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Hochschild homology of some quantum algebras[1]

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Abstract

For a type of quantum algebras we obtain a chain complex, simpler than the canonical one, whose homology is the Hochschild homology of the algebra. We applied this result to some concrete examples, as $U_q(sl(2,k))$ and $\mathcal{O}_q(M(2,k))$. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let k be an arbitrary commutative ring and $0 \le r \le n$ be two integers. Every pair of families of parameters $\mathbf{P}$ and $\mathbf{Q}$ verifying suitable hypothesis has associated a k-algebra $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$, which is a quantum deformation of $k[x_1,\dots,x_r,x_{r+1}^{\pm 1},\dots,x_n^{\pm 1}]$. For instance, when $\mathbf{P}$ is null we obtain, taking $r=n$, the quantum multiparametric affine space, and taking $r=0$, the quantum multiparametric torus. On the other hand, taking n even and $\mathbf{P}$ appropriate, we obtain a quantum version of the Weyl algebra, which can be considered as an algebra of differential operators on the quantum multiparametric affine space. The Hochschild homology of this type of algebras has been studied in several papers, such as [1,2,4,7–9]. Many algebras related to quantum groups, for instance $U_q(sl(2,k))$, $\mathcal{O}_q(M(2,k))$ and $\mathcal{O}_{q^2}(sok^3)$, are Ore extensions of an algebra $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$. Other examples can be found in [3,5,6], etc. The main purpose of this work is to study the Hochschild homology and cohomology of this type of algebras.

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In the first section, for every $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$-module M we build up complexes $X_*(M)$ and $X^*(M)$, simpler than the canonical ones, whose homologies are respectively the Hochschild homology and cohomology of $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$ with coefficients in M. This result was also obtained (at least when k is a characteristic zero field) in [8, 9]. Our method is close to Wambst's method in his work on quantization of the Koszul complex. The main difference is that we give explicit quasi-isomorphisms

$$\theta_* : X_*(M) \to (M \otimes \overline{A}^{\otimes^*}, b_*) \quad \text{and} \quad \tau_* : (M \otimes \overline{A}^{\otimes^*}, b_*) \to X_*(M) \qquad (*)$$

and also for cohomology. This allows us to compute the De Rham cohomology of these algebras.

In Section 2, by using the quasi-isomorphisms $(*)$ we obtain complexes simpler than the canonical ones, whose homologies are respectively the Hochschild homology and cohomology of an Ore extension of $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$. We finish our paper studying some concrete examples.

1. The Hochschild (co)homology of $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$

Let k be an arbitrary commutative ring and $0 \le r \le n$ be two integers, $\mathbf{Q} = (q_{ij})_{1 \le i,j \le n}$ and $\mathbf{P} = (p_{ij})_{1 \le i < j \le n}$ two families of elements of k verifying $q_{ii} = 1$, $q_{ij}q_{ji} = 1$ for all $i < j$ and $p_{ij} = 0$ for all $j > r$. Let us denote $A = S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$ the k-algebra generated by $x_1, \dots, x_n, x_{r+1}^{-1}, \dots, x_n^{-1}$ and the relations $x_j x_j^{-1} = 1 = x_j^{-1} x_j$ $(r < j \le n)$ and $x_j x_i = q_{ij} x_i x_j + p_{ij}$ $(1 \le i < j \le n)$ and $A^e = A \otimes A^{\mathrm{op}}$ the enveloping algebra of A. As usual, we consider A as a left and a right A^e-module with the actions given by $(P_0 \otimes P_1).P = P_0 P P_1$ and $P.(P_0 \otimes P_1) = P_1 P P_0$, respectively. In this section we find a free resolution $X'_*(A)$ of A as a left A^e-module, simpler than the Hochschild resolution $(A \otimes \overline{A}^{\otimes^*} \otimes A, b'_*)$ and homotopy equivalence maps $\theta'_* : X'_*(A) \to (A \otimes \overline{A}^{\otimes^*} \otimes A, b'_*)$ and $\tau'_* : (A \otimes \overline{A}^{\otimes^*} \otimes A, b'_*) \to X'_*(A)$. By applying these results we obtain, for every A-bimodule M, a complex $X_*(M)$ whose homology is the Hochschild homology of A with coefficients in M, and quasi-isomorphisms $\theta_* : X_*(M) \to (M \otimes \overline{A}^{\otimes^*}, b_*)$ and $\tau_* : (M \otimes \overline{A}^{\otimes^*}, b_*) \to X_*(M)$. We use these explicit construction in the following section. When $r = 0$ and k is a characteristic zero field, the complexes $X'_*(A)$ and $X_*(M)$ and the morphisms θ'_* and θ_* were found in [8].

The resolution $X'_*(A)$. Let V be the graded k-module freely generated by the homogeneous elements

$$y_1, \dots, y_n, \quad y_{r+1}^{-1}, \dots, y_n^{-1}, \quad z_1, \dots, z_n, \quad z_{r+1}^{-1}, \dots, z_n^{-1}, \quad e_1, \dots, e_n,$$

where the degree of the e_i's is 1 and the degree of other elements is 0. Let us consider the free graded differential k-algebra $X'_*(A) = (X'_*, \partial'_*)$ generated by V and the relations

$$y_j y_i = q_{ij} y_i y_j + p_{ij}, \qquad z_j z_i = q_{ij} z_i z_j + p_{ij}, \qquad z_j y_i = q_{ij} y_i z_j + p_{ij},$$
$$y_i^{-1} y_i = y_i y_i^{-1} = z_i^{-1} z_i = z_i z_i^{-1} = 1,$$
$$e_j e_i = -q_{ij} e_i e_j, \qquad e_i^2 = 0, \qquad e_j y_i = q_{ij} y_i e_j, \qquad e_j z_i = q_{ij} z_i e_j,$$

for $i<j$, and boundary map defined by $\partial_1'(e_i)=z_i-y_i$. The action of A^e in $X_*'(A)$ given by $(x_i\otimes 1).P=y_iP$ and $(1\otimes x_i).P=Pz_i$ for all $P\in X_*'$, gives $X_*'(A)$ the structure a left graded differential A^e-module.

Note. In Theorem 1.6 we show that the complex $X_*(A)=A\otimes_{A^e}X_*'(A)$ gives the Hochschild homology of A. In this complex the e_i's play the role of the differentials forms dx_i's. This is particularly true in the classical case $A=k[x_1,\dots,x_n]$.

Notations 1.1. We use the following notations:

(1) Let $\mathbf{m}=(m_1,\dots,m_n)$ be an n-tuple built up of non-negative integers $m_1,\dots,m_r$ and arbitrary integers $m_{r+1},\dots,m_n$. For every integer j between 1 and n we set

$$\mathbf{Y}^{\mathbf{m}}=y_1^{m_1}\dots y_n^{m_n},\qquad \mathbf{Y}^{\mathbf{m}}_{<i}=y_1^{m_1}\dots y_{i-1}^{m_{i-1}},$$
$$\mathbf{Y}^{\mathbf{m}}_{>i}=y_{i+1}^{m_{i+1}}\dots y_n^{m_n},\qquad \mathbf{Y}^{\mathbf{m}}_{\le i}=\mathbf{Y}^{\mathbf{m}}_{<i}y_i^{m_i},\qquad \mathbf{Y}^{\mathbf{m}}_{\ge i}=y_i^{m_i}\mathbf{Y}^{\mathbf{m}}_{>i}.$$

Similarly we define $\mathbf{X}^{\mathbf{m}}$, $\mathbf{Z}^{\mathbf{m}}$, $\mathbf{X}^{\mathbf{m}}_{<i}$, $\mathbf{Z}^{\mathbf{m}}_{<i}$, etc.

(2) Given $1\le i_j<\dots<i_s\le n$ we set $e_{i_s\dots i_j}=e_{i_s}\dots e_{i_j}$.

Lemma 1.2. *Let*

$$\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}})=\begin{cases}\sum_{l=0}^{m_i-1}\mathbf{Y}^{\mathbf{m}}_{<i}y_i^le_iz_i^{m_i-l-1}\mathbf{Z}^{\mathbf{m}}_{>i} & \text{if } m_i>0,\\ 0 & \text{if } m_i=0,\\ -\sum_{l=m_i}^{-1}\mathbf{Y}^{\mathbf{m}}_{<i}y_i^le_iz_i^{m_i-l-1}\mathbf{Z}^{\mathbf{m}}_{>i} & \text{if } m_i<0.\end{cases}$$

The following results hold:

(1) $\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}}y_j)=\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}})z_j$ *if* $i<j$,

(2) $\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}}y_i)=\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}})z_i+\mathbf{Y}^{\mathbf{m}}_{\le i}\mathbf{Z}^{\mathbf{m}}_{>i}e_i$,

(3) $\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}}y_j)=\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}})y_j$ *if* $i>j$,

(4) $\partial_1'(\sum_{j>i}\widetilde{\mathrm{T}}_j(\mathbf{Y}^{\mathbf{m}}))=\mathbf{Y}^{\mathbf{m}}-\mathbf{Y}^{\mathbf{m}}_{\le i}\mathbf{Z}^{\mathbf{m}}_{>i}$.

