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Well posedness and smoothing effect of Schrödinger-Poisson equation

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In this work we take under consideration the Cauchy problem for the Schrödinger-Poisson type equation $i\partial_t u = -\partial_x^2 u + V(u)u - f(|u|^2)u$, where f represents a local non-linear interaction (we take into account both attractive and repulsive models) and V is taken as a suitable solution of the Poisson equation $V = 1/2|x| * (C - |u|^2)$, $C \in C_c^\infty$ is the doping profile or *impurities*. We show that this problem is locally well posed in the weighted Sobolev spaces $\mathcal{H}^s := \{\varphi \in H^s(\mathbb{R}) : \int (1+x^2)^{1/2} |\varphi|^2 < \infty\}$ with $s \geq 1$, which means the local existence, uniqueness, and continuity of the solution with respect to the initial data. Moreover, under suitable assumptions on the local interaction, we show the existence of global solutions. Finally, we establish that for $s \geq 1$ local in time and space, smoothing effects are present in the solution; more precisely, in this problem there is locally a gain of half a derivative. © 2007 American Institute of Physics. [DOI: 10.1063/1.2776844]

I. INTRODUCTION

In this paper we are mainly concerned with a kind of smoothing effect which is present in the solutions of the one-dimensional (1D) Schrödinger-Poisson equation. In addition, and since our local existence analysis is developed under weaker assumptions on the decay at infinity of the solutions, we will also consider the related well-posedness problem.

Our starting point is the one-dimensional (1D) (unscaled) Schrödinger-Poisson problem

$$i\partial_t u = -\partial_x^2 u + V(u)u - f(|u|^2)u, \quad x \in \mathbb{R}, \quad (1.1)$$

$$u(x, 0) = \phi(x), \quad (1.2)$$

$$\partial_x^2 V = C - |u|^2, \quad x \in \mathbb{R}. \quad (1.3)$$

Here, $C(x)$ denotes the fixed positively charged background ions or *impurities* (will be referred to as *the doping profile* in the sequel) and it is assumed to be a (positive) regular function with compact support (for further details in semiconductor models, see Ref. 12 and references therein). The term $f(|u|^2)u$ represents a local interaction which is intended to take into account the Pauli exclusion principle for fermions (see Refs. 18, 11, and 6). Among the several models proposed to include the *exchange effects* of charged particles (say, electrons), we may cite the Schrödinger-Poisson- $X\alpha$ model $f(|u|^2)u = \alpha|u|^{2/N}u$ (notice that in dimension 1, the $X\alpha$ model yields the “focusing cubic nonlinear Schrödinger” (NLS)); actually, one rather considers $|u|^q u$ with a subcritical exponent q (which in one dimension reads $0 < q < 4$) (for further details on the Schrödinger-Poisson- $X\alpha$ model, see Refs. 13 and 1 and references therein). Before going into the details on the coupling with the Poisson equation, let us mention that our results allow rather general interactions, say, $f \in C^\infty(\mathbb{R})$ (in the focusing case, we also impose a subcritical exponent condition). Let

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$W(x)$ be any fundamental solution of the Poisson equation in Eq. (1.3); thus the solution must be written as a Hartree-type potential $V=W(x) * (C-|u|^2)$. Equation (1.1) becomes

$$i\partial_t u = -\partial_x^2 u + (W(x) * (C - |u|^2))u - f(|u|^2)u, \quad x \in \mathbb{R}. \tag{1.4}$$

The H^1 theory of the Cauchy problem given by Eq. (1.4) is widely developed in the work of Cazenave.³ However, the general result presented there (see Theorem 3.3.1) does not apply in lower dimensions (1 or 2) due to the fact that Green’s function $W(x)$ is unbounded and therefore both the kernel $W(x)$ and the related “external” potential $(W * C)(x)$ are not contained in $L^p + L^\infty$ for any p (see Examples 3.2.1 and 3.2.8).

The well posedness of the 1D Schrödinger-Poisson equation without the doping profile $C(x)$ was given first by Steinrück¹⁹ [who also neglected the local interaction $f(|u|^2)u$] and then by Stimming²⁰ (who adapted the proof given by the former in order to include the exchange potential) by means of semigroup theory using $\Sigma := \{\phi \in H^1 : x\phi \in L^2\}$ as a work space. In addition they discussed the related semiclassical limit (which falls into the Wigner-functions approach, which is out of the scope of our work). Nevertheless, any kind of smoothing effect is taken under consideration.

Finally, the choice of Green’s function $W(x)$ deserves some comments. Following Ref. 3, Example 3.2.8, the kernel of the Hartree-type potential is chosen to be an even function and this leads to

$$V = \frac{|x|}{2} * (C - |u|^2). \tag{1.5}$$

Despite the fact that other choices are also used in the literature, [for instance, Zhang *et al.*²⁴ took $W(x) = x + |x|$], in our work this symmetry is crucial in the choice of both the operator and the work space, and therefore in the improvement of the decay at infinity assumption. More precisely, from decomposition $V = b(\phi)(1+x^2)^{1/2} + V_\infty$ [where $b(\phi)$ is a constant which depends on the size of initial data ϕ], the (linear) operator is taken as $-\partial_x^2 + b(\phi)(1+x^2)^{1/2}$ defined in $\mathcal{H}^1 = \{\varphi \in H^1 : \int (1+x^2)^{1/2} |\varphi|^2 < +\infty\}$. Since for $b(\phi) > 0$ the associated quadratic form is positive, this operator indeed generates a semigroup. Moreover, both $\|\cdot\|_{\mathcal{H}^1}$ and the related norm are equivalent, from where \mathcal{H}^1 appears as the energy space associated to this operator. This shows that \mathcal{H}^1 could be seen as a *natural* space for the problem in Eqs. (1.1), (1.2), and (1.5). Furthermore, we are also interested in showing a kind of smoothing effect which, roughly speaking, can be expressed as *to gain half a derivative*, so we need to investigate the well posedness of this problem in the spaces $\mathcal{H}^s := \{\varphi \in H^s : \int (1+x^2)^{1/2} |\varphi(x)|^2 dx < \infty\}$, with $s \geq 1$. This means the local existence, uniqueness, and continuity of the solution with respect to the initial data.

Let us now turn to the smoothing effect. From the mathematical point of view, the Schrödinger equation appears as a delicate problem, since it possesses a mixture of properties of parabolic and hyperbolic equations. Indeed, it is *almost reversible* and it has conservation laws and also some dispersive properties such as the Klein-Gordon equation, but it has infinite speed of propagation. On the other hand, the Schrödinger equation has a kind of smoothing effect shared by parabolic problems but the *time reversibility* prevents it from generating an analytic semigroup. Despite the fact that the expression *smoothing effect* is used when referring to the loss of singularity (e.g., Strichartz estimates²¹), in this work, will denote the gain of derivatives. The first result in this direction (concerning dispersive equations) was given by Kato,⁹ who showed that the solutions of the 1D Korteweg-de Vries equation $\partial_t w + \partial_x^3 w + w\partial_x w = 0$ satisfy

$$\int_{-T}^T \int_{-R}^R |\partial_x w(x,t)|^2 dx dt \leq C(T,R, \|w(x,0)\|_{L^2}),$$

which means that the solution $w(x,t)$ gains (locally in time and space) one derivative.

The corresponding version of the above estimate for the free Schrödinger group $\{e^{it\partial_x^2}\}_{t \in \mathbb{R}}$,

$$\int_{-T}^T \int_{-R}^R |(1 - \partial_x^2)^{1/4} e^{it\partial_x^2} \phi_0|^2 dx dt \leq C(T, R, \|\phi_0\|_{L^2}), \tag{1.6}$$

was simultaneously established by Constantin and Saut,⁵ Sjölin,¹⁷ and Vega.²² A sharp version for the 1D case can be found in Ref. 10. We refer also to Ref. 2 for further developments. Actually, similar results for the Benjamin-Ono equation and the derivative NLS equation has been shown by Ponce¹⁴ and Rial,¹⁶ respectively. Anyway, in the linear Schrödinger equation, there is locally a gain of half a derivative and this is the kind of smoothing effect we will extend to other typical nonlinearities.

