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Comment on three-point function in *AdS*(3)/CFT(2)

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Recently, exact agreement has been found between bulk and boundary three-point functions in $AdS_3 \times S^3 \times T^4$ with Neveu-Schwarz-Neveu-Schwarz (NSNS) fluxes. This represents a nontrivial check of *AdS*/CFT correspondence beyond the supergravity approximation as it corresponds to an exact worldsheet computation. When taking a closer look at this computation, one notices that a crucial point for the bulk-boundary agreement to hold is an intriguing mutual cancellation between worldsheet contributions corresponding to the AdS_3 and to the S^3 pieces of the geometry, that results in a simple factorized form for the final three-point function. In this note we review this cancellation and clarify some points about the analytic relation between the $SU(2)$ and the $SL(2,\mathbb{R})$ structure constants. In particular, we dicuss the connection to the Coulomb gas representation. We also make some comments on the four-point function. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Exact agreement has been observed between boundary and bulk three-point functions in $AdS_3 \times S^3 \times T^4$ with Neveu-Schwarz-Neveu-Schwarz (NSNS) fluxes. In Refs. 1 and 2, Gaberdiel and Kirsch, and Dabholkar and Pakman, computed three-point functions of certain chiral primary states for type IIB string theory on $AdS_3 \times S^3 \times T^4$ in the tree-level approximation, and the resulting expressions were compared with the corresponding correlators in the dual two-dimensional conformal field theory at the orbifold point. As a result, exact agreement was found between bulk and boundary observables at large N. In Ref. 3 the analysis of this holographic agreement was extended to the case of chiral $\mathcal{N}=4$ operators, and the operators of spectral flowed sectors were considered in Ref. 4. The agreement was also studied from the supergravity point of view in Ref. 5.

The exact agreement found in Refs. 1–3 not only represents a highly nontrivial check of *AdS*/CFT correspondence beyond the supergravity approximation, but it can also be seen as evidence that a new nonrenormalization theorem holds for string theory in this background. This is because the bulk and boundary computations are performed at different points of the moduli space. This nonrenormalization mechanism was recently studied in Ref. 6.

When going through the worldsheet computation of Refs. 1 and 2, one immediately notices that a crucial point to find agreement between bulk and boundary observables is the surprising cancellation of all the factors in the worldsheet three-point functions that mix the momenta of vertex operators. Since the superstring σ-model in $AdS_3 \times S^3 \times T^4$ with NSNS fluxes corresponds to the $\mathcal{N}=1$ Wess–Zumino–Novikov–Witten (WZNW) on $SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$, it turns out that such cancellation gets translated into a remarkable simplification that happens between $SL(2,\mathbb{R})$ and $SU(2)$ structure constants when both are brought together.

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To those who are familiarized with the minimal Liouville gravity (MLG) (or, say the minimal string theory), the cancellation between $SL(2,\mathbb{R})$ and $SU(2)$ structure constants could seem reminiscent of the simplification that happens between three-point functions in Liouville field theory (LFT) and the three-point function in the generalized minimal models (GMMs). It was pointed out by Zamolodchikov that even though the analytic relation between GMM and LFT might give rise to the idea that GMM observables are simply an analytic continuation of the LFT quantities for pure imaginary values of the Liouville parameter b, it is not actually the case. It was shown in Ref. 7 that the GMM structure constants are not the mere analytic continuation of the LFT ones. In fact, contrary to one's expectation, GMM structure constants turn out to be, up to a proper renormalization of the vertex operators, the inverse of LFT structure constants, in the sense that the product of both quantities yields a remarkably simple factorized expression such as $\sim\Pi_{i=1}^{3}f(a_i)$, where a_i are the momenta of the Liouville vertex operators.

It was noticed in Ref. 2 that the cancellation that takes place between the $SU(2)$ and the $SL(2,\mathbb{R})$ supersymmetric structure constants when computing three-string amplitudes in $AdS_3\times S^3\times T^4$ is similar to what happens between GMM and LFT observables. This observation is correct, but, if not interpreted correctly, it might lead to the wrong conclusion that $SU(2)$ observables cannot be obtained as the analytic continuation of the analogous $SL(2,\mathbb{R})$ observables for negative values of the WZNW level k. What we want to point out in this note is that, unlike what happens in the $\mathcal{N}=1$ supersymmetric WZNW model, where the product of $SU(2)$ and $SL(2,\mathbb{R})$ three-point functions yields a simple factorized form as in MLG, the relation between bosonic $SU(2)$ structure constants and bosonic $SL(2,\mathbb{R})$ structure constants is different, and it does admit to be seen as an analytic continuation in k. Such analytic continuation is actually what one considers in the Coulomb gas approach to the nonrational WZNW theory.

The paper is organized as follows. In Sec. II, we discuss correlation functions in both $SL(2,\mathbb{R})$ and $SU(2)$ WZNW theory. In Sec. III, we review the calculation of three-point amplitudes of chiral states in $AdS_3\times S^3\times T^4$. We focus our attention on the cancellations that take place between the AdS_3 and the S^3 contributions. We discuss the analytic relation between $SL(2,\mathbb{R})$ and $SU(2)$ structure constants. Section IV contains some concluding remarks. In particular, we make some comments on the four-point function.

II. CORRELATION FUNCTIONS IN WZNW THEORY

A. $SL(2,\mathbb{R})_k$ WZNW correlators from Liouville theory

The $\mathcal{N}=1$ supersymmetric $SL(2,\mathbb{R})_{\hat{k}}$ WZNW model describes the superstring σ-model on the AdS_3 piece of the space-time, where the relation between the AdS_3 radius l and the string length scale l_s is given by $\hat{k}=l^2/l_s^2$, so that the semiclassical limit corresponds to $\hat{k}$ large. This interpretation is consistent with the value of the central charge of the theory,

$$c_{sl(2)} = 3 + 6/\hat{k},$$

which tends to 3 when $\hat{k}$ goes to infinity.

The supersymmetric affine algebra of the WZNW theory is generated by the supercurrent $\psi^a(z)+\theta J^a(z)$, where $a=1,2,3$, θ is a Grassman variable, and $\psi^a(z)$ represent three free fermions. The currents J^a generate the affine algebra $\widehat{sl}(2)$ of level $\hat{k}$, which is realized by the following operator product expansion (OPE):

$$J^a(z)J^b(w) \sim \frac{\eta^{ab}\hat{k}/2}{(z-w)^2} + \frac{i\varepsilon^{abc}J_c(w)}{(z-w)} + \cdots ,$$

where $\varepsilon^{abc}=1$ and $\eta^{ab}=\mathrm{diag}(++-)$, with $a,b,c=1,2,3$. The generators $J^a(z)$ can be written as

$$J^a(z) = j^a(z) - \frac{i}{\hat{k}}\varepsilon^{abc}\psi_b(z)\psi_c(z),$$

where, in turn, the bosonic currents $j^a(z)$ generate $sl(2)_k$ of level $k=\hat{k}+2$. The OPE between the currents $J^a(z)$ and the fermions $\psi^a(z)$ reads

$$J^a(z)\psi^b(w) \sim \frac{i\varepsilon^{abc}\psi_c(w)}{(z-w)} + \cdots, \quad \psi^a(z)\psi^b(w) \sim \frac{\eta^{ab}\hat{k}/2}{(z-w)} + \cdots.$$

The Sugawara construction yields the stress tensor

$$T(z) = \frac{1}{\hat{k}}\eta_{ab}(J^a(z)J^b(z) - \psi^a(z)\,\partial\,\psi^b(z)) \tag{1}$$

that generates the worldsheet Virasoro algebra.

