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All the solutions of the form $M_2 \times_W \Sigma_{d-2}$ for Lovelock gravity in vacuum in the Chern-Simons case

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In this paper we classify a certain family of solutions of Lovelock gravity in the Chern-Simons (CS) case, in arbitrary (odd) dimension, $d \geq 5$. The spacetime is characterized by admitting a metric that is a warped product of a two-dimensional spacetime M_2 and an (*a priori*) arbitrary Euclidean manifold Σ_{d-2} of dimension $d - 2$. We show that the solutions are naturally classified in terms of the equations that restrict Σ_{d-2}. According to the strength of such constraints we found the following branches in which Σ_{d-2} has to fulfill: a Lovelock equation with a single vacuum (Euclidean Lovelock Chern-Simons in dimension $d - 2$), a single scalar equation that is the trace of an Euclidean Lovelock CS equation in dimension $d - 2$, or finally a degenerate case in which Σ_{d-2} is not restricted at all. We show that all the cases have some degeneracy in the sense that the metric functions are not completely fixed by the field equations. This result extends the static five-dimensional case previously discussed in Dotti *et al*. [Phys. Rev. D **76**, 064038 (2007)], and it shows that in the CS case, the inclusion of higher powers in the curvature does not introduce new branches of solutions in Lovelock gravity. Finally, we comment on how the inclusion of a non-vanishing torsion may modify this analysis. *© 2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4795258]

I. INTRODUCTION

Gravity in higher dimensions has proved to be an interesting arena to test how generic are the notions gained in four-dimensional gravitational physics. Even in higher dimensional General Relativity (GR), properties as uniqueness and stability of solutions in vacuum may depart completely from their four-dimensional counterpart (for a recent summary of the state of the art see Ref. 1). Maintaining the second order character of the field equations in higher dimensions, it is possible to consider a more general setup than the one defined by Einstein's gravity, since as proved by Lovelock in Ref. 2 the most general parity-even Lagrangian in arbitrary dimension d, that gives second order field equations for the metric is given by an arbitrary linear combination of the dimensional continuations of all the lower dimensional Euler densities. This gives rise to the so-called Lovelock gravity, the simplest case after GR being the Einstein-Gauss-Bonnet (EGB) gravity. In this theory, in addition to the Einstein-Hilbert and cosmological terms, one includes a term which is quadratic in the curvature and gives non-trivial field equations in dimensions grater than four. This quadratic combination is very precise, in such a way that the possible higher derivative terms cancel each other and one gets second order field equations. Since the field equations come from a diffeomorphism invariant action, their divergence vanishes identically.

Departing from the family of metrics (1), looking for exact rotating solutions is a difficult problem beyond GR. Considering for example the Kerr-Schild ansatz that naturally gives rise to the Myers-Perry solution with cosmological constant in GR,[4] one finds that in order to have a

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non-trivial solution in EGB, the coupling constants must be fixed as in the Chern-Simons case[5] and even more the solution turns out to be non-circular,[6] making the analysis of the causal structure more cumbersome (for some perturbative and numerical solutions see also Ref. 7).

To find exact and analytic solutions of these theories is a non-trivial problem when one departs from spherical symmetry. For example, a problem that is solved in a very simple manner in GR, corresponds to finding the most general solution of the form

$$ds_d^2 = -f^2(t,r)\,dt^2 + \frac{dr^2}{g^2(t,r)} + r^2 d\Sigma_{d-2}^2, \tag{1}$$

where Σ_{d-2} is an arbitrary Euclidean manifold of dimension $d-2$. Einstein equations plus a cosmological constant in vacuum

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2}$$

imply that the metric functions do not depend on t, and are given by

$$f^2 = g^2 = -\frac{2\Lambda r^2}{(d-1)(d-2)} - \frac{\mu}{r^{d-2}} + \gamma, \tag{3}$$

where μ is an arbitrary integration constant and Σ_{d-2} must be an Einstein manifold fulfilling the equation

$$\tilde{R}_{ij} = (d-3)\,\gamma\,\tilde{g}_{ij}\ . \tag{4}$$

Here $\tilde{R}_{ij}$ is the Ricci tensor of Σ_{d-2} and $\tilde{g}_{ij}$ its metric.[3]