Main results

We begin with the local well posedness in \mathcal{H}^s of the Cauchy problem in Eqs. (1.1) and (1.2). **Theorem 3.1:** *Assume $f \in C^\infty(\mathbb{R})$ and let $s \geq 1$. Assume further $\phi \in \mathcal{H}^s$ is such that $\|\phi\|_{L^2}^2 \leq \|C\|_{L^1}$. Then there exist $T_* = T_*(\phi) > 0$, $T^* = T^*(\phi) > 0$, and a unique maximal solution $u \in C((-T_*, T^*), \mathcal{H}^s) \cap C^1((-T_*, T^*), H^{s-2})$ of the problem in Eqs. (1.1) and (1.2). u is maximal in the following sense: if $T_* < +\infty$ or $T^* < +\infty$, then $\|u(t)\|_{\mathcal{H}^s} \rightarrow +\infty$ as $t \rightarrow -T_*$ or $t \rightarrow T^*$.*

We then present a global result which is valid for local interactions with *subcritical* exponent. **Theorem 4.1:** *Assume $f \in C^\infty(\mathbb{R})$ is such that $|f(r)| \leq r^\sigma$ with $0 \leq \sigma < 2$ and let $s \geq 1$. Assume further $\phi \in \mathcal{H}^s$ is such that $\|\phi\|_{L^2}^2 \leq \|C\|_{L^1}$; then $T(\phi) = +\infty$.*

Finally, we show that some smoothing effect is present in the solution, more precisely, the following theorem.

Theorem 5.1: *Let $u(t)$ be the unique maximal solution of the problem in Eqs. (1.1) and (1.2) given by Theorem 1.1 then*

$$u \in L_{loc}^2((-T, T), H_{loc}^{s+1/2}).$$

Remark 1.1: The above result implies, for $s \in [3/2, 2)$, that the equation is realized in $L_{loc}^2((-T, T) \times \mathbb{R})$. This is a significant improvement of the result in Theorem 1.1, from which the equation holds in $C((-T, T), H^{s-2})$.

II. BASIC RESULTS

In this section we fix some notations and give a list of results which are useful in the development of our work.

A. Notation

- Bessel potential of order s : J^s .
- $J^s(u) := \mathcal{F}^{-1}(1+k^2)^{s/2} \hat{u}(k)$.
- $L^p(I, X)$, L^p functions from some interval I to some Banach space X .
- $L_\mu^2 := \{\varphi \in L^2 : \int \varphi^2 \mu < \infty\}$, where $\mu(x) := (1+x^2)^{1/2}$. We also consider $L_\mu^1 := \{\varphi \in L^1 : \int \varphi \mu < \infty\}$.
- $H^s := \{\varphi \in L^2 : J^s(\varphi) \in L^2\}$.
- $\mathcal{H}^s := H^s \cap L_\mu^2$.
- $\langle \cdot; \cdot \rangle$, complex inner product of L^2 .
- The related real inner product $\{ \cdot; \cdot \} := \text{Re} \langle \cdot; \cdot \rangle$.
- $\|\cdot\|_s$ norm in H^s . Recall that $\|\cdot\|_{L^2} = \|\cdot\|_0$, and both notations will be used.
- $\|\cdot\|_X$ norm in X , a Banach space different from H^s .
- $\|\cdot\|_{H^s}^2 := \|\cdot\|_s^2 + \|\cdot\|_{L_\mu^2}^2$ norm in H^s .

B. Results

Throughout this work we will make use of the following lemmas.

Lemma 2.1: Gagliardo-Nirenberg inequality. See Cazenave,³ Theorem 1.3.7.

Let $1 \leq p, q, r \leq \infty$, and $j, m \in \mathbb{N}$ such that $0 \leq j < m$. If $\theta \in [j/m, 1]$ is such that

$$\frac{1}{p} = \frac{j}{N} + \theta \left(\frac{1}{r} - \frac{m}{N} \right) + \frac{(1-\theta)}{q},$$

then there exists $C=C(\theta, p, q, r, j, m, N)$ such that

$$\sum_{|k|=j} \|\partial_x^k \varphi\|_{L^p} \leq C \left(\sum_{|k|=m} \|\partial_x^k \varphi\|_{L^r} \right)^\theta \|\varphi\|_{L^q}^{1-\theta},$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

Remark 2.1: Setting $p=2(\sigma+1)$, $j=0$, $m=N=1$, and $q=r=2$, standard estimates lead to

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \varepsilon \|u\|_1^2 + C(\varepsilon, \sigma) \|u\|_0^{(2\sigma+4)/(2-\sigma)}. \tag{2.1}$$

Lemma 2.2: Calderón first commutator theorem. see Ponce,¹⁴ Theorem 2.6, p. 531.

Let $\mathcal{A}: \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function and let H be the Hilbert transform

$$(H\varphi)(x) := \text{p.v.} \frac{1}{\pi} \int \frac{\varphi(y)}{x-y} dy.$$

Then the operator $[H; \mathcal{A}] \partial_x$ maps L^2 into L^2 with

$$\|[H; \mathcal{A}] \partial_x \varphi\|_0 \leq C \|\partial_x \mathcal{A}\|_{L^\infty} \|\varphi\|_0.$$

Lemma 2.3: See Coiffman and Meyer,⁴ Theorem III, p. 111. Let $\mathcal{A}: \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ function. Let b be such that $|b^{(j)}(k)| \leq C_j |k|^{-j}$ for $j=0, 1, 2, \dots$ and let B be its related pseudodifferential operator. Then the operator $[B; \mathcal{A}] \partial_x$ maps L^2 into L^2 and verifies

$$\|[B; \mathcal{A}] \partial_x \varphi\|_0 \leq C \|\partial_x \mathcal{A}\|_{L^\infty} \|\varphi\|_0.$$

Lemma 2.4: See Folland,⁷ Lemma 6.16. Let $\omega \in S$. Then there exists $C=C(\omega)$ such that

$$\|[J^{1/2}; \omega] J^{1/2} \varphi\|_0 \leq C \|\varphi\|_0,$$

for all $\varphi \in L^2$.

Lemma 2.5: Assume $s \geq 1$. Then there exists a bounded operator $B \in \mathcal{B}(L^2)$ such that

$$J^s - 1 = \partial_x \circ J^{s-1} B.$$

Proof: It suffices to show that the operator given by $\mathcal{F}((\partial_x J^{s-1})^{-1}(J^s - 1))$ belongs to $\mathcal{B}(L^2)$. Since this is a pseudodifferential operator with symbol

$$b(k) = -i \frac{(1+k^2)^{s/2} - 1}{k(1+k^2)^{(s-1)/2}}$$

and b belongs to $L^\infty(\mathbb{R})$, it follows that $B = \mathcal{F}^{-1} b \in \mathcal{B}(L^2)$. ■

The following result plays a major role in our present work since it allows to lift results from $s=1$ to $s > 1$. The proof is given for lack of a suitable reference.