The vertex operators $\Phi_j(x|z)$ representing states of the worldsheet theory are given by Virasoro primary fields with respect to (1) and expand representations of $SL(2,\mathbb{R})$. The index j labels such representation of $SL(2,\mathbb{R})$, while x is an auxiliary complex variable that allows for the following realization of the algebra:

$$j^a(z)\Phi_j(x|w) = -\frac{\mathcal{D}_x^a\Phi_j(x|w)}{(z-w)} + \cdots,$$

with the differential operators

$$\mathcal{D}_x^+ = x^2\partial_x - 2jx, \quad \mathcal{D}_x^- = \partial_x, \quad \mathcal{D}_x^3 = x\partial_x - j,$$

where, as usual, the notation $a=+,-,3$ corresponds to the generators $J^{\pm}(z)=J^1(z)\pm iJ^2(z)$.

The conformal dimension of vertex operators $\Phi_j(x|z)$ is given by

$$\Delta_{sl(2)} = -\frac{j(j+1)}{k-2}, \quad \text{with } k=\hat{k}-2.$$

Here, we are interested in correlation functions of these vertex operators. The four-point correlation function in the $SL(2,\mathbb{R})_k$ WZNW theory can be written in terms of the five-point function in LFT as follows:[8]

$$\left\langle \prod_{i=1}^{4}\Phi_{j_i}(x_i|z_i)\right\rangle_{sl(2)} = \mathcal{X}_k(j_1,j_2,j_3,j_4|x,z) \times \left\langle \prod_{i=1}^{5} V_{a_i}(z_i)\right\rangle_{\text{LFT}}, \tag{2}$$

where $2a_1=-b(j_1+j_2+j_2+j_4+1)$, $2a_{5>i>1}=-b(j_1+2j_i-j_2-j_3-j_4-b^{-2}-1)$, $2a_5=-b^{-1}$, $b^{-2}=k-2$, $z_1=z$, $z_2=0$, $z_3=1$, $z_4=\infty$, and on the right hand side also holds that $z_5=x$. The correlation function on the right hand side involves five exponential vertex operators of LFT [see (6) below]. The function $\mathcal{X}_k(j_1,j_2,j_3,j_4|x,z)$ is given by

$$\mathcal{X}_k(j_1,j_2,j_3,j_4|x,z) = \frac{|z|^{4(a_1a_2-b^2j_1j_2)}|z-1|^{4(a_1a_3-b^2j_1j_3)}}{|x|^{2a_2b^{-1}}|x-1|^{2a_3b^{-1}}|x-z|^{2a_1b^{-1}}}X_k(j_1,j_2,j_3,j_4),$$

with[1]

[1]When compareing with Ref. 8 take into account the relations $\Upsilon_W(x)=\Upsilon(-xb)=G_k^{-1}(x)b^{-b^2x^2-(b^2+1)x}$.

$$X_k(j_1,j_2,j_3,j_4) = \frac{\pi C_W^2(b)}{b^{5+4b^2}\Upsilon_0^2}\frac{(\nu(b))^s}{(\pi\mu\gamma(b^2)b^4)^{2j_1}} \times \frac{G_k\left(2+\sum_{i=1}^4 j_i\right)\prod_{n=2}^4 G_k\left(-1-j_1-2j_n+\sum_{i=2}^4 j_i\right)\gamma\left(b^2\left(j_1+2j_n-\sum_{i=2}^4 j_i\right)\right)}{\gamma\left(-b^2\sum_{i=1}^4 j_i-2b^2\right)\prod_{t=1}^4 G_k(2j_t+1)}, \tag{3}$$

where $s=1+\Sigma_{i=1}^4 j_i$, $\gamma(x)=\Gamma(x)/\Gamma(1-x)$, $\nu(b)=-b^2\gamma(-b^2)$, and where the special function $G_k(x)$ obeys the functional relations

$$G_k(x) = G_k(x-1)\gamma(-b^2x), \quad G_k(x) = G_k(-1-x-b^{-2}), \tag{4}$$

see Ref. 9 and references therein. The overall factor $\pi C_W^2(b)/b^2\Upsilon_0^2$ in (3) is a b-dependent function (namely, a factor independent of j_i), and it can be found in Ref. 8. The $SL(2,\mathbb{R})_k$ structure constants can be obtained from (3) in the limit $j_1=n=0$.

Equation (2) relates correlation functions of two different nonrational theories. It follows from the remarkable observation, originally due to Fateev and Zamolodchikov,[10] that the Knizhnik–Zamolodchikov equation[11] satisfied by the WZNW four-point function generates a solution to the Belavin–Polyakov–Zamolodchikov equation[12] satisfied by the five-point function that involves a degenerate field of momentum $a_5=-1/2b$.

Relation (2) permits to understand several nontrivial properties of the pole structure of $SL(2,\mathbb{R})_k$ WZNW four-point function: In Ref. 13 it was shown that the logarithmic dependences in the AdS_3 amplitudes, which can be understood in terms of AdS_3/CFT_2 as in Refs. 14 and 15, are ultimately associate with the OPE $V_{(b+1/b)/2}(z_i)V_{-1/2b}(x)$ when $a_i=(b+1/b)/2$ for $i=2,3,4$. Representation (2) is also useful to understand the origin of poles at the point $z=x$ that are associated with worldsheet instantons.[9] While from the perspective of WZNW theory such poles are unexpected as they are located in the middle of the moduli space, in terms of LFT these are naturally understood as emerging in the coincidence limit of two operators $V_{-sb/2}(z_1)V_{-1/2b}(x)$.

The normalization $X_k(j_1,j_2,j_3,j_4)$ in (2) is compatible with crossing symmetry of WZNW theory.[8] It can be also shown that $X_k(j_1,j_2,j_3,j_4)$ leads to a nice realization of the Hamiltonian reduction, which here corresponds to the limit $x\to z$.[16,17] In this limit, and considering the OPE,

$$V_{a_i}(z_i)V_{-1/2b}(x) = |x-z|^{2\xi_-}V_{-1/2b+a_i}(z_i) + (\pi\mu\gamma(b^2))^{b^{-2}}\frac{\gamma(2a_ib^{-1}-1-b^{-2})}{b^4\gamma(2a_ib^{-1})}|x-z|^{2\xi_+}V_{-1/2b-a_i}(z_i),$$

with $\xi_\pm=(\Delta_{a_i\pm 1/2b}-\Delta_{1/2b}-\Delta_{a_i})$ and $\Delta_a=a(b+b^{-1}-a)$, one finds

$$\left\langle \prod_{i=1}^4 \Phi_{j_i}(x_i|z_i)\right\rangle_{sl(2)} \sim \prod_{i=1}^4 \gamma(1+b^2(2j_i+1)) \times \left\langle \prod_{i=1}^4 V_{-bj_i}(z_i)\right\rangle_{\mathrm{LFT}} + \cdots, \tag{5}$$

where the symbol $\sim$ stands for a b-dependent factor and a singular factor $|x-z|^{(\cdots)}$, while the ellipses stand for subleading contribution, provided the Seiberg bound $a_i>(b+b^{-1})/2$ is obeyed. Notice that factors $\gamma(1+b^2(2j_i+1))$ in (5) can be absorbed in the normalization of Liouville vertices. Expression (5) can be proven by using formulas (1.28)–(1.29) of Ref. 18, together with the kind of tricks used in Appendix B of Ref. 13.