Solving exactly the same problem in Lovelock gravity is more complicated. For example, in the EGB theory for the static case, the work[8] solves this problem in arbitrary dimension finding a rich set of causal structures. For arbitrary values of the coupling constants of the theory, the analysis done in Ref. 8 reduces to the done previously reported in Ref. 9, where it was proved that if one assumes Σ_{d-2} to be Einstein, then one can show that it must also obey a quadratic restriction on the Weyl tensor which includes a new parameter θ. That parameter appears in the lapse function and even more, it modifies the asymptotic behavior of the metric (see also Ref. 10).

For arbitrary Σ, beyond the EGB case not much is known. The static solution in the spherically symmetric case was found in Ref. 11. When Σ_{d-2} is a constant curvature manifold, a Birkhoff's theorem was proved in Ref. 12 (see also Ref. 13). Reference 12 also shows that Birkhoff's theorem is not valid when the coupling constants are fixed in a precise way and some degeneracies may appear since in such cases, some of the metric functions are not determined by the field equations (for some particular cases, this was previously observed in Ref. 14). Lovelock theory, being a gravity theory with higher powers in the curvature, could have more than one maximally symmetric solution, and the mentioned degeneracies appear precisely at the regions in the space of couplings in which some of these vacua coincide (for some static black hole solutions, with constant curvature horizons in this case see Ref. 16, as well as Ref. 15 for some solutions of the EGB theory in the case in which there is no maximally symmetric solution at all).

It would be interesting therefore to classify all the solutions of the form (1) in higher curvature Lovelock theories. In this work we focus on the odd-dimensional case, when the highest possible power of the curvature is present in the Lagrangian and all the vacua coincide. This theory is known as Lovelock-Chern-Simons (LCS) theory (for a recent review see Ref. 17).

The action for a general Lovelock theory can be written as

$$I = \kappa \int \sum_{p=0}^{\left[\frac{d-1}{2}\right]} \alpha_p \varepsilon_{a_1\ldots a_{2p}a_{2p+1}\ldots a_d} \overbrace{R^{a_1a_2}\ldots R^{a_{2p-1}a_{2p}}}^{p-times} e^{a_{2p+1}}\ldots e^{a_d}, \tag{5}$$

where κ and α_p are arbitrary (dimensionfull) coupling constants, $\varepsilon_{a_1\ldots a_d}$ is the Lorentz invariant Levi-Civita tensor, $R^{ab} := d\omega^{ab} + \omega^{ac}\omega_c^b$ is the curvature two-form written in terms Lorentz connection one-form ω^{ab}, and e^a is the vielbein. $[x]$ stands for the integer part of x. Wedge exterior product between differential forms is understood. Finally, Latin indices $\{a_i, b_i\}$ run from 0 to $d-1$.

The term with $p = 0$ in (5), corresponds to a volume term that gives the contribution of the cosmological constant, for $p = 1$ one gets the Einstein-Hilbert term, while for $p = 2$ the Lagrangian reduces to the Gauss-Bonnet term. As mentioned before, here we will focus on the case $d = 2n + 1$ and the coefficients α_p are given by

$$\alpha_p := \frac{1}{2n-2p+1}\binom{n}{p}\frac{1}{l^{2(n-p)}} , \tag{6}$$

where l^2 is the squared curvature radius of the unique (AdS) maximally symmetric solution. For simplicity we will focus on the case $l^2 > 0$, nevertheless the de Sitter case is trivially obtained by analytically continuing $l \to il$, while the flat limit (up to some subtleties that will be mentioned when necessary) can be obtained by taking $l \to \infty$.

