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Journal of Number Theory 113 (2005) 339–379
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

A formula for the central value of certain Hecke L-functions

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Received 15 June 2004; revised 16 October 2004

Available online 5 February 2005

Communicated by David Goss

Abstract

Let $N \equiv 1 \bmod 4$ be the negative of a prime, $K = \mathbb{Q}(\sqrt{N})$ and $\mathcal{O}_K$ its ring of integers. Let $\mathcal{D}$ be a prime ideal in $\mathcal{O}_K$ of prime norm congruent to $3 \bmod 4$. Under these assumptions, there exists Hecke characters $\psi_{\mathcal{D}}$ of K with conductor $(\mathcal{D})$ and infinite type $(1, 0)$. Their L-series $L(\psi_{\mathcal{D}}, s)$ are associated to a CM elliptic curve $\mathfrak{A}(N, \mathcal{D})$ defined over the Hilbert class field of K. We will prove a Waldspurger-type formula for $L(\psi_{\mathcal{D}}, s)$ of the form $L(\psi_{\mathcal{D}}, 1) = \Omega \sum_{[\mathcal{A}], I} r(\mathcal{D}, [\mathcal{A}], I) m_{[\mathcal{A}], I}([\mathcal{D}])$ where the sum is over class ideal representatives I of a maximal order in the quaternion algebra ramified at $|N|$ and infinity and $[\mathcal{A}]$ are class group representatives of K. An application of this formula for the case $N = -7$ will allow us to prove the non-vanishing of a family of L-series of level $7|D|$ over K.

MSC: 11G40

Keywords: Hecke L-functions

1. Introduction

Given an elliptic curve E over $\mathbb{Q}$, and a fundamental discriminant D, a formula of Waldspurger relates the value of $L(E \otimes D, 1)$, the twist of E by D, with the coefficients

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[1] This paper is the author's Ph.D. thesis presented at the University of Texas in August 2003. He would like to thank Dr. Fernando Rodriguez-Villegas and the University of Texas at Austin.

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doi:10.1016/j.jnt.2004.12.004

of a 3/2 modular form (see [18]). The purpose of this work is to get a formula for quadratic twists of a family of elliptic curves with complex multiplication not defined over the rationals.

Given an imaginary quadratic field K the theory of complex multiplication (see [14]) gives a relation between elliptic curves with CM given by an order of K and L-series associated to Hecke characters ψ on K. The simplest case is when $K = \mathbb{Q}(\sqrt{N})$ with $N \equiv 1 \bmod 4$ the negative of a prime and ψ is a character of conductor $\sqrt{N}$. In this case the L-series corresponds to a CM elliptic curve $\mathfrak{A}(N)$ studied by Gross in [4], defined over H, the Hilbert class field of K. A formula for the central value of $L(\psi, 1)$ was given by Villegas in [12].

In this paper we will study the central value of the L-series corresponding to the CM elliptic curves $\mathfrak{A}(N, \mathcal{D})$, given by twists of $\mathfrak{A}(N)$ by the quadratic character of conductor $\sqrt{N}\mathcal{D}$ where $\mathcal{D}$ is a prime ideal of K prime to $\sqrt{N}$ and with prime norm congruent to $3 \bmod 4$. If we denote h the class number of K, the prime ideal $\mathcal{D}$ has h Hecke characters $\psi_{\mathcal{D}}$ of conductor $\mathcal{D}$ associated to it. The relation between the L-series of $\mathfrak{A}(N, \mathcal{D})$ and $L(\psi_{\mathcal{D}}, s)$ is given explicitly by

$$L(\mathfrak{A}(N, \mathcal{D})/H, s) = \prod_{\psi_{\mathcal{D}}} L(\psi_{\mathcal{D}}, s) L(\overline{\psi_{\mathcal{D}}}, s),$$

where H is the Hilbert class field of K and the product is over the h Hecke characters associated to $\mathcal{D}$ (see [4, formula (8.4.4), Theorem 18.1.7]). If we define $\mathfrak{B}$ be the Weil restriction of scalars of $\mathfrak{A}(N, \mathcal{D})$ to K, then $\mathfrak{B}$ is a CM abelian variety, and $L(\mathfrak{A}(N, \mathcal{D})/H, s) = L(\mathfrak{B}/K, s)$.

Let B be the quaternion algebra ramified at $|N|$ and infinity. Given an element $x \in B$ we denote $\mathrm{N}(x) := x\bar{x}$ its norm and $\mathrm{Tr}(x) := x + \bar{x}$ its trace. To the ideal $\mathcal{D}$ and an element $[\mathcal{A}]$ of $Cl(\mathcal{O}_K)$ we will associate a maximal order $O_{[\mathcal{A}],[\mathcal{D}]}$ in B depending only on $[\mathcal{A}]$ and the class of $\mathcal{D}$. If $\{I\}$ are representatives for left $O_{[\mathcal{A}],[\mathcal{D}]}$-ideals, the main theorem (Theorem 6) gives the formula $L(\psi_{\mathcal{D}}, 1) = \Omega \sum_{[\mathcal{A}], I} r(\mathcal{D}, [\mathcal{A}], I) m_{[\mathcal{A}], I}([\mathcal{D}])$ where the sum is over the ideals $\{I\}$ and ideal representatives of $\mathcal{O}_K$, Ω is a period, $r(\mathcal{D}, [\mathcal{A}], I)$ is a rational integer and the numbers $m_{[\mathcal{A}], I}([\mathcal{D}])$ are algebraic integers.

The paper consists of four sections besides the introduction. In the second section we give the basic definitions and derive a first formula for the value of the L-series at 1 (following Hecke's work on L-series, see [7]). Later we relate theta functions of quadratic forms to theta functions on the Siegel space. In the third section we introduce the period Ω and using Shimura's theory in Complex Multiplication we compute the field where the algebraic integers $m_{[\mathcal{A}], I}$ belong to. In the fourth section we study the problem of deciding whether two points in the Siegel space are equivalent or not in our specific case. For this purpose we introduce quaternion algebras, and relate special points with left $O_{[\mathcal{A}],[\mathcal{D}]}$-ideals. In the last section we study in detail the case when the class number of K is one. In this case the elliptic curve $\mathfrak{A}(N)$ is defined over $\mathbb{Q}$ and the numbers $m_{[\mathcal{A}], I}$ turn out to be rational integers. In the case $N = -7$ using the fact that the quaternion algebra has class number 1 for maximal ideals, we prove that the CM elliptic curves $\mathfrak{A}(N, \mathcal{D})$ defined over K have a non-vanishing L-series for all primes $\mathcal{D}$.

We finish this work with a remarkable relation between the numbers $m_{[\mathcal{A}],I}$ and the coordinates of the eigenvector of the modular form associated to $\mathfrak{A}(N)$ represented in the Brandt matrices of level N^2.

2. L-series

2.1. L-series definition

Given a number field K, we will denote $\mathcal{O}_K$ its ring of integers, $Cl(\mathcal{O}_K)$ its class group and h its class number.

Let $N \equiv 1 \bmod 4$ be the negative of a prime, $N \neq -3$ and $K := \mathbb{Q}(\sqrt{N})$. Let $D \equiv 1 \bmod 4$ be the negative of a prime such that the ideal generated by D splits completely in K, i.e. $(D) = (\mathcal{D})(\bar{\mathcal{D}})$. We will denote $L := \mathbb{Q}(\sqrt{D})$. Since the rings $\mathcal{O}_K/\mathcal{D}$ and $\mathbb{Z}/|D|\mathbb{Z}$ are isomorphic we define $\varepsilon_{\mathcal{D}}$ by

$$\begin{array}{ccc} (\mathcal{O}_K/\mathcal{D})^{\times} & \xrightarrow{\varepsilon_{\mathcal{D}}} & \pm 1 \\ & \searrow \quad \nearrow_{\left(\frac{\cdot}{|D|}\right)} & \\ & (\mathbb{Z}/|D|\mathbb{Z})^{\times} & \end{array}$$

where $\left(\frac{\ }{|D|}\right)$ is the Kronecker symbol. The character $\varepsilon_{\mathcal{D}}$ induces a Hecke character $\psi_{\mathcal{D}}$ on principal ideals by $\psi_{\mathcal{D}}(\langle\alpha\rangle) = \varepsilon_{\mathcal{D}}(\alpha)\alpha$.

Proposition 1. *The character $\psi_{\mathcal{D}}$ on principal ideals is well defined.*

Proof. Since 1 and -1 are the only units in K, we must check that $\varepsilon_{\mathcal{D}}(\alpha)\alpha = -\varepsilon_{\mathcal{D}}(-\alpha)\alpha$. This follows from the fact that $\varepsilon_{\mathcal{D}}$ is multiplicative and $|D| \equiv 3 \bmod 4$, hence $\varepsilon_{\mathcal{D}}(-1) = -1$. $\square$

Given $\mathcal{D}$ let $\sigma_{\mathcal{D}}$ denote an element in $Gal(K^{\mathrm{ab}}/K)$ corresponding to $\mathcal{D}$ via the Artin–Frobenius map, where K^{ab} denotes the abelian closure of K. We can define $\varepsilon_{\mathcal{D}}$ in a different way:

Proposition 2. *If $\alpha \notin \mathcal{D}$, $\varepsilon_{\mathcal{D}}(\alpha) = (\sqrt{\alpha})^{\sigma_{\mathcal{D}}-1}$.*

Proof. It is clear that $\sqrt{\alpha}^{\sigma_{\mathcal{D}}} = \xi_2\sqrt{\alpha}$ where $\xi_2 = \pm 1$. By definition given $\mathfrak{D}$ an ideal of $\bar{K}$ lying above $\mathcal{D}$, $\sigma_{\mathcal{D}}$ satisfies $\xi_2\sqrt{\alpha} = \sqrt{\alpha}^{\sigma_D} \equiv \sqrt{\alpha}^{|D|} \bmod \mathfrak{D}$. But $\sqrt{\alpha}^{|D|} = \alpha^{\frac{|D|-1}{2}}\sqrt{\alpha}$ hence $\alpha^{\frac{|D|-1}{2}} \equiv \xi_2 \bmod \mathcal{D}$. In particular $\varepsilon_{\mathcal{D}}(\alpha) = \varepsilon_{\mathcal{D}}(\xi_2) = \xi_2$ since $|D| \equiv 3 \bmod 4$. $\square$

The character actually depends on the choice of $\mathcal{D}$ (i.e. we have one character associated to $\mathcal{D}$ and another one associated to $\bar{\mathcal{D}}$). Abusing notation ψ will denote the character associated to $\mathcal{D}$ if it makes no confusion.

The character ψ defined on principal ideals extends to h Hecke characters on $I(\mathcal{O}_K)$ the set of ideals of $\mathcal{O}_K$. We fix an extension once and for all and we call it ψ. Then $\psi : I(\mathcal{O}_K) \longrightarrow T_\psi$, where T_ψ is the degree h field extension of K.

Definition 3. The L-series associated to ψ is

$$L(\psi, s) := \sum_{\mathcal{A}} \frac{\psi(\mathcal{A})}{N\mathcal{A}^s}, \tag{1}$$

where the sum is over all ideals $\mathcal{A}$ of $\mathcal{O}_K$.

By Hecke's work we know that $L(\psi, s)$ extends to an analytic function in the upper half plane, and satisfies the functional equation:

$$\left(\frac{2\pi}{\sqrt{ND}}\right)^{-s} \Gamma(s) L(\psi, s) = w_\psi \left(\frac{2\pi}{\sqrt{ND}}\right)^{s-2} \Gamma(2-s) L(\bar{\psi}, 2-s),$$

where w_ψ is the root number. The character ψ defines a weight 2 modular form given for z in the upper half plane by $f_\psi(z) = \sum_{\mathcal{A}} \psi(\mathcal{A}) e^{2\pi i z N\mathcal{A}}$, which has level ND. The root number is given by $w_\psi = f_\psi(\frac{i}{\sqrt{ND}}) / \overline{f_\psi(\frac{i}{\sqrt{ND}})}$.

Proposition 4. *Let α be a generator of $\mathcal{D}^h$. The root number in the functional equation for $\psi_{\mathcal{D}}$ is $w_\psi = \xi_2 \left(\frac{2}{|N|}\right) i \frac{\alpha}{|\alpha|}$, where ξ_2 is -1 if 2 is ramified in $K(\sqrt{\alpha\sqrt{N}})$ and 1 if not.*

Proof. See [1, Proposition 10.6, p. 20]. This is equivalent to saying that if α is the generator of $\mathcal{D}^h$ such that $K(\sqrt{\alpha\sqrt{N}})$ is a quadratic extension of K of conductor $\sqrt{N}\mathcal{D}$ then $w_\psi = -\left(\frac{2}{|N|}\right) i \frac{\alpha}{|\alpha|}$. □

The characters ψ are associated to a CM elliptic curve $\mathfrak{A}(N, \mathcal{D})$ defined over H, the Hilbert class field of K, by the formula:

$$L(\mathfrak{A}(N, \mathcal{D})/H, s) = \prod_{\psi_{\mathcal{D}}} L(\psi_{\mathcal{D}}, s) L(\overline{\psi_{\mathcal{D}}}, s).$$

See [4, formula (8.4.4), Theorem 18.1.7].

2.2. Choosing characters in a consistent way

Let $\mathcal{D}$ and $\mathcal{D}'$ be prime ideals of K as before (i.e. they have prime norm congruent to 3 mod 4). While extending the Hecke character $\psi_{\mathcal{D}}$ to $I(\mathcal{O}_K)$ we get a field extension $T_{\psi_{\mathcal{D}}}$. If we extend the Hecke character associated to $\mathcal{D}'$ in an arbitrary way, the image of both characters will lie in different fields. There is a natural way of defining a Hecke character $\psi_{\mathcal{D}'}$ associated to $\mathcal{D}'$ such that $\psi_{\mathcal{D}'}(I(\mathcal{O}_K)) \subset T_{\psi_{\mathcal{D}}}$. Any ideal of K raised to the h-power is principal, hence for all ideals $\mathcal{A}$ prime to $\mathcal{D}\mathcal{D}'$ we define:

$$\psi_{\mathcal{D}'}(\mathcal{A}) = \psi_{\mathcal{D}}(\mathcal{A})\frac{\varepsilon_{\mathcal{D}'}(\mathcal{A}^h)}{\varepsilon_{\mathcal{D}}(\mathcal{A}^h)}. \tag{2}$$

There is some abuse of notation on this definition since although $\mathcal{A}^h$ is principal, it has two generators α and $-\alpha$. But $\varepsilon_{\mathcal{D}}(-\alpha) = -\varepsilon_{\mathcal{D}}(\alpha)$ and $\varepsilon_{\mathcal{D}'}(-\alpha) = -\varepsilon_{\mathcal{D}'}(\alpha)$ hence the quotient is well defined.

Proposition 5. *There exists a Hecke character associated to $\mathcal{D}'$ taking values in T_ψ and defined as above on ideals prime to $\mathcal{D}\mathcal{D}'$.*

Proof. We start by proving that the character defined above is a Hecke character on ideals prime to $\mathcal{D}\mathcal{D}'$. If $\mathcal{A}$ is principal, say $\mathcal{A} = \langle\alpha\rangle$, then $\psi_{\mathcal{D}'}(\alpha) = \varepsilon_{\mathcal{D}}(\alpha)\alpha\frac{\varepsilon_{\mathcal{D}'}(\alpha)^h}{\varepsilon_{\mathcal{D}}(\alpha)^h}$. Since h is odd, and ε takes the values ± 1, we get that $\psi_{\mathcal{D}'}(\alpha) = \varepsilon_{\mathcal{D}'}(\alpha)\alpha$, hence it is a Hecke character.

Let $\mathfrak{q}$ be a prime ideal in the same equivalence class as $\mathcal{D}$ and prime to $\mathcal{D}\mathcal{D}'$ (there exists such an ideal by the Tchebotarev density theorem), say $\mathfrak{q}\beta = \mathcal{D}$. Then $\psi_{\mathcal{D}'}(\mathcal{D}) = \psi_{\mathcal{D}'}(\mathfrak{q}\beta) = \psi_{\mathcal{D}'}(\mathfrak{q})\psi_{\mathcal{D}'}(\beta) = \psi_{\mathcal{D}'}(\mathfrak{q})\varepsilon_{\mathcal{D}'}(\beta)\beta$. In this way we can extend the character to all ideals prime to $\mathcal{D}'$ and clearly this is well defined, taking values in T_ψ. □

From now on given two different characters $\psi_{\mathcal{D}}$ and $\psi_{\mathcal{D}'}$ we will always assume that they are chosen in a consistent way.

Given a quadratic imaginary field $\mathbb{Q}[\sqrt{-d}]$ we denote w_d the number of units in its ring of integers. For $z \in \mathfrak{h}$, we recall the definition:

$$\eta(z) = e^{2\pi iz/24}\prod_{n=1}^{\infty}(1 - e^{2\pi inz}).$$

While choosing ideal class representatives $\{[\mathcal{A}]\}$ for K we will assume they are prime to the ideal (6) and that they are written as $\mathcal{A} = \langle a, \frac{b+\sqrt{N}}{2}\rangle$ with $b \equiv 3 \bmod 48$. We define $\eta(\mathcal{A}) := \eta(\frac{b+\sqrt{N}}{2a})$. Our main theorem is the following:

Theorem 6. *Given $\mathcal{D}$ a prime ideal of K of prime norm congruent to $3\, mod\, 4$ let $\psi_{\mathcal{D}}$ be a Hecke character as before. Let B be the quaternion algebra over $\mathbb{Q}$ ramified at*

$|N|$ and infinity. For each ideal class representative $[\mathcal{A}]$ of K there exists $O_{[\mathcal{A}],[\mathcal{D}]}$ a maximal order in B such that

$$L(\psi_{\mathcal{D}},1)=\frac{2\pi}{w_{|D|}\sqrt{|D|}}\eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K)\left(\sum_{[\mathcal{A}]}\sum_{I} r(\mathcal{D},[\mathcal{A}],I)m_{[\mathcal{A}],I}([\mathcal{D}])\right), \tag{3}$$

where $\{I\}$ is a set of left $O_{[A],[\mathcal{D}]}$-ideal representatives, $r(\mathcal{D},[\mathcal{A}],I)\in\mathbb{Z}$ and $m_{[\mathcal{A}],I}([\mathcal{D}])$ are algebraic integers lying in a finite field extension of $\mathbb{Q}$ (see Diagram 1).