Lemma 2.6: Assume $0 \leq r \leq 1$, and $k \in \mathbb{N}$. Set $\mu := (1+x^2)^{1/2}$. Then for all $\varphi \in H^k$,

$$\|[J^{k+r}; \mu] \varphi\|_0 \leq C \|\varphi\|_k.$$

Proof: Let us rely on an induction argument, since $[J^r; \mu] = J^{r-1}[J; \mu] + [J^{r-1}; \mu]J$. The inductive step is treated as follows. Let L be the linear operator given by $L := [J; \mu]$. From Lemmas 2.5, and 2.3 and standard interpolation results, it follows that $L: H^\sigma \rightarrow H^\sigma$ is a bounded operator, from where $\|J^\sigma [J; \mu] \varphi\|_0 \leq C \|\varphi\|_\sigma$ holds. On the other hand, it remains to show that $[J^r; \mu] \in \mathcal{B}(L^2)$ for

small values of r . Then take $0 \leq r \leq 1$ and let B be the pseudodifferential operator with symbol $b(k) = k^{-1}((1+k^2)^{r/2} - 1)$. Since $J^r = 1 + \partial_x B$ and b satisfies the assumptions of Lemma 2.3, the result is proved. ■

Lemma 2.7: See Ponce,¹⁴ Lemma 2.7, p. 532. Let $J^s := (1 - \partial_x^2)^{s/2}$ be the Bessel potential of order s . If $s \geq 1$ and $n = 1$, then

$$\|[J^s; \varphi]\psi\|_0 \leq C \|\partial_x \varphi\|_s \|\psi\|_{s-1}.$$

Lemma 2.8: Moser's inequality. See Ref. 15, Theorem 6.5, p. 87. Let $\mathcal{A} \in C^\infty(\mathbb{C}; \mathbb{C})$ (in the real sense) with $\mathcal{A}(0) = 0$; then there is a smooth function $\tilde{A}: [0, +\infty) \rightarrow [0, +\infty)$ so that for all $u \in H^s$,

$$\|A(u)\|_s \leq \tilde{A}(\|u\|_{L^\infty}) \|u\|_s.$$

Previous results on harmonic analysis yield the following quite remarkable property of the Schrödinger group $\{e^{it(\partial_x^2 - A\mu)}\}_{t \in \mathbb{R}}$.

Lemma 2.9: For each (fixed) $A > 0$ the Schrödinger group $\{e^{it(\partial_x^2 - A\mu)}\}_{t \in \mathbb{R}}$ is well defined in \mathcal{H}^s and satisfies

- $\|e^{it(\partial_x^2 - A\mu)} \phi\|_0 = \|\phi\|_0$ and
- $\|e^{it(\partial_x^2 - A\mu)} \phi\|_{\mathcal{H}^s} \leq (1 + C(A, \mu)t^{|s|-1}) \|\phi\|_{\mathcal{H}^s}$.

Proof: Set $T_A := -\partial_x^2 + A\mu$ and $U_A(t) := e^{-itT_A}$. Since T_A is a real operator it follows the conservation of charge. Set now $H_A(\phi) := 1/2 \|\partial_x \phi\|_{L^2}^2 + A/2 \|\phi\|_{L^2}^2$; a direct computation yields the conservation of H_A : $H_A(U_A(t)\phi) = H_A(\phi)$. Since $A > 0$ it is easy to see that there exist constants $C_1(A), C_2(A)$ such that

$$C_1(A)H_A(\phi) \leq \|\phi\|_{H^1}^2 \leq C_2(A)H_A(\phi). \tag{2.2}$$

This leads to the estimate $\|U_A(t)\phi\|_{\mathcal{H}^1}^2 \leq C(A)\|\phi\|_{\mathcal{H}^1}^2$, and the result is true for $s = 1$.

As it was stated before, Lemma 2.6 will be used to lift previous result to $s > 1$. Setting $\varphi := U_A(t)\phi$, one has

$$\frac{1}{2} \partial_t \{J^s \varphi; J^s \varphi\} = \{\partial_t \varphi; J^s J^s \varphi\} = \{i\partial_x^2 \varphi; J^s J^s \varphi\} - \{iA\mu \varphi; J^s J^s \varphi\} = -\{iA[J^s; \mu] \varphi; J^s \varphi\}. \tag{2.3}$$

From Lemma 2.6, the estimate, $\|[J^s; \mu]\varphi\|_0 \leq C\|\varphi\|_{s-1}$ follows, which leads to $\partial_t \|\varphi\|_s^2 \leq CA\|\varphi\|_s \|\varphi\|_{s-1}$ and therefore to $\partial_t \|\varphi\|_s \leq CA\|\varphi\|_{s-1}$. The result follows from standard inductive argument. ■

Consider now the case $A = 0$ and the related group $U_0 := e^{it\partial_x^2}$.

Remark 2.2: In this case, the L_μ^2 norm control given by $\|U_A(t)\phi\|_{L_\mu^2}^2 \leq 1/A \|\phi\|_{H^1}^2 + \|\phi\|_{L_\mu^2}^2$ is lost.

Lemma 2.10: $U_0(t)$ is well defined in \mathcal{H}^s and verifies

- $\|U_0(t)\phi\|_s = \|\phi\|_s$, valid for all $s \in \mathbb{R}$, and
- $\|U_0(t)\phi\|_{L_\mu^2} \leq C(t, \mu) \|\phi\|_{H^1 \cap L_\mu^2}$.

Proof: The first assertion follows immediately from $[J^s; \partial_x^2] = 0$. From

$$\begin{aligned} \|U_0(t)\phi\|_{L_\mu^2}^2 &\leq \|\phi\|_{L_\mu^2}^2 + 2 \int_0^t |\{iU_0(t')\phi; \partial_x(\mu)\partial_x U_0(t')\phi\}| dt' \\ &\leq \|\phi\|_{L_\mu^2}^2 + 2\|\partial_x \mu\|_\infty \int_0^t \|U_0(t')\phi\|_0 \|\partial_x U_0(t')\phi\|_0 dt' \\ &\leq \|\phi\|_{L_\mu^2}^2 + Ct \|\partial_x \mu\|_\infty \|\phi\|_1^2, \end{aligned} \tag{2.4}$$

the second claim follows. ■

Finally, the continuous dependence on the initial data requires some continuity of the family U_A with respect to the parameter A which is given by the following lemma.

Lemma 2.11: Let $T > 0$ and $\phi \in \mathcal{H}^s$ be fixed. The map $U_{(\cdot)}(t)\phi: [0; +\infty) \rightarrow C([0; T]; \mathcal{H}^s)$ is continuous.

Proof: Assume first $\phi \in \mathcal{D}(\mathbb{R})$, and let $g(t) = U_B(-t)U_A(t)\phi - \phi$. Taking time derivatives yields $g'(t) = U_B(-t)(iT_B - iT_A)U_A(t)\phi$, where $T_A = -\partial_x^2 + A\mu$ (see Lemma 2.9). Since $g(0) = 0$ and $T_A - T_B = (A - B)\mu$, it follows that $g(t) = -i(A - B)\int_0^t U_B(-\tau)\mu U_A(\tau)\phi d\tau$. Taking the norm in \mathcal{H}^s and using that $U_B(\cdot)$ is unitary, one has the estimate $\|g(t)\|_{\mathcal{H}^s} \leq |A - B|\int_0^t \|\mu U_A(\tau)\phi\|_{\mathcal{H}^s} d\tau$. The assumption $\phi \in \mathcal{D}(\mathbb{R})$ ensures that $\mu U_A(\tau)\phi \in \mathcal{H}^s$ for any $\tau \in [0; T]$, then there exists a constant $C(T, \phi)$ such that

$$\sup_{t \in [0; T]} \|g(t)\|_{\mathcal{H}^s} \leq |A - B|C(T, \phi).$$

Taking into account the identity $U_A(t)\phi - U_B(t)\phi = U_B(t)g(t)$, the general result follows from a $\varepsilon/3$ argument. ■

III. WELL POSEDNESS OF THE CAUCHY PROBLEM

This section is concerned with the local existence of the Cauchy problem

$$i\partial_t u = -\partial_x^2 u + V(u)u - f(|u|^2)u, \tag{3.1}$$

$$u(x, 0) = \phi(x), \tag{3.2}$$

posed in the weighted Sobolev Spaces $\mathcal{H}^s = H^s \cap L^2_\mu$ with $s \geq 1$, where the local interaction satisfies $f \in C^\infty(\mathbb{R})$, the doping profile $C \in C_c^\infty$, and $V(u)$ is given by

$$V(u) = \frac{|x|}{2} * (C - |u|^2). \tag{3.3}$$

Setting

$$V_\infty(u) := \int \frac{|x - y| - \mu(x)}{2} (C(y) - |u(y)|^2) dy, \tag{3.4}$$

$$A(u) := \|C\|_{L^1} - \|u\|_0^2, \tag{3.5}$$

the potential V shall be written as

$$V(u) = \frac{1}{2}\mu(x)A(u) + V_\infty(u). \tag{3.6}$$

Some remarks are listed below.