When torsion vanishes, the field equations coming from (5) with the couplings given by (6) can be written as

$$E_a := \varepsilon_{aa_1\ldots a_{2n}}\overbrace{\bar{R}^{a_1a_2}\ldots\bar{R}^{a_{2n-1}a_{2n}}}^{n-times} = 0 , \tag{7}$$

where we have defined the concircular curvature two-form as $\bar{R}^{ab} := R^{ab} + \frac{1}{l^2}e^ae^b$. In terms of tensors, if we use the generalized Kronecker delta of strength one denoted by $\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p}$, by defining the concircular curvature tensor $\bar{R}^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} + \frac{1}{l^2}\delta^{\alpha\beta}_{\gamma\delta}$, the field equations (7) read

$$E^{\alpha}{}_{\beta} := \delta^{\alpha\alpha_1\ldots\alpha_{2n}}_{\beta\beta_1\ldots\beta_{2n}}\overbrace{\bar{R}^{\beta_1\beta_2}{}_{\alpha_1\alpha_2}\ldots\bar{R}^{\beta_{2n-1}\beta_{2n}}{}_{\alpha_{2n-1}\alpha_{2n}}}^{n-times} = 0 . \tag{8}$$

In Sec. II we will prove that all the solutions of the form (1), for the field equation (7) (or equivalently (8)) fall into one of the following three different classes:

Case I: The manifold Σ_{d-2} is arbitrary and the metric reads

$$ds^2 = -\left(\frac{r^2}{l^2} - \mu\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \mu} + r^2 d\Sigma^2_{d-2} , \tag{9}$$

where μ is an integration constant.

Case II: For $\xi \neq 0$, if the manifold Σ_{d-2} satisfies the following (scalar) restriction

$$\varepsilon_{i_1\ldots i_{2n-2}}\overbrace{\left(\tilde{R}^{i_1i_2} - \xi e^{i_1i_2}\right)\ldots\left(\tilde{R}^{i_{2n-3}i_{2n-2}} - \xi e^{i_{2n-3}i_{2n-2}}\right)}^{(n-1)-times} = 0 , \tag{10}$$

where $\tilde{R}^{ij}$ is the curvature two-form intrinsically defined on Σ_{d-2} and the indices $\{i, j\}$ run on Σ_{d-2}, then the metric reads

$$ds^2 = -\left(c_1(t)r + c_2(t)\sqrt{\frac{r^2}{l^2} + \xi}\right)^2 dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + \xi} + r^2 d\Sigma^2_{d-2} , \tag{11}$$

with $c_1(t)$ and $c_2(t)$ arbitrary integration functions. In the flat limit ($l \to \infty$) the metric reduces to

$$ds^2 = -\left(c_1(t)r + c_2(t)\right)^2 dt^2 + \frac{dr^2}{\xi} + r^2 d\Sigma^2_{d-2}.$$

In the case $\xi = 0$ (which does not exist in the limit $l \to \infty$) the restriction on Σ_{d-2} is obtained by setting $\xi = 0$ in (10) and the metric reads

$$ds^2 = -\left(c_1(t)r + \frac{c_2(t)}{r}\right)^2 dt^2 + \frac{l^2dr^2}{r^2} + r^2 d\Sigma^2_{d-2} , \tag{12}$$

where again $c_1(t)$ and $c_2(t)$ are arbitrary integration functions. Note that in all of these cases, by redefining the time coordinate, one can gauge away one of the two integration functions, but not both simultaneously.

Case III: The manifold Σ_{d-2} satisfies the following tensor restriction

$$\varepsilon_{ji_1\ldots i_{2n-2}}\overbrace{\left(\tilde{R}^{i_1i_2}-\xi e^{i_1i_2}\right)\ldots\left(\tilde{R}^{i_{2n-3}i_{2n-2}}-\xi e^{i_{2n-3}i_{2n-2}}\right)}^{(n-1)-times}=0 \tag{13}$$

and the metric reads

$$ds^2=-f^2(t,r)\,dt^2+\frac{dr^2}{\frac{r^2}{l^2}+\xi}+r^2d\Sigma_{d-2}^2\,, \tag{14}$$

with $f(t, r)$ an arbitrary function.