The term $\Omega=\frac{2\pi}{w_{|D|}\sqrt{|D|}}\eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K)$ on (3) corresponds to a period of the abelian variety $\mathfrak{B}$ and the number $r(\mathcal{D},[\mathcal{A}],I)$ is counting some special points with a ± 1 weight (see Section 4.3 for details). The rest of this paper will be a constructive proof of Theorem 6.

2.3. Computing the L-series value at 1

Given $\mathcal{A}$ an ideal of K, we will denote $[\mathcal{A}]$ its class in the class group. We can decompose the L-series as

$$L(\psi,s)=\sum_{[\mathcal{A}]}\sum_{\mathcal{B}\sim\mathcal{A}}\frac{\psi(\mathcal{B})}{N\mathcal{B}^s}. \tag{4}$$

Proposition 7. *All integral ideals equivalent to $\mathcal{A}$ are of the form $c\mathcal{A}$ for some $c\in\mathcal{A}^{-1}$.*

Proof. Easy to check. □

Since the only units in $\mathcal{O}_K$ are 1 and -1,

$$\sum_{\mathcal{B}\sim\mathcal{A}}\frac{\psi(\mathcal{B})}{N\mathcal{B}^s}=\frac{1}{2}\sum_{c\in\bar{\mathcal{A}}}\frac{\psi(c)\psi(\mathcal{A})}{\psi(N\mathcal{A})}\frac{N\mathcal{A}^s}{Nc^s}=\frac{1}{2}N\mathcal{A}^s\frac{\psi(\mathcal{A})}{\psi(N\mathcal{A})}\sum_{c\in\bar{\mathcal{A}}}\frac{\psi(c)}{Nc^s}.$$

Since ψ is multiplicative $\psi(\mathcal{A})\psi(\bar{\mathcal{A}})=\psi(N\mathcal{A})$, then $\frac{\psi(\mathcal{A})}{\psi(N\mathcal{A})}=\frac{1}{\psi(\bar{\mathcal{A}})}$. Using the fact that $N\mathcal{A}=N\bar{\mathcal{A}}$ it follows that $\sum_{\mathcal{B}\sim\mathcal{A}}\frac{\psi(\mathcal{B})}{N\mathcal{B}^s}=\frac{1}{2}\frac{N\bar{\mathcal{A}}^s}{\psi(\bar{\mathcal{A}})}\sum_{c\in\bar{\mathcal{A}}}\frac{\psi(c)}{Nc^s}$ and we can write the L-series as

$$L(\psi,s)=\frac{1}{2}\sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)}\frac{N\mathcal{A}^s}{\psi(\mathcal{A})}\sum_{c\in\mathcal{A}}\frac{c\varepsilon_{\mathcal{D}}(c)}{Nc^s}. \tag{5}$$

Without loss of generality, we may assume that $\mathcal{A}=a\mathbb{Z}+\frac{b+\sqrt{N}}{2}\mathbb{Z}$ and $\mathcal{D}=|D|\mathbb{Z}+\frac{b+\sqrt{N}}{2}\mathbb{Z}$, hence $\mathcal{A}\mathcal{D}=a|D|\mathbb{Z}+\frac{b+\sqrt{N}}{2}\mathbb{Z}$ (see [12, Section 2.3, p. 552]). If $c\in\mathcal{A}$

then $c = ma + n\frac{b+\sqrt{N}}{2}$, and $\varepsilon_{\mathcal{D}}(c) = \varepsilon_{\mathcal{D}}(ma + n\frac{b+\sqrt{N}}{2})$. Since $n\frac{b+\sqrt{N}}{2} \in \mathcal{D}$, $\varepsilon_{\mathcal{D}}(c) = \varepsilon_{\mathcal{D}}(a)\varepsilon_{\mathcal{D}}(m) = \varepsilon_{\mathcal{D}}(N\mathcal{A})\varepsilon_{\mathcal{D}}(m)$. We will denote $z_{\mathcal{A}}$ the point $\frac{b+\sqrt{N}}{2a}$ (respectively $z_{\mathcal{D}}$ the point $\frac{b+\sqrt{N}}{2|D|}$ and $z_{\mathcal{A}\mathcal{D}}$ the point $\frac{b+\sqrt{N}}{2a|D|}$). Also we denote by $\sum'$ the sum removing the zero element (or zero vector depending on the context). We have

$$L(\psi, s) = \frac{1}{2} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_K)} \frac{N\mathcal{A}^{1-s}\varepsilon_{\mathcal{D}}(N\mathcal{A})}{\psi(\mathcal{A})} \sum_{m,n\in\mathbb{Z}}{}' \frac{\varepsilon_{\mathcal{D}}(m)(m + z_{\mathcal{A}\mathcal{D}}|D|n)}{\mathrm{N}(m + z_{\mathcal{A}\mathcal{D}}|D|n)^s}. \tag{6}$$

If we change m by $-m$ in the sum, since $\varepsilon_{\mathcal{D}}(-1) = -1$, the term in the inner sum can be written as $\frac{\varepsilon_{\mathcal{D}}(m)}{(m+(-\bar{z}_{\mathcal{A}\mathcal{D}})|D|n)|m+(-\bar{z}_{\mathcal{A}\mathcal{D}})|D|n|^{2s-2}}$, where the point $-\bar{z}_{\mathcal{A}\mathcal{D}}$ is in the upper half plane. This sum is related to Eisenstein series that we define below:

Definition 8. Let p be a prime integer and $\varepsilon(m) := \left(\frac{m}{p}\right)$. We define the Eisenstein series associated to ε by $E_1(z, s) = \sum'_{m,n\in\mathbb{Z}} \frac{\varepsilon(m)}{(m+zpn)|m+zpn|^{2s}}$.

By (6) taking $p = |D|$ we get the relation:

$$L(\psi, s) = \frac{1}{2} \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_K)} \frac{N\mathcal{A}^{1-s}\varepsilon_{\mathcal{D}}(N\mathcal{A})}{\psi(\mathcal{A})} E_1(-\bar{z}_{\mathcal{A}\mathcal{D}}, s-1). \tag{7}$$

$E_1(z, s)$ turns out to be a modular form of weight 1 with a character. We need to compute its value at $s = 0$ for a point z in the upper half plane. This was done by Hecke and its value (given in formula (11)) can be found in [7, formulas (26), (27), p. 475]. For the reader convenience we re-derive the formula.

The series of $E_1(z, s)$ converges only for $\Re(s) > \frac{3}{2}$, but it can be analytically continued to the whole plane and satisfy a functional equation. We will compute its value at $s = 0$ using Hecke's trick. Since ε is a character of conductor p, we break the sum over m as

$$E_1(z, s) = \sum_{m\in\mathbb{Z}}{}' \frac{\varepsilon(m)}{m} + 2\sum_{n=1}^{\infty} \sum_{r \bmod p} \varepsilon(r) \sum_{m\in\mathbb{Z}} \frac{1}{(zpn + r + mp)|zpn + r + mp|^{2s}} \tag{8}$$

and dividing the last sum by p^{2s+1} we get

$$E_1(z, s) = 2L(\varepsilon, s) + 2\sum_{n=1}^{\infty} \sum_{r \bmod p} \frac{\varepsilon(r)}{p^{2s+1}} \sum_{m\in\mathbb{Z}} \frac{1}{\left(\frac{zpn + r}{p} + m\right)\left|\frac{zpn + r}{p} + m\right|^{2s}}. \tag{9}$$

For z in the upper half plane we define:

$$H(z,s)=\sum_{m\in\mathbb{Z}}\frac{1}{(z+m)|z+m|^{2s}}.$$

Lemma 9. *Let $z=x+iy$ be a point in the upper half plane, then*

$$\sum_{m=-\infty}^{\infty}(z+m)^{-(s+1)}(\bar{z}+x)^{-s}=\sum_{n=-\infty}^{\infty}\tau_n(y,s+1,s)e^{2\pi inx},$$

where $\tau_n(y,s+1,s)$ is given by

$$\tau_n(y,s+1,s)\frac{i\Gamma(s+1)\Gamma(s)}{(2\pi)^{2s+1}}=\begin{cases} n^{2s}e^{-2\pi ny}\sigma(4\pi ny,s+1,s) & (n>0),\\ |n|^{2s}e^{-2\pi|n|y}\sigma(4\pi|n|y,s,s+1) & (n<0),\\ \Gamma(2s)(4\pi y)^{-2s} & n=0,\end{cases}$$

and $\sigma(y,\alpha,\beta)=\int_0^\infty(t+1)^{\alpha-1}t^{\beta-1}e^{-yt}\,dt$.

Proof. This is Lemma 1, p. 84 [15]. □

The right-hand side of Lemma 9 equality converges for any $s>0$, so we can compute the limit when s tends to 0 of $\tau_n(y,s+1,s)$ in the different cases:

- Case $n=0$: $\lim_{s\to 0}\frac{(2\pi)^{2s+1}}{i\Gamma(s+1)}\frac{\Gamma(2s)}{\Gamma(s)}(4\pi y)^{-2s}=-i\pi$.
- Case $n<0$: $\lim_{s\to 0}\frac{(2\pi)^{2s+1}}{i\Gamma(s+1)\Gamma(s)}|n|^{2s}e^{2\pi|n|y}\int_0^\infty(t+1)^{s-1}t^se^{-4\pi|n|yt}\,dt=0$.
- Case $n>0$: $\lim_{s\to 0}\frac{(2\pi)^{2s+1}n^{2s}}{i\Gamma(s+1)}e^{-2\pi ny}\frac{1}{\Gamma(s)}\int_0^\infty(t+1)^st^{s-1}e^{-4\pi nyt}\,dt$.

We just need to compute $\lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^1(t+1)^st^{s-1}e^{-4\pi nyt}dt$. Doing integration by parts:

$$\int_0^1(t+1)^st^{s-1}e^{-4\pi nyt}\,dt=\frac{2^se^{-4\pi ny}}{s}-\int_0^1t^s(t+1)^{s-1}e^{-4\pi nyt}\,dt$$
$$-\frac{1}{s}\int_0^1t^s(t+1)^se^{-4\Pi nyt}(-4\pi nyt)\,dt.$$

The function $\Gamma(z)$ has a simple pole at $z=0$ with residue 1. Dividing the integral by $\Gamma(s)$ and taking the limit when s tends to zero we get

$$\lim_{s\to 0}\tau_n(y,s+1,s)=-2\pi ie^{-2\pi ny}. \tag{10}$$

We just prove:

Lemma 10. $\lim_{s\to 0} H(s,z) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n$.

Eq. (9) can be written as

$$E_1(z,s) = 2L(\varepsilon,s) + 2\sum_{n=1}^{\infty}\sum_{r \bmod p} \frac{\varepsilon(r)}{p^{2s+1}} H\left(\frac{zpn+r}{p}, s\right),$$

which by Lemma 9 is the same as

$$E_1(z,s) = 2L(\varepsilon,s) + 2\sum_{n=1}^{\infty}\sum_{r \bmod p} \frac{\varepsilon(r)}{p^{2s+1}} \sum_{k\in\mathbb{Z}} \tau_k(yn, s+1, s) e^{2\pi i k(\frac{xpn+r}{p})}.$$

Let $G(\varepsilon) := \sum_{r \bmod p} \varepsilon(r)\zeta_p^r$ be the Gauss sum associated to the quadratic character ε. Let $\zeta_p = e^{\frac{2\pi i}{p}}$. If we take the limit as s tends to zero and use Lemma 10 in the inner sum we get:

$$\sum_{r \bmod p} \frac{\varepsilon(r)}{p}\left(-\pi i - 2\pi i \sum_{k=1}^{\infty} q^{nk}\zeta_p^{rk}\right) = -\frac{2\pi i}{p} G(\varepsilon) \sum_{k=1}^{\infty} \varepsilon(k) q^{nk}.$$

If p is congruent to 3 mod 4 it is a well-known result that $G(\varepsilon) = i\sqrt{p}$, then

$$\lim_{s\to 0} E_1(z,s) = 2L(\varepsilon,1) + \frac{4\pi}{\sqrt{p}} \sum_{n=1}^{\infty}\left(\sum_{d|n} \varepsilon(d)\right) q^n. \tag{11}$$

Applying this to Eq. (7) (with $p = |D|$) we get the value of $L(\psi, 1)$.

We will write this number in terms of theta functions so as to relate the value for different ideals $\mathcal{D}$. Let $\mathcal{B}$ be any ideal of L. For z in the upper half plane, we define $\Theta_{\mathcal{B}}(z) = \sum_{\lambda\in\mathcal{B}} e^{2\pi i z \frac{N\lambda}{N\mathcal{B}}} = 1 + \sum_{n=1}^{\infty} r_{\mathcal{B}}(n) q^n$ where $r_{\mathcal{B}}(n)$ is the number of elements $\lambda \in \mathcal{B}$ of norm $nN\mathcal{B}$. Clearly if two ideals of L are equivalent, their theta functions are the same.

Lemma 11. *Let $w_{|D|}$ be the number of roots of unity in L, and z a point in the upper half plane. Then* $\frac{w_{|D|}\sqrt{|D|}}{4\pi} E_1(z,0) = \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \Theta_{[\mathcal{B}]}(z)$.

Proof. We need to check that the q-expansion on both sides is the same. The constant term on the right-hand side is h, the class number of $\mathbb{Q}(\sqrt{D})$. On the left-hand side we have $\frac{L(\varepsilon,1)w_{|D|}\sqrt{D}}{2\pi}$ which by the class number formula is h. Since the constant term is the same, we can apply the Mellin transform on both sides. Dividing by $w_{|D|}$ we need to prove the equality:

$$\sum_{n=1}^{\infty} \frac{\sum_{d|n} \varepsilon(d)}{n^s} = \frac{1}{w} \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \sum_{n=1}^{\infty} \frac{r_{[\mathcal{B}]}(n)}{n^s}. \tag{12}$$

Given a number field L the zeta function associated to it is

$$\zeta_L(s) = \sum_{\mathcal{B}} \frac{1}{N\mathcal{B}^s},$$

where the sum is over all integral ideals of L. It follows easily from the definition that $\zeta_L(s) = \frac{1}{w}\sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \sum_{n=1}^{\infty} \frac{r_{[\mathcal{B}]}(n)}{n^s}$ which is the right-hand side of (12).

It is a classical result that $\zeta_L(s) = \zeta(s)L(\varepsilon, s)$ (see for example [19, Theorem 4.3, p. 33]), then $\zeta_L(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{m=1}^{\infty} \frac{\varepsilon(m)}{m^s}\right)$ which is the left-hand side of (12). $\square$

Note that $-\bar{z}_{\mathcal{A}\mathcal{D}} = z_{\bar{\mathcal{A}}\bar{\mathcal{D}}}$, hence by Eq. (7) and Lemma 11 we get

$$L(\psi, 1) = \frac{2\pi}{w\sqrt{|D|}} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)} \frac{\varepsilon_{\mathcal{D}}(N\mathcal{A})}{\psi(\mathcal{A})} \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \Theta_{\mathcal{B}}(z_{\bar{\mathcal{A}}\bar{\mathcal{D}}}).$$

By Eq. (2) $\psi_{\bar{\mathcal{D}}}(\mathcal{A}) = \psi_{\mathcal{D}}(\mathcal{A})\varepsilon_{\bar{\mathcal{D}}}(\mathcal{A}^h)\varepsilon_{\mathcal{D}}(\mathcal{A}^h) = \psi_{\mathcal{D}}(\mathcal{A})\left(\frac{N\mathcal{A}}{|D|}\right)^h$. Since h is odd it follows that $\frac{\varepsilon_{\mathcal{D}}(N\mathcal{A})}{\psi_{\mathcal{D}}(\mathcal{A})} = \frac{1}{\psi_{\bar{\mathcal{D}}}(\mathcal{A})}$.

Theorem 12. *The value at $s = 1$ of $L(\psi, s)$ is given by*

$$L(\psi, 1) = \frac{2\pi}{w_{|D|}\sqrt{|D|}} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)} \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \frac{\Theta_{\mathcal{B}}(z_{\mathcal{A}\bar{\mathcal{D}}})}{\psi_{\bar{\mathcal{D}}}(\bar{\mathcal{A}})}.$$

2.4. Theta functions in several variables

The goal now is to write the identity of Theorem 12 in terms of theta functions in two variables so as to relate the L-function values for different primes D. Given an element $(\vec{z}, \Omega)$ in $\mathbb{C}^2 x \mathfrak{h}_2$ (the Siegel space of dimension 2), the generalized theta function is defined by

$$\theta(\vec{z}, \Omega) = \sum_{\vec{n}\in\mathbb{Z}^2} \exp(\pi i \vec{n}^t \Omega \vec{n} + 2\pi i \vec{n}^t . \vec{z}).$$

It satisfies a functional equation for the group Γ_{12} (following Igusa notation), which is defined to be: $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ such that $A^t C$ and $B^t D$ have even diagonal. In particular,

$$\theta(\vec{0}, -(Q\tau)^{-1}) = \sqrt{\det(Q)}\,(-i)\tau\theta(\vec{0}, Q\tau), \tag{13}$$

$$\theta(\vec{z}, \Omega + B) = \xi_\alpha \theta(\vec{z}, \Omega), \tag{14}$$

where Q and B are symmetric, integral and even diagonal two-by-two matrices, Q corresponds to a positive definite quadratic form, τ is a point in the upper half plane and ξ_α is a root of unity. (see [8, Section 5, p. 189].)

L is an imaginary quadratic field, so given an ideal $\mathcal{B}$ of $Cl(\mathcal{O}_L)$ we can associate to it a quadratic form of discriminant D via the group isomorphism between $Cl(\mathcal{O}_L)$ and {equivalence classes of quadratic forms of discriminant D}. More specifically, given a quadratic form of discriminant D, say $[a, b, c]$ where $b^2 - 4ac = D$, we associate the ideal $\langle a, \frac{b+\sqrt{D}}{2}\rangle$; conversely given any primitive ideal (i.e. not divisible by any rational integer greater than 1) $\mathcal{B}$, we can chose a pair of generators of the form $\mathcal{B} = \langle a, \frac{b+\sqrt{D}}{2}\rangle$, and associate to it the quadratic form $[a, b, c]$ where $c = (b^2 - D)/(4a)$. We will denote $Q_\mathcal{B}$ the matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ associated to the quadratic form $[a, b, c]$.