The Liouville correlation functions in (3) are defined by

$$\left\langle \prod_{i=1}^5 V_{a_i}(z_i)\right\rangle_{\mathrm{LFT}} = \int D\varphi e^{-S_L[\varphi;\mu]}\prod_{i=1}^5 e^{\sqrt{2}a_i\varphi(z_i)}, \tag{6}$$

with

$$S_L[\varphi;\mu]=\frac{1}{4\pi}\int d^2z(\partial\varphi\bar{\partial}\varphi+(b+b^{-1})R\varphi/2\sqrt{2}+4\pi\mu e^{\sqrt{2}b\varphi}), \tag{7}$$

where R is the scalar curvature of the worldsheet and μ is a real parameter.[19] By integrating out the zero mode of φ, (6) can be expanded as

$$\left\langle\prod_{i=1}^{5}V_{a_i}(z_i)\right\rangle_{\text{LFT}}=\Gamma(-n)b^{-1}\mu^n\delta\left(nb+\sum_{i=1}^{5}a_i-b-b^{-1}\right)\times\int D\varphi e^{-S_L[\varphi;\mu=0]}\prod_{i=1}^{5}e^{\sqrt{2}a_i\varphi(z_i)}\prod_{r=1}^{n}e^{\sqrt{2}b\varphi(w_r)}, \tag{8}$$

where now the path integral is understood as not including the zero mode.[20]

It is important to notice that expression above admits an integral representation of the form

$$\left\langle\prod_{i=1}^{4}\Phi_{j_i}(x_i|z_i)\right\rangle_{sl(2)}=X_k(j_1,j_2,j_3,j_4)|z|^{-4b^2j_1j_2}|1-z|^{-4b^2j_1j_3}\Gamma(-n)b^{-1}\mu^n$$
$$\times\int\prod_{r=1}^{n}d^2w_r\prod_{r=1}^{n}|w_r|^{-4a_2b}|w_r-1|^{-4a_3b}|w_r-z|^{-4a_1b}|w_r-x|^2\prod_{r<t}^{n}|w_r-w_t|^{-4b^2}, \tag{9}$$

where

$$n=b+b^{-1}\left(1-\sum_{i=1}^{5}\alpha_i\right)=2j_1. \tag{10}$$

As mentioned, for the particular case $j_1=0$ we would obtain the structure constants $C_{sl(2)}\times(j_2,j_3,j_4)\sim X_k(0,j_2,j_3,j_4)$. Replacing $j_1=0$ in the equation above it yields

$$\mathcal{X}_k(0,j_2,j_3,j_4)=-\frac{\gamma(-b^2)}{2\pi^2}(\nu(b))^{j_2+j_3+j_4+1}\frac{G_k(1+j_2+j_3+j_4)}{G_k(-1)}$$
$$\times\frac{G_k(-j_2+j_3+j_4)G_k(j_2-j_3+j_4)G_k(j_2+j_3-j_4)}{G_k(2j_2+1)G_k(2j_3+1)G_k(2j_4+1)}, \tag{11}$$

where the overall factor $C_W^2(b)G_k(-1)/\Upsilon_0^2G_k(1)$ has been replaced by $(b^{1+4b^2}/2\pi^3)\gamma(1-b^2)$, taking into account that in the limit $j_1\to 0$ one finds $(G_k(-1)/G_k(1))\Gamma(-n)=(b^2/2)\gamma(1+b^2)$.

Now, let us move on and consider the four-point function in the $SU(2)_k$ model.

B. $SU(2)_k$ WZNW correlators from minimal models

The $\mathcal{N}=1$ supersymmetric $SU(2)_{\hat{k}}$ WZNW theory has central charge,

$$c_{su(2)}=3-6/\hat{k}.$$

The affine symmetry is generated by the current algebra $\widehat{su}(2)_{\hat{k}}$, realized by the OPE,

$$K^a(z)K^b(w)\sim\frac{\delta^{ab}\hat{k}/2}{(z-w)^2}+\frac{i\varepsilon^{abc}K_c(w)}{(z-w)}+\cdots,$$

where $\varepsilon^{abc}=1$ and now $\delta^{ab}=\text{diag}(+++)$, with $a,b,c=1,2,3$. It also holds that

$$K^a(z)\chi^b(w)\sim\frac{i\varepsilon^{abc}\chi_c(w)}{(z-w)}+\cdots,\quad\chi^a(z)\chi^b(w)\sim\frac{\delta^{ab}\hat{k}/2}{(z-w)}+\cdots.$$

As in the case of $SL(2,\mathbb{R})_k$, the generators can be written as

$$K^a(z) = k^a(z) - \frac{i}{\hat{k}}\varepsilon^{abc}\chi_b(z)\chi_c(z),$$

where the bosonic currents $k^a(z)$ generate the algebra $\widehat{su}(2)_{k'}$ of level $k'=\hat{k}-2$, and $\chi^a(z)$ represent three free fermions.

The vertex operators $\Psi_{j'}(y|z)$ are Virasoro primaries of conformal dimension

$$\Delta_{su(2)} = \frac{j(j+1)}{k'+2}, \quad \text{with } k' = \hat{k} - 2,$$

where j' now labels representation of $SU(2)$, and where, again, y is an auxiliary complex variable, such that

$$k^a(z)\Psi_{j'}(y|w) = -\frac{\mathcal{K}_y^a\Psi_{j'}(y|w)}{(z-w)} + \cdots,$$

with

$$\mathcal{K}_y^+ = y^2\partial_y - 2j'y, \quad \mathcal{K}_y^- = -\partial_y, \quad \mathcal{K}_y^3 = y\partial_y - j',$$

and with $K^{\pm}(z)=K^1(z)\pm iK^2(z)$.

Four-point correlation function in the $SU(2)_{k'}$ WZNW theory can be written in terms of the five-point function in GMM as follows:

$$\left\langle \prod_{i=1}^{4} \Psi_{j_i'}(y_i|z_i) \right\rangle_{su(2)} = \mathcal{Y}_{k'}(j_1,j_2,j_3,j_4'|y,z) \times \left\langle \prod_{i=1}^{5} W_{\alpha_i}(z_i) \right\rangle_{\mathrm{GMM}}, \tag{12}$$

where $2\alpha_1=\beta(j_1+j_2+j_2+j_4+1)$, $2\alpha_{5>i>1}=\beta(j_1+2j_i-j_2-j_3-j_4+k'+1)$, $2\alpha_5=\beta^{-1}$, $\beta^{-2}=k'+2$, $z_1=z$, $z_2=0$, $z_3=1$, $z_4=\infty$, and on the right hand side also holds that $z_5=y$. Equation (12) is the $SU(2)$ analog of (2). Function $\mathcal{Y}_k(j_1,j_2,j_3,j_4|y,z)$ is given by

$$\mathcal{Y}_{k'}(j_1',j_2',j_3',j_4'|y,z) = \frac{|z|^{4(\beta^2 j_1' j_2' - \alpha_1\alpha_2)}|z-1|^{4(\beta^2 j_1' j_3' - \alpha_1\alpha_3)}}{|y|^{-2\alpha_2\beta^{-1}}|y-1|^{-2\alpha_3\beta^{-1}}|y-z|^{-2\alpha_1\beta^{-1}}} Y_{k'}(j_1',j_2',j_3',j_4'),$$

with

$$Y_{k'}(j_1',j_2',j_3',j_4') = (\gamma(\beta^2))^{2j_1'+1}P_{k'}\left(\sum_{a=1}^{4} j_a' + 1\right)\prod_{i=1}^{4}\frac{\sqrt{\gamma(1-\beta^2(2j_i'+1))}}{P_{k'}(2j_i')}$$
$$\times\prod_{n=2}^{4} P_{k'}\left(\sum_{l=2}^{4} j_l' - 2j_n' - j_1'\right), \tag{13}$$

where

$$P_{k'}(x) = \prod_{n=1}^{x}\gamma(n\beta^2), \quad x \geq 1,$$

while $P_{k'}(0)=1$. Normalization factor (13) is consistent with the fusion rules of the algebra.[21]

Expression (12) above also admits an integral representation,[10,22,23] namely,

$$\left\langle \prod_{i=1}^{4} \Psi_{j_i'}(y_i|z_i) \right\rangle_{su(2)} = Y_{k'}(j_1',j_2',j_3',j_4')|z|^{4\beta^2 j_1' j_2'}|1-z|^{4\beta^2 j_1' j_3'}$$
$$\times \int \prod_{r=1}^{2j_1'} d^2 t_r \prod_{r=1}^{2j_1'} |t_r|^{-4\alpha_2\beta}|t_r-1|^{-4\alpha_3\beta}|t_r-z|^{-4\alpha_1\beta}|t_r-y|^2 \prod_{r<l} |t_r-t_l|^{4\beta^2}.$$

This completes the parallelism with the formula (9) for $SL(2,\mathbb{R})$. Now, we are ready to discuss string amplitudes in $AdS_3 \times S^3$ in terms of correlation functions of the $SL(2,\mathbb{R}) \times SU(2)$ theory.