This result extends the static five-dimensional case previously analyzed in Ref. 18. In Case I, we see that the manifold Σ_{d-2} is arbitrary, i.e., it is not fixed by the field equations. In Case II, the manifold Σ_{d-2} is fixed by a single scalar equation which, even after using diffeomorphism invariance, in general it is not enough to determine a metric on it. Finally in Case III, we see that the lapse function $f^2(t, r)$ is left arbitrary by the field equations. Therefore we conclude that, in the previously mentioned sense, all the cases have some degeneracy.

II. PROOF OF THE CLASSIFICATION

To develop the proof of the classification it is useful to have the components of the curvature two-form with respect to some basis for the metric (1). If we define the components of the vielbein as

$$e^0=f\,dt,\, e^1=\frac{dr}{g}\text{ and } e^i=r\tilde{e}^i\,, \tag{15}$$

where $\tilde{e}^i$ is the vielbein intrinsically defined on Σ_{d-2}, then the nontrivial components of the concircular curvature two-form $\bar{R}^{ab}$ read

$$\bar{R}^{01}=Ae^0e^1,\,\bar{R}^{0i}=Be^0e^i+Ce^1e^i,\,\bar{R}^{1i}=Fe^1e^i-Ce^0e^i\text{ and }\bar{R}^{ij}=\tilde{R}^{ij}+Je^ie^j\,, \tag{16}$$

where $\tilde{R}^{ij}$ is the curvature two-form intrinsically defined on Σ_{d-2} and A, B, C, F, and J are functions of t and r defined by

$$A=A(t,r):=-\frac{g}{f}\left[\left(\frac{\dot{g}}{g^2f}\right)^{\cdot}+\left(gf'\right)'\right]+\frac{1}{l^2}\,, \tag{17}$$

$$B=B(t,r):=-g^2\frac{f'}{rf}+\frac{1}{l^2}\,,\ C:=C(t,r)=\frac{\dot{g}}{fr}, \tag{18}$$

$$F=F(t,r):=-\frac{\left(g^2\right)'}{2r}+\frac{1}{l^2}\,, \tag{19}$$

$$J=J(t,r):=-\frac{g^2}{r^2}+\frac{1}{l^2}\,. \tag{20}$$

Primes denote derivation with respect to r while dots derivation with respect to t.

There are three kinds of equations depending on whether the free index in (7) goes along the time direction, radial direction, or along the manifold Σ_{d-2}, which, respectively, reduce to

$$E_0 := 2n\varepsilon_{01i_1...i_{2n-1}}\bar{R}^{1i_1}\overbrace{\bar{R}^{i_2i_3}...\bar{R}^{i_{2n-2}i_{2n-1}}}^{n-1-times}=0\ ,$$

$$E_1 := 2n\varepsilon_{10i_1...i_{2n-1}}\bar{R}^{0i_1}\overbrace{\bar{R}^{i_2i_3}...\bar{R}^{i_{2n-2}i_{2n-1}}}^{n-1-times}=0\ ,$$

$$E_j := 2n\varepsilon_{j01i_1...i_{2n-2}}\bar{R}^{01}\overbrace{\bar{R}^{i_1i_2}...\bar{R}^{i_{2n-3}i_{2n-2}}}^{n-1-times}+2n\left(2n-2\right)\varepsilon_{j0i_11i_2i_3...i_{2n-2}}\bar{R}^{0i_1}\bar{R}^{1i_2}\overbrace{\bar{R}^{i_3i_4}...\bar{R}^{i_{2n-3}i_{2n-2}}}^{n-2-times}=0\ .$$