Let $\mathcal{B}$ be a primitive ideal representing a class in $Cl(\mathcal{O}_L)$, say $\mathcal{B} = \langle a, \frac{b+\sqrt{D}}{2}\rangle$ with $a = \mathrm{N}(\mathcal{B})$. If $\alpha \in \mathcal{B}$ then it can be written uniquely as $\alpha = ma + n\left(\frac{b+\sqrt{D}}{2}\right)$. Hence $\mathrm{N}(\alpha) = a(am^2 + mnb + n^2\frac{b^2-D}{4a})$ and

$$\Theta_\mathcal{B}(z) = \sum_{(m,n)\in\mathbb{Z}^2} \exp\left[\pi i z (m, n) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}\right]. \tag{15}$$

Since $z \in \mathfrak{h}$ and $Q_\mathcal{B}$ is symmetric, $zQ_\mathcal{B} \in \mathfrak{h}_2$. Hence $\Theta_\mathcal{B}(z) = \theta(\vec{0}, zQ_\mathcal{B})$. So we can rewrite the main formula of Theorem 12 as

$$L(\psi, 1) = \frac{2\pi}{w_{|D|}\sqrt{|D|}} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)} \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \frac{\theta(\vec{0}, z_{\mathcal{A}\bar{\mathcal{D}}} Q_\mathcal{B})}{\psi_{\bar{\mathcal{D}}}(\bar{\mathcal{A}})}. \tag{16}$$

3. Normalization of the theta function

Given a point $z_{\mathcal{A}\mathcal{D}}$, we define the normalizer:

$$\Upsilon(z_{\mathcal{A}\mathcal{D}}) := \eta(\mathcal{D})\eta(\mathcal{O}_K)\psi_\mathcal{D}(\bar{\mathcal{A}}).$$

Then the main formula (16) can be written as

$$L(\psi_{\mathcal{D}}, 1) = \frac{2\pi}{w\sqrt{|D|}} \left(\sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)} \sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} \frac{\theta(\vec{0}, z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}})}{\Upsilon(z_{\mathcal{A}\bar{\mathcal{D}}})} \right) \eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K). \tag{17}$$

We are interested in studying the number: $n_{\mathcal{A},\mathcal{B},\bar{\mathcal{D}}} = \theta(\vec{0}, z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}})/\Upsilon(z_{\mathcal{A}\bar{\mathcal{D}}})$. The normalizer Υ is chosen so as to make $n_{\mathcal{A},\mathcal{B},\bar{\mathcal{D}}}$ an algebraic integer as we will see later.

3.1. Complex multiplication

Let $\mathcal{F}_M$ be the field of all modular functions of level M whose q-expansion at every cusp has coefficients in $\mathbb{Q}(\xi_M)$ where ξ_M is any primitive Mth root of unity. Let $K(M)$ denote the ray class field of K mod M, and for a prime ideal $\mathfrak{p}$ in K relatively prime to M (say of norm p), $\sigma(\mathfrak{p})$ denotes the Frobenius automorphism of $K(M)/K$ corresponding to $\mathfrak{p}$.

Following Stark's notation if A is an integral matrix of determinant relatively prime to M, we denote $f \circ A$ the action of A on f which is characterized by the two properties:

- $(f \circ A)(z) = f(Az)$ if $A \in Sl_2(\mathbb{Z})$,
- $(f \circ A)(z) = \sigma_d(f)(z)$ if $A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ where $\sigma_d \in Gal(\mathbb{Q}(\xi_M)/\mathbb{Q})$ is defined by $\sigma_d(\xi_M) = \xi_M^d$. We extend this action to f by acting on the coefficients of the q-expansion at infinity.

Theorem 13. *Let $f(z)$ be in $\mathcal{F}_M$ and suppose that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ in K where p is a rational prime such that $(p, NM) = 1$. Suppose that $\mathcal{A} = [\mu, \nu]$ is a fractional ideal of K with $\vartheta = \mu/\nu$ in $\mathfrak{h}$ and let $B\binom{\mu}{\nu}$ be a basis for $\bar{\mathfrak{p}}\mathcal{A}$. Then $f(\vartheta)$ is in $K(M)$ and $f(\vartheta)^{\sigma(\mathfrak{p})} = [f \circ (pB^{-1})](B\vartheta)$.*

If in addition f is analytic in the interior of $\mathfrak{h}$ and has algebraic integer coefficients in its q-expansion at every cusp, then $f(\vartheta)$ is an algebraic integer.

Proof. This is Theorem 3 of [16, p. 213]. □

Proposition 14. *Following the previous notation, $\theta(\vec{0}, \frac{z}{a|D|} Q_{\mathcal{B}})/\eta(\frac{z}{|D|})\eta(z)$ is in $\mathcal{F}_{24aD^2}$.*

For the proof we need the elementary result:

Lemma 15. *If $f(z)$ is a modular form of weight k and level N and D is a positive integer then $f(\frac{z}{D})$ is a modular form of weight k and level at most ND.*

Proof of Proposition 14. Let $\mathcal{B}$ be the ideal $\mathcal{B} := \mathbb{Z}a + \mathbb{Z}\frac{b+\sqrt{D}}{2}$. Then the quadratic form associated to $\mathcal{B}$ is $[a, b, c]$ with $b^2 - 4ac = D$ and the matrix of the bilinear form is $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$. The theta series $\theta_{\mathcal{B}}$ is the theta series associated to this quadratic form hence it has level $|D|$, weight 1 and a character $\varepsilon(d) = \left(\frac{D}{d}\right)$ (see [9, Theorem 20, p. VI–25]). Using the previous lemma, we have that $\theta_{\mathcal{B}}(\frac{z}{a|D|})$ is a modular form of weight 1 and level aD^2.

The eta function is a modular form of weight 1/2 and level 24, then $\eta(\frac{z}{|D|})$ has weight 1/2 and level $24|D|$, so the product of the two eta functions has weight 1 and level $24|D|$. Hence the quotient has weight 0 and level at most $24aD^2$. We do not need a sharp estimate of the q-expansion, hence the minimum level is not important.

From the q-expansion of the functions $\theta_{\mathcal{B}}$, and η it is clear that the q-expansion at infinity of $\theta(\vec{0}, \frac{z}{a|D|} Q_{\mathcal{B}})/\eta(\frac{z}{|D|})\eta(z)$ is in $\mathbb{Q}(\xi_{24aD^2})$, hence we just need to check this condition at the other cusps. For that purpose we will study the q-expansion of each form separately.

Since the theta function $\theta_{\mathcal{B}}$ is a modular form for $\Gamma_0(|D|)$, there are just two inequivalent cusps which may be taken to be 0 and ∞. One transformation that send infinity to zero is given by the matrix $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sending z to $-1/z$.

The functional equation (13) reads as

$$\theta\left(\vec{0}, Q_{\mathcal{B}}^{-1}(-1/z)\right) = \det(Q_{\mathcal{B}})^{1/2}(-i)z\theta(\vec{0}, Q_{\mathcal{B}}z) = \sqrt{|D|}(-i)z\theta(0, Q_{\mathcal{B}}z). \tag{18}$$

Since $Q_{\mathcal{B}}^{-1} = Adj\,(Q_{\mathcal{B}})/|D|$, replacing z by $z/|D|$ we get

$$\theta\left(\vec{0}, Adj\,(Q_{\mathcal{B}})(-1/z)\right) = (-i)z/\sqrt{|D|}\,\theta(\vec{0}, Q_{\mathcal{B}}z/|D|). \tag{19}$$

Replacing $Q_{\mathcal{B}}$ by its adjoint matrix, we see that the q-expansion at 0 includes a 4th root of unity and the square root of $|D|$ (the z factor actually cancels out a factor coming from the eta function). Since $\sqrt{D} \in \mathbb{Q}(\xi_D)$, the q-expansion of $\theta(0, Q_{\mathcal{B}})$ has coefficients in $\mathbb{Q}(\xi_{8D})$ at all cusps. Replacing z by $z/a|D|$ we add at most (aD^2)th roots of unity to the q-expansions, hence the q-expansion of $\theta(0, \frac{z}{a|D|} Q_{\mathcal{B}})$ has coefficients in $\mathbb{Q}(\xi_{24aD^2})$ at all cusps.

We will use the following explicit version of the transformation formula for η, which can be found in [12, p. 560]:

Lemma 16. *Let* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathbb{Z})$ with γ *even*, δ *positive* (*and odd*), *and* $\tau \in \mathfrak{h}$. *Then*

$$\eta\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \left(\frac{\gamma}{\delta}\right) e_{24}(\kappa)\sqrt{\gamma\tau + \delta}\,\eta(\tau), \tag{20}$$

where $\kappa = 3(\delta - 1) + \delta(\beta - \gamma) - (\delta^2 - 1)\gamma\alpha$.

For any matrix in $\Gamma_0(2)$, the modular form η changes by a 24th root of unity, hence its q-expansion at the equivalent cusps modulo $\Gamma_0(2)$ have coefficients in $\mathbb{Q}(\xi_{24})$ and the q-expansion of $\eta(\frac{z}{|D|})$ has coefficients in $\mathbb{Q}(\xi_{24aD^2})$. But in modulo $\Gamma_0(2)$ there are just two inequivalent cusps which may be taken to be zero and infinity also. The eta function satisfies the functional equation $\eta(-1/z) = \sqrt{z/i}\,\eta(z)$. Hence its q-expansion at zero has coefficients in $\mathbb{Q}(\xi_8)$ and $\eta(\frac{z}{|D|})$ certainly has a q-expansion with coefficients in $\mathbb{Q}(\xi_{24aD^2})$ at zero. □

3.2. Field of definition

Theorem 17. *The number $\theta(\vec{0}, z_{\mathcal{A}\bar{\mathcal{D}}}Q_{\mathcal{B}})/\eta(z_{\bar{\mathcal{D}}})\eta(\mathcal{O}_K)$ is an algebraic integer in H, the Hilbert class field of K.*

Proof. The eta function does not vanish in the upper half plane so we can apply Theorem 13 and $\theta(\vec{0}, \frac{z_0}{a|D|}Q_{\mathcal{B}})/\eta(\frac{z_0}{|D|})\eta(z_0)$ is an algebraic integer in F (some field extension of K containing H) where $z_0 = \frac{b+\sqrt{N}}{2}$ corresponds to the ideal $\mathcal{O}_K$.

Let $g(z) := \theta(\vec{0}, \frac{z}{a|D|}Q_{\mathcal{B}})/\eta(\frac{z}{|D|})\eta(z)$. Given an element σ of $Gal(F/K)$ by complex multiplication theory there exists a prime ideal $\mathfrak{p}$ in K such that $\sigma = \sigma_p$, where σ_p is the element in $Gal(F/K)$ corresponding to $\mathfrak{p}$ via the Artin–Frobenius map. We want to prove that the quotient is in H hence we take $\mathfrak{p}$ to be principal and using the Tchebotarev density theorem we may assume that $\mathfrak{p}\bar{\mathfrak{p}}$ is prime to $\mathcal{A}, \bar{\mathcal{D}}$ and the ideal (6).

Since $\bar{\mathfrak{p}}$, $\mathcal{A}$ and $\bar{\mathcal{D}}$ are prime to each other, it easily seen that b can be chosen such that $\bar{\mathfrak{p}} = \langle\frac{b+\sqrt{N}}{2}, p\rangle$, $\mathcal{A} = \langle\frac{b+\sqrt{N}}{2}, a\rangle$, $\bar{\mathcal{D}} = \langle\frac{b+\sqrt{N}}{2}, |D|\rangle$ and $\mathcal{O}_K = \langle\frac{b+\sqrt{N}}{2}, 1\rangle$. Let z_0 denote the point $\frac{b+\sqrt{N}}{2}$. Then $\bar{\mathfrak{p}}\mathcal{A}\bar{\mathcal{D}} = \langle\frac{b+\sqrt{N}}{2}, pa|D|\rangle$, and on these basis the matrix B of Theorem 13 is given by $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Now $Bz_0 = \frac{z_0}{p}$ and $pB^{-1} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = S^{-1}BS$. By Theorem 13, $g(z_0)^{\sigma(\mathfrak{p})} = [g \circ (pB^{-1})](Bz_0)$.

Let $g^\star(z) = g \circ S(z) = g(-1/z) = \theta(\vec{0}, -1/(a|D|z)Q_{\mathcal{B}})/\eta(\frac{-1}{|D|z})\eta(\frac{-1}{z})$. If in (19) we replace z by $za|D|$ and $Q_{\mathcal{B}}$ by $Adj\,(Q_{\mathcal{B}})$, we get the equation

$$\theta(\vec{0}, Q_{\mathcal{B}}(-1/a|D|z)) = (-i)\sqrt{|D|}az\theta(\vec{0}, Adj\,(Q_{\mathcal{B}})az). \tag{21}$$

The eta function satisfies the functional equation $\eta(-1/z) = \sqrt{z/i}\,\eta(z)$. Replacing z by $|D|z$ and multiplying both equations:

$$\eta(-1/z)\eta(-1/(|D|z)) = \sqrt{|D|}\frac{z}{i}\eta(z)\eta(|D|z).$$

Hence we get

$$g(-1/z) = a\frac{\theta(\vec{0}, Adj\,(Q_{\mathcal{B}})az)}{\eta(z)\eta(|D|z)}.$$

The q-expansion of this quotient has rational coefficients hence it is fixed by the action of σ_p, i.e. $g^\star \circ \sigma_p = g^\star$. Then $[g \circ (pB^{-1})] = g$ and $(g(z_0))^{\sigma_p} = g(z_0/p)$.

Proposition 18. *With the notation as above, if* $\mathfrak{p}$ *is principal,* $g(z_0)^{\sigma_p} = g(z_0)$.

Proof. The proposition reduces to proving that $g(z_0/p) = g(z_0)$ if $\mathfrak{p}$ is principal of norm p which follows from the next two lemmas. This completes the proof of Theorem 17 since it implies that $g(z_0)^{\sigma_p} = g(z_0)$ for all principal ideals $\mathfrak{p}$. □

Lemma 19. *Let* $\bar{\mathfrak{p}} = \langle \mu \rangle$ *be a principal ideal prime to* $\mathcal{A}$ *and* $\bar{\mathcal{D}}$ *of norm p. Then the theta function* $\Theta_{\mathcal{B}}$ *satisfies the formula*:

$$\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2ap|D|}\right) = \bar{\mu}\varepsilon_{\bar{\mathcal{D}}}(\mu)\left(\frac{p}{|D|}\right)\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2a|D|}\right).$$

Note 1. *Since* $\varepsilon_{\bar{\mathcal{D}}}(\mu)\varepsilon_{\bar{\mathcal{D}}}(\bar{\mu}) = \left(\frac{p}{|D|}\right)$, the formula may be written as $\Theta_{\mathcal{B}}(\frac{b+\sqrt{N}}{2ap|D|}) = \psi_{\bar{\mathcal{D}}}(\bar{\mu})\Theta_{\mathcal{B}}(\frac{b+\sqrt{N}}{2a|D|})$.

Proof. $\Theta_{\mathcal{B}}$ is a modular form of weight 1 for $\Gamma_0(|D|)$ with a quadratic character. We chose b such that $\bar{\mathfrak{p}}\mathcal{A}\bar{\mathcal{D}} = \langle \frac{b+\sqrt{N}}{2}, pa|D|\rangle = \langle \mu\frac{b+\sqrt{N}}{2}, \mu a|D|\rangle$. Hence there exists a change of basis matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ such that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \frac{b+\sqrt{N}}{2} \\ ap|D| \end{pmatrix} = \begin{pmatrix} \mu\frac{b+\sqrt{N}}{2} \\ \mu a|D| \end{pmatrix}$.

If $\mu = \frac{m+n\sqrt{N}}{2}$, an easy computation shows that $\delta = \frac{m-nb}{2p}$ and $\gamma = n|D|a$. In particular, M is in $\Gamma_0(|D|)$ and by modularity of $\Theta_{\mathcal{B}}$ we have

$$\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2a|D|}\right) = \Theta_{\mathcal{B}}\left(M\frac{b+\sqrt{N}}{2ap|D|}\right) = \left(\gamma\frac{b+\sqrt{N}}{2ap|D|}+\delta\right)\chi(\delta)\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2ap|D|}\right).$$

And the formula

$$\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2a|D|}\right) = \frac{\mu}{p}\chi(\delta)\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2ap|D|}\right), \tag{22}$$

where $\chi(d) = \left(\frac{D}{q}\right)$ for any prime q which is sufficiently large and satisfies $q \equiv d \bmod |D|$ [9, Theorem 20, Chapter VI, p. 25]. Let q be a prime congruent to 1 mod 4 and congruent to $\delta \bmod |D|$. Then $\chi(\delta) = \left(\frac{D}{q}\right) = \left(\frac{|D|}{q}\right) = \left(\frac{q}{|D|}\right) = \left(\frac{\frac{m-nb}{2p}}{|D|}\right) =$

$\left(\frac{\frac{m-nb}{2}}{|D|}\right)\left(\frac{p}{|D|}\right)$. Then the proof follows from the definition of $\varepsilon_{\bar{\mathcal{D}}}$ and the fact that $\frac{\mu}{p} = (\bar{\mu})^{-1}$. $\square$

Lemma 20. *With the same assumptions as above, the eta function satisfies the equation* $\eta(\frac{b+\sqrt{N}}{2p|D|})\eta(\frac{b+\sqrt{N}}{2p}) = \bar{\mu}\varepsilon_{\bar{\mathcal{D}}}(\mu)\left(\frac{p}{|D|}\right)\eta(\frac{b+\sqrt{N}}{2|D|})\eta(\frac{b+\sqrt{N}}{2})$.