Proof. The first and fourth equality are easily checked. We prove the second by induction on $s=|m_{i+1}|+\dots+|m_n|$ and leave the third to the reader. When $s=0$ the result can be easily proved by a direct computation. Suppose that $s>0$ and write $\mathbf{Y}^m=y_1^{m_1}\dots y_u^{m_u}$ with $m_u\ne 0$. First of all consider the case $m_u>0$. Let $\mathbf{Y}^{\mathbf{m}'}=y_1^{m_1}\dots y_u^{m_u-1}$. Because of (1) and the inductive hypothesis we have

$$\begin{aligned}\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}}y_i)&=\widetilde{\mathrm{T}}_i(q_{iu}\mathbf{Y}^{\mathbf{m}'}y_iy_u+p_{iu}\mathbf{Y}^{\mathbf{m}'})\\&=q_{iu}\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}'}y_i)z_u+p_{iu}\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}'})\\&=q_{iu}\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}'})z_iz_u+q_{iu}\mathbf{Y}^{\mathbf{m}'}_{\le i}\mathbf{Z}^{\mathbf{m}'}_{>i}e_iz_u+p_{iu}\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}'})\\&=\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}'})z_uz_i+\mathbf{Y}^{\mathbf{m}'}_{\le i}\mathbf{Z}^{\mathbf{m}'}_{>i}z_ue_i\\&=\widetilde{\mathrm{T}}_i(\mathbf{Y}^{\mathbf{m}})z_i+\mathbf{Y}^{\mathbf{m}}_{\le i}\mathbf{Z}^{\mathbf{m}}_{>i}e_i.\end{aligned}$$

The case $m_u<0$ is similar. □

Proposition 1.3. *Let $X'_*(A)$ be as defined above and $\mu : X'_0 \to A$ the morphism of algebras given by $\mu(y_i) = \mu(z_i) = x_i$. The complex*

$$A \xleftarrow{\mu} X'_0 \xleftarrow{\partial'_1} X'_1 \xleftarrow{\partial'_2} X'_2 \xleftarrow{\partial'_3} X'_3 \xleftarrow{\partial'_4} \dots$$

is contractible as a complex of left A^{Op}-modules, with contracting homotopy $\varepsilon_0 : A \to X'_0$ and $\varepsilon_s : X'_{s-1} \to X'_s$ $(s > 0)$, given by

$$\varepsilon_0(1) = 1, \qquad \varepsilon_1(\mathbf{Y^m}) = \sum_{i=1}^{n} \widetilde{\mathrm{T}}_i(\mathbf{Y^m}),$$

$$\varepsilon_{s+1}(\mathbf{Y^m} e_{i_s \dots i_1}) = \sum_{i_{s+1} > i_s} \widetilde{\mathrm{T}}_{i_{s+1}}(\mathbf{Y^m}) e_{i_s \dots i_1} \quad (s \geq 1,\ 1 \leq i_1 < \dots < i_s \leq n).$$

Proof. We must prove that $\mu \circ \varepsilon_0 = id$, $\varepsilon_0 \circ \mu + \partial'_1 \circ \varepsilon_1$ and $\varepsilon_s \circ \partial'_s = \partial'_{s+1} \circ \varepsilon_{s+1}$. We prove the last one and leave the other to the reader. In order to simplify notation we set $\mathbf{E}_j = e_{i_s \dots i_{j+1}}(z_{i_j} - y_{i_j}) e_{i_{j-1} \dots i_1}$. By the above lemma we have

$$\varepsilon_s \circ \partial'_s(\mathbf{Y^m} e_{i_s \dots i_1}) = \varepsilon_s \left(\sum_{j=1}^{s} (-1)^{s-j} \mathbf{Y^m} \mathbf{E}_j \right)$$

$$= \sum_{j=1}^{s} (-1)^{s-j} \sum_{i_{s+1} > i_s} \widetilde{\mathrm{T}}_{i_{s+1}}(\mathbf{Y^m}) \mathbf{E}_j + \mathbf{Y}^{\mathbf{m}}_{\leq i_s} \mathbf{Z}^{\mathbf{m}}_{> i_s} e_{i_s \dots i_1}$$

and

$$\partial'_{s+1} \circ \varepsilon_{s+1}(\mathbf{Y^m} e_{i_s \dots i_1}) = \partial'_{s+1} \left(\sum_{i_{s+1} > i_s} \widetilde{\mathrm{T}}_{i_{s+1}}(\mathbf{Y^m}) e_{i_s \dots i_1} \right)$$

$$= (\mathbf{Y^m} - \mathbf{Y}^{\mathbf{m}}_{\leq i_s} \mathbf{Z}^{\mathbf{m}}_{> i_s}) e_{i_s \dots i_1} - \sum_{i_{s+1} > i_s} \sum_{j=1}^{s} (-1)^{s-j} \widetilde{\mathrm{T}}_{i_{s+1}}(\mathbf{Y^m}) \mathbf{E}_j. \qquad \square$$

Let us consider the families of morphisms of left A^e-modules

$$\theta'_* : X'_* \to A \otimes \overline{A}^{\otimes^*} \otimes A \quad \text{and} \quad \tau'_* : A \otimes \overline{A}^{\otimes^*} \otimes A \to X'_*,$$

recursively defined by

$$\theta'_0(1 \otimes 1) = 1 \otimes 1,$$

$$\theta'_{s+1}(e_{i_{s+1} \dots i_1}) = 1 \otimes (\theta'_s \circ \partial'_{s+1}(e_{i_{s+1} \dots i_1})),$$

$$\tau'_0(1 \otimes 1) = 1 \otimes 1,$$

$$\tau'_{s+1}(1 \otimes \mathbf{X}^{\mathbf{m}(1)} \otimes \dots \otimes \mathbf{X}^{\mathbf{m}(s+1)} \otimes 1)$$
$$= \varepsilon_{s+1} \circ \tau'_s \circ b'_{s+1}(1 \otimes \mathbf{X}^{\mathbf{m}(1)} \otimes \dots \otimes \mathbf{X}^{\mathbf{m}(s+1)} \otimes 1).$$

By induction we can show that

$$\theta'_*: X'_*(A) \to (A\otimes \overline{A}^{\otimes^*} \otimes A,\ b'_*) \quad \text{and} \quad \tau'_*: (A\otimes \overline{A}^{\otimes^*} \otimes A,\ b'_*) \to X'_*(A),$$

are chain homomorphisms.

Proposition 1.4. *Given a permutation* $\sigma\in S_s$ *let us write* $\mathrm{sg}_q(\sigma)=\prod_{h>j,\sigma(h)<\sigma(j)}\times(-q_{i_{\sigma(h)}i_{\sigma(j)}})$. *We have*:
(1) $\theta'_s(e_{i_s\dots i_1})=\sum_{\sigma\in S_s}\mathrm{sg}_q(\sigma)\otimes x_{i_{\sigma(s)}}\otimes\cdots\otimes x_{i_{\sigma(1)}}\otimes 1 \quad (1\le i_1<\cdots<i_s\le n)$,
(2) $\tau'_s(1\otimes \mathbf{X}^{\mathbf{m}(s)}\otimes\cdots\otimes\mathbf{X}^{\mathbf{m}(1)}\otimes 1)=\sum_{1\le i_1<\cdots<i_s\le n}\widetilde{\mathrm{T}}_{i_s}(\mathbf{Y}^{\mathbf{m}(s)})\dots\widetilde{\mathrm{T}}_{i_1}(\mathbf{Y}^{\mathbf{m}(1)})$.

Proof. (1) Assuming the equation valid for θ'_s we conclude that $1\otimes\theta'_s(e_{i_s\dots i_1}\mathbf{Z}^{\mathbf{m}})=0$ for every $\mathbf{Z}^{\mathbf{m}}$. Using this fact we obtain

$$\begin{aligned}
\theta'_{s+1}(e_{i_{s+1}\dots i_1}) &= \sum_{j=1}^{s+1} 1\otimes(\theta'_s((-1)^{s-j}e_{i_{s+1}\dots i_{j+1}}(y_{i_j}-z_{i_j})e_{i_{j-1}\dots i_1}))\\
&= \sum_{j=1}^{s+1} 1\otimes\left(\prod_{l=j+1}^{s+1}(-q_{i_j i_l})\theta'_s(y_{i_j}e_{i_{s+1}\dots\widehat{i_j}\dots i_1})\right)\\
&\quad -\sum_{j=1}^{s+1} 1\otimes\left((-1)^{s-l}\prod_{l=1}^{j-1}q_{i_l i_j}\theta'_s(e_{i_{s+1}\dots\widehat{i_j}\dots i_1}z_{i_j})\right)\\
&= \sum_{\sigma\in S_{s+1}}\mathrm{sg}_q(\sigma)\otimes x_{i_{\sigma(s+1)}}\otimes\cdots\otimes x_{i_{\sigma(1)}}\otimes 1.
\end{aligned}$$

(2) Assuming that the equation is valid for τ'_s it is easy to show that $\varepsilon_{s+1}\circ\tau'_s(1\otimes \mathbf{X}^{\mathbf{m}(s)}\otimes\cdots\otimes\mathbf{X}^{\mathbf{m}(1)}\otimes 1)=0$, for all $\mathbf{X}^{\mathbf{m}(1)},\dots,\mathbf{X}^{\mathbf{m}(s)}$. Using this fact and the equation (3) of Lemma 1.2 we obtain that

$$\begin{aligned}
&\tau'_{s+1}(1\otimes\mathbf{X}^{\mathbf{m}(s+1)}\otimes\cdots\otimes\mathbf{X}^{\mathbf{m}(1)}\otimes 1)\\
&= \varepsilon_{s+1}\circ\tau'_s\circ b'_{s+1}(1\otimes\mathbf{X}^{\mathbf{m}(s+1)}\otimes\cdots\otimes\mathbf{X}^{\mathbf{m}(1)}\otimes 1)\\
&= \varepsilon_{s+1}\circ\tau'_s(\mathbf{X}^{\mathbf{m}(s+1)}\otimes\cdots\otimes\mathbf{X}^{\mathbf{m}(1)}\otimes 1)\\
&= \varepsilon_{s+1}\left(\sum_{1\le i_1<\cdots<i_s\le n}\mathbf{Y}^{\mathbf{m}(s+1)}\widetilde{\mathrm{T}}_{i_s}(\mathbf{Y}^{\mathbf{m}(s)})\dots\widetilde{\mathrm{T}}_{i_1}(\mathbf{Y}^{\mathbf{m}(1)})\right)\\
&= \sum_{1\le i_1<\cdots<i_{s+1}\le n}\widetilde{\mathrm{T}}_{i_{s+1}}(\mathbf{Y}^{\mathbf{m}(s+1)})\dots\widetilde{\mathrm{T}}_{i_1}(\mathbf{Y}^{\mathbf{m}(1)}). \qquad \square
\end{aligned}$$

1.1. *The Hochschild (co)homology of A*

Remark 1.5. Let M be an A-bimodule. We consider the complexes $X_*(M)=M\otimes_{A^e} X'_*(A)$ and $X^*(M)=\mathrm{Hom}_{A^e}(X'_*(A),M)$ and the morphisms

$$\theta_* = id_M \otimes_{A^e} \theta'_*, \qquad \tau_* = id_M \otimes_{A^e} \tau'_*, \qquad \theta^* = \mathrm{Hom}_{A^e}(\theta'_*, M) \quad \text{and}$$
$$\tau^* = \mathrm{Hom}_{A^e}(\tau'_*, M).$$