III. STRING AMPLITUDES IN $AdS_3 \times S^3$

A. AdS_3/CFT_2 correspondence and three-point function

According to the AdS_3/CFT_2 correspondence, correlation functions of dimension-h operators in the boundary CFT correspond to string amplitudes on AdS_3, namely,

$$\prod_{i=3}^{N} \int d^2 z_i \left\langle \prod_{i=1}^{N} \Phi_{j_i}(x_i|z_i) \right\rangle_{\text{worldsheet}} \times \cdots = \left\langle \prod_{i=1}^{N} O_{h_i}(x_i) \right\rangle_{\text{boundary}}, \tag{14}$$

where the ellipses reflect the contribution of the internal space.[2]

The indices j_i, which label the representations of $SL(2,\mathbb{R})$, are related to the conformal dimension h_i of vertex operators in the dual theory by the simple relation

$$h_i = -j_i. \tag{15}$$

This can be seen, for instance, by looking at the x-dependence of three-point functions in the $SL(2,\mathbb{R})_k$ WZNW model, which goes like

$$\langle \Phi_{j_1}(x_1|0)\Phi_{j_2}(x_2|1)\Phi_{j_3}(x_3|\infty)\rangle_{sl(2)} = |x_{12}|^{2(j_1+j_2-j_3)}|x_{23}|^{2(j_2+j_3-j_1)}|x_{13}|^{2(j_3+j_1-j_2)}C_{sl(2)}(j_1,j_2,j_3),$$

where $|x_{ij}|=|x_i-x_j|$. From this we also observe that auxiliary complex variables x_i acquire now a physical meaning, as these are interpreted as the coordinates of the boundary, where the dual CFT_2 is defined on.

Unitarity of the worldsheet theory in AdS_3 also demands the bound

$$1-k < 2j < -1, \quad k > 2, \tag{16}$$

as well as the introduction of the spectral flowed sectors of the $\widehat{sl}(2)_{\hat{k}}$ algebra, which represent winding strings states in AdS_3; see Ref. 9 and references therein.

The boundary two-dimensional conformal field theory that is dual to the type IIB string theory in $AdS_3 \times S^3 \times T^4$ is some deformation of the symmetric product orbifold $\mathrm{Sym}^N(\widetilde{T}^4)$ of N copies of $\widetilde{T}^4$,[24] where $\widetilde{T}^4$ is closely related to T^4. This three-dimensional example of holographic correspondence is motivated by the near horizon limit of the $D1/D5$ system, where the geometry $AdS_3 \times S^3 \times T^4$ is seen to emerge. In the S-dual picture, this configuration corresponds to the setting of $Q_5=\hat{k}$ NS5-branes and Q_1 fundamental strings, where the number of copies of $\widetilde{T}^4$ is given by $N=Q_1Q_5$. The six-dimensional string coupling constant is given by $g_6^2=Q_5/Q_1$, and thus the string perturbative theory is reliable in the large Q_1 limit, or $N=Q_5Q_1 \gg Q_5$. In this limit, string states in the bulk are mapped to twisted states in $\mathrm{Sym}^N(\widetilde{T}^4)$ that are associated with conjugancy classes with a single nontrivial cycle of length n. The relation between n and the worldsheet momentum is[4,25]

[2]More precisely, the complete prescription for AdS_3/CFT_2 would also include contributions coming from disconnected worldsheet diagrams.[38]

$$n=2h-1+\hat{k}\omega, \quad 2h=2,3,4,\ldots,k, \quad \omega=0,1,2,\ldots,$$

where h is associated with index j of the representations of $SL(2,\mathbb{R})$ by (15), while ω labels the spectral flow sector of $SL(2,\mathbb{R})$ the representation belongs to. Here we will consider the sector $\omega=0$.

We are interested in worldsheet vertex operators that represent chiral string states in $AdS_3 \times S^3 \times T^4$. As an example, let us consider the worldsheet vertex operators of the form

$$\mathcal{O}_j(x|z) = \psi(x|z) \times \Phi_j(x|z) \times \Psi_{-1-j}(x|z), \tag{17}$$

where the fermionic contributions take the form $\psi(x|z)=-\psi^{+}(z)+2x\psi^{3}(z)-x^2\psi^{-}(z)$. This is a worldsheet vertex operator associated with chiral string states of the NS sector, written in the picture -1. In order to compute a three-point function we also need the expression for such a state in the picture 0. This is obtained by reading off the coefficient of the single pole of the OPE between the worldsheet supercurrent $G(z)$ and the vertex $\mathcal{O}_j(x|z)$. It yields the following form for the vertex in the picture 0:

$$\tilde{\mathcal{O}}_j(x|z) = \left(J(x|z) + \frac{2}{\hat{k}}\psi(x|z)\psi_a(z)\mathcal{D}_x^a + \frac{2}{\hat{k}}\psi(x|z)\chi_a(z)\mathcal{K}_x^a \right)\mathcal{O}_j(x|z), \tag{18}$$

where $J(x|z)=-J^{+}(z)+2xJ^{3}(z)-x^2J^{-}(z)$. From this, we see that the computation of the three-point amplitude $\langle \mathcal{O}_{j_1}(x_1|z_1)\mathcal{O}_{j_2}(x_2|z_2)\tilde{\mathcal{O}}_{j_3}(x_3|z_3)\rangle$ also requires to compute correlators of the form $i\varepsilon_{cd}^{f}\langle \psi^{a}(z_1)\psi^{b}(z_2)\psi^{c}(z_3)\psi^{d}(z_3)\rangle$ as well as correlators that involve the insertion of current operator $J^{a}(z_3)$.

According to (14), worldsheet operators $\mathcal{O}_{j_i}(x_i|z_i)$ are associated with operators $O_{h_i}(x_i)$ in the boundary CFT. The relation between $SL(2,\mathbb{R})$ spin j and $SU(2)$ spin j' in (17) is such that the bosonic contribution to the conformal dimension corresponding to the $AdS_3\times S^3$ piece[3] of the σ-model gives $\Delta_{sl(2)}+\Delta_{su(2)}=0$.

In Refs. 1 and 2, three-point functions of chiral operators (17) were shown to agree with three-point functions in the symmetric product at the orbifold point. In Refs. 2 and 4, the computation of all the cases is discussed in detail. The worldsheet three-point function of chiral operators (17) [with one of them written in the picture 0, as in (18)] takes the form

$$\langle \mathcal{O}_{j_1}(0|0)\mathcal{O}_{j_2}(1|1)\tilde{\mathcal{O}}_{j_3}(\infty|\infty)\rangle_{\text{worldsheet}} = \hat{k}^2|2-h_1-h_2-h_3|^2 C(-h_2,-h_3,-h_4), \tag{19}$$

where $C(j_2,j_3,j_4)$ is given by the product of $SL(2,\mathbb{R})$ and $SU(2)$ structure constants, namely, $C(j_2,j_3,j_4)=C_{sl(2)}(j_2,j_3,j_4)C_{su(2)}(-1-j_2,-1-j_3,-1-j_4)$; that is,

$$C(j_2,j_3,j_4) \sim X_k(0,j_2,j_3,j_4)Y_{k-4}(0,-1-j_2,-1-j_3,-1-j_4). \tag{20}$$

When all the pieces are brought together, and after some manipulation, expression (19) can be seen to agree with the three-point functions of the boundary theory.[26,27] This agreement exhibited by bulk and boundary observables is exact, and several steps through the computations combine in such a subtle form that no doubt remains about this is a highly nontrivial check of AdS/CFT conjecture. The roles played by the picture-changing operator in the three-point function and by the precise normalization of the two-point functions are crucial ingredients in the calculation. Nevertheless, the most striking feature in the calculation is, so far, the fact that all the dependences that mix the momenta j_i in the three-point function $C_{sl(2)}(j_2,j_3,j_4)$ cancel out against analogous dependences coming from $C_{su(2)}(-1-j_2,-1-j_3,-1-j_4)$. In the next subsections we will review these cancellations and, more interestingly, we will explain why this fact does not confront the

[3]Also notice that the relation between k' and k is such that the total central charge of the worldsheet theory saturates $c=3k/(k-2)-3k'/(k'+2)+9=3+6/\hat{k}+3-6/\hat{k}+4+5=15$, where the contribution of the T^4 factor and of the free fermions were included.

analytic relation that exists between $SL(2,\mathbb{R})$ and $SU(2)$ structure constants.