After introducing explicitly in these equations the components of the concircular curvature two-form (17)–(20), we get the following three equations

$$\mathcal{G}_0 := \left(Fe^1e^{i_1}-Ce^0e^{i_1}\right)\varepsilon_{01i_1...i_{2n-1}}\overbrace{\left(\tilde{R}^{i_2i_3}+Jr^2\tilde{e}^{i_2}\tilde{e}^{i_3}\right)...\left(\tilde{R}^{i_{2n-2}i_{2n-1}}+Jr^2e^{i_{2n-2}}e^{i_{2n-1}}\right)}^{n-1-times}=0\ ,$$

$$\mathcal{G}_1 := \left(Be^0e^{i_1}+Ce^1e^{i_1}\right)\varepsilon_{01i_1...i_{2n-1}}\underbrace{\left(\tilde{R}^{i_2i_3}+Jr^2\tilde{e}^{i_2}\tilde{e}^{i_3}\right)...\left(\tilde{R}^{i_{2n-2}i_{2n-1}}+Jr^2e^{i_{2n-2}}e^{i_{2n-1}}\right)}_{n-1-times}=0$$

and

$$\mathcal{G}_j := A\varepsilon_{ji_1...i_{2n-2}}\overbrace{\left(\tilde{R}^{i_1i_2}+Jr^2\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}}+Jr^2e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-1-times}$$
$$+2\left(n-1\right)\left(BF+C^2\right)r^4\varepsilon_{ji_1...i_{2n-2}}\tilde{e}^{i_1}\tilde{e}^{i_2}\overbrace{\left(\tilde{R}^{i_3i_4}+Jr^2\tilde{e}^{i_3}\tilde{e}^{i_4}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}}+Jr^2e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-2-times}=0,$$

where we have defined $\varepsilon_{i_1...i_{2n-1}} := \varepsilon_{01i_1...i_{2n-1}}$.

Considering the combinations $e^0\mathcal{G}_0+e^1\mathcal{G}_1=0$ and $e^1\mathcal{G}_0+e^0\mathcal{G}_1=0$ one, respectively, gets

$$\left(F-B\right)\varepsilon_{01i_1...i_{2n-1}}\overbrace{\left(\tilde{R}^{i_1i_2}+Jr^2\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}}+Jr^2e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-1-times}\tilde{e}^{i_{2n-1}}=0, \tag{21}$$

$$2C\varepsilon_{01i_1...i_{2n-1}}\overbrace{\left(\tilde{R}^{i_1i_2}+Jr^2\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}}+Jr^2e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-1-times}\tilde{e}^{i_{2n-1}}=0. \tag{22}$$

This immediately splits the analysis in two cases defined by the (would be) constraint on Σ_{d-2}

$$\varepsilon_{i_1...i_{2n-1}}\overbrace{\left(\tilde{R}^{i_1i_2}+Jr^2\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}}+Jr^2\tilde{e}^{i_{2n-3}}\tilde{e}^{i_{2n-2}}\right)}^{n-1-times}\tilde{e}^{i_{2n-1}}=0. \tag{23}$$

If (23) does not hold, then we need to impose $F = B$ and $C = 0$, the later implies that $g(t, r) = g(r)$, while the former implies $f(t, r) = S(t)g(r)$. The function $S(t)$ can be set to 1 without lost of generality by means of a redefinition of the time coordinate. Therefore in this branch (i.e., provided (23) does not hold), we have that (21) and (22) imply $f(t, r) = g(t, r) = f(r) = g(r)$. If (23) holds, then $\mathcal{G}_0 = 0 = \mathcal{G}_1$ without imposing any restriction on the function f and g at the moment. Note that the quantities with tilde on top depend only on the coordinates in Σ_{d-2}, while the combination Jr^2, could depend on both t and r. At the moment this is not relevant since Eqs. (21) and (22) are factorized in any case, but later we will see that the consistency of Eq. (23) strongly constraints the metric functions.

If we consider now equation $\mathcal{G}_0 = 0$, in the case in which (23) does not hold, since C is already vanishing in this branch, we need to impose the vanishing of the function F which in turn implies that $g^2 = \frac{r^2}{l^2} - \mu$, where μ is an integration constant. As mentioned, in this branch we also have $f^2 = \frac{r^2}{l^2} - \mu$ (since $f^2 = g^2$), and therefore one can see by direct evaluation that A identically vanishes also. Therefore $C = F = A = 0$ and then equation $\mathcal{G}_i = 0$ is also trivially satisfied without imposing any restriction on Σ_{d-2}. This concludes the proof of Case I outlined in the Introduction.