Remark 3.1: Since $\mathcal{H}^s \hookrightarrow L^2_\mu$, both potentials V_∞ and V are well defined.

Remark 3.2: Since $V_\infty \in L^\infty$ (see Lemma 3.1 below) if the initial data ϕ are such that $A(\phi) \neq 0$, then $V(u) = \mathcal{O}(|x|)$ for $|x| \rightarrow +\infty$.

In the sequel we will consider initial data ϕ such that $A(\phi) \geq 0$. The special case given by $A(\phi) = 0$ leads to the identity $V_\infty = V$; therefore such potential is bounded. Both instances $A(\phi) > 0$ and $A(\phi) = 0$ will be called, respectively, *the subcritical and critical cases*.

A. Local existence. Critical and subcritical cases

Since the results of this subsection are obtained by means of the fixed point techniques, some estimates are needed.

Lemma 3.1: Let $\varphi \in \mathcal{H}^s$ and let V_∞ be given by Eq. (3.4). Then the following properties hold:

- $\|V_\infty(\varphi)\|_{L^\infty} \leq \|C - |\varphi|^2\|_{L^1_\mu}$.

- $\|V_\infty(\varphi_1) - V_\infty(\varphi_2)\|_{L^\infty} \leq \| |\varphi_1|^2 - |\varphi_2|^2 \|_{L^1} \leq (\|\varphi_1\|_{L^2_\mu} + \|\varphi_2\|_{L^2_\mu}) \|\varphi_1 - \varphi_2\|_{L^2_\mu}$.
- $\|\partial_x V_\infty(\varphi)\|_{\mathcal{H}^s} \leq C(\|C\|_{L^1}, \|\varphi\|_{\mathcal{H}^{s-1}}) \|C - |\varphi|^2\|_{\mathcal{H}^s}$.
- $\|\partial_x V_\infty(\varphi_1) - \partial_x V_\infty(\varphi_2)\|_{\mathcal{H}^s} \leq C(\|\varphi_1\|_{\mathcal{H}^{s-1}}, \|\varphi_2\|_{\mathcal{H}^{s-1}}) \|\varphi_1 - \varphi_2\|_{\mathcal{H}^s}$.

Proof: From $|x-y| - \mu(x) \leq \mu(y)$ and $|u|^2 - |v|^2 = \bar{u}(u-v) + v(\bar{u}-\bar{v})$, the first and second assertions follow.

Take $j \in H^s \cap L^1_\mu$ and consider $1/2 := \int [x-y| - \mu(x)] j(y) dy$. Taking derivatives with respect to x yields

$$\partial_x V(j) = \int_{-\infty}^x j(y) dy - \frac{1 + \mu'}{2} \int_{y \in \mathbb{R}} j(y) dy. \tag{3.7}$$

Consider now $x \rightarrow -\infty$, then $1 + \mu'(x) \leq 1/2x^2$, which is in L^2_μ . In addition,

$$\begin{aligned} \int_{-\infty}^0 \mu(x) \left| \int_{-\infty}^x j(y) dy \right|^2 dx &\leq \int_{-\infty}^0 \left(\int_{-\infty}^x \int_{-\infty}^x \mu(x) |j(y)| |j(z)| dy dz \right) dx \\ &\leq \int_{-\infty}^0 \int_z^0 \int_{-\infty}^0 \mu(y) |j(y)| |j(z)| dy dx dz \\ &\leq \int_{-\infty}^0 \int_{-\infty}^0 \mu(z) \mu(y) |j(y)| |j(z)| dy dz \\ &\leq \|j\|_{L^1_\mu((-\infty; 0])}^2. \end{aligned}$$

Since similar results hold for $x \rightarrow +\infty$, one has the estimate $\|\partial_x \mathcal{V}(j)\|_{L^2_\mu} \leq C \|j\|_{L^1_\mu}$.

In order to prove the Lipschitz property, consider $s \geq 1$. From Lemma 2.5 there exists a pseudodifferential operator B -which satisfies the assumptions of Lemma 2.3 such that $J^s = 1 + B\partial_x J^{s-1}$. It follows that $J^s(\partial_x \mathcal{V}(j)) = \partial_x \mathcal{V}(j) + B J^{s-1} (j - \frac{1}{2} \mu'' \int_{y \in \mathbb{R}} j(y) dy)$, and this yields

$$\begin{aligned} \|\partial_x V(j)\|_s &\leq \|\partial_x V(j)\|_{L^2} + \|B\|_{B(L^2)} (\|j\|_{s-1} + C(\mu) \|j\|_{L^1_\mu}), \\ &\leq C(s) \|j\|_{H^{s-1} \cap L^1_\mu}. \end{aligned}$$

The third and fourth claims are obtained by taking, respectively, $j = C - |\phi|^2$ and $j = |\phi_1|^2 - |\phi_2|^2$. ■

The following conservation law will be useful in the sequel.

Lemma 3.2: Charge conservation. Let N be a real function and $u \in C((a, b), H^s) \cap C^1((a, b), H^{s-2})$ a solution of $i\partial_t u = -\partial_x^2 u + N(u)u$, with $s \geq 1$. Then, $\|u(\cdot, t)\|_{L^2}$ is a constant.

Proof: Taking the time derivative, it follows that

$$\partial_t \|u(t)\|^2 = \{\partial_t u; u\} = \{i\partial_x^2 u; u\} + \{iN(u)u; u\} = -\{i\partial_x u; \partial_x u\} = 0.$$

Since N is real we get $\text{Re}(iN(u)) = 0$. On the other hand, since $s \geq 1$ the boundary term in the integration by parts vanishes. ■

In the sequel we introduce the following auxiliary problem where $A \geq 0$ is fixed:

$$i\partial_t u = -\partial_x^2 u + A\mu u + V_\infty(u)u - f(|u|^2)u, \tag{3.8}$$

$$u(x, 0) = \phi(x). \tag{3.9}$$

Property 3.1: Assume that $f \in C^\infty(\mathbb{R})$ and let $s \geq 1$. If $\phi \in \mathcal{H}^s$, then there exist $T = T(\phi) > 0$ and a unique maximal solution $u \in C([0, T(\phi)), \mathcal{H}^s) \cap C^1([0, T(\phi)), H^{s-2})$ of the problem in Eqs. (3.8) and (3.9). u is maximal in the following sense: if $T(\phi) < +\infty$ then $\|u(t)\|_{\mathcal{H}^s} \rightarrow +\infty$ as $t \rightarrow T(\phi)$.

Proof: Let $U_A(t)$ be the unitary group generated by the operator $i\partial_x^2 - iA\mu$ and let F_A be the operator yielded by Duhamel's formula

$$F_A(u) = U_A(t)\phi - i \int_0^t U_A(t-s)(V_\infty(u)u - f(|u|^2)u)ds. \tag{3.10}$$

Thus the fixed point problem is given by $F_A(u)=u$. In the sequel, it will be shown that F is well defined and is a contraction in \mathcal{H}^s (from now on the subscript will be omitted).