An easy computation shows that

(1) $X_*(M)$: $M \xleftarrow{\partial_1} X_1 \xleftarrow{\partial_2} X_2 \xleftarrow{\partial_3} \cdots \xleftarrow{\partial_n} X_n \longleftarrow 0$, where

$$X_s = \bigoplus_{1\le i_1<\cdots<i_s\le n} Me_{i_s\ldots i_1},$$

$$\partial_s(me_{i_s\ldots i_1}) = \sum_{j=1}^{s}(-1)^{s-j}\left(\prod_{l=1}^{j-1} q_{i_l i_j}\, x_{i_j} m - \prod_{l=j+1}^{s} q_{i_j i_l}\, m x_{i_j}\right) e_{i_s\ldots\widehat{i_j}\ldots i_1},$$

(2) $X^*(M)$: $M \xrightarrow{\partial^1} X^1 \xrightarrow{\partial^2} X^2 \xrightarrow{\partial^3} \cdots \xrightarrow{\partial^n} X^n \longrightarrow 0$, where

$$X^s = \bigoplus_{1\le i_1<\cdots<i_s\le n} Me_{i_s\ldots i_1},$$

$$\partial^{s+1}(me_{i_s\ldots i_1}) = \sum_{j=0}^{s}\sum_{h=i_j+1}^{i_{j+1}-1}(-1)^{s-j}\left(\prod_{l=1}^{j} q_{i_l h}\, m x_h - \prod_{l=j+1}^{s} q_{h,i_l}\, x_h m\right) e_{i_s\ldots i_{j+1} h i_j \ldots i_1},$$

(3) $\theta_* : X_*(M) \to (M\otimes \overline{A}^{\otimes^*}, b_*)$, $\tau_* : (M\otimes\overline{A}^{\otimes^*}, b_*) \to X_*(M)$, $\theta^* : \mathrm{Hom}_{A^e}((A\otimes \overline{A}^{\otimes^*}\otimes A, b'_*), M) \to X^*(M)$ and $\tau^* : X^*(M) \to \mathrm{Hom}_{A^e}((A\otimes\overline{A}^{\otimes^*}\otimes A, b'_*), M)$ are given by

$$\theta_s(me_{i_s\ldots i_1}) = \sum_{\sigma\in S_s} \mathrm{sg}_q(\sigma) m\otimes x_{i_{\sigma(s)}}\otimes\cdots\otimes x_{i_{\sigma(1)}},$$

$$\theta^s(f) = \sum_{1\le i_1<\cdots<i_s\le n}\left(\sum_{\sigma\in S_s} f(\mathrm{sg}_q(\sigma)\otimes x_{i_{\sigma(s)}}\otimes\cdots\otimes x_{i_{\sigma(1)}}\otimes 1)\right) e_{i_s\ldots i_1},$$

$$\tau_s(m\otimes \mathbf{X}^{\mathbf{m}(s)}\otimes\cdots\otimes \mathbf{X}^{\mathbf{m}(1)}) = \sum_{1\le i_1<\cdots<i_s\le n} \gamma_s(m\otimes \widetilde{\mathrm{T}}_{i_s}(\mathbf{Y}^{\mathbf{m}(s)})\ldots\widetilde{\mathrm{T}}_{i_1}(\mathbf{Y}^{\mathbf{m}(1)})),$$

$$\tau^s(me_{i_s\ldots i_1})(1\otimes \mathbf{X}^{\mathbf{m}(s)}\otimes\cdots\otimes \mathbf{X}^{\mathbf{m}(1)}\otimes 1) = \gamma^s(m)(\widetilde{\mathrm{T}}_{i_s}(\mathbf{Y}^{\mathbf{m}(s)})\ldots\widetilde{\mathrm{T}}_{i_1}(\mathbf{Y}^{\mathbf{m}(1)})),$$

where $\gamma_s : M\otimes X'_s(A)\to X_s(M)$ is the morphism sending the element $m\otimes \mathbf{Y}^{\mathbf{m}_1} e_{i_1\ldots i_s}\mathbf{Z}^{\mathbf{m}_2}$ to $\mathbf{X}^{\mathbf{m}_2} m \mathbf{X}^{\mathbf{m}_1} e_{i_1\ldots i_s}$ and $\gamma^s(m) : X'_s(A)\to M$ is the morphism of A^e-modules sending the element $e_{i_s\ldots i_1}$ to m and $e_{i'_s\ldots i'_1}$ to 0 if some i'_j is different from i_j.

Theorem 1.6. *Let M be an A-bimodule and let $X_*(M)$, $X^*(M)$, θ_*, θ^*, τ_* and τ^* be as defined above. We have:*

(1) *The Hochschild homology* $\mathrm{H}_*(A,M)$ *of* A *with coefficients in* M *is the homology of* $X_*(M)$. *Moreover,* θ_* *and* τ_* *are chain maps that induce isomorphisms in homology which are inverse of each other,*

(2) *The Hochschild cohomology* $\mathrm{H}^*(A,M)$ *of* A *with coefficients in* M *is the cohomology of* $X^*(M)$. *Moreover* θ^* *and* τ^* *are chain maps that induce isomorphisms in homology which are inverse of each other.*

Proof. It is an immediate consequence of Propositions 1.3 and 1.4. □

1.2. *The De Rham cohomology of A*

Let $\widetilde{B}_*: X_*(A) \to X_*(A)[-1]$ the morphism of complexes given by

$$\widetilde{B}_s(X^{\mathbf{m}}e_{i_s\dots i_1}) = \sum_{j=0}^{s}(-1)^{s-j}\prod_{l=1,\,h=j+1}^{j,s} q_{i_l i_h}\sum_{v=i_j+1}^{i_{j+1}-1}\overline{\mathrm{T}}_v^{i_s\dots i_1}(\mathbf{X}^{\mathbf{m}})e_{i_s\dots i_{j+1}v i_j\dots i_1},$$

where

$$\overline{\mathrm{T}}_v^{i_s\dots i_1}(\mathbf{X}^{\mathbf{m}}) = \begin{cases} \sum_{l=0}^{m_v-1} Q_{v,l}^{i_s\dots i_1}x_v^{m_v-l-1}X^{\mathbf{m}}_{>v}X^{\mathbf{m}}_{<v}x_v^l & \text{if } m_v>0,\\ 0 & \text{if } m_v=0,\\ -\sum_{l=m_v}^{-1} Q_{v,l}^{i_s\dots i_1}x_v^{m_v-l-1}X^{\mathbf{m}}_{>v}X^{\mathbf{m}}_{<v}x_v^l & \text{if } m_v<0,\end{cases}$$

with

$$Q_{v,l}^{i_s\dots i_1} = \prod_{i_t>v}\left(q_{v,i_t}^l\prod_{h=1}^{v-1}q_{h,i_t}^{m_h}\right)\prod_{i_t<v}\left(q_{i_t v}^{m_v-l-1}\prod_{h=v+1}^{n}q_{i_t h}^{m_h}\right).$$

Corollary 1.7. *The De Rham cohomology* $\mathrm{H}^*_{DR}(A)$ *of* A *is the cohomology of the following complex*:

$$0\to \mathrm{H}_0(X_*(A)) \xrightarrow{\tilde{B}_0} \mathrm{H}_1(X_*(A)) \xrightarrow{\tilde{B}_1} \mathrm{H}_2(X_*(A)) \xrightarrow{\tilde{B}_2} \cdots,$$

where the map $\tilde{B}_*: \mathrm{H}_*(X_*(A)) \to \mathrm{H}_{*+1}(X_*(A)[-1])$ *is induced by* $\tilde{B}_*: X_*(A) \to X_*(A)[-1]$.

Proof. By definition, the De Rham cohomology of A is the cohomology of

$$0\to \mathrm{HH}_0(A) \xrightarrow{B_0} \mathrm{HH}_1(A) \xrightarrow{B_1} \mathrm{HH}_2(A) \xrightarrow{B_2} \cdots,$$

where $B_*: \mathrm{HH}_*(A) \to \mathrm{HH}_{*+1}(A)$ is induced by the map of complexes $B_*: (A\otimes \overline{A}^{\otimes^*}, b_*) \to (A\otimes\overline{A}^{\otimes^*}, b_*)[-1]$ given by

$$B_s(a_0\otimes\cdots\otimes a_s) = \sum_{i=0}^{s}(-1)^{is}\otimes a_i\otimes\cdots\otimes a_s\otimes a_0\otimes\cdots\otimes a_{i-1}.$$

The corollary follows then from the equality $\tilde{B}_* = \tau_{*+1}\circ B_*\circ\theta_*$. □

Some concrete calculations. Now, we shall apply the results above to compute homology in the concrete examples below. The first example was studied in [8, 9] under the hypothesis that k is a characteristic zero field and $r=0$ or $r=n$, and in [1] when k is an arbitrary field and $r=0$. Following the ideas of [8] we study here the case where k is an arbitrary field and $0\le r\le n$. The second example was studied in [2].

Example 1.8. Let A be the k-algebra generated by the elements $x_1,\dots,x_n,x_{r+1}^{-1},\dots,x_n^{-1}$ and the relations $x_jx_j^{-1}=1=x_j^{-1}x_j$ $(r<j\le n)$ and $x_jx_i=q_{ij}x_ix_j$ $(1\le i<j\le n)$ (i.e. $p_{ij}=0$ for all i,j). Write C for the set of those n-tuples $(m_1,\dots,m_n)\in(\mathbf{N}\cup\{0\})^r\times\mathbf{Z}^{n-r}$ satisfying the following two conditions:

(i) If $i\le r$, then $m_i=0$ or $\prod_{l=1}^n q_{il}^{m_l}=1$.

(ii) If $i>r$, then $\prod_{l=1}^n q_{il}^{m_l}=1$.

