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Collective dynamics of solitons and inequivalent quantizations

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The collective dynamics of solitons with a coset space G/H as moduli space is studied. It is shown that the collective band for a vibrational state is given by the inequivalent coset space quantization corresponding to the representation of H carried by the vibration. To leading order the collective dynamics is free motion in G/H coupled to background gauge fields determined by the vibrational state.

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I. INTRODUCTION

Solitons arise as static finite energy solutions to the equations of motion of nonlinear field theories. In general, a given soliton depends upon a set of parameters or moduli, and is a point in the manifold of solutions of equal energy, or moduli space. In many cases this manifold is simply an homogeneous or coset space G/H , where G is the group of symmetries of the action and $H \subset G$ is the symmetry of the solitonic solution.

Around a soliton there are two kinds of quantum excitations. The first corresponds to collective motion in the moduli space. The second are vibrational (intrinsic) excitations out of it. If the energy for the collective excitations is much lower than that for the vibrational ones the low energy spectrum can be approximately described by collective bands associated with each vibrational state. These bands can be described by an effective quantum mechanical problem given by the motion of a particle in the moduli space. However, as is well known from molecular and nuclear rotational bands, different vibrational states may have different collective bands. It is the purpose of this paper to show that in the case when the moduli space of the soliton is a coset space G/H a simple description of the collective bands of vibrational states can be given in terms of inequivalent coset space quantizations introduced by Mackey,¹ and more recently studied by Landsman and Linden² and McMullan and Tsutsui,³ among others.

Since the soliton is invariant under the subgroup H , vibrational excitations fit into irreducible representations (irreps) of H . Below we show that the collective band corresponding to a vibrational state in a representation χ of H realizes a representation of G induced by χ . This representation of G is reducible, and when it is broken into irreducible representations the whole collective band is obtained. This is equivalent to saying that the collective band for a vibrational state is given by the inequivalent quantization of G/H corresponding to the irrep χ of H carried by the vibration. In this way we find that collective motion is a physical example of the inequivalent coset space quantizations.

The lowest energy collective band of the soliton is that of the ground state of the vibrations. If the ground state is in the trivial representation of H , our results provide nothing new for

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determining this band, which is obtained from the usual (trivial) quantization of free motion in G/H . However, although the soliton is invariant under H , the quantum ground state need not be in the trivial representation because of topological factors which may arise when adiabatically performing the operations of H . For example, in the $SU(3)$ Skyrmion the Wess–Zumino–Witten term⁴ contributes with an extra phase. The same happens with the $SU(2)$ Skyrmion⁵ when it is quantized as a fermion. For the $B=1$ Skyrmion the result is the well-known fact that the ground state has spin an isospin $1/2$.

The outline of this paper is as follows: In Sec. II we introduce a model⁶ which generically describes a soliton model. The action is invariant under a group G of transformation of the fields but the soliton only under a subgroup H , so that the moduli space is a coset space G/H . In Sec. III we summarize Mackey’s¹ approach to the quantization on a coset space, and the more recent results by Landsman and Linden² and McMullan and Tsutsui.³ Section IV is the main section of this paper. There we demonstrate the relation between the wave functions and energies of the collective bands of vibrational states and the corresponding ones of the inequivalent coset space quantizations. Finally, in Sec. V we give our conclusions. We also discuss an example of a possible application of our results.

II. THE MODEL

Let us consider a general bosonic field theory in $d+1$ dimensions. The fields $\phi^s(\mathbf{x}, t)$ are real and take their values in an internal space \mathcal{T} . The variables \mathbf{x} and t are space and time coordinates, and $s=1\dots\dim(\mathcal{T})$ is an internal index. At any given time the fields $\phi^s(\mathbf{x}, t)$ determine a configuration of the system, given by a map from R^d (or a compactification of it) into \mathcal{T} . The set of these maps corresponds to the configuration space \mathcal{C} of the theory. The action is taken to be quadratic in the time derivatives of the fields, namely,

$$S = \int dt d^d x \left(\frac{1}{2} G_{st} \dot{\phi}^s(\mathbf{x}, t) \dot{\phi}^t(\mathbf{x}, t) - V \right), \tag{1}$$

where sums over repeated indices are implicit and,

$$G_{st} = G_{st}[\phi(\mathbf{x})] = G_{st}(\phi^{s1}, \partial_{i_1} \phi^{s2}, \dots, \partial_{i_1 \dots i_n} \phi^{sn}),$$

