can take only a finite number of values, and the totally ordered group H is archimedean, there is a natural number n such that for any $x \in \mathsf{ms}(g)$, there is $y \geq x$ and $nh(y) > g(x)$. It follows that $nh \geq g$, and since $nh \in I$, we have that $g \in I$. Finally, since each function in G is the difference of two positive functions, we conclude that $J(Z(I)) \subseteq I$. □

Theorem 3.3. *Given a completely normal generalized spectral space X satisfying Property* (Id), *there is a lattice-ordered abelian group G such that* $\mathsf{Spec}(G)$ *is homeomorphic to X.*

Proof. Suppose G is the Priestley power of a nontrivial totally ordered archimedean group over X. Then by the above lemma, the correspondence $A \mapsto J(X \setminus A)$ defines an isomorphism of $\mathcal{O}(X)$ onto the lattice of ℓ-ideals of G, and then by Corollary 1.3, $\mathsf{Spec}(G)$ is homeomorphic to X. □

4. Spectral root systems

A *spectral root system* is a root system $(X, \leq)$ fulfilling the following two conditions:

RS1 Each totally ordered subset of X has supremum and infimum in X.

RS2 If x, y are elements of X such that $x < y$, then there are s, t in X such that $x \leq s < t \leq y$, and there is no element of X between s and t.

It is well known that the set of prime ℓ-ideals of an ℓ-group, ordered by inclusion, is a spectral root system.

Given a spectral root system $(X, \leq)$, the set of elements of X having a successor will be denoted by X^-. Note that each $x \in X^-$ has exactly one successor. We call X^* the set $X^- \cup \max X$, where $\max X$ denotes the set of maximal elements of $(X, \leq)$.

It is not hard to see that X, endowed with the topology generated by the sets $\{y\}\downarrow$, for $y \in X^*$, is a generalized spectral space that satisfies the property (Id). Moreover, the specialization order induced by this topology coincides with the original order of X (and this topology is the finest one that one can define on X inducing this order). Then X is completely normal.

Consequently, the following well known result (see [4; 12, Theorem 3.4]) is a particular case of Theorem 3.3.

Corollary 4.1. *A partially ordered set is isomorphic to the set of prime ℓ-ideals of an ℓ-group, ordered by inclusion, if and only if it is a spectral root system.*

Remark. Let $X = \mathsf{S}(\omega)$ be the spectral space considered in Example 3.1. The Priestley power $\mathsf{Cont}_0(X, \mathbb{Z})$ can be identified with the ℓ-subgroup of $\mathbb{Z}^\omega$ of all sequences that are constant on a final segment of ω. Therefore $\mathsf{Cont}_0(X, \mathbb{Z})$ is a hyperarchimedean ℓ-group [2, Cor. 14.1.4] and consequently all its prime ℓ-ideals are maximal [2, Theorem 14.1.2]. Therefore $\mathsf{Spec}(\mathsf{Cont}_0(X, \mathbb{Z}))$ is not homeomorphic to X.

5. A spectrum without the (Id) property

We are going to finish the paper with an example of an ℓ-group G such that $\mathsf{Spec}(G)$ does not satisfy (Id).

Let $S = \mathsf{S}(\omega)$ as in Example 3.1. We shall write the elements of the Hahn power $\mathsf{V}(S, \mathbb{Z})$ in the form $((x_n)_{n \in \omega}, x_\alpha, x_\beta)$.

Let G be the subgroup of $\mathsf{V}(S,\mathbb{Z})$ generated by the functions

$$a=((n)_{n\in\omega},0,1),\quad b=((1,1,\ldots,1,\ldots),1,0)$$

and the so-called "finite" functions, by which we mean those whose support is a finite subset of ω.

The elements of G are of the form $g=f+pa+qb$, where f is finite and p and q are integers. We are going to see now that for each $g\in G$, $g^+\in G$. By (1), we have that $\mathsf{supp}(g^+)=\mathsf{supp}_+(g)$ and $g^+(x)=g(x)$ for each $x\in\mathsf{supp}(g^+)$. We consider the following four possible cases:

Case 1: The coefficients p and q are both nonnegative. In this case, $\mathsf{ms}_+(g)$ and $\mathsf{ms}(g)$ differ by a finite subset of ω. Hence, $\mathsf{supp}_+(g)$ differs from $\mathsf{supp}(g)$ by this same subset. Therefore, g and g^+ can differ at most by a finite function, and then $g^+\in G$.

Case 2: $p>0$ and $q<0$. In this case $\mathsf{ms}_+(g)=\{n\in\omega\mid n>-q/p\}\cup\{\beta\}$ (modulo the support of a finite element), and then $g^+=f'+pa+qb$ for some finite f'. Therefore $g^+\in G$.

Case 3: $p<0$. In this case $np+q>0$ if and only if $n<-q/p$, whence g^+ is finite.

Case 4: $p=0$ and $q<0$. In this case g^+ is finite.

Since in all possible cases $g^+\in G$, we conclude that G is a ℓ-subgroup of $\mathsf{V}(S,\mathbb{Z})$.

Consider now the prime ℓ-ideals of G.

For each $x\in\omega\cup\{\beta\}$, $I_x=\{g\in G\mid g(x)=0\}$ is a prime ℓ-ideal of G (in fact, a maximal ℓ-ideal). It is plain that the set A of all finite functions is an ℓ-ideal of G. To see that it is prime, consider two orthogonal elements g and h. Since a non-finite element has a cofinal support (in ω), we conclude that at least one of those two elements belongs to A. Therefore A is a prime ℓ-ideal of G.

We are going to show now that there are no other prime ℓ-ideals in G. Indeed, suppose (for the sake of a contradiction) that I is a prime ℓ-ideal of G, such that $I\neq I_x$ for all $x\in\omega\cup\{\beta\}$ and $I\neq A$. We have two possible cases:

Case 1: All the elements of I are finite, i.e., $I\subseteq A$. In this case we obtain the contradiction $I=A$. For, suppose there exists a finite g such that $g\notin I$. Let n_0 be the greatest integer in $\mathsf{supp}(g)\subseteq\omega$, and let $h=((x_n)_{n\in\omega},1,0)$, where $x_n=0$ if $n\leq n_0$ and $x_n=1$ if $n>n_0$. Then h is an element of G that does not belong to I and is orthogonal to g. Since neither g nor h belong to I we conclude that I is not prime. Therefore $I=A$.

Case 2: I contains an element $g:=f+pa+qb$ (for some $f\in A$ and $p,q\in\mathbb{Z}$) of infinite support – i.e., either p or q (or both) are nonzero. In fact, I contains such a g with $p\neq 0$, since $I\neq I_\beta$ implies $I\nsubseteq I_\beta$. We may assume $p>0$. Then the set $N:=\{n\in\omega\mid g(n)\leq 0\}$ is finite. For each $n\in N$, pick $f_n\in I_n$ such that $f_n(n)>0$ (using the fact that $I\neq I_n$ implies $I\nsubseteq I_n$); and then pick $m_n\in\mathbb{Z}$ such that $m_nf(n)>|g(n)|$. Then $0<a<g\vee\bigvee_{n\in N}m_nf_n\in I$. Since a is a strong order unit of G, this implies that I is not proper, i.e., $I=G$, a contradiction.

Hence, the correspondence $x\mapsto I_x$, for each $x\in\omega\cup\{\beta\}$, and $\alpha\mapsto A$ is a one-to-one function from S onto $\mathsf{Spec}(G)$, and it is easy to check that it is a homeomorphism.

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