Proposition 1.9. *Let A be as in the example above. Then*:

(1) *The Hochschild homology* $\mathrm{HH}_*(A)$ *of A is*

$$\mathrm{HH}_s(A)=\bigoplus_{\mathbf{m}_{i_1\dots i_s}\in C} k.x_1^{m_1}\cdots x_n^{m_n}e_{i_s\dots i_1}$$

where $\mathbf{m}_{i_1\dots i_s}$ *is the element of* $(\mathbf{N}\cup\{0\})^r\times\mathbf{Z}^{n-r}$ *whose jth coordinate is* m_j *if* $j\notin\{i_1,\dots,i_s\}$ *and* m_j+1 *if* $j\in\{i_1,\dots,i_s\}$.

(2) *Let $p\ge 0$ be the characteristic of k. Let us write* $\overline{C}=\{\mathbf{m}\in C: p/m_i\ (i=1,\dots,n)\}$. *The De Rham cohomology* $\mathrm{H}^*_{DR}(A)$ *of A is*

$$\mathrm{H}^s_{DR}(A)=\bigoplus_{\mathbf{m}_{i_1\dots i_s}\in\overline{C}} k.x_1^{m_1}\dots x_n^{m_n}e_{i_s\dots i_1}.$$

Proof. (1) By Theorem 1.6 $\mathrm{HH}_*(A)$ is the homology of the complex

$$X_*(A):\quad A\xleftarrow{\partial_1}X_1\xleftarrow{\partial_2}X_2\xleftarrow{\partial_3}\cdots\xleftarrow{\partial_n}X_n\longleftarrow 0,$$

where

$$X_s=\bigoplus_{1\le i_1<\dots<i_s\le n} Ae_{i_s\dots i_1},$$

$$\partial_s(x_1^{m_1}\cdots x_n^{m_n}e_{i_s\cdots i_1})=\sum_{j=1}^s Q_{i_j}^{\mathbf{m},i_s,\dots,i_1}x_1^{m_1}\dots x_{i_j}^{m_{i_j}+1}\dots x_n^{m_n}e_{i_s\dots\widehat{i_j}\dots i_1},$$

with

$$Q_{i_j}^{\mathbf{m},i_s,\dots,i_1}=(-1)^{s-j}\left(\prod_{l=1}^{i_j-1}q_{l,i_j}^{m_l}\prod_{l=1}^{j-1}q_{i_li_j}-\prod_{l=i_j+1}^{n}q_{i_jl}^{m_l}\prod_{l=j+1}^{s}q_{i_ji_l}\right).$$

The complex $X_*(A)$ has a $(\mathbf{N}\cup\{0\})^r\times\mathbf{Z}^{n-r}$-gradation if we set $dg(x_1^{m_1}\dots x_n^{m_n}e_{i_s\dots i_1})=\mathbf{m}_{i_1\dots i_s}$. For every multi-index $\overline{\alpha}=(\alpha_1,\dots,\alpha_n)\in(\mathbf{N}\cup\{0\})^r\times\mathbf{Z}^{n-r}$ let $X_*^{\overline{\alpha}}$ be the subcomplex of $X_*(A)$ with gradation $\overline{\alpha}$. If $\overline{\alpha} \in C$, then the boundary map of $X_*^{\overline{\alpha}}$ is zero. If $\overline{\alpha} \notin C$, then $X_*^{\overline{\alpha}}$ is exact. In fact, for every $1 \leq t \leq n$ we have $X_*(A)$ is the total complex of

$$X_{**}^t : \quad \begin{array}{ccccccccc} X_{10}^t & \xleftarrow{\partial_{11}^t} & X_{11}^t & \xleftarrow{\partial_{12}^t} & \cdots & \xleftarrow{\partial_{1n-1}^t} & X_{1n-1}^t & \longleftarrow & 0 \\ \downarrow{\scriptstyle \partial_1^v} & & \downarrow{\scriptstyle \partial_2^v} & & & & \downarrow{\scriptstyle \partial_{n-1}^v} & & \\ A & \xleftarrow{\partial_{01}^t} & X_{01}^t & \xleftarrow{\partial_{02}^t} & \cdots & \xleftarrow{\partial_{0n-1}^t} & X_{0n-1}^t & \longleftarrow & 0, \end{array}$$

where

$$X_{0s}^t = \bigoplus_{\substack{1 \leq i_1 < \cdots < i_s \leq n \\ t \notin \{i_1, \ldots, i_s\}}} A e_{i_s \cdots i_1}, \qquad X_{1s}^t = \bigoplus_{\substack{1 \leq i_1 < \cdots < i_s \leq n \\ t \in \{i_1, \ldots, i_s\}}} A e_{i_s \ldots i_1},$$

$$\partial_{0s}^t(x_1^{m_1} \ldots x_n^{m_n} e_{i_s \ldots i_1}) = \sum_{j=1}^{s} Q_{i_j}^{\mathbf{m}, i_s, \ldots, i_1} x_1^{m_1} \ldots x_{i_j}^{m_{i_j}+1} \ldots x_n^{m_n} e_{i_s \ldots \widehat{i_j} \ldots i_1},$$

$$\partial_{1s}^t(x_1^{m_1} \ldots x_n^{m_n} e_{i_s \ldots i_1}) = \sum_{\substack{j=1 \\ i_j \neq t}}^{s} Q_{i_j}^{\mathbf{m}, i_s, \ldots, i_1} x_1^{m_1} \ldots x_{i_j}^{m_{i_j}+1} \ldots x_n^{m_n} e_{i_s \ldots \widehat{i_j} \ldots i_1},$$

$$\partial_s^v(x_1^{m_1} \ldots x_n^{m_n} e_{i_s \ldots i_1}) = Q_t^{\mathbf{m}, i_s, \ldots, i_1} x_1^{m_1} \ldots x_t^{m_t+1} \ldots x_n^{m_n} e_{i_s \ldots \widehat{t} \ldots i_1}.$$

Choosing t such that $\prod_{l=1}^{n} q_{tl}^{\alpha_l} \neq 1$ we obtain that the columns of X_{**}^t are exact. Now, the proof can be easily finished.

(2) A direct computation using Corollary 1.6 shows that the De Rham cohomology of A is the cohomology of

$$0 \to \mathrm{HH}_0(A) \xrightarrow{\tilde{B}_0} \mathrm{HH}_1(A) \xrightarrow{\tilde{B}_1} \mathrm{HH}_2(A) \xrightarrow{\tilde{B}_2} \cdots,$$

where

$$\tilde{B}_s(X^{\mathbf{m}} e_{i_s \ldots i_1}) = \sum_{j=0}^{s} (-1)^{s-j} \sum_{v=i_j+1}^{i_{j+1}-1} m_v \prod_{h<v} q_{vh}^{m_h} \prod_{i_h<v} q_{v,i_h} X_{<v}^{\mathbf{m}} x_v^{m_v-1} X_{>v}^{\mathbf{m}} e_{i_s \ldots i_{j+1} v i_j \ldots i_1},$$

for $\mathbf{m}_{i_1 \ldots i_s} \in C$. The proof follows as in part 1. □

Example 1.10. Let $m > 1$ and q be an mth primitive root of unity. Let D_q be the k-algebra generated by x_1, x_1^{-1}, x_2 and the relations $x_1 x_1^{-1} = x_1^{-1} x_1 = 1$ and $x_2 x_1 - q x_1 x_2 = 1$. By Theorem 1.2 of [2], we have:

$$\mathrm{HH}_0(D_q) = \bigoplus_{i \in \mathbf{Z}} k.x_1^i \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} k.x_1^{um} x_2^{vm},$$

$$\mathrm{HH}_1(D_q) = \bigoplus_{i \in \mathbf{Z}} k.x_1^i e_1 \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} k.x_1^{um-1} x_2^{vm} e_1 \oplus \bigoplus_{u \in \mathbf{Z}, v > 0} k.x_1^{um} x_2^{vm-1} e_2,$$

$$\mathrm{HH}_2(D_q) = \bigoplus_{u \in \mathbf{Z}, v > 0} k.\sum_{s=1}^{m} x_1^{um-s} x_2^{vm-s} e_2 e_1.$$

Proposition 1.11. *Let D_q be as in the above example. Then*

(1) $B_0(x_1^i) = ix_1^{i-1}$ $(i \in \mathbf{Z})$

(2) $B_0(x_1^{um}x_2^{vm}) = umx_1^{um-1}x_2^{vm}e_1 + vmx_1^{um}x_2^{vm-1}e_2$ $(u \in \mathbf{Z},\ v > 0)$

(3) $B_1(x_1^i e_1) = 0$ $(i \in \mathbf{Z})$

(4) $B_1(x_1^{um-1}x_2^{vm}e_1) = vm\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1} x_1^{um-s}x_2^{vm-s}e_2e_1$ $(u \in \mathbf{Z},\ v > 0)$

(5) $B_1(x_1^{um}x_2^{vm-1}e_2) = um\sum_{s=1}^{m}\left(\frac{q}{1-q}\right)^{s-1} x_1^{um-s}x_2^{vm-s}e_2e_1$ $(u \in \mathbf{Z},\ v > 0)$

Proof. We prove (2) and (5), when $u > 0$. The other equalities follow in a similar way. It results from the equalities $x_2x_1^m = x_1^m x_2$ and $x_2^m x_1 = x_1 x_2^m$ that

$$B_0(x_1^{um}x_2^{vm}) = \sum_{l=0}^{um-1} x_1^{um-l-1}x_2^{vm}x_1^l e_1 + \sum_{l=0}^{vm-1} x_2^{vm-l-1}x_1^{um}x_2^l e_2$$

$$= umx_1^{um-1}x_2^{vm}e_1 + vmx_1^{um}x_2^{vm-1}e_2,$$

and

$$B_1(x_1^{um}x_2^{vm-1}e_2) = \sum_{l=0}^{um-1} q^l x_1^{um-l-1}x_2^{vm-1}x_1^l e_2e_1 = umx_1^{um-1}x_2^{vm-1}e_2e_1 + R,$$

where R is a linear combination of terms of the type $x_1^i x_2^j$, with $i < um-1$ and $j < vm-1$. Now, using that $X_3(D_q) = 0$ and $\mathrm{HH}_2(D_q) = \bigoplus_{u \in \mathbf{Z}, v>0} k.\sum_{s=1}^{m} x_1^{um-s}x_2^{vm-s}e_2e_1$ we obtain that $B_1(x_1^{um}x_2^{vm-1}e_2) = um\sum_{s=1}^{m}(q/(1-q))^{s-1}x_1^{um-s}x_2^{vm-s}e_2e_1$. $\square$