From Lemma 2.9, $\|F(u)\|_{\mathcal{H}^s} \leq C(t)(\|\phi_0\|_{\mathcal{H}^s} + \|V_\infty(u)u - f(|u|^2)u\|_{\mathcal{H}^s})$. On the other hand, Lemma 2.8 and embedding $H^1 \hookrightarrow L^\infty$ lead to $\|f(|u|^2)u\|_s \leq \tilde{f}(\|u\|_1)\|u\|_s$ and $\|f(|u|^2)u\|_{L^\infty_\mu} \leq \|f(|u|^2)\|_{L^\infty}\|u\|_{L^\infty_\mu}$. This yields

$$\|f(|u|^2)u\|_{H^s} \leq C(f, \|u\|_1)\|u\|_{H^s}. \tag{3.11}$$

Since $J^s V_\infty(u)u = [J^s; V_\infty(u)]u + V_\infty(u)J^s u$, Lemmas 2.7 and 3.1 give the estimate

$$\|V_\infty(u)u\|_s \leq C(\|u\|_{s-1})\|u\|_s + C(\|u\|_{H^1})\|u\|_s.$$

In addition the L^∞_μ norm estimate $\|V_\infty(u)u\|_{L^\infty_\mu} \leq \|V_\infty(u)\|_{L^\infty}\|u\|_{L^\infty_\mu}$ leads to

$$\|V_\infty(u)u\|_{H^s} \leq C(\|u\|_{H^{s-1}})\|u\|_{H^s}. \tag{3.12}$$

Therefore, estimates (3.11) and (3.12) show well the definition of F .

Set now $w(u) := V_\infty(u)u - f(|u|^2)u$, since $F(u) - F(v) = -i \int_0^t U_A(t-s)(w(u) - w(v))ds$. Taking $\|\cdot\|_{\mathcal{H}^s}$ and applying Lemma 2.9 leads to the estimate

$$\|F(u) - F(v)\|_{H^s} \leq C(A, t)\|w(u) - w(v)\|_{H^s}.$$

Thus, it only remains to show that the nonlinear term $w(u)$ is a locally Lipschitz function in \mathcal{H}^s . Take then $u, v \in \mathcal{H}^s$ such that $\|u\|_{\mathcal{H}^s}, \|v\|_{\mathcal{H}^s} \leq R$.

Since

$$\begin{aligned} f(|u|^2)u - f(|v|^2)v &= \int_0^1 \frac{d}{ds}(f(|z(s)|^2)z(s))ds \\ &= \int_0^1 f(|z(s)|^2)(u - v)ds + \int_0^1 2f'(|z(s)|^2)z(s)\text{Re}(z(s)(u - v))ds, \end{aligned}$$

where $z(s) = su + (1-s)v$, Lemma 2.8 leads to

$$\|f(|u|^2)u - f(|v|^2)v\|_s \leq C(f, R)\|u - v\|_s.$$

Since $f(\cdot) \in C^\infty$, it follows that $\|f(|u|^2) - f(|v|^2)\|_{L^\infty_\mu} \leq C(f, R)\|u - v\|_{L^\infty_\mu}$. A suitable arrangement of terms leads to

$$\begin{aligned} \|f(|u|^2)u - f(|v|^2)v\|_{L^\infty_\mu} &\leq \|f(|u|^2)(u - v)\|_{L^\infty_\mu} + \|(f(|u|^2) - f(|v|^2))v\|_{L^\infty_\mu} \\ &\leq \|f(|u|^2)\|_{L^\infty}\|u - v\|_{L^\infty_\mu} + \|v\|_{L^\infty}\|f(|u|^2) - f(|v|^2)\|_{L^\infty_\mu} \\ &\leq C(f, R)\|u - v\|_{L^\infty_\mu}. \end{aligned}$$

Therefore,

$$\|f(|u|^2)u - f(|v|^2)v\|_{H^s} \leq C(f, R)\|u - v\|_{H^s}. \tag{3.13}$$

By taking into consideration the identity

$$J^s(V_\infty(u)u - V_\infty(v)v) = [J^s; V_\infty(u) - V_\infty(v)]u + (V_\infty(u) - V_\infty(v))J^s u + [J^s; V_\infty(v)](u - v) + V_\infty(v)J^s(u - v)$$

and using Lemma 3.1, one might bound each term separately as follows:

$$\|[J^s; (V_\infty(u) - V_\infty(v))]u\|_0 \leq C\|\partial_x V_\infty(u) - \partial_x V_\infty(v)\|_s \|u\|_0 \leq C(R)\|u - v\|_s,$$

$$\|(V_\infty(u) - V_\infty(v))J^s u\|_0 \leq C\|V_\infty(u) - V_\infty(v)\|_{L^\infty} \|u\|_s \leq C(R)\|u - v\|_s,$$

$$\|[J^s; V_\infty(v)](u - v)\|_0 \leq \|\partial_x V_\infty(u)\|_s \|u - v\|_s \leq C(R)\|u - v\|_s,$$

$$\|V_\infty(v)J^s(u - v)\|_0 \leq \|V_\infty(v)\|_{L^\infty} \|u - v\|_s \leq C(R)\|u - v\|_s.$$

Similar estimates as above lead to $\|V_\infty(u)u - V_\infty(v)v\|_{L^2_\mu} \leq C(R)\|u - v\|_{L^2_\mu}$, and this gives

$$\|V_\infty(u)u - V_\infty(v)v\|_{H^s} \leq C(\mu, R)\|u - v\|_{H^s}. \tag{3.14}$$

Therefore, collecting estimates (3.13) and (3.14), the Lipschitz property of the nonlinear term $V_\infty(u)u - f(|u|^2)u$ is shown. ■

The following lemma yields the continuous dependence of the iteration process given by F_A with respect to the parameter A .

Lemma 3.3: The map $F_{(\cdot)}(\cdot): [0; +\infty) \times C([0; T]; \mathcal{H}^s) \rightarrow C([0; T]; \mathcal{H}^s)$ given by Eq. (3.10) is continuous.

Proof: It is a direct consequence of Lemma 2.11 and estimates (3.11)–(3.14). ■

The previous result leads to the well posedness of the Cauchy problem in Eqs. (3.1) and (3.2) when initial data satisfy $A(\phi) \geq 0$.

Theorem 3.1: Let $f \in C^\infty(\mathbb{R})$ and let $s \geq 1$. For every $\phi \in \mathcal{H}^s$, such that $\|\phi\|_0^2 \leq \|C\|_{L^1}$, there exists $T = T(\phi) > 0$ and there exists a unique, maximal solution $u \in C([0, T(\phi)), \mathcal{H}^s) \cap C^1([0, T(\phi)), H^{s-2})$ of the problem in Eqs. (3.1) and (3.2). u is maximal in the sense that if $T(\phi) < +\infty$, then $\|u(t)\|_{\mathcal{H}^s} \rightarrow +\infty$ as $t \rightarrow T(\phi)$.

Proof: Let ϕ be such that $A(\phi) \geq 0$ and take $A = \frac{1}{2}A(\phi)$. Applying Proposition 3.1 we get a solution $u \in C([0, T(\phi)), \mathcal{H}^s) \cap C^1([0, T(\phi)), H^{s-2})$ of the problem in Eqs. (3.8) and (3.9). From the conservation law of Lemma 3.2, it follows that $A(u) = A(\phi)$; finally, the definition of $A(u)$ and $V_\infty(u)$ [see Eqs. (3.5) and (3.4)] show that $V(u) = A\mu + V_\infty(u)$, from where we conclude that u is a solution of the problem in Eqs. (3.1) and (3.2). ■

Remark 3.3: Since the function $v(x, t) := \bar{u}(x, -t)$ solves the problem in Eqs. (3.1) and (3.2) in $(-T(\phi), 0]$, the related backward problem is also local in time well posed.

Remark 3.4: Since $A(\cdot)$ depends only on the initial data, uniqueness follows directly from the uniqueness of the auxiliary problem in Eqs. (3.8) and (3.9). On the other hand, the continuous dependence of u with respect to the initial data follows from Lemma 3.3.