B. Cancellations in the supersymmetric three-point function

Let us consider three-point amplitudes of chiral states in type IIB string theory in $AdS_3\times S^3 \times T^4$. The bosonic part corresponding to the six-dimensional piece $AdS_3\times S^3$ is the nontrivial contribution here. It is given by correlation functions of vertex operators $\Phi_j(x|z)\Psi_{-1-j}(x|z)$, which are the product of correlation functions in the $SL(2,\mathbb{R})_{\hat{k}+2}$ model and correlation functions in the $SU(2)_{\hat{k}-2}$ model, provided the relations $\hat{k}=k-2=k'+2$ and $j_i=-1-j_i'$.

Three-point function in $SL(2,\mathbb{R})_k$ WZNW model is obtained from (3) by taking the limit $j_1 \to 0$. This yields

$$X_k(0,j_2,j_3,j_4)=\frac{(\nu(b))^{j_2+j_3+j_4+1}}{2\pi^2\gamma(b^2)b^4}\frac{G_k(1+j_2+j_3+j_4)}{G_k(-1)}\prod_{i=2}^{4}\frac{G_k(j_2+j_3+j_4-2j_i)}{G_k(2j_i+1)}, \tag{21}$$

as we wrote in (11). On the other hand, knowing that $\beta=b$, and being aware that if x is a positive integer then the following identity holds:

$$P_{k'}(x)=\prod_{n=1}^{x}\gamma(n\beta^2)=\frac{G_k(-1)}{G_k(-1-x)}, \quad x\geq 1,$$

we obtain the explicit form for the three-point function in the $SU(2)_{k'}$ WZNW model,

$$Y_{k'}(0,-1-j_2,-1-j_3,-1-j_4)=\frac{\sqrt{\gamma(b^2)}G_k(-1)}{G_k(1+j_2+j_3+j_4)}\prod_{i=2}^{4}\frac{\sqrt{\gamma(1+b^2(2j_i+1))}G_k(2j_i+1)}{G_k(j_2+j_3+j_4-2j_i)}. \tag{22}$$

Rewriting this in a more convenient way and putting both (21) and (22) together, we find

$$X_k(0,j_2,j_3,j_4)Y_{k'}(0,-1-j_2,-1-j_3,-1-j_4)=\frac{1}{2\sqrt{\pi}}\prod_{i=2}^{4}\sqrt{B(j_i)}, \tag{23}$$

where $B(j)$ is given by the $SL(2,\mathbb{R})_k$ reflection coefficient,

$$\langle\Phi_{j_1}(x_1|0)\Phi_{j_1}(x_2|1)\rangle=|x_{12}|^{4j_1}B(j_1), \quad \text{with } B(j)=(\nu(b))^{2j+1}\frac{1}{\pi b^2}\gamma(1+(2j_i+1)b^2).$$

From (23) we observe that the contributions that mixed the momenta j_i have disappeared. Functions G_k coming from both $SL(2,\mathbb{R})_k$ and $SU(2)_{k'}$ factors cancel against each other, yielding a rather simplified factorized form. Therefore, we have reproduced the computation of Refs. 1 and 2 in a very succinct way, showing that the three-point function of chiral states in $AdS_3\times S^3\times T^4$ simplifies in such a way that the dependence of the momenta appears completely factorized.

Nevertheless, it is worth mentioning that the way we obtained (23) is not particularly useful, as it is almost the same that working out the expressions for both $SU(2)_{k'}$ and $SL(2,\mathbb{R})_k$ structure constants directly, as in Refs. 1, 2, and 28. However, what does represent an actual advantage is looking at the four-point function in terms of this minimal gravity representation [see (47) below].

C. Two relations between $SL(2,\mathbb{R})_k$ and $SU(2)_k$ structure constants

We have just seen that in the supersymmetric theory, the three-point functions of the $SL(2,\mathbb{R})_k$ model and that of $SU(2)_{k'}$ model are related by

$$X_k(0,j_2,j_3,j_4) \sim \frac{\prod_{i=2}^{4}\sqrt{B(j_i)}}{Y_{k'}(0,-1-j_2,-1-j_3,-1-j_4)}, \tag{24}$$

with $k'+2=k-2$. That is, all the contributions that mix the momenta j_i in (3) and (13) disappeared in (23). As mentioned, this striking simplification yielding the factorized form (24) is crucial to find agreement between bulk and boundary observables.

Expression (24) is due to the relations $j_i'=-1-j_i$ and $k-2=k'+2$. Roughly speaking, (24) expresses that supersymmetric $SL(2,\mathbb{R})_k$ structure constants are the *inverse* of supersymmetric $SU(2)_{k'}$ ones, provided the precise relations between j_i' and j_i. In turn, (24) is analog to the relation between three-point functions in GMM and three-point functions in LFT.[7]

Then, a natural question arises: Does not this inverse proportionality relation confront the fact that one can analytically continue the expressions from $SU(2)_k$ to get its noncompact analog $SL(2,\mathbb{R})_k$ (instead of its inverse)? That is, naively one would expect to find the expression for $SL(2,\mathbb{R})_k$ correlators by reversing the sign of k in the formulas for $SU(2)_k$ and performing some analytic extension; getting something like

$$X_k(0,j_2,j_3,j_4) \sim Y_{-k}(0,j_2,j_3,j_4)\prod_{i=2}^{4}\sqrt{B(j_i)}. \tag{25}$$

We will see in Sec. III D that this is actually the case. That is, one can analytically continue the expressions and prove a relation like (25). We emphasize that this is not in contradiction with relation (24) as it is commonly asserted.

D. Analytic continuation in *k* and the bosonic three-point function

Three-point functions in minimal models coupled to Liouville gravity were computed by Dotsenko in Ref. 29. When going through the computation of these correlators, which is based on the Coulomb gas approach, one needs to make sense of expressions typically given by formal products of the form

$$\prod_{n=1}^{x} f(n) \tag{26}$$

for negative values of the upper index x. We will see below that similar expressions appear when trying to extend the $SU(2)_k$ structure constants for negative values of j_i' and k. In order to propose a reasonable extension for products like (26) when $x<0$, one can start by noticing that for positive x it holds

$$\Pi_f(x) = \prod_{n=1}^{x} f(n) = \frac{\prod_{n=1}^{\infty} f(n)}{\prod_{n=x+1}^{\infty} f(n)} = \frac{\prod_{n=1}^{\infty} f(n)}{\prod_{n=1}^{\infty} f(n+x)}. \tag{27}$$

After that, in a quite natural way, the following extension for the $\Pi_f(x)$ function with negative argument is proposed,[29]

$$\Pi_f(-x) = \prod_{n=0}^{x-1} f^{-1}(-n). \tag{28}$$

Now, consider this analytic extensions for the products $P_k(x)$ standing in (13). It yields

$$\prod_{n=1}^{-|l|}\gamma(nb^2)=b^{4(|l|-1)}\frac{\gamma(|l|)}{\gamma(|l|b^2)}\prod_{n=1}^{+|l|}\gamma(nb^2) \tag{29}$$

being l an integer and where we used $\gamma(x)\gamma(1-x)=1$, $\gamma(1+x)=-x^2\gamma(x)$. This permits to make sense of the following expression:

$$\prod_{n=-|l|}^{+|l|}\gamma(nb^2)=\gamma(-|l|)b^{-4|l|-2}, \tag{30}$$

which will be rederived later in an alternative way.