2. The Hochschild (co)homology of an Ore extension of $S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$

Let A be a k-algebra, $\alpha : A \to A$ a morphism of algebras and $\delta : A \to A$ an α-derivative (i.e. δ is a k-linear map verifying $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$ for each pair a, b of elements of A). The Ore extension $A[t, \alpha, \delta]$ associated to (A, α, δ) is the left A-module $A[t]$ consisting of polynomials in t with coefficients in A, with a structure of k-algebra given by $ta = \alpha(a)t + \delta(a)$ $(a \in A)$. Now let $A = S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$ as in the previous section and $E = A[t, \alpha, \delta]$ be an Ore extension of A such that $\alpha(x_i) = \overline{q}_i^{-1}x_i$ with $\overline{q}_i \in k$ $(1 \le i \le n)$. Given an E-bimodule M, positive integers $1 \le i_1 < \cdots < i_s \le n$ and $1 \le v \le n$, denote by $\overline{\mathrm{T}}_v^{i_s \dots i_1} : M \times A \to M$ and $\overline{\mathrm{T}}_v^{i_s \dots i_1} : A \times M \to M$ the bilinear maps defined by

$$\overline{\mathrm{T}}_v^{i_s \dots i_1}(m, \mathbf{X}^{\mathbf{m}}) = \begin{cases} \sum_{l=0}^{m_v-1} Q_{v,l}^{i_s \dots i_1} x_v^{m_v-l-1} X_{>v}^{\mathbf{m}} m X_{<v}^{\mathbf{m}} x_v^l & \text{if } m_v > 0, \\ 0 & \text{if } m_v = 0, \\ -\sum_{l=m_v}^{-1} Q_{v,l}^{i_s \dots i_1} x_v^{m_v-l-1} X_{>v}^{\mathbf{m}} m X_{<v}^{\mathbf{m}} x_v^l & \text{if } m_v < 0 \end{cases}$$

and

$$\overline{\mathrm{T}}_v^{i_s\dots i_1}(\mathbf{X}^{\mathbf{m}},m)=\begin{cases}\sum_{l=0}^{m_v-1} Q_{v,l}^{i_s\dots i_1} X_{<v}^{\mathbf{m}} x_v^l m x_v^{m_v-l-1} X_{>v}^{\mathbf{m}} & \text{if } m_v>0,\\ 0 & \text{if } m_v=0,\\ -\sum_{l=m_v}^{-1} Q_{v,l}^{i_s\dots i_1} X_{<v}^{\mathbf{m}} x_v^l m x_v^{m_v-l-1} X_{>v}^{\mathbf{m}} & \text{if } m_v<0,\end{cases}$$

where

$$Q_{v,l}^{i_s\dots i_1}=\prod_{i_t>v}\left(q_{v,i_t}^l\prod_{h=1}^{v-1} q_{h,i_t}^{m_h}\right)\prod_{i_t<v}\left(q_{i_tv}^{m_v-l-1}\prod_{h=v+1}^{n} q_{i_th}^{m_h}\right).$$

We obtain below complexes $X_{**}(M)$ and $X^{**}(M)$ whose homologies are, respectively, the Hochschild homology and cohomology of E with coefficients in M. Finally, using this result, we study the homology of some k-algebras, like $\mathcal{O}_q(M(2,k))$, $\mathcal{O}_{q^2}(sok^3)$ and $U_q(sl(2,k))$, that appear naturally in the theory of quantum groups.

Notation 2.1. Let E and M be as above. We use the following notations:

(1) Let $X_{**}(M)$ be the diagram

$$X_{**}(M):\quad\begin{array}{ccccccccccc} Mw & \xleftarrow{\partial_{11}} & X_1w & \xleftarrow{\partial_{12}} & X_2w & \xleftarrow{\partial_{13}} & \cdots & \xleftarrow{\partial_{1n}} & X_nw & \longleftarrow & 0\\ \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_n} & & \\ M & \xleftarrow{\partial_{01}} & X_1 & \xleftarrow{\partial_{02}} & X_2 & \xleftarrow{\partial_{03}} & \cdots & \xleftarrow{\partial_{0n}} & X_n & \longleftarrow & 0,\end{array}$$

where

$$X_s=\bigoplus_{1\le i_1<\dots<i_s\le n} Me_{i_s\dots i_1},\qquad X_sw=\bigoplus_{1\le i_1<\dots<i_s\le n} Me_{i_s\dots i_1}w,$$

$$\partial_{0s}(me_{i_s\dots i_1})=\sum_{j=1}^{s}(-1)^{s-j}\left(\prod_{l=1}^{j-1} q_{i_li_j}x_{i_j}m-\prod_{l=j+1}^{s} q_{i_ji_l}mx_{i_j}\right)e_{i_s\dots\widehat{i_j}\dots i_1},$$

$$\partial_{1s}(me_{i_s\dots i_1}w)=\sum_{j=1}^{s}(-1)^{s-j}\left(\overline{q}_{i_j}\prod_{l=1}^{j-1} q_{i_li_j}x_{i_j}m-\prod_{l=j+1}^{s} q_{i_ji_l}mx_{i_j}\right)e_{i_s\dots\widehat{i_j}\dots i_1}w,$$

$$\begin{aligned}\varphi_s(me_{i_s\dots i_1}w)=(-1)^s\Bigg(&\left[\prod_{l=1}^{s}\overline{q}_{i_l}mt-tm\right]e_{i_s\dots i_1}+\sum_{j=1}^{s}\sum_{u<j}(-1)^{j-u}\sum_{v=i_j+1}^{i_{j+1}-1}\\ &\times\prod_{l=u+1}^{j} q_{i_ui_l}\overline{\mathrm{T}}_v^{i_s\dots\widehat{i_u}\dots i_1}(m,\delta(x_{i_u}))e_{i_s\dots i_{j+1}vi_j\dots\widehat{i_u}\dots i_1}\\ &+\sum_{j=1}^{s}\sum_{v=i_{j-1}+1}^{i_{j+1}-1}\overline{\mathrm{T}}_v^{i_s\dots\widehat{i_j}\dots i_1}(m,\delta(x_{i_j}))e_{i_s\dots i_{j+1}vi_{j-1}\dots i_1}+\sum_{j=1}^{s}\sum_{u>j}(-1)^{u-j}\\ &\times\sum_{v=i_{j-1}+1}^{i_j-1}\prod_{l=j}^{u-1} q_{i_li_u}\overline{\mathrm{T}}_v^{i_s\dots\widehat{i_u}\dots i_1}(m,\delta(x_{i_u}))e_{i_s\dots\widehat{i_u}\dots i_jvi_{j-1}\dots i_1}\Bigg).\end{aligned}$$

(2) Let $X^{**}(M)$ be the diagram

$$X^{**}(M):\quad \begin{array}{ccccccccccc} Mw & \xrightarrow{\partial^{11}} & X^1w & \xrightarrow{\partial^{12}} & X^2w & \xrightarrow{\partial^{13}} & \cdots & \xrightarrow{\partial^{1n}} & X^nw & \longrightarrow & 0 \\ \uparrow{\scriptstyle\varphi^0} & & \uparrow{\scriptstyle\varphi^1} & & \uparrow{\scriptstyle\varphi^2} & & & & \uparrow{\scriptstyle\varphi^n} & & \\ M & \xrightarrow{\partial^{01}} & X^1 & \xrightarrow{\partial^{02}} & X^2 & \xrightarrow{\partial^{03}} & \cdots & \xrightarrow{\partial^{0n}} & X^n & \longrightarrow & 0, \end{array}$$

where

$$X^s = \bigoplus_{1\le i_1<\cdots<i_s\le n} Me_{i_s\dots i_1}, \qquad X^sw = \bigoplus_{1\le i_1<\cdots<i_s\le n} Me_{i_s\dots i_1}w,$$

$$\partial^{0s}(me_{i_s\dots i_1}) = \sum_{j=0}^{s}(-1)^{s-j}\sum_{h=i_j+1}^{i_{j+1}-1}\left(\prod_{l=1}^{j} q_{i_l h}mx_h - \prod_{l=j+1}^{s} q_{h,i_l}x_h m\right)e_{i_s\dots i_{j+1}hi_j\dots i_1},$$

$$\partial^{1s}(me_{i_s\dots i_1}w) = \sum_{j=0}^{s}(-1)^{s-j}\sum_{h=i_j+1}^{i_{j+1}-1}\left(\overline{q}_h\prod_{l=1}^{j} q_{i_l h}mx_h - \prod_{l=j+1}^{s} q_{h,i_l}x_h m\right)$$
$$\times e_{i_s\dots i_{j+1}hi_j\dots i_1}w,$$

$$\varphi^s(me_{i_s\dots i_1}) = (-1)^s\Bigg(\left[\prod_{l=1}^{s}\overline{q}_{i_l}tm - mt\right]e_{i_s\dots i_1}w + \sum_{j=1}^{s}\sum_{u<j}(-1)^{j-u}$$
$$\times\sum_{v=i_{u-1}+1}^{i_u-1}\prod_{l=u}^{j-1} q_{v,i_l}\overline{\mathrm{T}}_{i_j}^{i_s\dots i_1}(\delta(x_v),m)e_{i_s\dots\widehat{i_j}\dots i_u v i_{u-1}\dots i_1}$$
$$+\sum_{j=1}^{s}\sum_{v=i_{j-1}+1}^{i_{j+1}-1}\overline{\mathrm{T}}_{i_j}^{i_s\dots i_1}(\delta(x_v),m)e_{i_s\dots i_{j+1}vi_{j-1}\dots i_1}$$
$$+\sum_{j=1}^{s}\sum_{u>j}(-1)^{u-j}\sum_{v=i_v+1}^{i_{v+1}-1}\prod_{l=j+1}^{u} q_{i_l v}\overline{\mathrm{T}}_{i_j}^{i_s\dots i_1}(\delta(x_v),m)$$
$$\times e_{i_s\dots i_{u+1}vi_u\dots\widehat{i_j}\dots i_1}\Bigg).$$