B. Conservation laws

The problem in Eqs. (3.1) and (3.2) presents a conserved quantity, which is given by the energy functional (see Cazenave³)

$$E(\varphi) = \frac{1}{2}\|\partial_x \varphi\|^2 + \frac{1}{2}\{V(\varphi)\varphi; \varphi\} + \frac{1}{4}\left\{\left(\frac{|x|}{2} * |\varphi|^2\right)\varphi; \varphi\right\} - \frac{1}{2}\int_{x \in \mathbb{R}} F(|\varphi^2|), \tag{3.15}$$

where $F' = f$ and $F(0) = 0$.

Lemma 3.4: Conservation of energy. Let $A \geq 0$, $s \geq 1$, and $f \in C^\infty$. If $u \in C((-T_*, T^*), \mathcal{H}^s) \cap C^1((-T_*, T^*), H^{s-2})$ is the unique solution of the problem in Eqs. (3.1) and (3.2), then $E(u(t)) = E(u(0))$ for all $t \in (-T_*, T^*)$.

Proof: Let $\varphi \in H^s$; since $s \geq 1$, it follows that $\partial_x \varphi \in L^2$. From embedding $H^1 \hookrightarrow L^\infty$, it follows that $\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{H^1}$ and this leads to $|\int_{x \in \mathbb{R}} F(|\varphi|^2)| \leq \|f\|_{L^\infty[0; \|\varphi\|_1]} \|\varphi\|_0^2$.

From identity (3.6), one has $|\{V(\varphi)\varphi; \varphi\}| \leq A/2\|\varphi\|_{L^2}^2 + \|V_\infty(\varphi)\|_{L^\infty} \|\varphi\|_0^2$.

Remark 3.5: the previous argument also yields $|\{(|x|^\mu * |\varphi|^2)\varphi; \varphi\}| \leq \|\varphi\|_{L^\mu}^2 \|\varphi\|_0^2$.

This shows that E is a well defined functional in \mathcal{H}^s .

Taking into consideration identity (3.3) and calling $\mathcal{U}(x) = |x|/2 * \mathcal{C}$, one might write

$$E(\varphi) = \frac{1}{2}\|\partial_x \varphi\|^2 + \frac{1}{2}\{\mathcal{U}(x)\varphi; \varphi\} - \frac{1}{4}\left\{\left(\frac{|x|}{2} * |\varphi|^2\right)\varphi; \varphi\right\} - \frac{1}{2}\int_{x \in \mathbb{R}} F(|\varphi|^2). \tag{3.16}$$

A straightforward computation yields

$$\begin{aligned} \partial_t E = & \{-\partial_x^2 u; \partial_t u\} + \{\mathcal{U}(x)u; \partial_t u\} - \frac{1}{2}\left\{\left(\frac{|x|}{2} * |u|^2\right)u; \partial_t u\right\} - \frac{1}{4}\left\{\partial_t \left(\frac{|x|}{2} * |u|^2\right)u; u\right\} \\ & - \int_{x \in \mathbb{R}} f(|u|^2)\text{Re}(u\partial_t \bar{u})dx. \end{aligned}$$

On the other hand, since $\int_{x \in \mathbb{R}} f(|u|^2)\text{Re}(u\partial_t \bar{u})dx = \{f(|u|^2)u; \partial_t u\}$, one may write

$$\partial_t E = \{\partial_x^2 u + L(u); \partial_t u\} - \frac{1}{4}\left\{\partial_t \left(\frac{|x|}{2} * |u|^2\right)u; u\right\} + \frac{1}{2}\left\{\left(\frac{|x|}{2} * |u|^2\right)u; \partial_t u\right\}.$$

The first term vanishes since u solves the equation; the remaining terms are equal since $|x|$ is an even function. This shows that E is constant along the trajectory $u(t)$. ■

IV. GLOBAL EXISTENCE IN \mathcal{H}^s

This section is devoted to establishing a global existence result for the problem in Eqs. (3.1) and (3.2).

Since the existence of global solutions is strongly related with the critical exponent (see Weinstein²³), some control on the local interaction must be made. This suggests the following two basic assumptions:

$$f > 0, \quad f(r) \leq r^\sigma \quad \text{with } 0 \leq \sigma < 2 \text{ [assumption (A1)]}, \tag{4.1}$$

$$f < 0, \quad f \in C(\mathbb{R}) \text{ [assumption (A2)]}. \tag{4.2}$$

Remark 4.1: Since $f < 0$ means repulsive interaction, no further assumption shall be made. On the opposite, since the local interaction is given by $f(|u|^2)u$, the case $\sigma=2$ corresponds to the critical exponent (see Refs. 8 and 23). Therefore, in the attractive model, assumption (A1) means subcritical exponent.

Theorem 4.1: Assume that assumption (A1) or (A2) holds. Let $A \geq 0$, $\phi \in \mathcal{H}^s$, and $s \geq 1$ and let $u \in C((-T_*, T^*), \mathcal{H}^s) \cap C^1((-T_*, T^*), H^{s-2})$ be the unique (local) solution of the problem in Eqs. (3.1) and (3.2); then $T^* = T^* = +\infty$ (i.e., u is globally defined in \mathcal{H}^s).

Proof: Take $s=1$ and let $u \in C((-T, T), H^1) \cap C^1((-T, T), H^{-1})$ be the (local) solution of the problem in Eqs. (3.1) and (3.2). Using that both terms $\{(|x|/2 * |u|^2)u; u\}$ and $A\{\mu u; u\}$ are non-negative, one has

$$E(\phi) = E(u) \geq \frac{1}{2}\|\partial_x u\|_0^2 + \frac{1}{2}\{V_\infty(u)u; u\} - \frac{1}{2}\int_{x \in \mathbb{R}} F(|u|^2).$$

Thus, if assumption (A1) holds, one has

$$\left| \frac{1}{2} \int_{x \in \mathbb{R}} F(|u|^2) \right| \leq \frac{1}{\sigma + 1} \|u\|_{L^{2\sigma+2}}^{2\sigma+2}, \tag{4.3}$$

By means of estimate (2.1) (see the Gagliardo-Nirenberg inequality), this yields

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \varepsilon \|u\|_1^2 + C(\varepsilon, \sigma) \|u\|_0^{(2\sigma+4)/(2-\sigma)}, \tag{4.4}$$

which, together with Lemma 3.1, leads to

$$\frac{1}{2} \|\partial_x u\|_0^2 \leq E(\phi_0) + \|\mu C\|_{L^1} \|u\|_0^2 + \|u\|_{L^\mu}^2 \|u\|_0^2 + \varepsilon \|u\|_1^2 + C(\varepsilon, \sigma) \|u\|_0^{(2\sigma+4)/(2-\sigma)}.$$

Taking $\varepsilon < 1/2$ and using the conservation law of Lemma 2 one has the following estimate:

$$\|u\|_1 \leq C(\phi)(1 + \|u\|_{L^\mu}). \tag{4.5}$$

On the other hand, following assumption (A2), the term $1/2 \int_{x \in \mathbb{R}} F(|u|^2)$ is non-negative and therefore the previous estimate also holds in this case.

Take $t \in [0, T)$; since $\|u(\cdot, t)\|_{L^2}^2 = \|\phi\|_{L^2}^2 + \int_0^t d/dt' (\|u(t')\|_{L^2}^2) dt'$ and u is a solution of Eq. (3.1), it follows that $\|u(\cdot, t)\|_{L^\mu}^2 = \|\phi\|_{L^\mu}^2 + 2 \int_0^t \{i u \partial_x^2 u; \mu u\} dt'$; note that $V(u)$ and $f(|u|^2)$ are real functions so their related terms vanish, this leads to

$$\begin{aligned} \|u(t)\|_{L^\mu}^2 &\leq \|\phi\|_{L^\mu}^2 + 2 \int_0^t |\{iu; \partial_x \mu \partial_x u\}| dt' \leq \|\phi\|_{L^\mu}^2 + 2 \|\partial_x \mu\|_\infty \int_0^t \|u(t')\|_0 \|\partial_x u(t')\|_0 dt' \\ &\leq \|\phi\|_{L^\mu}^2 + 2 \|\partial_x \mu\|_\infty \|\phi_0\|_0 \int_0^t \|\partial_x u(t')\|_0 dt'. \end{aligned}$$

Now, estimate (4.5) leads to $\|u(t)\|_{L^2}^2 \leq C(\phi)(1 + t + \int_0^t \|u(t')\|_{L^2} dt')$. A standard ordinary differential equation argument yields $\|u(t)\|_{L^2} \leq C(\phi)(1 + t^2)$.