On the other hand if (23) holds (as mentioned before) $\mathcal{G}_0$ and $\mathcal{G}_1$ vanish identically and then at the moment, the functions $f(t, r)$ and $g(t, r)$ are not restricted. Before continuing to equation $\mathcal{G}_i = 0$, let us go back to the problem of the consistency of Eq. (23). Assuming $(Jr^2)' \neq 0$ and $(Jr^2)^{\cdot} \neq 0$, by taking derivatives of Eq. (23) with respect to t and r one eventually gets

$$\frac{\partial \left(Jr^2\right)}{\partial r}\varepsilon_{i_1...i_{2n-1}}\tilde{e}^{i_1}...\tilde{e}^{i_{2n-1}} = 0\,, \tag{24}$$

$$\frac{\partial \left(Jr^2\right)}{\partial t}\varepsilon_{i_1...i_{2n-1}}\tilde{e}^{i_1}...\tilde{e}^{i_{2n-1}} = 0\,, \tag{25}$$

which imply the vanishing of the volume form of Σ_{d-2}, arriving to a contradiction. We have proved then that in the case when (23) holds, the consistency of Eq. (23) implies that $(Jr^2)' = (Jr^2)^{\cdot} = 0$ which in turn implies that $g^2 = \frac{r^2}{l^2} + \xi$ with ξ an integration constant, and consequently (23) reads

$$\varepsilon_{i_1...i_{2n-1}}\overbrace{\left(\tilde{R}^{i_1i_2} - \xi\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}} - \xi\tilde{e}^{i_{2n-3}}\tilde{e}^{i_{2n-2}}\right)}^{n-1-times}\tilde{e}^{i_{2n-1}} = 0, \tag{26}$$

which now depends only on the coordinates of Σ_{d-2}.

The remaining structure comes from the analysis of equation $\mathcal{G}_i = 0$. Note that when $g^2 = \frac{r^2}{l^2} + \xi$, $C = 0 = F$, therefore $\mathcal{G}_j$ reduces to

$$A\varepsilon_{ji_1...i_{2n-2}}\overbrace{\left(\tilde{R}^{i_1i_2} - \xi\tilde{e}^{i_1}\tilde{e}^{i_2}\right)...\left(\tilde{R}^{i_{2n-3}i_{2n-2}} - \xi e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-1-times} = 0. \tag{27}$$

If $\xi \neq 0$, the equation $A = 0$ allows to integrate $f(t, r)$, which in this case reads

$$f^2 = \left(c_1(t)r + c_2(t)\sqrt{\frac{r^2}{l^2} + \xi}\right)^2, \tag{28}$$

while for $\xi = 0$ it integrates as

$$f^2 = \left(c_1(t)r + \frac{c_2(t)}{r}\right)^2. \tag{29}$$

The latter case is not defined in the flat limit ($l \to \infty$) while in such a limit, when $g^2 = \xi$, the equation $A = 0$ gives the following expression for f:

$$f^2(t, r) = (c_1(t)r + c_2(t))^2\,.$$

In all of these expressions $c_1(t)$ and $c_2(t)$ are arbitrary integration functions and note that one of them can be gauged away by a redefinition of the time coordinate.