Theorem 2.2. *Let $A = S^r_{\mathbf{Q},\mathbf{P}}(\mathbf{X})$ as in the previous section, $E = A[t,\alpha,\delta]$ an Ore extension of A such that $\alpha(x_i) = \overline{q}_i^{-1}x_i$ with $\overline{q}_i \in k$ $(1\le i\le n)$ and M be an E-bimodule. The diagrams $X_{**}(M)$ and $X^{**}(M)$ are double complexes. Moreover the Hochschild homology $\mathrm{H}_*(E,M)$ of E with coefficients in M is the homology of $X_{**}(M)$ and the Hochschild cohomology $\mathrm{H}^*(E,M)$ of E with coefficients in M is the cohomology of $X^{**}(M)$.*

Proof. By Theorem 1.4 of [1] the Hochschild homology $H_*(E,M)$ of E with coefficients in M is the homology of the complex

$$
X_{**}(M):\quad
\begin{array}{ccccccc}
M & \xleftarrow{b_{11}} & M\otimes \overline{E} & \xleftarrow{b_{12}} & M\otimes \overline{E}^{\otimes^2} & \xleftarrow{b_{13}} & \cdots\\
\downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \\
M & \xleftarrow{b_{01}} & M\otimes \overline{E} & \xleftarrow{b_{02}} & M\otimes \overline{E}^{\otimes^2} & \xleftarrow{b_{03}} & \cdots,
\end{array}
$$

where the vertical and horizontal maps are defined by

$$
\begin{aligned}
b_{0s}(a_0\otimes\cdots\otimes a_s) &= \sum_{i=0}^{s-1}(-1)^i a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_s\\
&\quad +(-1)^s a_s a_0\otimes a_1\otimes\cdots\otimes a_{s-1},\\
b_{1s}(a_0\otimes\cdots\otimes a_s) &= \sum_{i=0}^{s-1}(-1)^i a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_s\\
&\quad +(-1)^s \alpha^{-1}(a_s)a_0\otimes a_1\otimes\cdots\otimes a_{s-1},\\
\psi_s(a_0\otimes\cdots\otimes a_s) &= (-1)^s\Bigg(a_0 t\otimes\alpha^{-1}(a_1)\otimes\cdots\otimes\alpha^{-1}(a_s) - t a_0\otimes a_1\otimes\cdots\otimes a_s\\
&\quad -\sum_{j=1}^{s} a_0\otimes\cdots\otimes a_{j-1}\otimes\delta\circ\alpha^{-1}(a_j)\\
&\quad \otimes\alpha^{-1}(a_{j+1})\otimes\ldots\otimes\alpha^{-1}(a_s)\Bigg).
\end{aligned}
$$

A direct computation shows that $\varphi_* = \tau_*\circ\psi_*\circ\theta_*$. To prove the assertion for Hochschild homology, note that there is a diagram of quasi-isomorphisms

$$X_{**}(M) \xleftarrow{\sim} Z_{**}(M) \xleftarrow{\sim} Y_{**}(M),$$

where

$$
Z_{**}(M):\quad
\begin{array}{ccccccc}
Mw & \xleftarrow{\partial_{11}} & X_1 w & \xleftarrow{\partial_{12}} & X_2 w & \xleftarrow{\partial_{13}} & \cdots\\
\downarrow{\scriptstyle \psi_0\circ\theta_0} & & \downarrow{\scriptstyle \psi_1\circ\theta_1} & & \downarrow{\scriptstyle \psi_2\circ\theta_2} & & \\
M & \xleftarrow{b_{01}} & M\otimes \overline{A} & \xleftarrow{b_{02}} & M\otimes \overline{A}^{\otimes^2} & \xleftarrow{b_{03}} & \cdots
\end{array}
$$

To prove our cohomology assertion, proceed as follows: Note $X_{**}(E^e)$ is a free resolution of E as a left E^e-module. Consequently, the Hochschild cohomology of E with coefficients in M is the cohomology of $\mathrm{Hom}_{E^e}(X_{**}(E^e),M)$. One checks that the latter complex is isomorphic to $X^{**}(M)$. This concludes the proof. □

2.3. A filtration of $X_{**}(E)$. Now we consider a filtration of $X_{**}(E)$ which can be useful for the computing. Let $E=A[t,\alpha,\delta]$ be as in Theorem 2.2. For each $v\geq 0$ denote $F^v_{**}(E)$ the subcomplex of $X_{**}(E)$ defined by

$$F^v_{0s}(E)=\bigoplus_{1\leq i_1<\cdots<i_s\leq n}\left(\bigoplus_{j=0}^{v}At^j\right)e_{i_s\dots i_1}$$

and

$$F^v_{1s}(E)=\bigoplus_{1\leq i_1<\cdots<i_s\leq n}\left(\bigoplus_{j=0}^{v-1}At^j\right)e_{i_s\dots i_1}w.$$

Remark 2.3.1. $F^0_{**}(E)\subseteq F^1_{**}(E)\subseteq\dots$ is a filtration of $X_{**}(E)$ verifying

$$X_{**}(E)=\bigcup_{v\geq 0}F^v_{**}(E).$$

The graded complex $G_{**}(E)=\bigoplus_{v\geq 0}F^v_{**}(E)/F^{v-1}_{**}(E)$ associated to this filtration is $G_{**}(E)=X_{**}(Gr(E))$, where $Gr(E)=A[t,\alpha,0]$. That is

$$G^0_*(E)=A\xleftarrow{\partial^0_1}Y_1\xleftarrow{\partial^0_1}Y_2\xleftarrow{\partial_3}\cdots\xleftarrow{\partial^0_n}Y_n\longleftarrow 0,$$

where

$$Y_s=\bigoplus_{1\leq i_1<\cdots<i_s\leq n}Ae_{i_s\dots i_1},$$

$$\partial^0_s(\mathbf{X}^{\mathbf{m}}e_{i_s\dots i_1})=\sum_{j=1}^{s}(-1)^{s-j}\left(\prod_{l=1}^{j-1}q_{i_li_j}x_{i_j}\mathbf{X}^{\mathbf{m}}-\prod_{l=j+1}^{s}q_{i_ji_l}\mathbf{X}^{\mathbf{m}}x_{i_j}\right)e_{i_s\dots\widehat{i_j}\dots i_1}$$

and for $v>0$,

$$G^v_{**}(E)=\begin{array}{ccccccccccc}
Aw & \xleftarrow{\partial^{v-1}_{11}} & Y_1w & \xleftarrow{\partial^{v-1}_{12}} & Y_2w & \xleftarrow{\partial^{v-1}_{13}} & \cdots & \xleftarrow{\partial^{v-1}_{1n}} & Y_nw & \longleftarrow & 0\\
\downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_n} & & \\
A & \xleftarrow{\partial^{v}_{01}} & Y_1 & \xleftarrow{\partial^{v}_{02}} & Y_2 & \xleftarrow{\partial^{v}_{03}} & \cdots & \xleftarrow{\partial^{v}_{0n}} & Y_n & \longleftarrow & 0,
\end{array}$$

where

$$Y_s=\bigoplus_{1\leq i_1<\cdots<i_s\leq n}Ae_{i_s\dots i_1},\qquad Y_sw=\bigoplus_{1\leq i_1<\cdots<i_s\leq n}Ae_{i_s\dots i_1}w,$$

$$\partial_{0s}^{v}(\mathbf{X^m}e_{i_s\ldots i_1}) = \sum_{j=1}^{s}(-1)^{s-j}\left(\prod_{l=1}^{j-1} q_{i_l i_j}\, x_{i_j}\mathbf{X^m} - \overline{q}_{i_j}^{-v}\prod_{l=j+1}^{s} q_{i_j i_l}\,\mathbf{X^m}x_{i_j}\right) e_{i_s\ldots\widehat{i_j}\ldots i_1},$$

$$\partial_{1s}^{v-1}(\mathbf{X^m}e_{i_s\ldots i_1}w) = \sum_{j=1}^{s}(-1)^{s-j}\left(\overline{q}_{i_j}\prod_{l=1}^{j-1} q_{i_l i_j}\, x_{i_j}\mathbf{X^m} - \overline{q}_{i_j}^{-v-1}\prod_{l=j+1}^{s} q_{i_j i_l}\,\mathbf{X^m}x_{i_j}\right) e_{i_s\ldots\widehat{i_j}\ldots i_1}w,$$

$$\varphi_s^{v}(\mathbf{X^m}e_{i_s\ldots i_1}w) = (-1)^s\left[\prod_{l=1}^{s}\overline{q}_{i_l} - \prod_{l=1}^{s}\overline{q}_l^{\,-m_l}\right]\mathbf{X^m}e_{i_s\ldots i_1}.$$

Remark 2.3.2. When A is as in Example 1.8, then $Gr(E) = A[t,\alpha,0]$ is the k-algebra generated by the elements $x_1,\ldots,x_n,x_{r+1}^{-1},\ldots,x_n^{-1},t$ and the relations $x_jx_j^{-1} = 1 = x_j^{-1}x_j$ $(r < j \le n)$, $x_jx_i = q_{ij}x_ix_j$ $(1 \le i < j \le n)$ and $tx_i = \overline{q}_i^{-1}x_it$ $(1 \le i \le n)$. In this case, by Proposition 1.9, we have:

$$\mathrm{HH}_s(Gr(E)) = \bigoplus_{(\mathbf{m}_{i_1\ldots i_s},m)\in C'} k\,.x_1^{m_1}\ldots x_n^{m_n}t^m e_{i_s\ldots i_1}$$

$$\oplus \bigoplus_{(\mathbf{m}_{i_1\ldots i_{s-1}},m+1)\in C'} k\,.x_1^{m_1}\ldots x_n^{m_n}t^m e_{i_s\ldots i_1}w,$$

where $\mathbf{m}_{i_1\ldots i_u}$ is the element of $(\mathbf{N}\cup\{0\})^r \times \mathbf{Z}^{n-r}$ whose jth coordinate is m_j if $j \notin \{i_1,\ldots,i_u\}$ and $m_j + 1$ if $j \in \{i_1,\ldots,i_u\}$, and C' is the set of those $(n+1)$-tuples $(m_1,\ldots,m_n,m) \in (\mathbf{N}\cup\{0\})^r \times \mathbf{Z}^{n-r} \times (\mathbf{N}\cup\{0\})$ satisfying the following three conditions:

(i) If $i \le r$, then $m_i = 0$ or $\overline{q_i}^{-m}\prod_{l=1}^{n} q_{il}^{m_l} = 1$,
(ii) If $r < i \le n$, then $\overline{q_i}^{-m}\prod_{l=1}^{n} q_{il}^{m_l} = 1$,
(iii) $m = 0$ or $\prod_{l=1}^{n}\overline{q}_l^{m_l} = 1$.