$$V = V[\phi(\mathbf{x})] = V(\phi^{s1}, \partial_{i_1} \phi^{s2}, \dots, \partial_{i_1 \dots i_n} \phi^{sn}),$$

are functions of the fields and a finite number of its spatial derivatives. For example if the action corresponds to a nonlinear σ -model,

$$S = \int dt d^d x \left(\frac{1}{2} g_{st}(\phi) \partial^\mu \phi^s(\mathbf{x}, t) \partial_\mu \phi^t(\mathbf{x}, t) - v(\phi) \right), \tag{2}$$

the functionals G_{st} and V are given by,

$$G_{st} = g_{st}(\phi), \quad V = g_{st}(\phi) \partial \phi^s \partial \phi^t + v(\phi), \tag{3}$$

that is, the spatial derivatives are included in V . In other cases, for example the Skyrme model,⁵ the function G_{st} depend also on the spatial derivatives of the field. A useful way of thinking about the system is to consider it as a particle moving in the infinite dimensional space of configurations \mathcal{C} (i.e., of maps from R^d into \mathcal{T}) with a potential V and a metric given by $g_{st}(\mathbf{x}, \mathbf{y}) = G_{st} \delta^d(\mathbf{x} - \mathbf{y})$.

The equations of motion following from the action (1) have static solutions which satisfy

$$\frac{\delta}{\delta \phi^s(\mathbf{x})} \int d^d y V[\phi(\mathbf{y})] = \frac{\partial V}{\partial \phi^s(\mathbf{x})} - \partial_{i_1} \frac{\partial V}{\partial (\partial_{i_1} \phi^s(\mathbf{x}))} + \dots + (-)^n \partial_{i_1 \dots i_n} \frac{\partial V}{\partial (\partial_{i_1 \dots i_n} \phi^s(\mathbf{x}))} = 0. \tag{4}$$

For our present purpose we assume that Eqs. (4) have soliton solutions. We consider the case in which the action S is invariant under an unbroken finite-dimensional compact group G of transformations of the fields,

$$\phi^s(\mathbf{x}) \mapsto R_g^s[\phi(\mathbf{x})], \quad g \in G. \quad (5)$$

This group usually includes spatial rotations and internal transformations of the fields. The invariance of the action under these transformations is expressed as,

$$G_{st}[\phi(\mathbf{x})] \delta^3(x-y) = \int d^d z \frac{\delta R_g^u[\phi(\mathbf{z})]}{\delta \phi^s(\mathbf{x})} \frac{\partial R_g^v[\phi(\mathbf{z})]}{\partial \phi^t(\mathbf{y})} G_{uv}[\phi(\mathbf{z})], \quad (6)$$

$$\int d^d x V[\phi(x)] = \int d^d x V[R_g[\phi(x)]]. \quad (7)$$

The first condition can be rephrased saying that the transformations are isometries of the space \mathcal{C} .

Let us consider a soliton solution $\tilde{\phi}(\mathbf{x})$ of (4), i.e., a minimum of the potential $\int V[\phi] d^d x$. In general, $\tilde{\phi}(\mathbf{x})$ is only invariant under a subgroup H of the symmetry group G of the action. Due to the invariance of the potential under G , any configuration $R_g[\tilde{\phi}(\mathbf{x})]$ obtained by acting with G on $\tilde{\phi}(\mathbf{x})$ is also a solution. The whole family of solutions, known as the moduli space of the soliton, is thus given by the orbit of $\tilde{\phi}(\mathbf{x})$ under G , and is therefore an homogeneous or coset space,

$$\mathcal{M} = O_G(\tilde{\phi}) = G/H. \quad (8)$$

It is possible that there exist other zero modes not related to symmetries of the action, but we shall ignore that possibility here. Consider now fluctuations around the soliton, $\tilde{\phi}(\mathbf{x}) \rightarrow \tilde{\phi}(\mathbf{x}) + \varphi(\mathbf{x}, t)$. The linearized equations for the fluctuations read,

$$\tilde{G}_{st} \ddot{\varphi}^t(\mathbf{x}, t) + \int d^d y K_{st}(\mathbf{x}, \mathbf{y}) \varphi^t(\mathbf{y}, t) = 0, \quad (9)$$

where \tilde{G}_{st} is evaluated on the solution, $\tilde{G}_{st} = G_{st}[\tilde{\phi}(\mathbf{x})]$ and,

$$K_{st}(\mathbf{x}, \mathbf{y}) = \left. \frac{\delta^2 V}{\delta \phi^s(\mathbf{x}) \delta \phi^t(\mathbf{y})} \right|_{\phi = \tilde{\phi}}. \quad (10)$$