In term of ideals:

$$\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})\eta(\bar{\mathfrak{p}}) = \bar{\mu}\varepsilon_{\bar{\mathcal{D}}}(\mu)\left(\frac{p}{|D|}\right)\eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K). \tag{23}$$

Proof. Since we choose $|N| \equiv 3 \bmod 4$, and $|N| \neq 3$, the number of units in H is 2 (see [6, Tables 3, 4, p. 507]). Given a principal ideal $\langle u\rangle$ with $u \in \mathcal{O}_K$, prime to $\langle 6\rangle$ define:

$$\kappa(u) = \chi_4(N_{K/\mathbb{Q}}(u))\frac{1}{\bar{u}}\frac{\eta^2(u)}{\eta^2(\mathcal{O}_K)},$$

where $\chi_4(a) = \left(\frac{-1}{a}\right)$. Since the number of units in H is 2, κ is a quadratic character (see [6, Lemma 14]). We can write the left-hand side of (23) as

$$\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})\eta(\bar{\mathfrak{p}}) = \left(\frac{\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})}{\eta(\bar{\mathcal{D}})}\frac{\eta(\mathcal{O}_K)}{\eta(\bar{\mathfrak{p}})}\right)\frac{\eta^2(\bar{\mathfrak{p}})}{\eta^2(\mathcal{O}_K)}\eta(\mathcal{O}_K)\eta(\bar{\mathcal{D}}). \tag{24}$$

If μ is a generator of $\bar{\mathfrak{p}}$, $\frac{\eta^2(\bar{\mathfrak{p}})}{\eta^2(\mathcal{O}_K)} = \kappa(\mu)\bar{\mu}\chi_4(p)$. By Proposition 10 of [6] $\left(\frac{\eta(\bar{\mathfrak{p}})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{\mathcal{D}}} = \left(\frac{p}{|D|}\right)\frac{\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})}{\eta(\bar{\mathcal{D}})}$. Then we get

$$\left(\frac{\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})}{\eta(\bar{\mathcal{D}})}\frac{\eta(\mathcal{O}_K)}{\eta(\bar{\mathfrak{p}})}\right) = \left(\frac{p}{|D|}\right)\left(\frac{\eta(\bar{\mathfrak{p}})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{\mathcal{D}}-1} = \left(\frac{p}{|D|}\right)\left(\sqrt{\kappa(\mu)\bar{\mu}\chi_4(p)}\right)^{\sigma_{\mathcal{D}}-1}.$$

By Lemma 12 of [6], $\kappa(-1) = -1$. Since the right term of (23) remains unchanged replacing μ by $-\mu$, without loss of generality we can choose μ such that $\kappa(\mu) = \chi_4(p)$. Replacing each term on the right-hand side of (24) and using Proposition 2 we get

$$\eta(\bar{\mathfrak{p}}\bar{\mathcal{D}})\eta(\bar{\mathfrak{p}}) = \left(\frac{p}{|D|}\right)\varepsilon_{\mathcal{D}}(\bar{\mu})\,\bar{\mu}\,\eta(\mathcal{O}_K)\eta(\bar{\mathcal{D}}).$$

And the result follows since $\varepsilon_{\mathcal{D}}(\bar{\mu}) = \varepsilon_{\bar{\mathcal{D}}}(\mu)$. $\square$

Theorem 21. *The number* $n_{\mathcal{A},\mathcal{B},\bar{\mathcal{D}}}$ *is in the field* $\mathcal{M}_\psi = HT_\psi$. *It corresponds to the fields diagram* (*Diagram* 1):

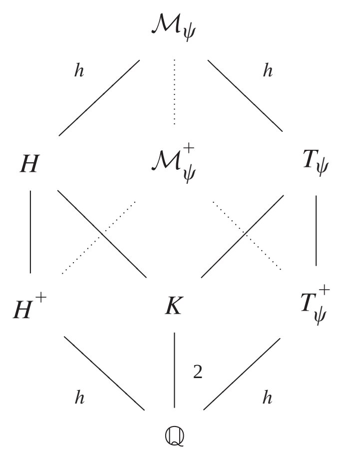

Diagram 1

Proof. By Theorem 17 the number $\theta(\vec{0}, z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}})/\eta(z_{\bar{\mathcal{D}}})\eta(\mathcal{O}_K)$ is in H and T_ψ contains the image of $\psi_{\bar{\mathcal{D}}}$ hence $n_{\mathcal{A},\mathcal{B},\bar{\mathcal{D}}}$ is in $\mathcal{M}_\psi$. $\square$

Proposition 22. *The quotient* $\theta_{Q_{\mathcal{B}}}(z_{\mathcal{A}\bar{\mathcal{D}}})/\psi_{\bar{\mathcal{D}}}(\bar{\mathcal{A}})$ *depends only on the class of* $\mathcal{B}$ *and the class of* $\mathcal{A}$.

Proof. Independence of $\mathcal{B}$ is clear since $\Theta_{\mathcal{B}}$ depends only in the class of $\mathcal{B}$.

To prove independence of $\mathcal{A}$, let $\alpha \in \mathcal{O}_K$ be an element with prime norm q such that $q \nmid 6a|D|$. By definition $\Theta_{\mathcal{B}}(z_{\alpha\mathcal{A}\mathcal{D}}) = \Theta_{\mathcal{B}}(\frac{b+\sqrt{N}}{2aq|D|})$. Then by Lemma 19:

$$\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2aq|D|}\right) = \psi_{\bar{\mathcal{D}}}(\bar{\alpha})\Theta_{\mathcal{B}}\left(\frac{b+\sqrt{N}}{2a|D|}\right). \qquad \square$$

We will denote by $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ the number $n_{\mathcal{A},\mathcal{B},\bar{\mathcal{D}}}$.

Proposition 23. *The number* $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ *is an algebraic integer*.

Proof. In Theorem 17 we proved that $\theta_{Q_{\mathcal{B}}}(z_{\mathcal{A}\bar{\mathcal{D}}})/(\eta(z_{\bar{\mathcal{D}}})\eta(z_{\mathcal{O}_K}))$ is an algebraic integer and the number $\psi_{\bar{\mathcal{D}}}(\bar{\mathcal{A}})$ has norm $N\mathcal{A}$. Since the quotient depends on the class of the ideal $\mathcal{A}$ but not $\mathcal{A}$ itself, using the Tchebotarev density theorem we can choose two prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ in the same class of $\mathcal{A}$ of prime norms p_1 and p_2. Looking at $\mathfrak{p}_1$ we see that the minimal polynomial of $n_{[\mathfrak{p}_1],[\mathcal{B}],\bar{\mathcal{D}}}$ has rational coefficients with only 1 or p_1 in the denominator. Considering $\mathfrak{p}_2$ we see that the minimal polynomial of $n_{[\mathfrak{p}_2],[\mathcal{B}],\bar{\mathcal{D}}}$ only has 1 or p_2 in the denominator. Since $n_{[\mathfrak{p}_1],[\mathcal{B}],\bar{\mathcal{D}}} = n_{[\mathfrak{p}_2],[\mathcal{B}],\bar{\mathcal{D}}}$ its minimal polynomial must have integer coefficients. □

Proposition 24. $n_{[\mathcal{A}],[\bar{\mathcal{B}}],\bar{\mathcal{D}}} = n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$.

Proof. It is easy to check that the theta function $\Theta_{\mathcal{B}}$ associated to $\mathcal{B}$ is the same as the theta function $\Theta_{\mathrm{Adj}\,B}$ associated to the adjoint matrix of $\mathcal{B}$. Note that $[\mathcal{B}^{-1}] = [\bar{\mathcal{B}}]$. Clearly the point $z_{\mathcal{A}\bar{\mathcal{D}}}$ and the number $\psi_{\bar{\mathcal{D}}}(\mathcal{A})$ are independent of $\mathcal{B}$. □

Lemma 25. *The character $\psi_{\bar{\mathcal{D}}}$ satisfy*: $\overline{\psi_{\mathcal{D}}(\bar{\mathcal{A}})} = \psi_{\bar{\mathcal{D}}}(\mathcal{A})$.

Proof. $\overline{\psi_{\mathcal{D}}(\bar{\mathcal{A}})}\psi_{\mathcal{D}}(\bar{\mathcal{A}}) = N\mathcal{A}$, and $N\mathcal{A} = \psi_{\mathcal{D}}(\bar{\mathcal{A}})\psi_{\mathcal{D}}(\mathcal{A})\varepsilon_{\mathcal{D}}(N\mathcal{A})$ hence $\overline{\psi_{\mathcal{D}}(\bar{\mathcal{A}})} = \left(\frac{N\mathcal{A}}{|D|}\right)\psi_{\mathcal{D}}(A)$. We chose the characters so that $\psi_{\bar{\mathcal{D}}}(\mathcal{A}) = \psi_{\mathcal{D}}(\mathcal{A})\varepsilon_{\bar{\mathcal{D}}}(\mathcal{A}^h)\varepsilon_{\mathcal{D}}(\mathcal{A}^h) = \psi_{\mathcal{D}}(\mathcal{A})\left(\frac{(N\mathcal{A})^h}{|D|}\right)$ (see (2)). Since $|N|$ is prime, h is odd. □

Proposition 26. $\overline{n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}} = n_{[\bar{\mathcal{A}}],[\mathcal{B}],\mathcal{D}}$.

Proof. It is clear from their definition that $\overline{\Theta_{\mathcal{B}}(z_{\mathcal{A}\bar{\mathcal{D}}})} = \Theta_{\mathcal{B}}(-\overline{z_{\mathcal{A}\bar{\mathcal{D}}}})$ and $\overline{\eta(z_{\mathcal{A}\bar{\mathcal{D}}})} = \eta(-\overline{z_{\mathcal{A}\bar{\mathcal{D}}}})$. Since $-\overline{z_{\mathcal{A}\bar{\mathcal{D}}}} = z_{\bar{\mathcal{A}}\mathcal{D}}$ and $\overline{\psi_{\mathcal{D}}(\bar{\mathcal{A}})} = \psi_{\bar{\mathcal{D}}}(\mathcal{A})$, the result follows. □

Proposition 27. *If the ideal $\mathcal{D}$ is principal in $\mathcal{O}_K$*, $\overline{n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}} = n_{[\bar{\mathcal{A}}],[\mathcal{B}],\bar{\mathcal{D}}}$.

Proof. The proof of this proposition involves the same kind of techniques used in the previous ones (a little more tedious) so we omit the proof. □

In particular, this implies that if $\mathcal{A}$ and $\mathcal{D}$ are both principal then the number $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ lives in a subfield of $\mathcal{M}_\psi$ which we denote $\mathcal{M}_\psi^+$ (following [1] notation, see page 13) and corresponds to the previous field diagram (see Theorem 21, Diagram 3.3). We will be needing the next lemmas for the theorem relating the numbers $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ for different ideals $\mathcal{D}$.

Lemma 28. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two prime ideals of $\mathbb{Q}(\sqrt{N})$ with norm $|D|$ and $|D'|$, respectively, and let $\mu \in \mathbb{Q}(\sqrt{N})$ be such that $\mu\mathcal{D} = \mathcal{D}'$. Then* $\frac{\eta^2(\mathcal{A}\mathcal{D}')}{\eta^2(\mathcal{A}\mathcal{D})} = \bar{\mu}\kappa(\mu)\chi_4(N\mu)$.

Proof. Note that although κ is defined on integer elements, since it is a character on $(\mathcal{O}_K/12\mathcal{O}_K)^\times$, we can extend it multiplicatively to all elements in $\mathbb{Q}(\sqrt{N})$ with both numerator and denominator prime to 12. By definition $\kappa(\mu) = \frac{1}{\bar{\mu}}\chi_4(N\mu)\frac{\eta^2(\mu)}{\eta^2(\mathcal{O}_K)}$ then we are led to prove that $\frac{\eta^2(\mathcal{A}\mathcal{D}')}{\eta^2(\mathcal{A}\mathcal{D})}\frac{\eta^2(\mathcal{O}_K)}{\eta^2(\mu)} = 1$.

By Proposition 10 of [6] we can write the left-hand side as $\left(\frac{\eta^2(\mathcal{A}\mathcal{D})}{\eta^2(\mathcal{O}_K)}\right)^{\sigma_{(\bar{\mathcal{D}}'\bar{\mathcal{D}}^{-1})}-1}$.

Since $\frac{\eta^2(\mathcal{A}\mathcal{D})}{\eta^2(\mathcal{O}_K)}$ is in H (by Theorem 20 of [6]) then $\sigma_{\mathcal{A}}$ represents the classical Artin–Frobenius map from $Cl(\mathcal{O}_K)$ to $Gal(H/K)$, and since $\bar{\mathcal{D}}'\bar{\mathcal{D}}^{-1}$ is principal, $\sigma_{\bar{\mathcal{D}}'\bar{\mathcal{D}}^{-1}}$ is the identity. □

Lemma 29. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two prime ideals of $\mathbb{Q}(\sqrt{N})$ such that $\mathcal{D} \sim \mathcal{D}'$. Then* $\frac{\eta(\mathcal{A}\mathcal{D}')\eta(\mathcal{D})}{\eta(\mathcal{A}\mathcal{D})\eta(\mathcal{D}')} = \varepsilon_{\mathcal{D}}(\bar{\mathcal{A}}^h)\varepsilon_{\mathcal{D}'}(\bar{\mathcal{A}}^h)$.

Proof. By Proposition 10 of [6] we have

$$\frac{\eta(\mathcal{A}\mathcal{D}')\eta(\mathcal{D})}{\eta(\mathcal{D}')\eta(\mathcal{A}\mathcal{D})} = \left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{\bar{\mathcal{D}}'}} \left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{-\sigma_{\bar{\mathcal{D}}}} \left(\frac{a}{|D|}\right)\left(\frac{a}{|D'|}\right). \tag{25}$$

Since the Artin–Frobenius map is a homomorphism:

$$\left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{\bar{\mathcal{D}}'}-\sigma_{\bar{\mathcal{D}}}} = \left(\left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{(\bar{\mathcal{D}}'(\bar{\mathcal{D}})^{-1})}-1}\right)^{\sigma_{\bar{\mathcal{D}}}}.$$

But $\left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{(\bar{\mathcal{D}}'(\bar{\mathcal{D}})^{-1})}-1} = \pm 1$ (see the proof of Lemma 28), then $\sigma_{\bar{\mathcal{D}}}$ acts trivially on it. Let $\mu \in \mathbb{Q}(\sqrt{N})$ be such that $\bar{\mathcal{D}}'\bar{\mathcal{D}}^{-1}$ is the principal ideal generated by $\frac{\bar{\mu}}{|D|}$ then by Theorem 19 of [6]:

$$\left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{(\bar{\mathcal{D}}'(\bar{\mathcal{D}})^{-1})}-1} = \kappa\left(\frac{\mu}{|D|}\right)^{\frac{a-1}{2}}\left(\frac{\bar{\mu}|D|}{\bar{\mathcal{A}}}\right).$$

Since $|D|$ is prime to 12, and κ is a multiplicative quadratic character, $\kappa(\frac{\mu}{|D|}) = \kappa(\mu)\kappa(|D|)$. The character κ defined on $(\mathcal{O}_K/12\mathcal{O}_K)^\times$ factors as a product of two characters, κ_3 from $(\mathcal{O}_K/3\mathcal{O}_K)^\times$ to the group of third roots of unity and κ_4 from $(\mathcal{O}_K/4\mathcal{O}_K)^\times$ to the group of fourth roots of unity (see Lemma 14 of [6]). In our case $\kappa_3 = 1$ and the character is completely determined from the congruence mod 4. Then $\kappa(|D|) = \kappa(-1) = -1$. Using the quadratic reciprocity law,

$$\left(\frac{\eta(\mathcal{A})}{\eta(\mathcal{O}_K)}\right)^{\sigma_{(\bar{\mathcal{D}}'(\bar{\mathcal{D}})^{-1})}-1} = \kappa(\mu)^{\frac{a-1}{2}}\left(\frac{\bar{\mu}}{\bar{\mathcal{A}}}\right)\left(\frac{a}{|D|}\right). \tag{26}$$

Also since $\kappa(\mu)\kappa(\bar{\mu}) = \kappa(|D||D'|) = 1$, $\kappa(\mu) = \kappa(\bar{\mu})$ and we can write (25) as

$$\frac{\eta(\mathcal{A}\mathcal{D}')\eta(\mathcal{D})}{\eta(\mathcal{A}\mathcal{D})\eta(\mathcal{D}')} = \kappa(\bar{\mu})^{\frac{a-1}{2}}\left(\frac{\mu}{\mathcal{A}}\right)\left(\frac{a}{|D'|}\right).$$

Since $\bar{\mathcal{D}}\mathcal{D}'$ is the principal ideal generated by μ and ε is a multiplicative quadratic character,

$$\varepsilon_{\mathcal{D}}(\bar{\mathcal{A}}^h)\varepsilon_{\mathcal{D}'}(\bar{\mathcal{A}}^h) = \varepsilon_{\mathcal{D}}(\bar{\mathcal{A}}^h)\varepsilon_{\bar{\mathcal{D}}}(\bar{\mathcal{A}}^h)\varepsilon_{\bar{\mathcal{D}}\mathcal{D}'}(\bar{\mathcal{A}}^h) = \left(\frac{a}{|D|}\right)\left(\frac{\bar{\mathcal{A}}^h}{\mu}\right). \tag{27}$$

Using the reciprocity law in $\mathbb{Q}(\sqrt{N})$ (see for example Theorem 21 of [6]):

$$\left(\frac{\bar{\mathcal{A}}^h}{\mu}\right) = \left(\frac{\mu}{\bar{\mathcal{A}}^h}\right)\kappa(\bar{\mu})^{\frac{a-1}{2}} = \left(\frac{\mu}{\bar{\mathcal{A}}}\right)\kappa(\bar{\mu})^{\frac{a-1}{2}} = \kappa(\bar{\mu})^{\frac{a-1}{2}}\left(\frac{\mu}{\mathcal{A}}\right)\left(\frac{|D||D'|}{a}\right). \tag{28}$$

And the lemma follows from $\left(\frac{|D||D'|}{a}\right) = \left(\frac{a}{|D|}\right)\left(\frac{a}{|D'|}\right)$. $\square$

Lemma 30. *Let $A : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be the skew-symmetric form given by the matrix $A := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then the following data on $\mathbb{R}^{2n}$ are equivalent*:

(1) *A complex structure $U : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (i.e. a linear map with $U^2 = -I_n$) such that there exists a positive definite Hermitian form H for this complex structure with imaginary part A.*
(2) *An n-dimensional complex subspace of $\mathbb{C}^{2n}$ such that if we note $A_{\mathbb{C}}$ the complex linear extension of A, we have*:
  - *$A_{\mathbb{C}}(x, y) = 0$ for all x, y in the subspace.*
  - *$iA_{\mathbb{C}}(x, \bar{x}) < 0$ for all non-zero x in the subspace.*

(3) *A complex matrix Ω in $\mathfrak{h}_n$.*

These are three of the four equivalent conditions proved in Lemma 4.1 of [8]. The equivalence associates to $\Omega \in \mathfrak{h}_n$ the image of the map $X \mapsto (X, -\Omega X)$ as an n-dimensional subspace of $\mathbb{C}^{2n}$.