Now, let us use (28) to show how the $SL(2,\mathbb{R})_k$ structure constants can be obtained by analytic extension of the $SU(2)_{k'}$ quantities, provided the relation $k'=-k$. Although it might seem we have already shown this, it is worth noticing that what we showed before is something slightly different: We showed that, if $k'+2=k-2$ and $j_i'=-1-j_i$, then the $SL(2,\mathbb{R})_k$ structure constants are inversely proportional to $SU(2)_{k'}$ structure constants.

To derive $SL(2,\mathbb{R})_k$ structure constants from (13), we assume $k=-k'$, and then write

$$P_{k'}(x)=\prod_{n=1}^{x}\gamma(n\beta^2)=\prod_{n=1}^{x}\gamma(-nb^2)=\prod_{n=1}^{x}\gamma^{-1}(1+nb^2),$$

since now $\beta^2=1/(k'+2)=-b^2=-1/(k-2)$ (instead of $\beta^2=+b^2$ as before). According to (28), for $x<0$ we have

$$P_{k'}(x)=\frac{\gamma(|x|b^2)}{\Gamma(0)}\left(\prod_{n=1}^{|x|}\gamma(nb^2)\right)^{-1}=\frac{G_k(-1-|x|)}{\Gamma(0)G_k(-1)}\gamma(|x|b^2),\quad x<0. \tag{31}$$

That is, if $k=-k'$ and $x<0$ we get $P_{-k}(x)\sim G_k(x-1)\gamma(-xb^2)/G_k(-1)$, while if $k-2=k'+2$ and $x>0$, we get something different like $P_{-k}(x)\sim G_k(-1)/G_k(-1-x)$. Using expression (31) and $G_k(x)=G_k(-1+x)\gamma(-xb^2)$, one finds[4]

$$\begin{aligned}Y_{-k}(0,j_2,j_3,j_4)=&\frac{\sqrt{\gamma(-b^2)}G_k(-1)(\nu(b))^{j_2+j_3+j_4+2}}{b^3\sqrt{\pi^3}B(j_2)B(j_3)B(j_4)}\frac{G_k(1+j_2+j_3+j_4)}{G_k(-1)}\\&\times\frac{G_k(-j_2+j_3+j_4)G_k(j_2-j_3+j_4)G_k(j_2+j_3-j_4)}{G_k(2j_2+1)G_k(2j_3+1)G_k(2j_4+1)}.\end{aligned} \tag{32}$$

It is instructive to compare (32) with (11). This realizes (25), and this relation between $X_k(0,j_2,j_3,j_4)$ and $Y_{k'}(0,j_2,j_3,j_4)$ is somehow *the inverse* of that we found between (21) and (22).

In Sec. III E we will rederive relation (25) in a different way. In particular, it will allow us to show how the Coulomb gas representation emerges from the analytic extension of $Y_k(0,j_2,j_3,j_4)$ to negative values of k and j_i. In other words, we will show that this relation between $SL(2,\mathbb{R})$ and $SU(2)$ WZNW models is nothing but the same sort of analytic continuation that one considers in the free field representation of nonrational theories.

E. The Coulomb gas approach and Wakimoto representation

Here, we will reconsider the problem of how to recover $SL(2,\mathbb{R})_k$ structure constants from (13). That is, we want to obtain

[4]Here, we have omitted a divergent $\Gamma(0)$ factor; see discussion below.

$$\left\langle \prod_{i=2}^{4} \Phi_{j_i}(x_i|z_i) \right\rangle_{sl(2)} = \prod_{i<j} |x_{ij}|^{2(j_i+j_j-j_k)} |z_{ij}|^{-2(\Delta_i+\Delta_j-\Delta_k)} C_{sl(2)}(j_2,j_3,j_4),$$

with $C_{sl(2)}(j_2,j_3,j_4)=X_k(0,j_2,j_3,j_4)$, starting from the expression for $Y_k(0,j_2,j_3,j_4)$ in the $SU(2)$ case. So, let us consider the quantity

$$Y_{-k}(0,j_2,j_3,j_4)\prod_{i=2}^{4}\sqrt{B(j_i)} \sim (\nu(b))^s \gamma(-b^2)\prod_{i=2}^{4}\gamma(1+b^2(2j_i+1)) \times \frac{P_{-k}(s)P_{-k}(j_2+j_3-j_4)P_{-k}(j_2-j_3+j_4)P_{-k}(-j_2+j_3+j_4)}{P_{-k}(2j_2)P_{-k}(2j_3)P_{-k}(2j_4)}, \tag{33}$$

where $s=j_2+j_3+j_4+1$, and where the symbol $\sim$ stands for the omission of irrelevant b-dependent factors. Let us be also reminded of the definition $P_{-k}(x)=\prod_{n=1}^{x}\gamma(-nb^2)$ with $b^{-2}=k-2$. Notice also that a divergent factor $\Gamma(0)$ arises in (33), although we are omitting it here. This factor stands for the integration over the zero mode in the integral realization,[30] i.e., it corresponds to the factor $\Gamma(-n)=\Gamma(-2j_1)$ in (8), see (10). This factor is eventually cancelled out by another contribution $\Gamma^{-1}(0)$ arising when analytically extending expression (33); see (45) below.

The first step in rewriting (33) will be to consider the three factors of the form

$$\frac{P_{-k}(j_2+j_3+j_4-2j_a)}{P_{-k}(2j_a)} = \frac{\prod_{r=1}^{j_2+j_3+j_4-2j_a}\gamma(-b^2r)}{\prod_{r=1}^{2j_a}\gamma(-b^2r)}. \tag{34}$$

Let us write them by splitting the product. In turn, at least formally, we can write

$$\frac{P_{-k}(j_2+j_3+j_4-2j_a)}{P_{-k}(2j_a)} = \prod_{r=1}^{2j_a}\gamma^{-1}(-b^2r)\prod_{r=1}^{2j_a}\gamma(-b^2r)\prod_{r=2j_a+1}^{j_2+j_3+j_4-2j_a}\gamma(-b^2r) = \prod_{r=2j_a+1}^{j_2+j_3+j_4-2j_a}\gamma(-b^2r).$$

Again, let us split the product, basically extending what would be valid for the case $2j_a+1 < -2j_a-1 < j_2+j_3+j_4-2j_a$. Then, we write

$$\frac{P_k(j_2+j_3+j_4-2j_a)}{P_k(2j_a)} = \prod_{r=2j_a+1}^{-2j_a-1}\gamma(-b^2r)\prod_{r=-2j_a}^{j_2+j_3+j_4-2j_a}\gamma(-b^2r).$$

Now, we can replace the products $\prod_{r=-x}^{x}\gamma(-b^2r)$ appearing in the expression above by the quantity $(-b^2)^{-2x-1}\gamma(-x)$, using[5]

$$\prod_{r=2j_a+1}^{-2j_a-1}\gamma(-b^2r) = (-b^2)^{4j_a+1}\gamma(2j_a+1). \tag{35}$$

Then, (33) would take the form

$$Y_{-k}(0,j_2,j_3,j_4)\prod_{i=2}^{4}\sqrt{B(j_i)} \sim (\nu(b))^s\gamma(-b^2)(-b^2)^{4(1-s)}\prod_{a=2}^{4}\frac{\gamma(2j_a+1)}{\gamma(-b^2(2j_a+1))} \times \prod_{r=1}^{s}\gamma(-b^2r)\prod_{b=2}^{4}\prod_{r=-2j_b}^{j_2+j_3+j_4-j_b}\gamma(-b^2r). \tag{36}$$

[5]It follows from prescription (28), but it can be also heuristically motivated as follows: First consider the expansion $\prod_{r=-x}^{x}\gamma(-b^2r)=(\Gamma(b^2x)\Gamma(b^2(x-1))\cdots\Gamma(b^2)\Gamma(0)\Gamma(-b^2)\cdots\Gamma(-b^2x))/(\Gamma(1-b^2x)\Gamma(1-b^2(x-1))\cdots\Gamma(1-b^2)\Gamma(1)\Gamma(1+b^2)\cdots\Gamma(1+b^2x))$. Then, using $\Gamma(x+1)=x\Gamma(x)$ and replacing $\Gamma(0)=(-1)^{-x}\Gamma(-x)\Gamma(x+1)$, one finds (35).