Summarizing, in this branch we have that if $\xi \neq 0$, the metric reads

$$ds^2 = -\left(c_1(t)r + c_2(t)\sqrt{\frac{r^2}{l^2} + \xi}\right)^2 dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + \xi} + r^2 d\Sigma^2_{d-2}, \tag{30}$$

which in the limit $l \to \infty$ takes the form

$$ds^2 = -(c_1(t)r + c_2(t))^2 dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + \xi} + r^2 d\Sigma^2_{d-2},$$

while for $\xi = 0$ we have

$$ds^2 = -\left(c_1(t)r + \frac{c_2(t)}{r}\right)^2 dt^2 + \frac{l^2 dr^2}{r^2} + r^2 d\Sigma^2_{d-2}\,,$$

where $c_1(t)$ and $c_2(t)$ are arbitrary integration functions and Σ_{d-2} fulfills in both cases, the same scalar equation (26). Note that here the constant ξ appears in the restriction on Σ_{d-2} and can be scaled to ± 1 when it is non-vanishing. This concludes the proof of Case II outlined in the Introduction.

If $A \neq 0$, Eq. (27) implies a tensor restriction on Σ_{d-2}, which naturally, is stronger than its trace given by (26). When this tensor restriction holds, the metric reads

$$ds^2 = -f^2(t,r)\,dt^2 + \frac{dr^2}{\frac{r^2}{l^2}+\xi} + r^2 d\Sigma_{d-2}^2,$$

with Σ_{d-2} constrained by

$$\varepsilon_{ji_1\ldots i_{2n-2}}\overbrace{\left(\tilde{R}^{i_1i_2}-\xi\tilde{e}^{i_1}\tilde{e}^{i_2}\right)\ldots\left(\tilde{R}^{i_{2n-3}i_{2n-2}}-\xi e^{i_{2n-3}}e^{i_{2n-2}}\right)}^{n-1-times}=0,$$

and the function $f(t, r)$ is arbitrary. This concludes the proof of Case III outlined in the Introduction. This tensor restriction corresponds to an Euclidean Lovelock CS equation in dimension $d - 2 = 2n - 1$.

This concludes the proof of the classification.

III. DISCUSSION

A. On the causal structures

For the Case II and Case III, the (t, r)-part of the metrics obtained depend on arbitrary functions of the time coordinate, therefore the causal structure of this spacetimes is not fixed. Note that this dependence cannot be gauged away completely by a diffeomorphism. Nevertheless, a few comments on the causal structures are in order in all of the three cases when the integration functions are chosen to be constants, i.e., $c_1(t) = c_1$ and $c_2(t) = c_2$.

In Case I, the solution describes a black hole. This solution reduces to the one found in Ref. 19. In such case also, its thermodynamics and causal structure coincide with that of the three-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole[20] where, for generic values of μ, the causal structure singularity at $r = 0$ of the three-dimensional case is now replaced by a curvature singularity as can be seen by evaluating, for example, the Ricci scalar.

In Case II, the metric

$$ds^2 = -\left(c_1 r + c_2\sqrt{\frac{r^2}{l^2}+\xi}\right)^2 dt^2 + \frac{dr^2}{\frac{r^2}{l^2}+\xi} + r^2 d\Sigma_{d-2}^2, \qquad (31)$$

might describe the traversable wormhole found in Ref. 21, which is asymptotically AdS at both asymptotic regions. This is the case when $\xi = -1$ and $|\frac{c_2}{lc_1}| < 1$, which can be seen directly by performing the change of coordinates $r = l\cosh\rho$ and allowing the coordinate ρ to go from $-\infty$ to $+\infty$. In this case, the metric reduces to

$$ds^2 = l^2\left[-\cosh^2(\rho-\rho_0)\,dt^2 + d\rho^2 + \cosh^2\rho\, d\Sigma_{d-2}^2\right], \qquad (32)$$

where $\rho_0 = -\tanh^{-1}\left(\frac{c_2}{lc_1}\right)$ and we have properly rescaled the time coordinate. The conditions under which the propagation of a scalar field on this background is stable, was studied in Ref. 22, and some holographic properties of strings attached to the boundaries have been explored in Ref. 23. For $\xi = 0$ the metric reduces to

$$ds^2 = -\left(c_1 r + \frac{c_2}{r}\right)^2 dt^2 + \frac{l^2 dr^2}{r^2} + r^2 d\Sigma_{d-2}^2. \qquad (33)$$

When $c_1 \neq 0$ this spacetime is asymptotically locally AdS, while if $c_1 = 0$, the (t, r)-part of the metric reduces to a Lifshitz geometry (geometry with an anisotropic scaling symmetry), with a dynamic exponent equals to $z = -1$.