Theorem 31. *Let $z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}$ and $z_{\mathcal{A}\mathcal{D}'}Q_{\mathcal{B}'}$ be two points in $\mathfrak{h}_2$ such that they are equivalent mod Γ_{12} and $\mathcal{D} \sim \mathcal{D}'$ in $\mathbb{Q}(\sqrt{N})$. Then $n_{[\mathcal{A}],[\mathcal{B}],\bar{D}} = \pm n_{[\mathcal{A}],[\mathcal{B}'],\bar{D}'}$.*

Proof. For simplicity we will denote $\Omega_{\mathcal{D}} := z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}$ and $\Omega_{\mathcal{D}'} := z_{\mathcal{A}\mathcal{D}'}Q_{\mathcal{B}'}$. Since $\Omega_{\mathcal{D}}$ is equivalent to $\Omega_{\mathcal{D}'}$ there exists a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $Sp_4(\mathbb{Z})$ such that $\gamma \star (\Omega_{\mathcal{D}}) = \Omega_{\mathcal{D}'}$. By the previous lemma, giving a point $\Omega_{\mathcal{D}}$ in the Siegel space is equivalent to giving the subspace of $\mathbb{C}^4$ $(I_2, -\Omega_{\mathcal{D}})^t$ where the action of $Sp_4(\mathbb{Z})$ is given by multiplication on the left by $(\gamma^t)^{-1}$. Then

$$\gamma \star \begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} C\Omega_{\mathcal{D}} + D \\ -(A\Omega_{\mathcal{D}} + B) \end{pmatrix} = \begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}'} \end{pmatrix}(C\Omega_{\mathcal{D}} + D).$$

By the coherent way we chose characters, $\frac{\psi_{\mathcal{D}}(\mathcal{A})}{\psi_{\mathcal{D}'}(\mathcal{A})} = \varepsilon_{\mathcal{D}}(\mathcal{A}^h)\varepsilon_{\mathcal{D}'}(\mathcal{A}^h)$. Hence,

$$\frac{n_{[\mathcal{A}],[\mathcal{B}],\bar{D}}}{n_{[\mathcal{A}],[\mathcal{B}'],\bar{D}'}} = \frac{\theta(\Omega_{\mathcal{D}})}{\theta(\Omega_{\mathcal{D}'})}\frac{\eta(\mathcal{D}')}{\eta(\mathcal{D})}\varepsilon_{\mathcal{D}}(\bar{\mathcal{A}}^h)\varepsilon_{\mathcal{D}'}(\bar{\mathcal{A}}^h) = \frac{\theta(\Omega_{\mathcal{D}})}{\theta(\Omega_{\mathcal{D}'})}\frac{\eta(\mathcal{A}\mathcal{D}')}{\eta(\mathcal{A}\mathcal{D})}.$$

The last equality follows from Lemma 29. We claim that

$$\frac{\theta^2(\Omega_{\mathcal{D}})}{\theta^2(\Omega_{\mathcal{D}'})} = \mathrm{Det}(\mathrm{C}\Omega_{\mathcal{D}} + \mathrm{D})^{-1} = \frac{\eta^2(\mathcal{A}\mathcal{D})}{\eta^2(\mathcal{A}\mathcal{D}')}. \tag{29}$$

The first equality follows at once from the functional equation of the theta function. Since $|D|$ is prime and $\mathrm{Det}(\mathrm{Q}) = |\mathrm{D}|$ there exists matrices $U, V \in Sl_2(\mathbb{Z})$ such that $UQV = \begin{pmatrix} 1 & 0 \\ 0 & |D| \end{pmatrix}$ (respectively, U' and V' for Q'). Then,

$$\begin{pmatrix} V^{-1} & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}} \end{pmatrix} V = \begin{pmatrix} I_2 \\ -UQVz_{\mathcal{A}\mathcal{D}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -z_{\mathcal{A}\mathcal{D}} & 0 \\ 0 & -z_{\mathcal{A}} \end{pmatrix}.$$

Similarly,

$$\begin{pmatrix} V'^{-1} & 0 \\ 0 & U' \end{pmatrix}\begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}'} \end{pmatrix} V' = \begin{pmatrix} I_2 \\ -U'Q'V'z_{\mathcal{A}\mathcal{D}'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -z_{\mathcal{A}\mathcal{D}'} & 0 \\ 0 & -z_{\mathcal{A}} \end{pmatrix}.$$

We split into two cases:

- If $\mathcal{D}' = \bar{\mathcal{D}}$ we take basis $\mathcal{D} = \langle |D|, \frac{b+\sqrt{N}}{2}\rangle$ and $\mathcal{A} = \langle a, \frac{b+\sqrt{N}}{2}\rangle$. Let r be such that $r|D| \equiv b \bmod a$ then $\mathcal{D}' = \langle |D'|, \frac{(2r|D|-b)+\sqrt{N}}{2}\rangle$ and $\mathcal{A}\mathcal{D}' = \langle a|D'|, \frac{(2r|D|-b)+\sqrt{N}}{2}\rangle$. Let $\mu \in K$ be such that $\mu\mathcal{D} = \mathcal{D}'$, then $\mathcal{A}\mathcal{D}' = \langle a|D'|, \frac{(2r|D|-b)+\sqrt{N}}{2}\rangle$ $= \langle \mu a|D|, \mu(\frac{b+\sqrt{N}}{2})\rangle = \mu\mathcal{A}\mathcal{D}$ hence there exists a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ such that: $M\begin{pmatrix} \mu(\frac{b+\sqrt{N}}{2}) \\ \mu a|D| \end{pmatrix} = \begin{pmatrix} \frac{(2r|D|-b)+\sqrt{N}}{2} \\ a|D'| \end{pmatrix}$.
- If $\mathcal{D}' \neq \bar{\mathcal{D}}$, we may choose basis $\mathcal{D} = \langle |D|, \frac{b+\sqrt{N}}{2}\rangle$, $\mathcal{D}' = \langle |D'|, \frac{b+\sqrt{N}}{2}\rangle$ and $\mathcal{A} = \langle a, \frac{b+\sqrt{N}}{2}\rangle$. If μ is such that $\mu\mathcal{D} = \mathcal{D}'$, then $\mathcal{A}\mathcal{D}' = \langle a|D'|, \frac{b+\sqrt{N}}{2}\rangle =$

$\langle \mu a|D|, \mu(\frac{b+\sqrt{N}}{2})\rangle = \mu\mathcal{A}\mathcal{D}$ hence there exists a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ such that: $M \begin{pmatrix} \mu(\frac{b+\sqrt{N}}{2}) \\ \mu a|D| \end{pmatrix} = \begin{pmatrix} \frac{b+\sqrt{N}}{2} \\ a|D'| \end{pmatrix}$.

In both cases, let $N := \begin{pmatrix} \delta & 0 & -\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then it is clear that

$$N \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -z_{\mathcal{A}\mathcal{D}} & 0 \\ 0 & -z_{\mathcal{A}} \end{pmatrix} \begin{pmatrix} \frac{\mu|D|}{|D'|} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -z_{\mathcal{A}\mathcal{D}'} & 0 \\ 0 & -z_{\mathcal{A}} \end{pmatrix}.$$

Combining these results we get that

$$\begin{pmatrix} V' & 0 \\ 0 & U'^{-1} \end{pmatrix} N \begin{pmatrix} V^{-1} & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}} \end{pmatrix} V \begin{pmatrix} \frac{\mu|D|}{|D'|} & 0 \\ 0 & 1 \end{pmatrix} V'^{-1} = \begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}'} \end{pmatrix}$$

and

$$\begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}'} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} I_2 \\ -\Omega_{\mathcal{D}} \end{pmatrix} (C\Omega_{\mathcal{D}} + D)^{-1}.$$

Since both lattices have the same volume then $|\mathrm{Det}(\mathrm{C}\Omega_{\mathcal{D}} + \mathrm{D})|^{-1} = \frac{|\mu||\mathrm{D}|}{|\mathrm{D}'|}$.

By Lemma 28, $\frac{\eta^2(\mathcal{A}\mathcal{D})}{\eta^2(\mathcal{A}\mathcal{D}')} = \frac{1}{\bar{\mu}}\kappa(\mu) = \frac{\mu|D|}{|D'|}\kappa(\mu)$. Now $\mathrm{Det}(\mathrm{C}\Omega_{\mathrm{D}} + \mathrm{D})^{-1}$ and $\kappa(\mu)\frac{\mu|D|}{|D'|}$ have the same absolute value and both lie in $\mathbb{Q}(\sqrt{N})$ hence they differ by ± 1. Then

$$\left(\frac{\theta(\Omega_{\mathcal{D}})}{\theta(\Omega_{\mathcal{D}'})} \frac{\eta(\mathcal{A}\mathcal{D}')}{\eta(\mathcal{A}\mathcal{D})}\right)^2 = \mathrm{Det}(\mathrm{C}\Omega_{\mathcal{D}} + \mathrm{D})^{-1}\bar{\mu}\kappa(\mu) = \pm 1.$$

Taking square roots:

$$\sqrt{\pm 1} = \frac{\theta(\Omega_{\mathcal{D}})}{\theta(\Omega_{\mathcal{D}'})} \frac{\eta(\mathcal{A}\mathcal{D}')}{\eta(\mathcal{A}\mathcal{D})}.$$

By Theorem 21 we know that $\frac{\theta(\Omega_{\mathcal{D}})}{\eta(\mathcal{D})\eta(\mathcal{O}_K)}$ and $\frac{\theta(\Omega'_{\mathcal{D}})}{\eta(\mathcal{D}')\eta(\mathcal{O}_K)}$ are in H. Since $\sqrt{-1} \notin H$ the theorem follows. $\square$

It is not clear how to determine the sign a priori, and we are not able to give any answer in this direction.

4. Equivalence of special points

The problem of determining whether two points in $\mathfrak{h}_2$ are equivalent or not is complicated in general. For our case we will get this equivalence via ideals in quaternion algebras. A good reference for the basic definitions and some elementary facts about quaternion algebras is Pizer's paper [11].

Let B be a quaternion algebra over $\mathbb{Q}$. A lattice $\mathfrak{L}$ is a rank 4 $\mathbb{Z}$-module. An order O is a lattice that is a ring with unity. Given an order O a left O-ideal is a lattice $\mathfrak{L}$ such that $\mathfrak{L}_p := \mathfrak{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = O_p\alpha_p$ where α_p is an element in $B_p^\times$. Given a lattice $\mathfrak{L}$ we define its left order $O_l(\mathfrak{L}) := \{x \in B \mid x\mathfrak{L} \subset \mathfrak{L}\}$ (respectively the right order). We define $\mathrm{N}(\mathfrak{L})$ as the positive generator of the $\mathbb{Z}$-module $\langle \mathrm{N}(x) \mid x \in \mathfrak{L}\rangle$.

Proposition 32. *Let B be a quaternion algebra over $\mathbb{Q}$ ramified at $p_1, \ldots, p_n$ and $\mathfrak{L}$ be an ideal in B. Then $O_l(\mathfrak{L})$ is a maximal order if and only if disc $(\mathfrak{L})= (p_1 \ldots p_n)^2\mathrm{N}(\mathfrak{L})^4$.*

Proof. By definition $disc(\mathfrak{L})$ is the determinant of the bilinear form associated to $\mathfrak{L}$ on any basis. Since $\mathfrak{L}$ is locally principal at all primes, given a finite prime q, $\mathfrak{L}_q = O_l(\mathfrak{L})_q\alpha_q$. Clearly $disc(\mathfrak{L}_q) = \mathrm{N}(\alpha_q)^4 disc(O_q)$; then the statement follows from the fact that this proposition is true replacing $\mathfrak{L}$ by an order O and $\mathrm{N}(\mathfrak{L})$ by 1 (see [11, Proposition 1.1, p. 344]), and the fact that the norm of $\mathfrak{L}$ is the product over all primes q of $q^{v_q(N\alpha_q)}$ where $v_q(n)$ is the q-valuation. $\square$

We restrict ourselves to the case B a quaternion algebra over $\mathbb{Q}$ ramified at the prime $|N|$ and infinity.

Lemma 33. *Let O be a maximal order, $\{I_1, \ldots, I_h\}$ a set of left O-ideal representatives, and $\{R_1, \ldots, R_h\}$ be the right orders of $\{I_1, \ldots, I_h\}$, respectively. Then for a given $i = 1, \ldots, h$ the maximal order R_i appears twice on the list if and only if there is no embedding of $\mathbb{Z}[\sqrt{N}]$ into R_i.*

Proof. Although this is a well-known statement we give a proof since we will use it latter. An embedding of $\mathbb{Z}[\sqrt{N}]$ into R_i is determined by the image of $\sqrt{N}$. Hence giving such an embedding is equivalent to giving an element $\beta \in R_i$ of trace zero and norm $|N|$. Let $\mathcal{P}$ be the bilateral O-ideal of norm $|N|$. For a given left O-ideal I_j, the ideal $\mathcal{P}I_j$ is another left O-ideal. Note that if $\mathcal{P}_j$ is the bilateral R_j ideal of norm $|N|$, then $I_j^{-1}\mathcal{P}I_j = \mathcal{P}_j$ by the uniqueness of such a bilateral ideal. Then the ideals I_j and $\mathcal{P}I_j$ are equivalent if and only if there exists $\beta \in R_j^\times$ such that $I_j\beta = \mathcal{P}I_j$. Multiplying on the left by I_j^{-1} we see that $R_j\beta = I_j^{-1}\mathcal{P}I_j = \mathcal{P}_j$ hence $\mathcal{P}_j$ is principal, and the element β has norm $|N|$. Since $|N|$ is a ramified prime, i.e. $B_{|N|}$ is a division ring, it is easy to see that if $\mathrm{N}(\alpha) = |N|$ then $\mathrm{Tr}(\alpha) = 0$.

To see that this is the only way in which a maximal order R appears twice on the list of right orders, suppose that I and J are two non-equivalent left O-ideals with same right order R. Then $I^{-1}J$ is a non-principal bilateral ideal for R. Let $\mathcal{P}_R$ be the ideal of norm $|N|$ in R, then $\mathcal{P}_R$ is non-principal and J is equivalent to $\mathcal{P}I$. $\square$

4.1. Siegel space and applications

Definition 34. Let $\mathfrak{L}$ be a $\mathbb{Z}$ lattice of rank $2n$ and V the vector space $\mathfrak{L}\otimes\mathbb{R}$. We call a triple (P, J, U) a Siegel point if:

- P is a real $2n\times 2n$ symmetric matrix such that the associated quadratic form $P(x, y)$ is positive definite (that will correspond to the real part of H).
- J is a real $2n \times 2n$ non-degenerate skew symmetric matrix with associated form $J(x, y)$ (that will correspond to the imaginary part of H).
- $U \in \mathbb{R}^{2n\times 2n}$ is such that $U^2 = -I_{2g}$ (complex structure)

with the relation:

$$-JU = U^t J = P. \tag{30}$$

Via the matrix U we can put a complex structure on the vector space V. Let H be the bilinear form $H(x, y) := P(x, y) + iJ(x, y)$. Condition (30) implies that $H(ix, y) = iH(x, y)$. Since J is skew symmetric and P symmetric, it follows that $H(x, y) = \overline{H(y, x)}$. Then H defined in this way is a positive definite Hermitian form. Each choice of a reduced basis for J will give a point in the Siegel space (by Lemma 30) and different bases give equivalent points.

Given two lattices $\mathfrak{L}$ and $\mathfrak{L}'$, a *morphism* $\gamma : \mathfrak{L} \to \mathfrak{L}'$ is an $\mathbb{Z}$-linear map from $\mathfrak{L}$ to $\mathfrak{L}'$. Given $\gamma : \mathfrak{L}' \to \mathfrak{L}$ an isomorphism of lattices, we define an action of γ on a Siegel point (P, J, U) as $(\gamma^* P, \gamma^* J, \gamma^* U)$ where given $x, y \in \mathfrak{L}'$, $\gamma^* P(x, y) = P(\gamma(x), \gamma(y))$, $\gamma^* J(x, y) = J(\gamma(x), \gamma(y))$ and $\gamma^*(x) = \gamma^{-1}(U(\gamma(x)))$.

If we choose V_0 to be a skew symmetric reduced base for J, i.e. a base where J is of the form $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and γ is an automorphism sending a skew symmetric reduced basis to another one, then $\gamma \in Sp_{2n}(\mathbb{Z})$ and the action of γ on the Siegel point Ω associated to V_0 is the usual action of $Sp_{2n}(\mathbb{Z})$ on $\mathfrak{h}_n$.

4.2. Siegel points from quaternion algebras

Let N be the negative of a prime congruent to $3 \bmod 4$, and $B = (-1, N)$ the quaternion algebra ramified at N and infinity. Let O be a maximal order in B such that there exists an embedding (not necessarily optimal) of $\mathbb{Z} + \mathbb{Z}\sqrt{N}$ into O. Let $u \in O$ be the image of $\sqrt{N}$, i.e. $u^2 = N$ and $\mathrm{Tr}(u) = 0$. By I we will denote a left O-ideal for a maximal order O. To I we associate a Siegel point $(P, J, U)_I$ as follows:

- We take V the real vector space $V := B \otimes_{\mathbb{Q}} \mathbb{R}$.
- Define U acting on V as left multiplication by $\frac{u}{\sqrt{|N|}}$.
- We think of I as a full rank lattice in V.

- For $x, y \in I$ define $P(x, y) := \frac{1}{\sqrt{|N|}} \mathrm{Tr}(x\bar{y})/\mathrm{N}(I)$.
- For $x, y \in I$ define $J(x, y) := \mathrm{Tr}(u^{-1}x\bar{y})/\mathrm{N}(I)$.

Proposition 35. *The triple $(P, J, U)_I$ defined as above is a Siegel point.*

Proof. We start checking the properties of the matrices P, J and U:

- P is a real form. Since $\mathrm{Tr}(x\bar{y})$ is real, $\mathrm{Tr}(x\bar{y}) = \mathrm{Tr}(y\bar{x})$ which implies that $P(x, y)$ is symmetric. Clearly $P(x, x) = \frac{1}{\sqrt{|N|}} \mathrm{N}(x)/\mathrm{N}(I)$ is positive definite.
- J is a real form. Since u is pure imaginary, u^{-1} is also. Then $J(x, x) = \mathrm{Tr}(u^{-1}\mathrm{N}(x))/\mathrm{N}(I) = 0$. It is also clear that $J(x, y)$ is non-degenerate, since for any non-zero $x \in V$, $J(x, u^{-1}x) \neq 0$. Since $J(x, x) = 0$ for all x it follows that $J(x, y) = -J(y, x)$.
- Let $x \in V$, then $U^2(x) = U(\frac{u}{\sqrt{|N|}} x) = \frac{u^2}{|N|} x = -x$.