We provide below some examples of Theorem 2.2.

Example 2.4. Let A be the k-algebra generated by x_1, x_2, x_2^{-1} and the relations $x_1x_2 = q^2x_2x_1$ and $x_2x_2^{-1} = 1 = x_2^{-1}x_2$ and let E be the Ore extension $E = A[t,\alpha,\delta]$, where $\alpha(x_1) = x_1$, $\alpha(x_2) = q^{-2}x_2$, $\delta(x_1) = (x_2 - x_2^{-1})/(q - q^{-1})$ and $\delta(x_2) = 0$. This algebra is the quantum group $U_q(sl(2,k))$. Let us write $D_q(m,x_2) = [(m)_{q^{-2}}x_2 - (m)_{q^2}x_2^{-1}]/(q - q^{-1})$. A simple computation shows that

(1) $t^mx_2^n = q^{-2nm}x_2^nt^m$,
(2) $t^mx_1 = x_1t^m + D_q(m,x_2)t^{m-1}$,
(3) $tx_1^m = x_1^mt + x_1^{m-1}D_q(m,x_2)$.

By Theorem 2.2 we have the following, where in order to abbreviate notations we set $\mathbf{X^m} = x_1^{m_1}x_2^{m_2}t^{m_3}$, $w^0 = 1$ and $w^1 = w$.

Theorem 2.4.1. *The Hochschild homology* $\mathrm{HH}_*(E)$ *is the homology of the double complex*

$$X_{**}(E) = \begin{array}{ccccccc} Ew & \xleftarrow{\partial_{11}} & Ee_1w \oplus Ee_2w & \xleftarrow{\partial_{12}} & Ee_{21}w & \longleftarrow & 0 \\ \downarrow{\varphi_0} & & \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \\ E & \xleftarrow{\partial_{01}} & Ee_1 \oplus Ee_2 & \xleftarrow{\partial_{02}} & Ee_{21} & \longleftarrow & 0 \end{array}$$

where, for $v=0,1$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}}e_1w^v) = ((1-q^{-2m_2})x_1^{m_1+1}x_2^{m_2}t^{m_3} - x_1^{m_1}x_2^{m_2}D_q(m_3,x_2)t^{m_3-1})w^v,$$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}}e_2w^v) = (q^{-2m_1+2v} - q^{-2m_3})x_1^{m_1}x_2^{m_2+1}t^{m_3}w^v,$$

$$\partial_{v2}(\mathbf{X}^{\mathbf{m}}e_{21}w^v) = (q^{-2m_1-2+2v} - q^{-2m_3})x_1^{m_1}x_2^{m_2+1}t^{m_3}e_1w^v$$
$$+((q^{-2m_2-2}-1)x_1^{m_1+1}x_2^{m_2}t^{m_3} + q^{-2}x_1^{m_1}x_2^{m_2}D_q(m_3,x_2)t^{m_3-1})e_2w^v,$$

$$\varphi_0(\mathbf{X}^{\mathbf{m}}w) = (1-q^{-2m_2})x_1^{m_1}x_2^{m_2}t^{m_3+1} - x_1^{m_1-1}D_q(m_1,x_2)x_2^{m_2}t^{m_3},$$

$$\varphi_1(\mathbf{X}^{\mathbf{m}}e_1w) = ((q^{-2m_2}-1)x_1^{m_1}x_2^{m_2}t^{m_3+1} + x_1^{m_1-1}D_q(m_1,x_2)x_2^{m_2}t^{m_3})e_1$$
$$+\frac{1}{q-q^{-1}}(x_1^{m_1}x_2^{m_2+1}t^{m_3} + (q^{2m_1}+q^{2m_3})x_1^{m_1}x_2^{m_2-2}t^{m_3})e_2,$$

$$\varphi_1(\mathbf{X}^{\mathbf{m}}e_2w) = ((q^{-2m_2}-q^2)x_1^{m_1}x_2^{m_2}t^{m_3+1} + x_1^{m_1-1}D_q(m_1,x_2)x_2^{m_2}t^{m_3})e_2,$$

$$\varphi_2(\mathbf{X}^{\mathbf{m}}e_{21}w) = ((q^2-q^{-2m_2})x_1^{m_1}x_2^{m_2}t^{m_3+1} - x_1^{m_1-1}D_q(m_1,x_2)x_2^{m_2}t^{m_3})e_{21}.$$

Remark 2.4.2. From Theorem 2.4.1 it follows immediately that $\mathrm{HH}_n(E)=0$ for $n>3$.

Example 2.5. Let A be the k-algebra generated by x_1, x_2, and the relation $x_2x_1=q^{-2}x_1x_2$ and let E be the Ore extension $E=A[t,\alpha,\delta]$, where $\alpha(x_1)=x_1$, $\alpha(x_2)=q^{-2}x_2$, $\delta(x_1)=(q^{-1}-q)x_2^2$ and $\delta(x_2)=0$. This algebra is the quantum group $\mathcal{O}_{q^2}(sok^3)$. A simple computation shows that

(1) $t^mx_2^n=q^{-2nm}x_2^nt^m$,
(2) $t^mx_1=x_1t^m+(q^{-1}-q)(m)_{q^{-4}}x_2^2t^{m-1}$,
(3) $tx_1^m=x_1^mt+(q^{-1}-q)(m)_{q^{-4}}x_1^{m-1}x_2^2$.

By Theorem 2.2 we have the following, where in order to abbreviate notations we set $\mathbf{X}^{\mathbf{m}}=x_1^{m_1}x_2^{m_2}t^{m_3}$, $w^0=1$ and $w^1=w$.

Theorem 2.5.1. *The Hochschild homology* $\mathrm{HH}_*(E)$ *is the homology of the double complex*

$$X_{**}(E) = \begin{array}{ccccccc} Ew & \xleftarrow{\partial_{11}} & Ee_1w \oplus Ee_2w & \xleftarrow{\partial_{12}} & Ee_{21}w & \longleftarrow & 0 \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \\ E & \xleftarrow{\partial_{01}} & Ee_1 \oplus Ee_2 & \xleftarrow{\partial_{02}} & Ee_{21} & \longleftarrow & 0 \end{array}$$

where, for $v = 0, 1$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}} e_1 w^v) = ((1 - q^{-2m_2}) x_1^{m_1+1} x_2^{m_2} t^{m_3} + (q - q^{-1})(m_3)_{q^{-4}} x_1^{m_1} x_2^{m_2+2} t^{m_3-1}) w^v,$$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}} e_2 w^v) = (q^{-2m_1+2v} - q^{-2m_3}) x_1^{m_1} x_2^{m_2+1} t^{m_3} w^v,$$

$$\begin{aligned}\partial_{v2}(\mathbf{X}^{\mathbf{m}} e_{21} w^v) = {} & (q^{-2m_1-2+2v} - q^{-2m_3}) x_1^{m_1} x_2^{m_2+1} t^{m_3} e_1 w^v \\ & + ((q^{-2m_2-2} - 1) x_1^{m_1+1} x_2^{m_2} t^{m_3} \\ & + (q^{-3} - q^{-1})(m_3)_{q^{-4}} x_1^{m_1} x_2^{m_2+1} t^{m_3-1}) e_2 w^v,\end{aligned}$$

$$\varphi_0(\mathbf{X}^{\mathbf{m}} w) = (1 - q^{-2m_2}) x_1^{m_1} x_2^{m_2} t^{m_3+1} - (q^{-1} - q)(m_1)_{q^{-4}} x_1^{m_1-1} x_2^{m_2+2} t^{m_3},$$

$$\begin{aligned}\varphi_1(\mathbf{X}^{\mathbf{m}} e_1 w) = {} & ((q^{-2m_2} - 1) x_1^{m_1} x_2^{m_2} t^{m_3+1} + (q^{-1} - q)(m_1)_{q^{-4}} x_1^{m_1-1} x_2^{m_2+2} t^{m_3}) e_1 \\ & + (q^{-1} - q)(q^{-2m_1} + q^{-2m_3}) x_1^{m_1} x_2^{m_2+1} t^{m_3} e_2,\end{aligned}$$

$$\varphi_1(\mathbf{X}^{\mathbf{m}} e_2 w) = ((q^{-2m_2} - q^2) x_1^{m_1} x_2^{m_2} t^{m_3+1} + (q^{-1} - q)(m_1)_{q^{-4}} x_1^{m_1-1} x_2^{m_2+2} t^{m_3}) e_2,$$

$$\varphi_2(\mathbf{X}^{\mathbf{m}} e_{21} w) = ((q^2 - q^{-2m_2}) x_1^{m_1} x_2^{m_2} t^{m_3+1} - (q^{-1} - q)(m_1)_{q^{-4}} x_1^{m_1-1} x_2^{m_2+2} t^{m_3}) e_{21}.$$

Remark 2.5.2. From Theorem 2.5.1 it follows immediately that $\mathrm{HH}_n(E) = 0$ for $n > 3$. When q is not a root of unity, the study of the homology of $G_{**}(E)$ shows that $\mathrm{HH}_n(E) = 0$ for $n > 2$.