The fluctuations can be understood as infinitesimal vectors in the tangent space to \mathcal{C} at the point $\tilde{\phi}(\mathbf{x})$. The (infinite) set of those normal to the moduli space \mathcal{M} are massive excitations, since \mathcal{M} is the ‘‘valley’’ of minima of the static energy. They satisfy,

$$\int d^d y K_{st}(\mathbf{x}, \mathbf{y}) \varphi^t(\mathbf{y}, t) - \omega_n^2 \tilde{G}_{st} \psi_n^t(\mathbf{x}) = 0, \quad (11)$$

where n labels the modes, $\omega_n \neq 0$, and $\psi_n^s(\mathbf{x})$ are the corresponding normalized eigenfunctions, $\int d^d x \tilde{G}_{st} \psi_n^s(\mathbf{x}) \psi_m^t(\mathbf{x}) = \delta_{nm}$. The fluctuations tangent to \mathcal{M} are massless, and are usually known as zero modes. They satisfy,

$$\int d^d y K_{st}(\mathbf{x}, \mathbf{y}) \varphi^t(\mathbf{y}, t) \psi_\alpha^t(\mathbf{y}) = 0, \quad (12)$$

where $\alpha = 1, \dots, \dim(G/H)$ label the zero modes, which are given by the nonvanishing infinitesimal transformations (5) of the soliton, $\psi_\alpha^s(\mathbf{x}) = \delta_\alpha \bar{\phi}^s(\mathbf{x})$. The norm of the zero modes corresponds to the inertia tensor of the soliton,

$$\mathcal{I}_{\alpha\beta} = \int d^d x \tilde{G}_{st} \psi_\alpha^s(\mathbf{x}) \psi_\beta^t(\mathbf{x}). \tag{13}$$

The massive and zero modes satisfy the completeness relation,

$$\tilde{G}_{st} [\delta^{nm} \psi_n^s(\mathbf{x}) \psi_m^t(\mathbf{y}) + \mathcal{I}^{\alpha\beta} \psi_\alpha^s(\mathbf{x}) \psi_\beta^t(\mathbf{y})] = \delta^d(x - y), \tag{14}$$

where $\mathcal{I}^{\alpha\beta}$ is the inverse of the inertia tensor.

Quantum-mechanically, the massive modes correspond to vibrations of the soliton, while the zero modes to collective motion in \mathcal{M} . We are interested in the case in which the collective energy is much smaller than the vibrational one. Schematically, this is given by

$$\frac{\hbar^2}{\mathcal{I}} \ll \hbar \omega \Rightarrow \frac{\hbar}{\mathcal{I}\omega} \ll 1, \tag{15}$$

where \mathcal{I} is the inertia for the collective motion and ω the frequency for the intrinsic excitations. In this regime it is natural to treat the collective motion exactly and the vibrations in perturbation theory. The relation (15) implies that the small parameter is proportional to \hbar , and therefore the perturbative expansion is an expansion in loops (in powers of \hbar). In the following $\hbar = 1$, so that the expansion parameter is $1/\mathcal{I}\omega$.

In order to proceed with the quantization around the soliton $\bar{\phi}(\mathbf{x})$ it is necessary to treat differently the massive vibrations from the zero modes. One possible approach, which will prove useful to our purposes, is to introduce collective coordinates by performing a transformation of the fields $\phi^s(\mathbf{x}, t)$ with time dependent parameters,^{7-10,6}

$$\phi^s(\mathbf{x}, t) \mapsto R_{g(\alpha)}^s[\phi(\mathbf{x}, t)], \tag{16}$$

where $\alpha^a(t)$ ($a = 1, \dots, \dim(G)$) parameterize G . The action (1) is not symmetric under this time dependent transformations, and it changes to,

$$S \mapsto \int dt d^d x \left(\frac{1}{2} G_{st} D_0 \phi^s(\mathbf{x}, t) D_0 \phi^t(\mathbf{x}, t) - V \right). \tag{17}$$