Since in Case III the lapse function is arbitrary, the causal structure is also undefined even in the static case.

B. Does torsion help removing the degeneracy?

The field equations coming from the variation with respect to the spin connection in Lovelock theory, do not necessarily imply that torsion should vanish (for some explicit solutions see, e.g., Ref. 24). For example in five dimensions, in first order formalism for the Lovelock CS case, the field equations coming from the variation with respect to the vielbein and the spin connection are, respectively, given by

$$\varepsilon_{abcde}\left(R^{bc}+\frac{1}{l^2}e^b e^c\right)\left(R^{de}+\frac{1}{l^2}e^d e^e\right)=0, \tag{34}$$

$$\varepsilon_{abcde}\left(R^{cd}+\frac{1}{l^2}e^c e^d\right)T^e=0, \tag{35}$$

where we have introduced the torsion two-form $T^e := De^e := \frac{1}{2}e^e{}_\alpha T^\alpha_{\ \mu\nu}dx^\mu \wedge dx^\nu$. Therefore choosing the Levi-Civita connection is *ad hoc*. Then it is natural to wonder whether the equations coming from the torsion may help removing the degeneracy. Posing the question in a different manner one could ask: is there a non-degenerate branch of solutions of (34) and (35) in which the vielbein and the spin connection are compatible with the local isometries of Σ_{d-2}? It is clear that there are particular cases in which the torsion may not be vanishing and anyway the system is degenerated since, if for example we choose the (non-Riemannian) curvature to be constant $R^{ab}=-\frac{1}{l^2}e^a e^b$, then the torsion is left completely arbitrary by the field equations. Note also that, since this theory has an extra symmetry that mixes the spin connection and the vielbein (see Ref. 17), the arbitrariness in the torsion can be transformed into an arbitrariness of the line element constructed out from the corresponding vielbein. A thorough analysis with the inclusion of torsion will be presented elsewhere.[25]

C. Further comments

As studied for the static quadratic case in Ref. 8, when one considers Lovelock theories that do not belong to the subclass of Lovelock CS, but nevertheless the couplings are related in such a way that there is a unique vacuum, there are also sectors in which some of the metric functions are arbitrary. Therefore, this phenomenon seems to be more related to the fact of having degenerate maximally symmetric solution than with the appearance of an extra symmetry. In such non-Lovelock CS theories, as well as in the Lovelock CS ones, this degeneracy allows to have interesting causal structures as solutions (see, e.g., Ref. 26). Nevertheless in the former cases, there are more restrictions on Σ_{d-2}, which on one hand can be thought of as helping to remove the degeneracy, while in the other hand could be not compatible beyond the constant curvature case. A simple set of geometries beyond constant curvature manifolds (or their products) are product of the homogenous three-dimensional Thurston geometries, which have been recently found to provide simple examples of transverse sections of hairy black holes for some Lovelock theories in even dimensions.[27] In the context of compactifications of Lovelock CS theories, involving metrics that are products of constant curvature spaces, the degenerate behavior is also present as it was proved in Ref. 28 back in the early 1990s. The inclusion of matter fields seems to help removing the mentioned degeneracies (see, for example, Ref. 29).

If one departs from the underlying $(A)dS$ symmetry group, static spherically symmetric solutions of gravitational CS theories with matter fields have also been recently considered in Ref. 30. In this reference, the authors considered a Chern-Simons theory evaluated on a Lie algebra that is obtained by performing what the authors called a S-expansion procedure[31] from the AdS algebra and a particular semigroup S, which provides an approach to obtain GR in odd-dimensions from a CS theory. It would be interesting to study further the properties of these theories and to integrate them in the general ansatz (1) classifying the possible non-degenerate sectors.

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