By manipulating Γ-functions, we get

$$Y_{-k}(0,j_2,j_3,j_4)\prod_{i=2}^{4}\sqrt{B(j_i)} \sim (\nu(b))^s\gamma(-b^2)\frac{\gamma(j_2-j_3-j_4)}{\gamma(2j_2+1)} \times \frac{(-1)^s\mathcal{I}_k}{\pi^s\gamma^s(b^2)\Gamma(-s)\Gamma(1+s)}\prod_{a=2}^{4}\frac{\gamma(2j_a+1)}{\gamma(-b^2(2j_a+1))}, \tag{37}$$

where we have defined

$$\mathcal{I}_k = \Gamma(-s)\Gamma(s+1)\pi^s(-1)^s(-b^2)^{2s}(\gamma(b^2))^s\prod_{r=1}^{s}\gamma(-b^2r)\times\prod_{r=0}^{s-1}(\gamma(1-b^2(r-2j_2))\gamma(-b^2(r-2j_3)) \times\gamma(-b^2(r-2j_4))). \tag{38}$$

The reason why we preferred to write the expression for $Y_{-k}(0,j_2,j_3,j_4)\Pi_{i=2}^4\sqrt{B(j_i)}$ in its form (37) and (38) is that $\mathcal{I}_k$ can be identified as the contribution coming from a Dotsenko–Fateev integral,[12,31,32]

$$\mathcal{I}_k = \Gamma(-s)\prod_{r=1}^{s}\int d^2w_r\prod_{r=1}^{s}|w_r|^{4j_2b^2}|1-w_r|^{4j_3b^2-2}\prod_{r<t}^{s-1,s}|w_r-w_t|^{-4b^2}. \tag{39}$$

This follows from formula (B.9) of the Appendix of Ref. 22.

It is worth noticing that integral (39) is precisely the one that arises in the Wakimoto free field representation of three-point functions.[31] For instance, the exponent of $|1-w_r|^{-2+4j_3b^2}$ in (39) can be thought of as coming from the Wick contraction between a $SL(2,\mathbb{R})_k$ vertex operator and the rth screening operator in the Coulomb gas representation. The contributions $|w_r|^{+4j_2b^2}$ indicate the presence of highest weight states of discrete representations in the correlator.

Wakimoto free field representation follows from the considering the action

$$S[\phi,\beta,\gamma;\lambda] = \frac{1}{4\pi}\int d^2z(\partial\phi\bar{\partial}\phi - bR\phi/2\sqrt{2} + \bar{\beta}\,\partial\,\bar{\gamma} + \beta\bar{\partial}\gamma + 4\pi\lambda\beta\bar{\beta}e^{-\sqrt{2}b\phi}), \tag{40}$$

where λ is an arbitrary constant, $\beta(z)$ and $\gamma(z)$ form a commutative ghost system, and $\phi(z)$ is a boson field with background charge $-b=-1/\sqrt{k-2}$.[33] The nonvanishing propagators are

$$\langle\beta(w)\gamma(z)\rangle = \frac{1}{(w-z)}, \quad \langle\phi(w)\phi(z)\rangle = -2\log|w-z|. \tag{41}$$

In the large ϕ regime, which corresponds to the near boundary limit in AdS_3 space, the vertex operators take the form

$$\Phi_{j_i,m_i,\bar{m}_i}(z_i) = \gamma_{(z_i)}^{j_i+m_i}\gamma_{(\bar{z}_i)}^{j_i+\bar{m}_i}e^{\sqrt{2}j_ib\phi(z_i)} + B(j_i)\gamma_{(z_i)}^{-1-j_i+m_i}\gamma_{(\bar{z}_i)}^{-1-j_i+\bar{m}_i}e^{-\sqrt{2}(j_i+1)b\phi(z_i)} + \cdots,$$

with

$$\Phi_{j_i,m_i,\bar{m}_i}(z_i) = \int d^2x_i\Phi_{j_i}(x_i|z_i)x_i^{j_i+m_i}\bar{x}_i^{j_i+\bar{m}_i} \tag{42}$$

for $i=2,3,4$. On the other hand, the screening operators come from the perturbation term in (40), taking the form

$$\mathcal{S}(w_r) = \lambda\beta_{(w_r)}\bar{\beta}_{(w_r)}e^{-\sqrt{2}b\phi(w_r)}, \tag{43}$$

$r=1,2,\ldots,s$, with $s=j_2+j_3+j_4+1$.

This representation yields the integral expression (39) through the Wick contractions standing in

$$\lambda^s\Gamma(-s)\prod_{r=2}^{s}\int d^2w_r\langle\prod_{i=2}^{4}\gamma_{(z_i)}^{j_i+m_i}\bar{\gamma}_{(\bar{z}_i)}^{j_i+\bar{m}_i}e^{\sqrt{2}bj_i\phi(z_i)}\prod_{r=2}^{s}\beta_{(w_r)}\bar{\beta}_{(\bar{w}_r)}e^{-\sqrt{2}b\phi(w_r)}\rangle_{\lambda=0}=\lambda^s\mathcal{I}_k,$$

where the average $\langle\cdots\rangle_{\lambda=0}$ is the functional sum for the action (40) with $\lambda=0$.

The precise relation between three-point functions in the m-basis and those in the x-basis is discussed in Ref. 32. There, expressions like the right hand side of (37) were shown to lead to exact result[34] through analytic continuation. Moreover, in Ref. 32 [see Eqs. (2.45) and (2.63) therein], it was discussed how the Dotsenko–Fateev integral (39) could be formally continued to be also expressed in terms of special functions as follows:

$$\mathcal{I}_k=b^2\pi^s(\gamma(b^2))^s\gamma(-1-j_2-j_3-j_4)\gamma(2j_2+1)\gamma(-j_2-j_3+j_4)\gamma(-j_2+j_3-j_4)$$
$$\times\frac{G_k(-2-j_2-j_3-j_4)}{G_k(-1)}\prod_{a=2}^{4}\frac{G_k(-1-j_2-j_3-j_4+2j_a)}{G_k(-2j_a-1)}. \tag{44}$$

The way of proposing expression (44) is completely analog to what Zamolodchikov and Zamolodchikov did for LFT in Ref. 35, where the exact expression for Liouville structure constants was obtained from the analytic continuation of the formula of the residues corresponding to resonant correlators. Considering such analytic continuation, we can replace the piece

$$(-b^2)^{2s}\prod_{r=1}^{s}\gamma(-b^2r)\prod_{r=0}^{s-1}\gamma(1-b^2(r-2j_2))\gamma(-b^2(r-2j_3))\gamma(-b^2(r-2j_4))$$
$$=\frac{(-1)^s}{\Gamma(-s)\Gamma(s+1)\pi^s\gamma^s(b^2)}\mathcal{I}_k,$$

arising in (37), by the following contribution:

$$-\frac{(-b^2)^{-2s+1}\gamma(-1-j_2-j_3-j_4)\gamma(-j_2-j_3+j_4)\gamma(-j_2+j_3-j_4)}{\Gamma(0)}$$
$$\times\frac{\gamma(2j_2+1)G_k(-2-j_2-j_3-j_4)}{G_k(-1)}\prod_{a=2}^{4}\frac{G_k(-1-j_2-j_3-j_4+2j_a)}{G_k(-2j_a-1)}, \tag{45}$$

where the factor $\Gamma^{-1}(0)$ arises from writing $(-1)^{-s}\Gamma(-s)\Gamma(s+1)=\Gamma(0)$. As anticipated, this factor precisely cancels the divergent factor $\Gamma(-2j_1)=\Gamma(0)$ standing from evaluating $j_1=0$ in (8). Taking into account functional properties (4), one finds

$$Y_{-k}(0,j_2,j_3,j_4)\prod_{i=2}^{4}\sqrt{B(j_i)}\sim(\nu(b))^{j_2+j_3+j_4+1}\frac{G_k(1+j_2+j_3+j_4)}{G_k(-1)}\prod_{a=2}^{4}\frac{G_k(j_2+j_3+j_4-2j_a)}{G_k(2j_a+1)}.$$

That is, we recovered $SL(2,\mathbb{R})_k$ structure constants from the expression for $SU(2)_{k'}$ model with $k'=-k$; namely, $Y_{-k}(0,j_2,j_3,j_4)\Pi_{i=2}^{4}\sqrt{B(j_i)}\sim C_{sl(2)}(j_2,j_3,j_4)$. This is nothing but (25), what we proved in Sec. III D by means of relation (31).