The covariant time derivatives are given by,

$$D_0 \phi^s(\mathbf{x}, t) = \dot{\phi}^s(\mathbf{x}, t) + \dot{\alpha}^a(t) \zeta_a^b(\alpha(t)) \delta_b \phi^s(\mathbf{x}, t), \tag{18}$$

where $\zeta_a^b(\alpha)$ are components of the left-invariant Cartan–Maurer one-form $g^{-1} dg = d\alpha^a \zeta_a^b(\alpha) T_b$, with T_a being the infinitesimal group generators, and $\delta_a \phi^s(\mathbf{x}, t)$ correspond to the infinitesimal transformations (5). The transformed action (17) is invariant under gauge transformations, i.e., time dependent transformations of the fields and the collective coordinates,

$$\delta_\varepsilon \phi^s(\mathbf{x}, t) = \varepsilon^a(t) \delta_a \phi^s(\mathbf{x}, t), \quad \delta_\varepsilon \alpha^a(t) = -\varepsilon^b(t) \Theta_b^a(\alpha(t)), \quad (\Theta = \zeta^{-1}). \tag{19}$$

The transformed action (17) can be understood as describing the problem from an arbitrary moving frame of reference, its motion given by the collective coordinates.

The classical Hamiltonian corresponding to the action (17) is given by

$$\mathcal{H} = \int d^d x \left(\frac{1}{2} G^{st} [\phi(\mathbf{x}, t)] \pi_s(\mathbf{x}, t) \pi_t(\mathbf{x}, t) + V[\phi(\mathbf{x}, t)] \right) - \lambda_a(t) \Phi_a(t), \tag{20}$$

where $\pi_s(\mathbf{x},t)$ are the canonical momenta conjugate to $\phi^s(\mathbf{x},t)$, i.e., $\{\pi_s(\mathbf{x},t),\phi^t(\mathbf{y},t)\} = -\delta_s^t\delta^d(x-y)$, and $\lambda_a(t)$ are Lagrange multipliers which impose the first class constraints $\Phi_a=J_a-I_a$, which generate the gauge transformations (19).¹¹ We display also the explicit dependence of G_{st} and V on the fields. The operators J_a are the generators for the infinitesimal transformations of the fields,

$$J_a = \int d^d x \pi_s(\mathbf{x}) \delta_a \phi^s(\mathbf{x}), \tag{21}$$

and I_a are the right generators for the collective coordinates, $I_a = \Theta_a^b \mathcal{P}_b$, where \mathcal{P}_a are the conjugate momenta to the collective coordinates, $\{\alpha^a, \mathcal{P}_b\} = \delta_b^a$. While the gauge symmetry (19) acts from the right on the collective coordinates, the original symmetry G acts from the left. The operators that generate the left infinitesimal transformations of the collective coordinates are correspondingly given by $L_a = \hat{\Theta}_a^b \mathcal{P}_b$, where $\hat{\zeta}_a^b(\alpha)$ are components of the right-invariant Cartan–Maurer one-form $dg g^{-1} = d\alpha^a \hat{\zeta}_a^b(\alpha) T_b$, and $\hat{\Theta} = \hat{\zeta}^{-1}$.

The gauge algebra \mathfrak{g} , generated by the constraints $\Phi_a = J_a - I_a$, can be decomposed into $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$, where the subalgebra \mathfrak{h} is generated by $\Phi_i = J_i - I_i$ ($i = 1, \dots, \dim(H)$), while the complement \mathfrak{p} is generated by $\Phi_\alpha = J_\alpha - I_\alpha$ ($\alpha = \dim(H) + 1, \dots, \dim(G)$). The operators J_i generate the subgroup of transformations of the fields that leave the soliton $\tilde{\phi}(\mathbf{x})$ invariant, while J_α generate the transformations which change it. Since H is a subgroup, the operators J_i close under commutation, $[J_i, J_j] = i C_{ij}^k J_k$, where C_{ab}^c are the structure constants of G . Furthermore, as G is compact the generators J_α can be chosen in such a way that

$$[J_i, J_\alpha] = i C_{i\alpha}^\beta J_\beta, \tag{22}$$

which shows that J_α transform in a representation (possibly reducible) of H .