Example 2.6. Let A be the k-algebra generated by x_1, x_2, x_3 and the relations $x_2x_1 = qx_1x_2$, $x_3x_1 = qx_1x_3$, $x_3x_2 = x_2x_3$ and let E be the Ore extension $E = A[t, \alpha, \delta]$, where $\alpha(x_1) = x_1$, $\alpha(x_2) = qx_2$, $\alpha(x_3) = qx_3$, $\delta(x_1) = [(q^2 - 1)/q]x_2x_3$, $\delta(x_2) = 0$ and $\delta(x_3) = 0$. This algebra is the quantum group $\mathcal{O}_q(M(2,k))$. A simple computation shows that

(1) $t^m x_2^n = q^{nm} x_2^n t^m$,
(2) $t^m x_3^n = q^{nm} x_3^n t^m$,
(3) $t^m x_1 = x_1 t^m + \frac{q^{2m}-1}{q} x_2 x_3 t^{m-1}$,
(4) $t x_1^m = x_1^m t + \frac{q^{2m}-1}{q} x_1^{m-1} x_2 x_3$.

By Theorem 2.2 we have the following result, where, in order to abbreviate notations, we set $\mathbf{X}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} x_3^{m_3} t^{m_4}$, $w^0 = 1$ and $w^1 = w$.

Theorem 2.6.1. *The Hochschild homology* $\mathrm{HH}_*(E)$ *is the homology of* $X_{**}(E) =$

$$
\begin{array}{ccccccccc}
Ew & \xleftarrow{\partial_{11}} & \bigoplus\limits_{1\le i\le 3} Ee_i w & \xleftarrow{\partial_{12}} & \bigoplus\limits_{1\le i<j\le 3} Ee_{ji} w & \xleftarrow{\partial_{13}} & Ee_{321} w & \longleftarrow & 0 \\
\Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi_1} & & \Big\downarrow{\scriptstyle \varphi_2} & & \Big\downarrow{\scriptstyle \varphi_3} & & \\
E & \xleftarrow{\partial_{01}} & \bigoplus\limits_{1\le i<j\le 3} Ee_i & \xleftarrow{\partial_{02}} & \bigoplus\limits_{1\le i<j\le 3} Ee_{ji} & \xleftarrow{\partial_{03}} & Ee_{321} & \longleftarrow & 0
\end{array}
$$

where, for $v = 0, 1$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}} e_1 w^v) = \left((1-q^{m_2+m_3}) x_1^{m_1+1} x_2^{m_2} x_3^{m_3} t^{m_4} + \frac{1-q^{2m_4}}{q} x_1^{m_1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4-1} \right) w^v,$$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}} e_2 w^v) = (q^{m_1-v} - q^{m_4}) x_1^{m_1} x_2^{m_2+1} x_3^{m_3} t^{m_4} w^v,$$

$$\partial_{v1}(\mathbf{X}^{\mathbf{m}} e_3 w^v) = (q^{m_1-v} - q^{m_4}) x_1^{m_1} x_2^{m_2} x_3^{m_3+1} t^{m_4} w^v,$$

$$\begin{aligned}
\partial_{v2}(\mathbf{X}^{\mathbf{m}} e_{21} w^v) = {} & (q^{m_1+1-v} - q^{m_4}) x_1^{m_1} x_2^{m_2+1} x_3^{m_3} t^{m_4} e_1 w^v \\
& + ((q^{m_2+m_3+1} - 1) x_1^{m_1+1} x_2^{m_2} x_3^{m_3} t^{m_4} \\
& + (q^{2m_4} - 1) x_1^{m_1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4-1}) e_2 w^v,
\end{aligned}$$

$$\begin{aligned}
\partial_{v2}(\mathbf{X}^{\mathbf{m}} e_{31} w^v) = {} & (q^{m_1+1-v} - q^{m_4}) x_1^{m_1} x_2^{m_2} x_3^{m_3+1} t^{m_4} e_1 w^v \\
& + ((q^{m_2+m_3+1} - 1) x_1^{m_1+1} x_2^{m_2} x_3^{m_3} t^{m_4} \\
& + (q^{2m_4} - 1) x_1^{m_1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4-1}) e_3 w^v,
\end{aligned}$$

$$\partial_{v2}(\mathbf{X}^{\mathbf{m}} e_{32} w^v) = (q^{m_1-v} - q^{m_4})(x_1^{m_1} x_2^{m_2} x_3^{m_3+1} t^{m_4} e_2 w^v - x_1^{m_1} x_2^{m_2+1} x_3^{m_3} t^{m_4} e_3 w^v),$$

$$\begin{aligned}
\partial_{v3}(\mathbf{X}^{\mathbf{m}} e_{321} w^v) = {} & (q^{m_1+1-v} - q^{m_4})(x_1^{m_1} x_2^{m_2} x_3^{m_3+1} t^{m_4} e_{21} w^v - x_1^{m_1} x_2^{m_2+1} x_3^{m_3} t^{m_4} e_{31} w^v) \\
& + ((1 - q^{m_2+m_3+2}) x_1^{m_1+1} x_2^{m_2} x_3^{m_3} t^{m_4} \\
& + q(1 - q^{2m_4}) x_1^{m_1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4-1}) e_{32} w^v,
\end{aligned}$$

$$\varphi_0(\mathbf{X}^{\mathbf{m}} w) = (1 - q^{m_2+m_3}) x_1^{m_1} x_2^{m_2} x_3^{m_3} t^{m_4+1} - \frac{q^{2m_1} - 1}{q} x_1^{m_1-1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4},$$

$$\begin{aligned}
\varphi_1(\mathbf{X}^{\mathbf{m}} e_1 w) = {} & \left((q^{m_2+m_3} - 1) x_1^{m_1} x_2^{m_2} x_3^{m_3} t^{m_4+1} - \frac{q^{2m_1} - 1}{q} x_1^{m_1-1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4} \right) e_1 \\
& - (q - q^{-1})(q^{m_1+m_2} x_1^{m_1} x_2^{m_2} x_3^{m_3+1} t^{m_4} e_2 - q^{m_4} x_1^{m_1} x_2^{m_2+1} x_3^{m_3} t^{m_4} e_3),
\end{aligned}$$

$$\varphi_1(\mathbf{X}^{\mathbf{m}} e_2 w) = \left((q^{m_2+m_3} - q^{-1}) x_1^{m_1} x_2^{m_2} x_3^{m_3} t^{m_4+1} - \frac{q^{2m_1} - 1}{q} x_1^{m_1-1} x_2^{m_2+1} x_3^{m_3+1} t^{m_4} \right) e_2,$$

$$\varphi_1(\mathbf{X}^{\mathbf{m}}e_3w)=\left((q^{m_2+m_3}-q^{-1})x_1^{m_1}x_2^{m_2}x_3^{m_3}t^{m_4+1}-\frac{q^{2m_1}-1}{q}x_1^{m_1-1}x_2^{m_2+1}x_3^{m_3+1}t^{m_4}\right)e_3,$$

$$\varphi_2(\mathbf{X}^{\mathbf{m}}e_{21}w)=\left((q^{-1}-q^{m_2+m_3})x_1^{m_1}x_2^{m_2}x_3^{m_3}t^{m_4+1}-\frac{q^{2m_1}-1}{q}x_1^{m_1-1}x_2^{m_2+1}x_3^{m_3+1}t^{m_4}\right)e_{21}$$
$$-q^{m_3+m_4}(q^2-1)x_1^{m_1}x_2^{m_2+1}x_3^{m_3}t^{m_4}e_{32},$$

$$\varphi_2(\mathbf{X}^{\mathbf{m}}e_{31}w)=\left((q^{-1}-q^{m_2+m_3})x_1^{m_1}x_2^{m_2}x_3^{m_3}t^{m_4+1}-\frac{q^{2m_1}-1}{q}x_1^{m_1-1}x_2^{m_2+1}x_3^{m_3+1}t^{m_4}\right)e_{31}$$
$$-q^{m_1}(q-q^{-1})x_1^{m_1}x_2^{m_2}x_3^{m_3+1}t^{m_4}e_{32},$$

$$\varphi_2(\mathbf{X}^{\mathbf{m}}e_{32}w)=\left((q^{-2}-q^{m_2+m_3})x_1^{m_1}x_2^{m_2}x_3^{m_3}t^{m_4+1}-\frac{q^{2m_1}-1}{q}x_1^{m_1-1}x_2^{m_2+1}x_3^{m_3+1}t^{m_4}\right)e_{32},$$

$$\varphi_3(\mathbf{X}^{\mathbf{m}}e_{321}w)=\left((q^{m_2+m_3}-q^{-2})x_1^{m_1}x_2^{m_2}x_3^{m_3}t^{m_4+1}+\frac{q^{2m_1}-1}{q}x_1^{m_1-1}x_2^{m_2+1}x_3^{m_3+1}t^{m_4}\right)e_{321}.$$

Remark 2.6.2. From Theorem 2.6.1 it follows immediately that $\mathrm{HH}_n(E)=0$ for $n>4$. When q is not a root of unity, the study of the homology of $G_{**}(E)$ shows that $\mathrm{HH}_n(E)=0$ for $n>2$.

References

[1] J.A. Guccione, J.J. Guccione, Hochschild and cyclic homology of Ore extensions and some quantum examples, K-theory, to appear.

[2] J.A. Guccione, J.J. Guccione, Hochschild and cyclic homology of Q-difference operators, J. Kyoto University 35 (3) (1995) 413–422.

[3] D. Jordan, Iterated skew polynomial rings and quantum groups, J. Algebra 156 (1993) 194–218.

[4] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, Comm. Math. Phys. 146 (1992) 343–356.

[5] N. Reshetikhin, L. Takhtadzhyan, L. Faddeev, Quantization of Lie groups and Lie algebras, Algebra i Analiz 1 (1989) 178–206; English translation: Leningrad Math. J. 1 (1990) 193–225.

[6] S. Smith, Quantum groups, an introduction and survey for ring theorists, in: S. Montgomery, L.W. Small (Eds.), Noncommutative Rings, MSRI Publ., vol. 24, Springer, Berlin, 1992, pp. 131–178.

[7] L. Takhtadjian, Noncommutative homology of quantum tori, Funkt. Anal. Pril. 24 (2) (1989) 75–76; English translation: Funk. Anal. Appl. 23 (1989) 147–149.

[8] M. Wambst, Complexes de Koszul Quantiques, Ann. Inst. Fourier, Grenoble 43 (4) (1993) 1089–1156.

[9] M. Wambst, Homologie de Hochschild et homologie cyclique du tore quantique multiparamétré, Preprint.