IV. DISCUSSION

We have explained how the fact that three-point superstring amplitudes of chiral states in $AdS_3\times S^3$ lead to a factorized expression does not confront the fact that formulas of $SL(2,\mathbb{R})_k$ WZNW model can be obtained from those of $SU(2)_{k'}$ WZNW model by analytically continuing in k. This turns out to be related to the shifting of the Kac-Moody level k in the supersymmetric theory: While in the bosonic theory, an appropriate analytic continuation of $SU(2)_{k'}$ correlators leads to the expression of $SL(2,\mathbb{R})_k$ correlators (with $k'=-k$), in the supersymmetric theory both observables are, roughly speaking, one the inverse of the other (with $k'+2=k-2$). In this sense, it is fair to say that the computation in the superstring theory is more similar to the one in bosonic MLG than the one in bosonic WZNW model itself. It is the magic of supersymmetry that is behind the cancellation in the three-point function, and not merely the similarity between the Liouville theory and the $SL(2,\mathbb{R})$ WZNW theory. This cancellation in the three-point function is the key point for the matching between bulk and boundary observables,[1,2] and this was the motivation to revisit this calculation herein.

Before concluding, let us make some comments on the four-point function. First, let us recall the relation between Liouville momenta α_i and the spin variable j_i in the $SL(2,\mathbb{R})_k$ WZNW model ($k-2=b^{-2}$), namely,

$$a_1 = -\frac{b}{2}(j_1+j_2+j_3+j_4+1), \quad a_i = -\frac{b}{2}(j_1+2j_i-j_2-j_3-j_4-b^{-2}-1)$$

for $i=2,3,4$. On the other hand, the relation between the GMM momenta α_i and the $SU(2)_{k'}$ WZNW model ($k'+2=\beta^{-2}$) spin variables j'_i is the following:

$$\alpha_1 = \frac{\beta}{2}(j'_1,j'_2,j'_3,j'_4+1), \quad \alpha_i = \frac{\beta}{2}(j'_1+2j'_i-j'_2-j'_3-j'_4+\beta^{-2}-1)$$

for $i=2,3,4$. Then, talking into account that in the supersymmetric theory $k'+2=k-2$ and that chiral states obey $j_i=-1-j'_i$, we find

$$a_i = \alpha_i + b \tag{46}$$

for the five states $i=1,2,3,4,5$. Remarkably, (46) is exactly the relation between the momenta α_i and a_i in MLG, as it is necessary for the vertex operators $V_{a_i}\times W_{\alpha_i}$ to have conformal dimension one with respect to the full stress tensor $T_{\text{Liouville}}+T_{\text{minimal model}}$. In turn, restrictions on the momenta in the supersymmetric correlators in $AdS_3\times S^3\times T^4$ agree with requirements for conformal invariance in the MLG.

Using (46) we can show that the expression for the bosonic part of the worldsheet four-point functions $\langle \mathcal{O}_{j_1}\mathcal{O}_{j_2}\mathcal{O}_{j_3}\mathcal{O}_{j_4}\rangle$ simplifies in a remarkable way. Recalling

$$P_k(x) = \prod_{n=1}^{x} \gamma(n\beta^2) = \frac{G_k(-1)}{G_k(-1-x)}, \quad x>0,$$

and taking into account $j'_i=-1-j_i$ (for $i=1,2,3,4$), we can write the $SU(2)_{k'}$ four-point function as follows:

$$Y_{k'}(-1-j_1,-1-j_2,-1-j_3,-1-j_4) = \frac{(\gamma(b^2))^{-2j_1-1}}{G_k\left(2+\sum_{a=2}^{4} j_a\right)} \prod_{n=1}^{4} \frac{G_k(2j_n+1)}{\sqrt{\gamma(-(2j_n+1)b^2)}} \times \frac{1}{\prod_{n=2}^{4} G_k\left(-1-2j_n-j_1+\sum_{i=2}^{4} j_i\right)}. \tag{47}$$

Considering both (3) and (47) together, the final expression reads[6]

[6]Notice that there exists a remarkable similarity between this expression and Eq. (23).

$$\mathcal{X}_k(j_1,j_2,j_3,j_4)\mathcal{Y}_{k'}(-1-j_1,-1-j_2,-1-j_3,-1-j_4) = \frac{C_W^2}{\Upsilon_0^2}\frac{|z|^2|1-z|^2}{|x|^2|1-x|^2|z-x|^2} \times \frac{\pi^3}{b^{3+4b^2}}\prod_{i=1}^{4}\frac{\sqrt{B(j_i)}}{\gamma(2ba_i-b)},$$

where we have chosen $\mu\pi\gamma^2(b^2)b^{4-2b^2}=1$. Although the computation of worldsheet four-point function, in addition, would require to deal with the insertion of picture-changing operators in $\langle \mathcal{O}_{j_1}\mathcal{O}_{j_2}\tilde{\mathcal{O}}_{j_3}\tilde{\mathcal{O}}_{j_4}\rangle$, it is still encouraging that the bosonic piece of the correlator $\langle \mathcal{O}_{j_1}\mathcal{O}_{j_2}\mathcal{O}_{j_3}\mathcal{O}_{j_4}\rangle$ yields a very simple form in terms of MLG five-point functions. In fact, one gets

$$\left\langle \prod_{i=1}^{4}\Phi_{j_i}(x_i|z_i)\right\rangle_{sl(2)} \times \left\langle \prod_{i=1}^{4}\Psi_{-1-j_i}(x_i|z_i)\right\rangle_{su(2)} = \frac{\pi^3}{b^{3+4b^2}}\frac{C_W^2}{\Upsilon_0^2}\prod_{i=1}^{4}\frac{\sqrt{B(j_i)}}{\gamma(2ba_i-b)} \times \frac{|z|^2|1-z|^2}{|x|^2|1-x|^2|z-x|^2}\left\langle \prod_{i=1}^{5}U_{a_i}(z_i)\right\rangle_{\text{MLG}}, \tag{48}$$

where $\langle \Pi_{i=1}^5 U_{a_i}(z_i)\rangle_{\text{MLG}}$ on the right hand side refers to the five-point correlation function in MLG; that is,

$$\left\langle \prod_{i=1}^{5}U_{a_i}(z_i)\right\rangle_{\text{MLG}} = \left\langle \prod_{i=1}^{5}V_{a_i}(z_i)\right\rangle_{\text{LFT}} \times \left\langle \prod_{i=1}^{5}W_{a_i-b}(z_i)\right\rangle_{\text{GMM}}, \tag{49}$$

with $z_2=0$, $z_3=1$, $z_4=\infty$, while $z_1=z$, $z_5=x$. It is worth mentioning that N-point correlation numbers in MLG were recently computed[18,36,37] for particular values of $N-3$ of the N momenta a_i. Therefore, the fact one has access to these observables makes relation (48) quite interesting. For instance, one could raise the question whether holographic agreement for extremal four-point functions in $AdS_3\times S^3\times T^4$ is also observed as it happens in $AdS_5\times S^5$. To answer this kind of questions, we have to learn more about the nonrenormalization mechanism and, more importantly, we have to get more information about the boundary four-point function. Unfortunately, four-point functions in the symmetric product to compare with are not available; a computation of these observables would be a major progress.

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