As is usual in this kind of problems, the idea is to replace the zero mode excitations of the fields $\phi(\mathbf{x},t)$ by the collective coordinates $\alpha(t)$, by means of a suitable gauge fixing of action (17). It is important to note that the number of zero modes is equal to the dimension of G/H , while the number of collective variables is $\dim(G)$. By gauge fixing the constraints Φ_α we will eliminate the zero modes in favor of the collective coordinates α^α on G/H . We will make use of the remaining H -gauge symmetry to prove that the effective collective dynamics of the soliton can be understood in terms of inequivalent coset space quantizations.

III. INEQUIVALENT COSET SPACE QUANTIZATIONS

In this section we review briefly the approach developed by Mackey to the quantization of a system whose configuration space is a coset space G/H .¹ This must be distinguished from the quantization of G/H as a phase space, which is discussed for example in Ref. 12 (and references therein). It was shown by Mackey that when the configuration space is a coset space G/H there are many different quantizations not equivalent to each other by unitary transformations, which are labeled by the unitary irreps χ of the subgroup H . The wave functions in a given inequivalent quantization are vector valued, taking values in the representation space V_χ . They can be obtained from vector valued functions in G which satisfy (χ -equivariant condition),

$$f_\mu(g h) = \pi_{\nu\mu}^\chi(h) f_\nu(g), \tag{23}$$

where $g \in G$, $h \in H$, and $\pi^\chi(h)$ are the matrices of the representation χ of H . The left regular representation of G is defined on functions of G acting as,

$$f(g) \rightarrow f(g_1^{-1} g). \tag{24}$$

Equation (23) is invariant under this action. Therefore, the set of functions $f_\mu(g)$ satisfying condition (23) transforms under the left action of G in a representation called the representation of G induced by χ . See Refs. 1 and 13 for the definition and properties of induced representations.

Given a local section $\sigma:G/H\rightarrow G$ every element $g \in G$ can be uniquely written as,

$$g = \sigma(\xi(g))h_\sigma(g), \tag{25}$$

where $\xi \in G/H$ and $h \in H$. This allows to define a one-to-one correspondence between the set of functions $f_\mu(g)$ and functions $F_\mu:G/H\rightarrow V_\chi$ such that,

$$f_\mu(g) = \pi_{\nu\mu}^\chi(h(g))F_\nu(\xi(g)). \tag{26}$$

The functions $F_\mu(\xi)$ transform under G in the induced representation as,

$$F_\mu(\xi) \xrightarrow{g} \pi_{\nu\mu}^\chi(h_\sigma(g^{-1}\sigma(\xi)))F_\nu(g^{-1}\xi), \tag{27}$$

where $g^{-1}\xi = \xi(g^{-1}\sigma(\xi))$ defines the action of G on G/H .

Landsman and Linden² studied the dynamical consequences of the inequivalent quantizations for the motion of a particle in G/H . They discovered that in the nontrivial quantum sectors the particle couples to a background gauge field A_α , known as the H -connection, which takes values in the representation of the sub-algebra $\pi^\chi(\mathfrak{h})$. The Hamiltonian is given by,

$$\mathcal{H} = -\frac{1}{2}g^{\alpha\beta}(\nabla_\alpha + A_\alpha)(\partial_\beta + A_\beta), \tag{28}$$

where ∇_α is the covariant derivative constructed out of the metric $g_{\alpha\beta}$ on G/H . Due to the H -connection the Hamiltonian is matrix valued (in the trivial representation of H it reduces to minus one-half the Laplacian $-\frac{1}{2}\Delta_{G/H} = -\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\partial_\beta$).

McMullan and Tsutsui³ developed a different approach to the inequivalent quantizations. Using the fact that in Eq. (25) $h \in H$ can be further decomposed as $h = r(h)s(h)$, where s belongs to the Cartan subgroup of H , they showed that instead of Mackey's functions $f_\mu(g)$ one can use its highest weight component f_χ evaluated at $g = \sigma r$ (i.e., $s = 1$). The condition (23) implies that these functions are annihilated by the raising operators $E_{\varphi>0}$ in the Chevalley basis $\{H_\alpha, E_\varphi\}$ of \mathfrak{h} . In this case the wave functions are scalars, which allows for a simpler definition of the corresponding path integral.