Since $\|u(t)\|_{\gamma^s}^2 = \|u(t)\|_{L^2}^2 + \|u(t)\|_s^2$, it will be suffice to estimate $\|u(t)\|_s$ for $s > 1$. Moreover, in view of the inductive nature of estimate (3.12), it is enough to use induction on $[s]$.

Set first $1 < s \leq 2$. Including the nonlinear terms $(V_\infty(u) + f(|u|^2))u$ into identity (2.3) and using estimates (3.11) and (3.12), one has

$$\|u(t)\|_s^2 \leq C(\phi)(1 + t^2) \int_0^t \|u(t')\|_s dt' + C(\phi, \|u\|_1)(1 + t^2) \int_0^t \|u(t')\|_s^2 dt'.$$

Hence, Gronwall's lemma yields $\|u(t)\|_s \leq C(t, \|\phi_0\|_{\gamma^s})$.

Let now $2 \leq k \in \mathbb{N}$ and set $k < s \leq k+1$. The same computations as before yield

$$\|u(t)\|_s^2 \leq C(\phi)(1 + t^2) \int_0^t \|u(t')\|_s dt' + C(\phi, \|u\|_k)(1 + t^2) \int_0^t \|u(t')\|_s^2 dt'.$$

From the inductive step, one has $\|u(t)\|_k \leq C(t, \|\phi_0\|_{\gamma^k})$ and, therefore, the result proceeds from Gronwall's lemma. ■

V. SMOOTHING EFFECT

In this section we establish the kind of smoothing effect which is present in the solution of the Cauchy problem

$$i \partial_t u = - \partial_x^2 u + V(u)u - f(|u|^2)u, \tag{5.1}$$

$$u(x, 0) = \phi(x), \tag{5.2}$$

where $f \in C^\infty(\mathbb{R})$, $C \in C_c^\infty(\mathbb{R})$, and $V(u) := |x|/2 * (C - |u|^2)$. Such smoothing effect is detailed in the following theorem.

Theorem 5.1: Assume $s \geq 1$ and $\|\phi\|_{L^2}^2 \leq \|C\|_{L^1}$. Let $u(t)$ be the unique, maximal solution of Eqs. (5.1) and (5.2) provided by Theorem 3.1 then

$$u \in L_{loc}^2((-T, T), H_{loc}^{s+1/2}).$$

Remark 5.1: The above result implies that $u(t) \in H_{loc}^{s+1/2}$ for a.e. $t \in (-T, T)$. Thus $u(t) \in C_{loc}^{s-\varepsilon}$ for a.e. $t \in (-T, T)$ and any $\varepsilon > 0$. Moreover, for $s \in [3/2, 2)$ we have that the equation is realized in $L_{loc}^2((-T, T) \times \mathbb{R})$. This is a significant improvement of the result in Theorem (3.1), from which the equation holds in $C((-T, T), H^{s-2})$.

Remark 5.2: This kind of smoothing effect represents an extension (local in space and time) of the result in the linear case given by Constantin and Saut, Sjölin, and Vega in Refs. 5, 17, and 22.

Proof: The proof will be given in several steps.

First step. Let $R \in \mathbb{R}$ and T_1 such that $[-T_1; T_1] \subseteq (-T_*(\phi); T^*(\phi))$ and take $\omega \in C_c^\infty$ such that $\text{supp}(\omega) \subseteq [-R, R]$; then

$$\int_{-T_1}^{T_1} \int |\omega J^{s+1/2} u|^2 \leq \|\omega\|_{L^\infty} \int_{-T_1}^{T_1} \int_{-R}^R |J^{s+1/2} u|^2,$$

and this shows that one is allowed to restrict to functions $\omega \in C_c^\infty$ such $0 \leq \omega \leq 1$ and $\omega \equiv 1$ in $[-R, R]$.

The following identity will be useful in the sequel:

$$\begin{aligned} \{\omega J^{s+1/2} u; J^{s+1/2} u\} &= \{\omega J^{1/2}(J^s - 1)u; J^{1/2}(J^s - 1)u\} + 2\{\omega(J^s - 1)u; Ju\} + 2\{(J^s - 1)u; [J^{1/2}; \omega]J^{1/2}u\} \\ &\quad + \{\omega J^{1/2}u; J^{1/2}u\}. \end{aligned} \tag{5.3}$$

Take now $\Omega \in C^\infty$ such that $\omega := \partial_x \Omega$. Let H be the Hilbert transform (see Lemma 2.2) and let $P_\pm := 1/2(1 \pm iH)$ be the projection operators (see Rial¹⁶). We also consider the following identity:

$$\frac{1}{2} \partial_t \langle \Omega J^s P_\pm u; J^s P_\pm u \rangle = \{i\Omega J^s \partial_x^2 P_\pm u; J^s P_\pm u\} - \{i\Omega J^s P_\pm V(u)u; J^s P_\pm u\} + \{i\Omega J^s P_\pm f(u)u; J^s P_\pm u\}. \tag{5.4}$$

On the other hand, the linear term can be written as follows:

$$\{i\Omega J^s \partial_x^2 P_\pm u; J^s P_\pm u\} = \{i[\Omega; \partial_x] \partial_x J^s P_\pm u; J^s P_\pm u\} + \{i\Omega \partial_x J^s P_\pm u; \partial_x J^s P_\pm u\} = -\{i\omega \partial_x J^s P_\pm u; J^s P_\pm u\}. \tag{5.5}$$

Since the Hilbert transform satisfies $H^2 = -1$, $-HP_\pm = \pm P_\pm$, and $[H; J^s] = [H; \partial_x] = 0$, one can deduce

$$\begin{aligned} -\{i\omega \partial_x J^s P_\pm u; J^s P_\pm u\} &= \{i\omega H H \partial_x J^s P_\pm u; J^s P_\pm u\} \\ &= \{i[\omega; H] \partial_x H J^s P_\pm u; J^s P_\pm u\} + \{i\omega H \partial_x H J^s P_\pm u; J^s P_\pm u\} \\ &= \{B_1 H J^s P_\pm u; J^s P_\pm u\} \pm \{\omega H \partial_x J^s P_\pm u; J^s P_\pm u\}, \end{aligned} \tag{5.6}$$

where $B_1 := i[\omega; H] \partial_x$.

Remark 5.3: Since $[J^s - 1; P_\pm] = 0$, identities (5.4)–(5.6) are still valid if J^s is replaced by $J^s - 1$.

Remark 5.4: From Lemma 2.2 it follows that B_1 is a bounded operator in L^2 . Furthermore, one has the following estimate:

$$|\{B_1 H (J^s - 1) P_\pm u; (J^s - 1) P_\pm u\}| \leq C(\omega) \|u\|_s^2. \tag{5.7}$$

In addition, since $H\partial_x = D$ and $J = (1 + D^2)^{1/2}$, where D is the pseudodifferential operator with symbol $|k|$,

$$\begin{aligned} \{\omega H\partial_x v; v\} &= \{\omega Dv; v\} = \{\omega(D - J)v; v\} + \{\omega Jv; v\} \\ &= \{\omega(D - J)v; v\} + \{[\omega; J^{1/2}]J^{1/2}v; v\} + \{\omega J^{1/2}v; J^{1/2}v\} \\ &= \{B_2v; v\} + \{\omega J^{1/2}v; J^{1/2}v\}, \end{aligned} \tag{5.8}$$

where $B_2 := \omega(D - J) + [\omega; J^{1/2}]J^{1/2}$.