As for the relation, it is easy to check that $J(\frac{u}{\sqrt{|N|}} x, y) = P(x, y)$ and that $J(x, \frac{u}{\sqrt{|N|}} y) = -P(x, y)$. $\square$

Definition 36. Given a lattice $\mathfrak{L}$ in B we define its dual by $\mathfrak{L}^{\#} := \{b \in B : \mathrm{Tr}(b\mathfrak{L}) \subset \mathbb{Z}\}$. Given an order R we define its different by $R^{\imath} := NR^{\#}$.

Proposition 37. *If O is a maximal order, $O^{\imath}$ is a bilateral ideal for O of index N^2, and $\frac{1}{N} O \subset O^{\imath} \subset O$.*

Proof. See [17, Lemma 4.7, p. 24].

Proposition 38. *If $x, y \in I$ then $J(x, y) \in \mathbb{Z}$. Also the matrix of J on the basis given by I has determinant* 1.

Proof. Since we are considering the reduced norm, if V is the matrix associated to multiplication (on the left or on the right) by v, then $\mathrm{N}(v) = \sqrt{\det(V)}$. Let $W(x, y) := \mathrm{Tr}(x\bar{y})$ be the bilinear form of B. If we denote W the matrix of $W(x, y)$ on the basis given by I, $J = \frac{1}{\mathrm{N}(I)} (U^{-1})^t W$. Then $\det(J) = \mathrm{N}(I)^{-4} \mathrm{N}(u)^{-2} \det(W)$. By definition $\det(W) = disc(I)$, which is an ideal for a maximal order, then by Proposition 32 $disc(I) = N^2 \mathrm{N}(I)^4$ and $\det(J) = 1$.

Since the trace is linear, $J(x, y) = \mathrm{Tr}(u^{-1}x \frac{\bar{y}}{\mathrm{N}(I)})$. For ideals I with maximal left order it is true that $I^{-1} = \bar{I}/\mathrm{N}(I)$ and $II^{-1} = O$, hence $J(x, y) \in \mathbb{Z}$ for all $x, y \in I$ if and only if $\mathrm{Tr}(u^{-1}v) \in \mathbb{Z}$ for all $v \in O$. By Proposition 37 this is the same as $u^{-1} \in O^{\#}$. But $u^{-1} = -\frac{u}{N}$, and since $u \in O$ it follows that $\frac{u}{N} \in \frac{1}{N} O \subset O^{\#}$. $\square$

This gives a method for assigning to every left O-ideal a Siegel point. Note that choosing different skew symmetric reduced basis of I will give equivalent Siegel points. From now on we fixed a maximal order O with an embedding of $\mathbb{Z}[\sqrt{N}]$.

Proposition 39. *Let $u \in O$ with $\mathrm{N}(u) = |N|$ and $\mathrm{Tr}(u) = 0$, and denote by U the complex multiplication associated to u. If I, I' are two equivalent left O-ideals, then the Siegel points $(P, J, U)_I$ and $(P, J, U)_{I'}$ are equivalent.*

Proof. Since $I \sim I'$ there exists $\alpha \in B^\times$ such that $I = I'\alpha$. Let W denote the isomorphism of B given by $W(v) = v\alpha$. We claim that W is the isomorphism that makes the two Siegel points equivalent.

Since $W(I') = I$, we need to check that $W^*P = P'$, $W^*J = J'$ and $W^*U = U$.

- If $x, y \in I'$ by definition $(W^*P)(x, y) := P(W(x), W(y)) = P(x\alpha, y\alpha) = \frac{\mathrm{Tr}(x\alpha\bar{\alpha}\bar{y})}{\mathrm{N}(I)} = \frac{N\alpha}{\mathrm{N}(I)}\mathrm{Tr}(x\bar{y}) = P'(x, y)$.
- The equality $W^*J = J'$ follows from a similar argument.
- By definition U is given by multiplying on the left by $u/\sqrt{|N|}$ while W is given by multiplying on the right by α then clearly this maps commute with each other and $W^*U := W^{-1} \circ U \circ W = U$. $\square$

Lemma 40. *Let U be the complex multiplication given by u and $\alpha \in B$ an element such that $\alpha O \alpha^{-1} = O$. Define $I' = \alpha I \alpha^{-1}$ and $u' = \alpha u \alpha^{-1}$, then $(P, J, U)_I \sim (P', J', U')_{I'}$.*

Proof. Let $W : B \to B$ be the isomorphism defined by $W(x) = \alpha x \alpha^{-1}$. By hypothesis $W(R) = R$, $W(I) = I'$. It is easy to see that $W^*P = P'$ and $W^*J = J'$. If $x \in B$ then $W^{-1} \circ U \circ W(x) = W^{-1} \circ U(\alpha x \alpha^{-1}) = W^{-1}(u\alpha x\alpha^{-1})/\sqrt{|N|} = \alpha^{-1}u\alpha x/\sqrt{|N|} = U'(x)$. $\square$

This lemma suggests that we should consider not just elements u in O corresponding to $\sqrt{N}$ (i.e. $u^2 = N$ and $\mathrm{Tr}(u) = 0$) but modulo conjugation by the normalizer of O. It is clear that $Norm(O) = \{h \in B \mid Oh \text{ is bilateral}\}$. All bilateral ideals are principal, generated by $u^s m$ where $s = 0, 1$ and m is a rational number (see [3, Proposition 1, p. 92]). The generator of an ideal is well defined up to units in O, then $Norm(O) = \{\zeta u^s m \mid s = 0 \text{ or } 1, m \in \mathbb{Q} \text{ and } \zeta \in O \text{ is a unit}\}$.

Corollary 41. *If I and I' are left O-ideals with the same right order then the Siegel points $(P, J, U)_I$ and $(P, J, U)_{I'}$ are equivalent.*

Proof. If I and I' are equivalent this follows from Proposition 39. If I and I' are not equivalent, we know by Lemma 33 that $O_r(I)$ has no embedding of $\mathbb{Z}[\sqrt{N}]$. Let u be the element in O giving the complex multiplication. Then uI has the same left and right order as I but they are not equivalent, hence $uI \sim I' \sim uIu^{-1}$. By Proposition 40 the Siegel points $(P, J, U)_I$ and $(P, J, U')_{uI}$ are equivalent. Just note that U' is given by $u^{-1}uu = u$. $\square$

In particular, we should index the Siegel points not by the class number of ideals, but by the type number of maximal orders. We still have equivalent Siegel points coming from conjugation by units of O and these are all the possibilities for $Norm(O)$. For counting equivalent classes of Siegel points, fixed a maximal order O we have to count the number of embeddings of $\mathbb{Z}[\sqrt{N}]$ into O modulo conjugation by units of O.

Given a maximal ideal O, let $\mathbb{B} := \{I_1, \ldots, I_h\}$ be a set of left O-ideal representatives and $\mathbb{T} := \{R_1, \ldots, R_t\}$ the distinct right orders of the ideals in $\mathbb{B}$. We index the Siegel points by pairs (ϕ, R_i) where ϕ is an embedding from $\mathbb{Z}[\sqrt{N}]$ to some R_j and R_i is an order in $\mathbb{T}$. By this we mean the Siegel point obtained with the complex multiplication given by $\phi(\sqrt{N})$, and an ideal I with left order R_j and right order R_i.

If d is a negative discriminant we denote by $h(d)$ the class number of binary quadratic forms of discriminant d. Let $u(d) = 1$ unless $d = -3, -4$ when $u(d) = 3, 2$, respectively (half the number of units in the ring of integers of discriminant d). For $\mathfrak{d} > 0$ we define the Hurwitz's class number $H(\mathfrak{d})$ by

$$H(\mathfrak{d}) := \sum_{df^2=-d} \frac{h(d)}{u(d)} \tag{31}$$

if $\mathfrak{d}$ is a discriminant and by zero if not. A short table of the non-zero values is given by

$\mathfrak{d}$	$H(\mathfrak{d})$	$\mathfrak{d}$	$H(\mathfrak{d})$
3	1/3	11	1
4	1/2	12	4/3
7	1	15	2
8	1	16	3/2

If $-\mathfrak{d}$ is a discriminant we denote $\mathcal{O}_{-\mathfrak{d}}$ the order of discriminant $\mathfrak{d}$ in the imaginary quadratic field $\mathbb{Q}[\sqrt{-\mathfrak{d}}]$. For $p \in \mathbb{Z}$ prime we define $H_p(\mathfrak{d})$ to be the modified invariant as follows:

$$H_p(\mathfrak{d}) = \begin{cases} 0 & \text{if } -\mathfrak{d} \text{ is not a discriminant,} \\ 0 & \text{if } p \text{ splits in } \mathcal{O}_{-\mathfrak{d}}, \\ H(\mathfrak{d}) & \text{if } p \text{ is inert in } \mathcal{O}_{-\mathfrak{d}}, \\ \frac{1}{2}H(\mathfrak{d}) & \text{if } p \text{ is ramified in } \mathcal{O}_{-\mathfrak{d}} \text{ but does not divide} \\ & \text{the conductor of } \mathcal{O}_{-\mathfrak{d}}, \\ H_p(\mathfrak{d}/p^2) & \text{if } p \text{ divides the conductor of } \mathcal{O}_{-\mathfrak{d}}. \end{cases} \tag{32}$$

The number of embeddings of $\mathcal{O}_{-\mathfrak{d}}$ into any R_i $(i = 1, \ldots, n)$ modulo conjugation by $R_i^\times/\{\pm 1\}$ is $H_{|N|}(\mathfrak{d})$ (see [5, Proof of Proposition 1.9, p. 122]).

We want to compute the number of embeddings of $\mathbb{Z}[\sqrt{N}]$ into any R_i, i.e. choose $\mathfrak{d} = 4|N|$, then

$$H_{|N|}(4N) = \begin{cases} \frac{1}{2}h(4N) & \text{if } N \equiv 1 \bmod 4, \\ h(N) & \text{if } N \equiv 7 \bmod 8, \\ 2h(N) & \text{if } N \equiv 3 \bmod 8 \text{ and } N \geqslant 11. \end{cases} \tag{33}$$

Note that in the case $\mathfrak{d} = 4|N|$ an order R_i on $\mathbb{T}$ appears twice as a right order if and only if it has no embedding of $\mathcal{O}_{4N}$. In this case it does not contribute to the sum, and hence the number of embeddings of $\mathbb{Z}[\sqrt{N}]$ into the t orders in $\mathbb{T}$ is also $H_N(4N)$. With this we proved:

Proposition 42. *The number of non-equivalent Siegel points constructed is at most* $H_N(4N)t$.

Proposition 43. *Let B be a quaternion algebra over a commutative field K, and let $B_0 := \{\beta \in B \mid \mathrm{Tr}(\beta) = 0\}$. If $\psi : B_0 \to B_0$ is an isometry of K-vector spaces then there exists an element $\beta \in B^\star$ such that $\sigma(x) = \beta x \beta^{-1}$ or $\sigma(x) = -\beta x \beta^{-1} = \beta \bar{x} \beta^{-1}$.*

Proof. See [17, Theorem 3.3, p. 12]. □

Lemma 44. *Let $\psi : B \to B$ be an isomorphism of $\mathbb{Q}$-vector spaces (respectively $\sigma : B_q \to B_q$ an isomorphism of $\mathbb{Q}_q$-vector spaces) such that $\sigma(1) = 1$ and σ is an isometry. Then there exists an $\alpha \in B^\star$ (respectively $\alpha \in B_q^\star$) such that $\sigma(x) = \alpha x \alpha^{-1}$ or $\sigma(x) = \alpha \bar{x} \alpha^{-1}$.*

Proof. Since $\sigma(1) = 1$ and σ is a morphism, $\sigma(\mathbb{Q}) = \mathbb{Q}$. Denoting B_0 the trace zero elements, $\sigma(B_0) = B_0$ and $\sigma|_{B_0} : B_0 \to B_0$ is an isometry. By Proposition 43 we get two different cases:

(1) $\sigma_{B_0}(x) = \alpha \bar{x} \alpha^{-1}$ for some $\alpha \in B^\star$. Then σ is the antiautomorphism given by $\sigma(x) = \alpha \bar{x} \alpha^{-1}$.
(2) $\sigma_{B_0}(x) = \alpha x \alpha^{-1}$ for some $\alpha \in B^\star$. Then σ is an automorphism given by $\sigma(x) = \alpha x \alpha^{-1}$. □

Theorem 45. *The $H_N(4N)t$ Siegel points $\{(\phi, R_i)\}$ constructed above are non-equivalent.*

Proof. The proof breaks in two steps. First we will prove that for a fixed embedding of $\mathbb{Z}[\sqrt{N}]$ into R (say u is the image of $\sqrt{N}$), the t left R-ideals give non-equivalent points (P, J, U) where U is multiplication by $u/\sqrt{|N|}$. Then we will prove that different embeddings give non-equivalent Siegel points.

Let I_1, I_2 two left R-ideals. Abusing notation we will denote P_i the symmetric form P_{I_i} and analogously for J_i. Suppose there exists $W : V \to V$ an isomorphism making the Siegel points (P_1, J_1, U) and (P_2, J_2, U) equivalent. Let $\beta = W(1)$, σ the map $\sigma(v) = W(v)\beta^{-1}$ and V_0 the space of elements in V with trace zero. We claim that σ is an isometry.

By hypothesis $W^* P_1 = P_2$ then evaluating at $(1, 1)$ we have

$$(W^* P_1)(1, 1) = P_2(1, 1) = \frac{2}{\mathrm{N}(I_2)}.$$

By definition, $(W^*P_1)(1,1) = \frac{\mathrm{Tr}(W(1),\overline{W(1)})}{\mathrm{N}(I_1)} = 2\frac{\mathrm{N}(\beta)}{\mathrm{N}(I_1)}$ hence $\mathrm{N}(\beta) = \frac{\mathrm{N}(I_1)}{\mathrm{N}(I_2)}$. Then $\|x\|/\sqrt{N} = P_2(x,x)\mathrm{N}(I_2)/2 = W^*(P_1(x,x))\mathrm{N}(I_2)/2 = \frac{\|W(x)\|}{\mathrm{N}(I_1)}\mathrm{N}(I_2) = \frac{\|W(x)\|}{\|\beta\|} = \|\sigma(x)\|/\sqrt{N}$, i.e. σ is an isometry. Since σ is an isometry and $\sigma(1) = 1$, by Lemma 44 we have two different cases:

(1) $\sigma(x) = \alpha\bar{x}\alpha^{-1}$ for some $\alpha \in B^\times$, i.e. σ is an antiautomorphism and $W(x) = \alpha\bar{x}\alpha^{-1}\beta^{-1}$.
(2) $\sigma(x) = \alpha x\alpha^{-1}$ for some $\alpha \in B^\times$ and $W(x) = \alpha x\alpha^{-1}\beta^{-1}$.

We know that W preserves the complex multiplication, i.e. $W^{-1} \circ U \circ W(x) = U(x)$.

In the first case, $W^{-1}(x) = \alpha^{-1}\bar{\beta}\bar{x}\alpha$. Then $W^*U(x) = W^{-1}(u\alpha\bar{x}\alpha^{-1}\beta^{-1}) = \alpha^{-1}\bar{\beta}\bar{\beta}^{-1}\bar{\alpha}^{-1}x\bar{\alpha}\bar{u}\alpha = x\alpha^{-1}\bar{u}\alpha$. It must be the case that $ux = x\alpha^{-1}\bar{u}\alpha$ for all $x \in B$ (which is the same as saying that $ux\alpha^{-1} = x\alpha^{-1}\bar{u}$) which would imply that $u \in \mathbb{Q}$ and is not the case. Then we must be in the second case.

Since $W(I_1) = I_2$, $I_2 = \alpha I_1\alpha^{-1}\beta^{-1}$. In particular $\alpha R\alpha^{-1} = R$, i.e. $\alpha \in Norm(R)$. Then I_1 and I_2 have the same right order and represent the same class between the t left R-ideals we started with.

Assume that there is a left R-ideal I and a left R'-ideal I' such that R and R' are non-conjugate maximal orders and the Siegel points $(P, J, U)_I$ and $(P', J', U')_{I'}$ are equivalent. Then there exist an isomorphism $W : V \to V$ that sends one point to the other. Arguing as before we get the same two possible cases for W. In the first case, since $W^*U = U'$ we would get that $u'x\alpha^{-1} = x\alpha^{-1}\bar{u}$ for all $x \in V$. Taking $x = \alpha$ we would get that $u' = \bar{u}$ and it commutes with all elements of V, then it is rational which is not the case.

Then $W(x) = \alpha x\alpha^{-1}\beta^{-1}$ and $I' = \alpha I\alpha^{-1}\beta$. In particular the orders R and R' are conjugate which is a contradiction. □

4.3. Ideals associated to Siegel points

For finding relations between the numbers $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$, we will assign to each point $z_{\mathcal{A}\bar{\mathcal{D}}}Q_{\mathcal{B}}$ on the Siegel space $\mathfrak{h}_2$ a rank 4 $\mathbb{Z}$-lattice $I_z \in B$ and a basis of it such that the Siegel point $(P, J, U)_I$ on this basis is $z_{\mathcal{A}\bar{\mathcal{D}}}Q_{\mathcal{B}}$. We will then prove that the left order of I_z is a maximal order $O_{[\mathcal{A}],[\mathcal{D}]}$ with an embedding of $\mathbb{Z}[\sqrt{N}]$ into it. This will imply that the number of different values (up to a sign) for $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ is at most $h(\mathcal{O}_N)^2t$.

Proposition 46. *There exists u and v in B such that*:

- $\mathrm{Tr}(u\bar{v}) = 0$, $\mathrm{Tr}(u) = 0$ *and* $\mathrm{Tr}(v) = 0$.
- $\mathrm{N}(u) = |N|$.
- $\mathrm{N}(v) = |D|$.
- *u and v are in a maximal order R of B.*

Proof. Since $|N| \equiv 3 \bmod 4$, we can assume $B = (-1, N)$. Choosing $u = j$ it is clear that $\mathrm{Tr}(u) = 0$ and $\mathrm{N}(u) = |N|$, hence we are looking for v in B such that $\mathrm{Tr}(uv) = 0$,

$\mathrm{Tr}(v) = 0$ and $\mathrm{N}(v) = |D|$. This conditions forces v to have the form $v = xi + yk$ and we are looking for an integer solution of the quadratic equation:

$$x^2 + |N|y^2 - |D|z^2 = 0. \tag{34}$$

We can assume that the solution is primitive (i.e. $gcd(x, y, z) = 1$). If (x, y, z) is a solution, clearly $gcd(z, N) = 1 = gcd(x, N)$ and $gcd(x, D) = 1 = gcd(y, D)$.