IV. COLLECTIVE DYNAMICS

In this section we will prove that the collective band associated with an intrinsic vibrational state can be obtained from the inequivalent quantization corresponding to the representation of the subgroup H carried by the vibration. To see this we consider the canonical quantization of the action (17) introduced in Sec. II. This action has a G -gauge invariance due to the introduction of the collective coordinates as additional variables. Gauge fixing the G/H part of the gauge symmetry allows to eliminate the zero modes in favor of the collective coordinates. The remaining H -gauge invariance can be treated with the Dirac method of imposing the constraints on the wave functions.¹¹ In our case this becomes Mackey's condition (23) for the collective wave functions associated to a given vibrational state. Alternatively, the collective coordinates which parameterize H can be eliminated, fixing the H -gauge symmetry. In this case the wave functions become $F_\mu(x)$ of Eq. (27) and the Hamiltonian that of Landsman and Linden.

A. Elimination of the zero modes

We start by eliminating the zero modes in favor of collective coordinates on G/H .

Let us expand the fields in terms of fluctuations around the soliton. The fluctuations can be written as linear combinations of the normal modes (11) and (12),

$$\phi^s(\mathbf{x}, t) = \tilde{\phi}^s(\mathbf{x}) + \psi_n^s(\mathbf{x})q_n(t) + \psi_\alpha^s(\mathbf{x})q_\alpha(t), \quad (29)$$

where the time dependent coefficients $q_A(t)$ (A stands both for n and α) are now the dynamical degrees of freedom. The conjugate momenta to the fields read,

$$\pi_s(\mathbf{x}, t) = \tilde{G}_{st}[\psi_n^t(\mathbf{x})p_n(t) + \mathcal{I}^{\alpha\beta}\psi_\alpha^t(\mathbf{x})p_\beta(t)], \quad (30)$$

where p_A are conjugate to q_A . Inserting these expressions into the Hamiltonian (20), and making use of the equations for the normal modes (11), (12) and their orthogonality relations, we obtain up to quadratic order in the fluctuations,

$$\mathcal{H} = E_{\text{clas}} + \frac{1}{2}(p_n^2 + \omega_n^2 q_n^2) + \frac{1}{2}\mathcal{I}^{\alpha\beta}p_\alpha p_\beta - \lambda_a \Phi_a + O(q^3, pq^2). \quad (31)$$

The first term is the classical soliton energy $E_{\text{clas}} = \int d^d x V[\tilde{\phi}(\mathbf{x})]$. The second are the (infinite) harmonic oscillators corresponding to the vibrations of the soliton. The third term are the zero modes, and is purely kinetic. Anharmonic terms have been omitted.

Expanding similarly the constraints we get,

$$\Phi_\alpha = p_\alpha + (D_\alpha)_{AB} p_A q_B - I_\alpha + O(pq^2), \quad (32)$$

$$\Phi_i = (D_i)_{AB} p_A q_B - I_i + O(pq^2), \quad (33)$$

where we have kept up to quadratic terms in the fluctuations but the collective operators are treated exactly. The matrices $(D_a)_{AB}$ are defined as,

$$(D_a)_{AB} = \int d^d x \tilde{G}_{st} \psi_A^s(\mathbf{x}) \delta_a \psi_B^t(\mathbf{x}), \quad (34)$$

where $\delta_a \psi_A^t(\mathbf{x})$ are the infinitesimal transformations (5) of the eigenfunctions, and correspond to the representation of G under which the fluctuations transform.

We want to eliminate the degrees of freedom associated with the zero energy fluctuations (q_α, p_α) . We achieve this by choosing to fix the gauge invariance generated by Φ_α the following gauge fixing conditions,

$$q_\alpha = 0, \quad (\alpha = 1, \dots, \dim(G/H)), \quad (35)$$

which satisfy, $\{q_\alpha, \Phi_\beta\} \approx \delta_{\alpha\beta} + (D_\beta)_{\alpha n} q_n + O(q^2)$, where \approx indicates evaluated where (35) holds. This gauge condition *does not* fix the H -gauge invariance generated by Φ_i since, $\{q_\alpha, \Phi_i\} \approx 0$, where we have used the fact that massive and zero fluctuations do not mix under H and therefore $(D_i)_{\alpha n} = 0$.