Remark 5.5: Lemma 2.5 can be adapted to show that $D - J$ is a bounded operator. On the other hand, Lemma 2.4 shows that $[\omega; J^{1/2}]J^{1/2}$ is also a bounded operator and this yields the boundedness of B_2 . In addition one has the estimate

$$|\{B_2(J^s - 1)P_{\pm}u; (J^s - 1)P_{\pm}u\}| \leq C(\omega)\|u\|_s^2. \tag{5.9}$$

Setting $v = P_{\pm}(J^s - 1)u$ in Eq. (5.8), replacing J^s by $J^s - 1$ in Eqs. (5.4)–(5.6), and after a suitable rearrangement of terms, one has

$$\begin{aligned} \{\omega J^{1/2}(J^s - 1)P_{\pm}u; J^{1/2}(J^s - 1)P_{\pm}u\} &= \pm \frac{1}{2} \partial_x \langle \Omega(J^s - 1)P_{\pm}u; (J^s - 1)P_{\pm}u \rangle \\ &\quad \mp \{B_1 H(J^s - 1)P_{\pm}u; (J^s - 1)P_{\pm}u\} \\ &\quad - \{B_2(J^s - 1)P_{\pm}u; (J^s - 1)P_{\pm}u\} \pm \text{nonlinear terms}. \end{aligned} \tag{5.10}$$

Second step. In this step it will be proven that if the result is true for (both) projection operator $P_{\pm}u$, then it is true for u .

Taking into consideration the identity

$$\{\omega J^{1/2}v; J^{1/2}v\} = \{\omega J^{1/2}P_+v; J^{1/2}P_+v\} + \{\omega J^{1/2}P_-v; J^{1/2}P_-v\} + 2\{\omega J^{1/2}P_+v; J^{1/2}P_-v\},$$

it is clear that it only remains to show how to bound the last term. Since

$$\{\omega J^{1/2}P_+v; J^{1/2}P_-v\} = \{[J^{1/2}; \omega]J^{1/2}P_+v; P_-v\} + \{\omega JP_+v; P_-v\},$$

Lemma 2.4 yields

$$|\{[J^{1/2}; \omega]J^{1/2}P_+v; P_-v\}| \leq C(\omega)\|v\|_0^2.$$

On the other hand, since

$$\{\omega JP_+v; P_-v\} = \{[P_-; \omega]JP_+v; v\},$$

Lemma 2.5 yields

$$\{\omega JP_+v; P_-v\} = \{[P_-; \omega]\partial_x BP_+v; v\} + \{[P_-; \omega]P_+v; v\},$$

which, together with the replacement $v = J^s u$, give the estimates

$$|\{[P_-; \omega]\partial_x BP_+J^s u; J^s u\}| \leq C(\omega)\|u\|_s^2,$$

$$|\{[P_-; \omega]P_+J^s u; J^s u\}| \leq C(\omega)\|u\|_s^2,$$

from where we can conclude

$$\{\omega J^{1/2}v; J^{1/2}v\} \leq \{\omega J^{1/2}P_+v; J^{1/2}P_+v\} + \{\omega J^{1/2}P_-v; J^{1/2}P_-v\} + C(\omega)\|u\|_s^2. \tag{5.11}$$

Third step. This step is concerned with the boundedness of nonlinear terms. We start with the following identities:

$$\{i\Omega(J^s - 1)P_{\pm}f(|u|^2)u; (J^s - 1)P_{\pm}u\} = \{iP_{\pm}\Omega P_{\pm}(J^s - 1)f(|u|^2)u; (J^s - 1)u\},$$

$$\{i\Omega(J^s - 1)P_{\pm}V(u)u; (J^s - 1)P_{\pm}u\} = \{iP_{\pm}\Omega P_{\pm}(J^s - 1)V(u)u; (J^s - 1)u\}.$$

Setting $\Phi := P_{\pm}\Omega P_{\pm}$ produces

$$\{i\Phi(J^s - 1)V(u)u; (J^s - 1)u\} = \{i\Phi[J^s; V(u)]u; (J^s - 1)u\} + \{i\Omega[P_{\pm}; V(u)](J^s - 1)u; (J^s - 1)P_{\pm}u\},$$

which after the replacement $V(u) = (A/2)\mu + V_{\infty}(u)$ becomes

$$\begin{aligned} \{i\Phi(J^s - 1)V(u)u; (J^s - 1)u\} &= \{i\Phi[(J^s - 1); V_{\infty}(u)]u; (J^s - 1)u\} \\ &\quad + \{i\Omega[P_{\pm}; V_{\infty}(u)](J^s - 1)u; (J^s - 1)P_{\pm}u\} \\ &\quad + \frac{A}{2}\{i\Phi[(J^s - 1); \mu]u; (J^s - 1)u\} \\ &\quad + \frac{A}{2}\{i\Omega[P_{\pm}; \mu](J^s - 1)u; (J^s - 1)P_{\pm}u\}. \end{aligned} \tag{5.12}$$

From Lemmas 3.1 and 2.6 it follows that the first three terms of Eq. (5.12) are all bounded. The last term is bounded from Lemmas 2.2 and 2.5. Therefore we have the following estimate:

$$|\{i\Phi(J^s - 1)V(u)u; (J^s - 1)u\}| \leq C(\omega)\|u\|_s^2. \tag{5.13}$$

Finally, Lemma 2.8 yields the related estimate for the local term,

$$|\{i\Phi(J^s - 1)f(|u|^2)u; (J^s - 1)u\}| \leq C(\omega)\tilde{f}(\|u\|_s)\|u\|_s^2. \tag{5.14}$$

Conclusion. Applying the Cauchy-Schwarz's inequality in Eq. (5.3) yields

$$\begin{aligned} \{\omega J^{s+1/2}P_{\pm}u; J^{s+1/2}P_{\pm}u\} &\leq \{\omega J^{1/2}(J^s - 1)P_{\pm}u; J^{1/2}(J^s - 1)P_{\pm}u\} + 2\|u\|_s\|u\|_1 + C(\omega)\|u\|_s\|u\|_0 + \|u\|_1\|u\|_0 \\ &\leq \{\omega J^{1/2}(J^s - 1)P_{\pm}u; J^{1/2}(J^s - 1)P_{\pm}u\} + C(\omega)\|u\|_s\|u\|_1. \end{aligned}$$

Using expression (5.10) and estimates (5.13) and (5.14) and integrating in a compact interval $I \subseteq (-T^*, T^*)$ produces

$$\begin{aligned} \int_I \{\omega J^{s+1/2}P_{\pm}u; J^{s+1/2}P_{\pm}u\} dt &\leq C(\omega)\max_I\{\|u(\cdot, t)\|_s^2\} + C(\omega)\int_I \|u(\cdot, t')\|_s^2 dt' \\ &\quad + C(\omega)\int_I \tilde{f}(\|u(\cdot, t')\|_s)\|u(\cdot, t')\|_s^2 dt', \end{aligned}$$

from where

$$\int_I \{\omega J^{s+1/2}P_{\pm}u; J^{s+1/2}P_{\pm}u\} dt \leq C(|I|, \omega, \max_I\{\|u(\cdot, t)\|_s\}). \tag{5.15}$$

Remark 5.6: If in addition we take under consideration a priori estimates of Sec. III (with the further assumption (A1) or (A2)), we get

$$\int_I \{\omega J^{s+1/2}P_{\pm}u; J^{s+1/2}P_{\pm}u\} dt \leq C(|I|, \omega, \|\phi\|_{H^s}^2).$$

Anyway, collecting estimates (5.11) and (5.15) produces

$$\int_I \int |\omega J^{s+1/2} u|^2 \leq C(|I|, \omega, \max_I \{\|u(\cdot, t)\|_s\}),$$

which finishes the proof. ■

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