To prove the existence of such a solution we use the Hasse–Minkowski principle. Clearly (34) has a non-zero solution over $\mathbb{R}$, so we need to prove the existence of local non-zero solutions for all primes. We consider the different cases:

- For a prime $p \neq N$ and $p \neq D$ the quadratic form clearly has a local solution (see [13, Corollary 2, p. 6]).
- For the prime $|N|$ by Hensel's lemma it is enough to look for solutions of (34) modulo $|N|$:

  $$x^2 - |D|z^2 \equiv 0 \bmod |N| \text{ iff } (xz^{-1})^2 \equiv |D| \bmod |N|.$$

  This equation has solution if and only if $\left(\frac{|D|}{|N|}\right) = 1$. By the quadratic reciprocity law and the fact that $|N| \equiv 3 \bmod 4$ this last condition is equivalent to asking that $|D|$ splits in $\mathbb{Q}(\sqrt{N})$ which is the case.
- For the prime $|D|$, looking at (34) modulo $|D|$:

  $$x^2 + |N|y^2 \equiv 0 \bmod |D| \text{ iff } N \equiv (xy^{-1})^2 \bmod |D| \text{ iff } \left(\frac{N}{|D|}\right) = 1.$$

  Which is the case since $|D|$ splits in $\mathbb{Q}(\sqrt{N})$.

Given u and v as before, consider the rank 4 $\mathbb{Z}$-lattice $R = \langle 1, u, v, uv\rangle$. It is easy to see that R is actually an order, hence contained in a maximal one. □

Remark. If we define $R = \langle 1, \frac{1+j}{2}, v, \left(\frac{1+j}{2}\right) v\rangle$ it is easy to see that this is also an order. The advantage of this order is that it contains an embedding of the ring of integers of $\mathbb{Q}(\sqrt{N})$, but is not maximal.

Let $z_{\mathcal{A}\bar{\mathcal{D}}}Q_{\mathcal{B}} = (\frac{b_1+\sqrt{N}}{2a_1|D|}) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$, u and v as in Proposition 46 (choosing $u = j$). Define

$$I_z := \left\langle \left(\frac{b_1 - j}{2a_1|D|}\right) av, \left(\frac{b_1 - j}{2a_1|D|}\right)\left(\frac{|D| + bv}{2}\right), \frac{v - b}{2}, a \right\rangle. \tag{35}$$

If we denote ϕ the embedding of $\mathbb{Q}(\sqrt{N})$ into B and ψ the embedding of $\mathbb{Q}(\sqrt{D})$ into B with $\phi(\sqrt{N}) = u$ and $\psi(\sqrt{D}) = v$ and choosing the basis $\mathcal{B} = \langle v_1, v_2 \rangle$ (where in our notation $v_1 = \frac{b+\sqrt{D}}{2}$ and $v_2 = a$) then the ideal I_z was defined by

$$I_z = \left\langle \phi\left(\frac{b_1 - \sqrt{N}}{2a_1|D|}\right)\psi(\sqrt{D})\psi(\bar{v_2}), \phi\left(\frac{b_1 - \sqrt{N}}{2a_1|D|}\right)\psi(\sqrt{D})\psi(\bar{v_1}), \psi(\bar{v_1}), \psi(\bar{v_2}) \right\rangle.$$

If we forget the specific basis, and think of I_z just as a rank 4 $\mathbb{Z}$-lattice in B it is given by $I_z = \left\langle \phi\left(\frac{b_1-\sqrt{N}}{2a_1|D|}\right)\psi(\sqrt{D})\psi(\bar{\mathcal{B}}), \psi(\bar{\mathcal{B}}) \right\rangle$.

Proposition 47. *The element $\frac{1+j}{2}$ is in the left order of I_z.*

Proof. This is an easy but tedious computation. We will just give the coordinates of the product of $\frac{1+j}{2}$ with each element of the basis of I_z (given above) as a linear combination.

- $\left(\frac{1+j}{2}\right) a = [ba_1, -2aa_1, 0, \frac{b_1+1}{2}]$.
- $\left(\frac{1+j}{2}\right)\left(\frac{v-b}{2}\right) = [-2ca_1, ba_1, \frac{b_1+1}{2}, 0]$.
- $\left(\frac{1+j}{2}\right)\left(\frac{b_1-j}{2a_1|D|}\right) av = [\frac{1-b_1}{2}, 0, 2ac_1, bc_1]$.
- $\left(\frac{1+j}{2}\right)\left(\frac{b_1-j}{2a_1|D|}\right)\left(\frac{|D|+bv}{2}\right) = [0, \frac{1-b_1}{2}, bc_1, 2cc_1]$. □

Proposition 48. *The element $a_1 v$ is in the left order of I_z.*

Proof. Since $\mathcal{B}$ is an ideal, it is clear that $v\langle w_3, w_4\rangle \subset \langle w_3, w_4\rangle$. By the way we choose v, it satisfies $vj = -jv$, then

$$(a_1 v)\left(\frac{b_1 - j}{2a_1|D|}\right) = \left(\frac{b_1 - j}{2a_1|D|}\right)(-a_1 v) + \frac{b_1}{|D|}v. \tag{36}$$

For the part corresponding to the first two elements of I_z note that they can be written as $\left(\frac{b_1-j}{2a_1|D|}\right)v(a)$ and $\left(\frac{b_1-j}{2a_1|D|}\right)v\left(\frac{v-b}{2}\right)$. Since $\mathcal{B}$ is an ideal, $v\mathcal{B} \subset \mathcal{B}$ and the assertion follows from Eq. (36). □

Corollary 49. *The order $R = \langle 1, \frac{1+j}{2}, a_1 v, \frac{1+j}{2}a_1 v\rangle$ is contained in the left order of I_z and has discriminant $(a_1^2 N D)^2$ or index $a_1^2|D|$ in a maximal order.*

Proof. It is clear that R is in the left order of I_z by the previous two propositions. It is also clear that it is an order. To compute its discriminant, note that the bilinear matrix associated to it is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & \frac{1-N}{2} & 0 & 0 \\ 0 & 0 & 2a_1^2|D| & a_1^2|D| \\ 0 & 0 & a_1^2|D| & a_1^2|D|\frac{1-N}{2} \end{pmatrix}.$$

Then note that the index in a maximal order (which has discriminant N^2) is the square root of the discriminant. $\square$

Theorem 50. *Let U be the complex multiplication associated to $\frac{-j}{\sqrt{|N|}}$. Then the Siegel point $(P, J, U)_{I_z}$ associated to the ideal I_z in the given basis is $z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}}$.*

Proof. This is a straightforward computation so we omit the details. Just check that the given basis of I_z is symplectic, i.e. that the matrix $J(x, y)$ in the given basis is a multiple of the matrix $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ (since $J(x, y)$ is skewsymmetric there are half the conditions to check), and that the matrix U associated to the point $z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}}$ is the same as the complex multiplication matrix on I_z. $\square$

Theorem 51. *The lattice I_z is an ideal for a maximal order.*

Proof. The strategy is to prove that the quadratic form associated to the ideal I_z is locally equivalent to the maximal order one for all primes. We need the next lemma:

Lemma 52. *The quadratic form associated to the lattice I_z has discriminant N^2.*

Proof. The bilinear form is the same as the Siegel point $z_{\mathcal{A}\bar{\mathcal{D}}} Q_{\mathcal{B}}$ hence its bilinear form matrix is

$$B_I = \begin{pmatrix} 2c_1 Q_{\mathcal{B}} & b_1 I_2 \\ b_1 I_2 & 2a_1 D Q_{\mathcal{B}}^{-1} \end{pmatrix}.$$

Since $Q_{\mathcal{B}}$ has determinant D, it is an easy computation to prove that the determinant of this matrix is N^2 (using that $b_1^2 - 4a_1c_1|D| = N$). $\square$

A maximal order for $B = (-1, N)$ is given by $O = \langle \frac{1+j}{2}, \frac{i+k}{2}, j, k \rangle$ (see Proposition 5.2, p. 369 of [11]), then it is easy to compute the matrix of the quadratic form trace and to check that it has discriminant N^2, and is an improperly primitive integral form. Since the discriminant of both forms is a unit for all primes $p \neq |N|$ then they are locally equivalent (see Corollary of Theorem 3.1 of [2, p. 116]). Hence $(I_z)_p$ is locally principal for all primes $p \neq |N|$.

For the ramified prime, $D(I_z) = N^2$ hence it is locally principal. Locally principal ideals have the same discriminant as their left orders hence $O_l(I_z)$ is maximal. $\square$

4.4. Comparing Siegel points

If I is an ideal for a maximal order, and U a complex multiplication, the Siegel point associated to (U, I) is the same as the one associated to the point $(U, I\alpha)$ for any $\alpha \in B^\times$ (with the same choice of basis). Suppose two Siegel points z and z' have equivalent ideals I_z and $I_{z'}$, say $I_z = I_{z'}\alpha$ for some $\alpha \in B^\times$. Then since the complex multiplication is the same for all the ideals we constructed, the two Siegel points are equivalent by Proposition 39. Let M be the matrix in $Sp_4(\mathbb{Z})$ making the change of basis between I_z and $I_{z'}\alpha$.

Lemma 53. *The matrix M is in the subgroup $\Gamma_{1,2}$.*

Proof. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $z = \left(\frac{b_1+\sqrt{N}}{2a_1}\right) Q$ and $z' = \left(\frac{b_1'+\sqrt{N}}{2a_1'}\right) Q'$ where Q and Q' have even diagonal. Since M sends the bilinear form associated to the ideal I_z to the bilinear form associated to the ideal $I_{z'}\alpha$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 2c_2 Q & b_2 I_2 \\ b_2 I_2 & 2a_2 Q^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 2c_2' Q & b_2' I_2 \\ b_2' I_2 & 2a_2' Q'^{-1} \end{pmatrix}.$$

By the way we choose generators, $b_i \equiv 1 \bmod 4$, $i = 1, 2$ (also $b_i' \equiv 1 \bmod 4$, $i = 1, 2$) hence $2Q \equiv \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \bmod 4$. Let $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Looking at the first 2×2 matrix of the previous equality $\bmod 4$ we get: $2c_2 A^t J A + C^t A + A^t C + 2a_2 C^t J C \equiv 2J \bmod 4$. In particular 4 divides the diagonal.

If $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^t J A = \begin{pmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{pmatrix}$ hence 4 divides the diagonal of $2c_2 A^t J A$ and $2a_2 C^t J C$. Also $A^t C$ is symmetric hence $A^t C + C^t A = 2A^t C$ and we get that 2 divides the diagonal of $A^t C$. The proof for $B^t D$ is analogous looking at the last 2×2 matrix. $\square$

Proposition 54. *For fixed ideals $\mathcal{A}$ and $\mathcal{D}$, the left order of $I_{z_{\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}}$ is independent of the ideal $\mathcal{B}$.*

Proof. We know $I_z = \left\langle \phi\left(\frac{b_1-\sqrt{N}}{2a_1|D|}\right) \psi(\sqrt{D})\psi(\bar{\mathcal{B}}), \psi(\bar{\mathcal{B}}) \right\rangle$. The ideal $\mathcal{B}_q := \mathcal{B} \otimes \mathbb{Z}_q$ is principal, hence there exists an element $\delta_q \in L_q := \mathbb{Q}_q(\sqrt{D})$ such that $\mathcal{B}_q = \mathcal{O}_L \delta_q$. Then $I_z \otimes \mathbb{Z}_q = \left\langle \phi\left(\frac{b_1+\sqrt{N}}{2a_1|D|}\right) \psi(\sqrt{D})\psi(\mathcal{O}_L), \psi(\mathcal{O}_L) \right\rangle \bar{\delta}_q$, hence its left order is clearly independent of $\mathcal{B}$. $\square$

Proposition 55. *Let $\mathcal{A}$ and $\mathcal{A}'$ two equivalent ideals of $\mathcal{O}_K$ prime to $\mathcal{D}$, say $\mathcal{A}' = \alpha\mathcal{A}$. Then $\phi(\alpha^{-1}) I_{z_{\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}} = I_{z_{\mathcal{A}'\mathcal{D}Q_{\mathcal{B}}}}$.*

Proof. It is enough to prove that $I_{z_{\alpha\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}} \subseteq \phi(\alpha^{-1}) I_{z_{\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}}$. Then $I_{z_{\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}} \subseteq \phi(\alpha) I_{z_{\alpha\mathcal{A}\mathcal{D}Q_{\mathcal{B}}}}$ and the result follows.

Without loss of generality we may assume that $\mathcal{A}$ and $\mathcal{A}'$ are prime to each other, then we can choose basis such that $\bar{\mathcal{A}}\bar{\mathcal{D}} = \langle a|D|, \frac{b-\sqrt{N}}{2}\rangle$ and $\bar{\mathcal{A}}'\bar{\mathcal{D}} = \langle a'|D|, \frac{b-\sqrt{N}}{2}\rangle$.

Then there exists $M = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in Sl_2(\mathbb{Z})$ such that

$$x_1 a|D|\bar{\alpha} + x_2 \left(\frac{b-\sqrt{N}}{2}\right)\bar{\alpha} = a'|D|, \tag{37}$$

$$x_3 a|D|\bar{\alpha} + x_4 \left(\frac{b-\sqrt{N}}{2}\right)\bar{\alpha} = \frac{b-\sqrt{N}}{2}. \tag{38}$$

Claim. $D \mid x_2$. If $\bar{\alpha} = \frac{1}{a}\left(\frac{\alpha_1+\alpha_2\sqrt{N}}{2}\right)$ with $\alpha_i \in \mathbb{Z}$ looking at the imaginary parts of the above equalities we get that

$$\frac{\sqrt{N}}{4}\,(2x_1 a|D|\alpha_2 + x_2(b\alpha_2 - \alpha_1)) = 0,$$
$$\frac{\sqrt{N}}{4}\,(2x_3 a|D|\alpha_2 + x_4(b\alpha_2 - \alpha_1)) = \frac{-a\sqrt{N}}{2}.$$

This implies the claim. If $\mathcal{B} = \langle w_1, w_2\rangle$, $\phi(\alpha^{-1})I_{z_{\mathcal{A}\mathcal{D}}} = \left\langle \phi\left(\bar{\alpha}(\frac{b-\sqrt{N}}{2a|D|})\right)\psi(\sqrt{D}\bar{w}_1),\right.$ $\left.\phi\left(\bar{\alpha}(\frac{b-\sqrt{N}}{2a|D|})\right)\psi(\sqrt{D}\bar{w}_2), \phi(\alpha^{-1})\psi(\bar{w}_1), \phi(\alpha^{-1})\psi(\bar{w}_2)\right\rangle$. Since $a\bar{\alpha} = \alpha^{-1}a'$ Eq. (37) implies that

$$x_3\phi(\alpha^{-1}) + x_4\phi\left(\alpha^{-1}\left(\frac{b-\sqrt{N}}{2a|D|}\right)\right) = \phi\left(\frac{b-\sqrt{N}}{2a'|D|}\right).$$

Since $\mathcal{B}$ is an ideal, $\sqrt{D}w_i \in \mathcal{B}$ hence $\phi(\frac{b-\sqrt{N}}{2a'|D|})\psi(\sqrt{D}\bar{w}_i) \in \phi(\alpha^{-1})I_{z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}}$ for $i = 1, 2$. Since $D \mid x_2$ Eq. (38) can be written as

$$x_1\phi(\alpha^{-1}) + \frac{x_2}{|D|}\phi\left(\alpha^{-1}\left(\frac{b-\sqrt{N}}{2a|D|}\right)\right)\psi(\sqrt{D})^2 = 1,$$

which implies that $\psi(\bar{w}_i) \in \phi(\alpha^{-1})I_{z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}}$ for $i = 1, 2$. □

Corollary 56. *If $\mathcal{A}$, $\mathcal{A}'$ are two equivalent ideals in $\mathcal{O}_K$ prime to $\mathcal{D}$ then the ideal $I_{z_{\alpha\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}}$ and $I_{z_{\alpha\mathcal{A}'\mathcal{D}}Q_{\mathcal{B}}}$ have equivalent left orders.*

Proposition 57. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two split prime ideals of $\mathbb{Q}[\sqrt{N}]$ of norms $|D|$ and $|D'|$, respectively, such that $\mathcal{D}' = \mu\mathcal{D}$. Let $\mathcal{B}$ and $\mathcal{B}'$ be ideals of $\mathbb{Q}[\sqrt{D}]$ and of $\mathbb{Q}[\sqrt{D'}]$, respectively. Then the ideals $I_{z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}}$ and $I_{z_{\mathcal{A}\mathcal{D}'}Q_{\mathcal{B}'}}$ have the same left order if following the notation of Proposition* 46 *we take $v' = \mu v$.*

Proof. We are abusing notation while stating this theorem, since μ is an element of $\mathbb{Q}[\sqrt{N}]$. We will not distinguish between an element in B or in $\mathbb{Q}[\sqrt{N}]$ via the identification $\sqrt{N} \mapsto j$, and the case will be clear from the context.

By Proposition 54 it is enough to restrict to the case $\mathcal{B}$ and $\mathcal{B}'$ principal. In this case we will prove that the ideals associated to them are slightly different and use this to prove the proposition. We can choose basis such that $\mathcal{D} = \langle |D|, \frac{b_1+\sqrt{N}}{2}\rangle$ and $\mathcal{D}' = \langle |D'|, \frac{b_1+\sqrt{N}}{2}\rangle$. Let $\mu = \frac{\alpha}{|D|} + \frac{\beta}{|D|}\sqrt{N}$. Since $\mu\left(\frac{b_1+\sqrt{N}}{2}\right) \in \mathcal{D}'$ and $\mu^{-1}\left(\frac{b_1+\sqrt{N}}{2}\right) \in \mathcal{D}$, $\frac{\alpha+\beta b_1}{|D|} \in \mathbb{Z}$ and $\frac{\alpha-\beta b_1}{|D'|} \in \mathbb{Z}$.

Since $b = 1$ the definition of the ideals is

- $I_{\mathcal{D}} := I_{z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}} = \left\langle \left(\frac{b_1-j}{2a_1|D|}\right) v, \left(\frac{b_1-j}{2a_1|D|}\right)\left(\frac{v+|D|}{2}\right), \frac{v-1}{2}, 1 \right\rangle$,
- $I_{\mathcal{D}'} := I_{z_{\mathcal{A}\mathcal{D}'}Q'_{\mathcal{B}}} = \left\langle \left(\frac{b_1-j}{2a_1|D'|}\right) v', \left(\frac{b_1-j}{2a_1|D'|}\right)\left(\frac{v'+|D'|}{2}\right), \frac{v'-1}{2}, 1 \right\rangle$,

where v and v' are the elements of norm $|D|$ and $|D'|$, respectively, as in Proposition 46. We will write the elements of $I_{\mathcal{D}'}$ in the basis of $I_{\mathcal{D}}$, the other case follows from symmetry.

- $\frac{v'-1}{2} = [-a_1\beta, 0, \frac{\alpha+b_1\beta}{|D|}, \frac{\alpha+b_1\beta+D}{2|D|}]$,
- $\left(\frac{b_1-j}{2a_1|D'|}\right) v' = [\frac{\alpha-\beta b_1}{|D'|}, 0, 4\beta c, 2\beta c]$ which has integer coefficients,
- $\left(\frac{b_1-j}{2a_1|D'|}\right)\left(\frac{v'+|D'|}{2}\right) = [\frac{\alpha-\beta b_1-|D'|}{2|D'|}, 1, 2\beta c, \beta c]$.

We cannot say that the two ideals are the same, since the numbers α and β may have a 2 in the denominator, but $(I_{\mathcal{D}})_p = (I_{\mathcal{D}'})_p$ for all primes $p \neq 2$. In particular if we denote $O_{\mathcal{D}}$ and $O_{\mathcal{D}'}$ the left order of $I_{\mathcal{D}}$ and $I_{\mathcal{D}'}$ respectively, we get that $(O_D)_p = (O_{\mathcal{D}'})_p$ for all $p \neq 2$. Since the denominators are at most 2 it is easy to check that $4O_{\mathcal{D}} + \mathbb{Z} \subset O_{\mathcal{D}'}$, and has index at most 2^8. By Corollary 49, the order $R \subset O_{\mathcal{D}'}$ with index $a_1^2|D|$, which is odd. Then $4O_D + R = O_{\mathcal{D}'}$. Also $4O_{\mathcal{D}} + R = O_{\mathcal{D}}$ hence both orders are the same. $\square$

By Theorem 31 we know that the numbers $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ depend (up to multiplication by ± 1) on the equivalence class of $\mathcal{A}$, the equivalence class of $\mathcal{D}$ and the class of $z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}} \bmod \Gamma_{12}$. If we fix the class of $\mathcal{A}$ and the class of $\mathcal{D}$ we can associate ideals to the points $z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}}$ as in (35) and by Proposition 31 they all have the same left order. Then by Corollary 41 we get at most $t(B)$ different points in the Siegel space. This implies:

Theorem 58. *The number of different $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ up to multiplication by ± 1 in $\mathcal{M}$ is at most $h(\mathcal{O}_K)^2 t(B)$, where $t(B)$ is the type number for maximal orders.*

Note that this number is independent of the class number of $\mathcal{O}_L$. With all these results we return and finish the proof of Theorem 6:

Given $\mathcal{A}$ and $[\mathcal{D}]$ as before we associate to them a maximal order $O_{\mathcal{A},[\mathcal{D}]}$. For any left $O_{\mathcal{A},[\mathcal{D}]}$-ideal I we want to define the number $m_{[\mathcal{A}],I}([\mathcal{D}])$.

- If there exists a pair $(\mathcal{D}', \mathcal{B})$ where $\mathcal{D}' \in \mathcal{O}_K$ is a prime ideal of norm D' congruent to 3 mod 4, $\mathcal{D}' \sim \mathcal{D}$ and $\mathcal{B}$ is an ideal of $\mathbb{Q}(\sqrt{-D'})$ such that $I = I_{z_{\mathcal{A}\mathcal{D}'}Q_{\mathcal{B}}}$, we define $m_{\mathcal{A},I}([\mathcal{D}]) = \xi_2 n_{\mathcal{A},[\mathcal{B}],\bar{\mathcal{D}}'}$.
  The number ξ_2 is chosen such that $m_{\mathcal{A},I}([\mathcal{D}])$ is a complex number in the upper half plane union $\mathbb{R}_{\geqslant 0}$.
- If no such pair exists we define $m_{\mathcal{A},I}([\mathcal{D}]) = 0$.

Proposition 59. *This definition is "independent" of the equivalent class of the ideal* $\mathcal{A}$.

Proof. By Corollary 56 if two ideal $\mathcal{A}$, $\mathcal{A}'$ are equivalent (say $\mathcal{A}' = \alpha\mathcal{A}$), their left orders are conjugate. Furthermore a bijection between left $O_{\mathcal{A},[\mathcal{D}]}$-ideals and left $O_{\mathcal{A}',[\mathcal{D}]}$-ideals is given by multiplication on the right by $\phi(\alpha^{-1})$ (by Proposition 55). Since the number $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}'}$ is independent of the equivalent class of $\mathcal{A}$ this map preserves the numbers $\{m_{\mathcal{A},I}([\mathcal{D}])\}$. □

Hence we think of the numbers $m_{\mathcal{A},I}([\mathcal{D}])$ as defined on equivalence classes and denote them $m_{[\mathcal{A}],I}([\mathcal{D}])$.

Formula (17) says:

$$L(\psi_{\mathcal{D}}, 1) = \frac{2\pi}{w\sqrt{|D|}}\eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K)\left(\sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)}\sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}\right).$$

To the Siegel point $z_{\mathcal{A}\bar{\mathcal{D}}}Q_{\mathcal{B}}$ we associate the left $O_{[\mathcal{A}],[\mathcal{D}]}$-ideal $I_{\mathcal{B}}$ as in (35). Given I a left $O_{[\mathcal{A}],[\mathcal{D}]}$-ideal, we define

$$r(\mathcal{D}, [\mathcal{A}], I) = \begin{cases} \sum_{\{\mathcal{B}\in\mathcal{O}_L \mid I_{\mathcal{B}}\sim I\}} n_{[\mathcal{A}],[\mathcal{B}],\bar{D}}/m_{\mathcal{A},I}([\mathcal{D}]) & \text{if } m_{\mathcal{A},I}([\mathcal{D}]) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 53 and Theorem 31 imply that if the ideals $I_{\mathcal{B}}$ and $I_{\mathcal{B}'}$ are equivalent, $n_{[\mathcal{A}],[\mathcal{B}],\bar{D}} = \pm n_{[\mathcal{A}],[\mathcal{B}'],\bar{D}}$ hence $r(\mathcal{D}, [\mathcal{A}], I) \in \mathbb{Z}$. Rearranging the sum we get

$$L(\psi_{\mathcal{D}}, 1) = \frac{2\pi}{w\sqrt{|D|}}\eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K)\left(\sum_{[\mathcal{A}]\in Cl(\mathcal{O}_K)}\sum_{I} r(\mathcal{D}, [\mathcal{A}], I)m_{[\mathcal{A}],I}([\mathcal{D}])\right)$$

as claimed. □

Question. Is it true that for any left $O_{[\mathcal{A}],[\mathcal{D}]}$-ideal I there exists a pair $(\mathcal{D}', \mathcal{B})$ such that $I \sim I_{z_{\mathcal{A}\mathcal{D}'}Q_{\mathcal{B}}}$?

All the examples we computed show this is the case.

Proposition 60. *Let $\mathcal{A}$ be an ideal of $\mathbb{Q}(\sqrt{N})$, then $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ and $n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$ differ by a unit in a quadratic extension of $\mathcal{M}$.*

Proof. Let $\sigma_{\mathcal{A}}$ be the automorphism of H corresponding to the ideal $\mathcal{A}$ via the Artin–Frobenius map. Then we proved that $\left(\frac{\theta(z_{\mathcal{O}_K\mathcal{D}}Q_{\mathcal{B}})}{\eta(\mathcal{D})\eta(\mathcal{O}_K)}\right)^{\sigma_A} = \frac{\theta(z_{\mathcal{A}\mathcal{D}}Q_{\mathcal{B}})}{\eta(\mathcal{A}\mathcal{D})\eta(\mathcal{A})}$. Hence $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}} = \left(\frac{\eta(\mathcal{A})\eta(\mathcal{A}\mathcal{D})}{\eta(\mathcal{D})\eta(\mathcal{O}_K)\psi_{\bar{\mathcal{D}}}(\mathcal{A})}\right)(n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}})^{\sigma_{\mathcal{A}}}$. Note that the quotient of etas squared is in H while $\psi_{\bar{\mathcal{D}}}(\mathcal{A})$ is in T, hence $\zeta := \left(\frac{\eta(\mathcal{A})\eta(\mathcal{A}\mathcal{D})}{\eta(\mathcal{D})\eta(\mathcal{O}_K)\psi_{\bar{\mathcal{D}}}(\mathcal{A})}\right)$ is in a quadratic extension of $\mathcal{M}$. Clearly $\mathrm{N}(\zeta) = 1$ as required. □

5. The class number one case

We study now the case of imaginary quadratic fields with class number equal to one. In this case $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ are rational integers for any choice of $\mathcal{D}$. There are just six such cases (we exclude the case $N = -3$) so we can study all this cases by numerical computations. Here are some examples:

5.1. Case $N = -7$

This case is the easiest one since the class number in the quaternion algebra is also one. Then the numbers $n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$ are integers and differ by a unit.

Theorem 61. *Let $N = -7$ and $\mathcal{D}$ be any ideal of prime norm congruent to $3 \bmod 4$. Then $L(\psi_{\mathcal{D}}, 1) \neq 0$.*

Proof. By Proposition 24 we know that the number associated to an ideal $\mathcal{B}$ is the same as the one associated to $\bar{\mathcal{B}}$. For a prime ideal $\mathcal{D}$ let $\mathbf{\Omega} = \eta(\bar{\mathcal{D}})\eta(\mathcal{O}_K)\frac{2\pi}{w\sqrt{|D|}}$ where $-D = \mathrm{N}(\mathcal{D})$ and w is the number of units in $\mathbb{Q}[\sqrt{D}]$. Formula (17) for $L(\psi, 1)$ reads:

$$L(\psi, 1) = \left(\sum_{[\mathcal{B}]\in Cl(\mathcal{O}_L)} n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}\right)\Omega = \left(n_{[\mathcal{O}_K],[\mathcal{O}_L],\bar{\mathcal{D}}} + 2\sum_{[\mathcal{B}]\in\Phi} n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}\right)\Omega, \quad (39)$$

where Φ is a maximal subset of $Cl(\mathcal{O}_L)$ such that $[\mathcal{O}_L] \notin \Phi$ and if $[\mathcal{B}] \in \Phi$ then $[\bar{\mathcal{B}}] \notin \Phi$.

Taking the maximal order O as left O-ideal representative, we see that the number associated to it is 1 up to a sign, then $\frac{L(\psi,1)}{\Omega} \equiv 1 \bmod 2$. $\square$

In the next table, we list some of the numbers $n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$ to show the behavior of the sign.

D	$\mathcal{B}$	$n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$
11	[1, −1, 3]	1
23	[1, −1, 6]	1
23	[13, −17, 6]	−1
23	[13, 17, 6]	−1
43	[1, −1, 11]	−1
67	[1, −1, 17]	1
71	[1, −1, 18]	−1
71	[19, 9, 2]	−1
71	[19, −9, 2]	−1
71	[29, 33, 10]	1
71	[29, −33, 10]	1
71	[43, 141, 116]	−1
71	[43, −141, 116]	−1

5.2. *Case* $N = -11$

In this case the quaternion algebra has type number 2 for maximal orders, so we get two different integers associated to different $\mathcal{D}$'s. Each number $n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$ will be associated to an ideal class. Let $B = (-1, -11)$ be the quaternion algebra ramified at 11 and infinity. Let $O := \langle \frac{1}{2} + \frac{j}{2}, \frac{i}{2} + \frac{k}{2}, j, k \rangle$ be a maximal order and I a non-principal ideal. Here is a table of $n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$ for different values of D and $\mathcal{B}$, writing down the associated ideal also.

D	$\mathcal{B}$	$n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$	Ideal
23	[1, −1, 6]	2	I_1
23	[13, −17, 6]	0	O
23	[13, 17, 6]	0	O
31	[1, −1, 8]	−2	I_1
31	[5, 17, 16]	0	O
31	[5, −17, 16]	0	O
47	[1, −1, 12]	0	O
47	[7, −17, 12]	2	I_1
47	[7, 17, 12]	2	I_1
47	[17, −53, 42]	0	O
47	[17, 53, 42]	0	O

Note that the number 0 is associated to the principal ideal, while the number 2 is associated to I_1. With the same reasoning as in Theorem 61 we can get a partial result proving that the ideals $\mathcal{D}$ such that $z_{\mathcal{D}} Q_{\mathcal{O}_L}$ is associated to the ideal I_1 have a non-vanishing L-series.

Following the method described in [10], taking $\{O, I_1\}$ as representatives for the maximal order and constructing the Brandt matrices for level 11^2 we get that the eigenvector associated to the modular form of weight 2 and level 11^2 is $[0, 0, 0, 1, -1, 0, 0, 0, 1, -1]$. The first three zeros correspond to the principal ideal, and the ± 1 to I_1. Then the number associated to each ideal is the same as the one associated to it via $n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$, since the eigenvector is well defined up to a constant.

5.3. Case $N = -163$

Let $B = (-1, -163)$ be the quaternion algebra ramified at 163 and infinity. In this case, the class number for maximal orders is 14 while the type number is 8. Consider the maximal order $O := \langle 1, i, \frac{1}{2} + \frac{j}{2}, \frac{i}{2} + \frac{k}{2} \rangle$. A set of representatives of left O-ideals is given by $\{I_j\}_{j=1}^{14}$ with $I_1 = O$ and

- $I_2 := \langle 2, 2i, \frac{1}{2} + i + \frac{j}{2}, -1 + \frac{i}{2} + \frac{k}{2} \rangle$
- $I_3 := \langle 3, 3i, \frac{1}{2} + i + \frac{j}{2}, -1 + \frac{i}{2} + \frac{k}{2} \rangle$
- $I_4 := \langle 3, 3i, \frac{-1}{2} + i + \frac{j}{2}, -1 - \frac{i}{2} + \frac{k}{2} \rangle$
- $I_5 := \langle 6, 6i, \frac{1}{2} + i + \frac{j}{2}, -1 + \frac{i}{2} + \frac{k}{2} \rangle$
- $I_6 := \langle 6, 6i, \frac{-1}{2} + i + \frac{j}{2}, -1 - \frac{i}{2} + \frac{k}{2} \rangle$
- $I_7 := \langle 4, 4i, \frac{3}{2} + i + \frac{j}{2}, -1 + \frac{3i}{2} + \frac{k}{2} \rangle$
- $I_8 := \langle 4, 4i, \frac{-3}{2} + i + \frac{j}{2}, -1 - \frac{3i}{2} + \frac{k}{2} \rangle$
- $I_9 := \langle 6, 6i, \frac{5}{2} + i + \frac{j}{2}, -1 + \frac{5i}{2} + \frac{k}{2} \rangle$
- $I_{10} := \langle 6, 6i, \frac{-5}{2} + i + \frac{j}{2}, -1 - \frac{5i}{2} + \frac{k}{2} \rangle$
- $I_{11} := 5, 5i, \frac{1}{3} + 2i + \frac{j}{2}, -2 + \frac{i}{2} + \frac{k}{2} \rangle$
- $I_{12} := \langle 5, 5i, \frac{-1}{2} + 2i + \frac{j}{2}, -2 - \frac{i}{2} + \frac{k}{2} \rangle$
- $I_{13} := \langle 7, 7i, \frac{5}{2} + 3i + \frac{j}{2}, -3 + \frac{5i}{2} + \frac{k}{2} \rangle$
- $I_{14} := \langle 7, 7i, \frac{-5}{2} + 3i + \frac{j}{2}, -3 - \frac{i}{2} + \frac{k}{2} \rangle$

The pairs of ideals (I_{2j+1}, I_{2j+2}) with $j = 1, \ldots, 6$, have the same right order, hence each pair will have the same integer associated. For the table we consider the range of primes between 150 and 200 so as to get all the ideals $\{I_j\}$ associated to some number

$n_{[\mathcal{O}_K],[\mathcal{B}],\bar{\mathcal{D}}}$. The table is

D	$\mathcal{B}$	$n_{[\mathcal{A}],[\mathcal{B}],\bar{\mathcal{D}}}$	Ideal
151	[1, −1, 38]	20	I_2
151	[29, 9, 2]	14	I_8
151	[29, −9, 2]	14	I_8
151	[11, −5, 4]	8	I_{13}
151	[11, 5, 4]	8	I_{14}
151	[43, 137, 110]	4	I_{12}
151	[43, −137, 110]	4	I_{12}
167	[1, −1, 42]	0	I_1
167	[157, 33, 2]	−20	I_2
167	[157, −33, 2]	−20	I_2
167	[61, 65, 18]	−2	I_4
167	[61, −65, 18]	−2	I_3
167	[29, 93, 76]	−10	I_6
167	[29, −93, 76]	−10	I_5
167	[127, −177, 62]	−14	I_7
167	[127, 177, 62]	−14	I_8
167	[19, −21, 8]	−12	I_9
167	[19, 21, 8]	−12	I_{10}
179	[1, −1, 45]	0	I_1
179	[19, 45, 29]	2	I_3
179	[19, −45, 29]	2	I_4
179	[13, 17, 9]	4	I_{12}
179	[13, −17, 9]	4	I_{11}
199	[1, −1, 50]	0	I_1
199	[31, −69, 40]	−20	I_2
199	[31, 69, 40]	−20	I_2
199	[43, −133, 104]	−4	I_{12}
199	[43, 133, 104]	−4	I_{11}
199	[13, 29, 20]	−14	I_8
199	[13, −29, 20]	−14	I_7
199	[131, 453, 392]	−8	I_{14}
199	[131, −453, 392]	−8	I_{13}

The eigenvector for the Brandt matrices corresponding to the form of weight 2 and level 167^2 is given by the vector $[0, 10, 1, 1, 5, -5, 7, -7, -6, 6, 2, 2, -4, 4]$ with respect to the maximal order representatives $\{I_j\}$.

Considering all the class number 1 imaginary quadratic fields (the computations being the same in all cases), we can prove:

Theorem 62. *Let E be a CM elliptic curve over* $\mathbb{Q}$ *of level* p^2*. Then the coordinate of the eigenvector of the Brandt matrices associated to E on the place corresponding to an ideal I is given up to a sign by* $m_{[\mathcal{O}_K],I}([\mathcal{D}])$.

Acknowledgments

The author thank the referee for suggesting a better organization of this work. The author thank and dedicate this paper to Joana Terra for all her support.

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