We now replace the Poisson brackets by Dirac brackets,¹¹

$$\{A, B\}_D = \{A, B\} - \{A, q_\alpha\}(\delta_{\alpha\beta} - D_{\beta\alpha n} q_n)\{\Phi_\beta, B\}, \quad (36)$$

in order to treat the gauge conditions and constraints as operator identities. We are then able to solve p_α from the equations $\Phi_\alpha = 0$,

$$p_\alpha = I_\alpha - (D_\alpha)_{nm} p_n q_m + O(pq^2, Iq). \quad (37)$$

Replacing (q_α, p_α) by Eqs. (35), (37) in the Hamiltonian (31) we obtain,

$$\mathcal{H} = E_{\text{clas}} + \frac{1}{2}(p_n^2 + \omega_n^2 q_n^2) + \frac{1}{2}\mathcal{I}^{\alpha\beta} I_\alpha I_\beta - \mathcal{I}^{\alpha\beta} I_\alpha (D_\beta)_{nm} p_n q_m - \lambda_i(t) \Phi_i(t) + O(q^3, pq^2, Ipq^2). \quad (38)$$

The zero modes have been completely eliminated. Instead we have a kinetic term for the collective coordinates and a coupling between the collective coordinates and the vibrations. Anharmonic terms in the vibrations and higher order vibration-collective couplings have been omitted.

B. Collective energy eigenfunctions

The Hamiltonian (38) describes the dynamics of the system in the 1-loop approximation for the vibrations, while the collective motion is treated exactly. To this order, the spectrum corresponds to vibrational states and on top of each state a collective band. There is still the H -gauge symmetry.

Since the soliton is invariant under the subgroup H , the vibrations split into irreducible representations (irreps) of H . Consider now a given vibrational state classified by the irrep χ of H , and let's derive the *effective* collective dynamics for this state from the Hamiltonian (38). In other words, we restrict to energies which can only excite collective modes, but not any other vibrational ones. Taking into account this restriction, the wave functions for a state in the collective band has the general form,

$$\psi(q, \alpha) = \varphi_\mu(q) f_\mu(\alpha). \tag{39}$$

The functions $\varphi_\mu(q)$ ($\mu = 1, \dots, \dim(\chi)$) form a basis for the irrep χ of the vibration, while the collective functions $f_\mu(\alpha)$ are arbitrary and have to be determined. We still have to take into account the H -gauge invariance. As mentioned above, to deal with it we apply Dirac's method of imposing the constraints on the wave functions.¹¹ This restricts the wave functions (39) to satisfy $\Phi_i \psi(q, \alpha) = 0 \forall_i$, which imposes on the collective functions the conditions,

$$I_i f_\mu(\alpha) = i(T_i)_{\nu\mu}^\chi f_\nu(\alpha) \quad \forall_i. \tag{40}$$

Here $(T_i)_{\mu\nu}^\chi$ are the infinitesimal generators of H in the χ representation. This is the infinitesimal version of Eq. (23). This condition also holds for the discrete transformations $h \in H$, which together with (40) imply for all $h \in H$,

$$R_h f_\mu(\alpha) = \pi_{\nu\mu}^\chi(h) f_\nu(\alpha), \tag{41}$$

where R_h stands for the right action of h on the group element parameterized by α , $g(\alpha) \rightarrow g(\alpha)h$. Therefore, the collective functions $f_\mu(\alpha)$ satisfy Mackey's condition (23), i.e., they transform under the left action of G in the representation of G induced by the representation χ of H .¹³ In other words, the states in the collective band are those of the inequivalent quantization of G/H given by the representation χ of H carried by the intrinsic state.

In order to diagonalize the Hamiltonian (38) we can make use of the original symmetry G . The induced representation of G under which the collective functions in (39) transform is reducible. It can be broken into irreducible components. In fact, by the Peter–Weyl theorem¹³ the collective functions $f_\mu(\alpha)$ can be rewritten as,

$$f_\mu(\alpha) = C_\mu^{IMN} D_{MN}^I(\alpha), \tag{42}$$

where the sum is over all irreps I of G defined by the matrices $D_{MN}^I(\alpha)$. Under the left action of G each term of the sum transforms in the corresponding representation I of G . Under the right action of H each term in (42) transforms in the representation I of G considered as a representation of H . This representation of H is in general reducible and can be broken in irreducible pieces, $I|_H = \chi_1 + \dots + \chi_n$, where χ_i are irreps of H . In this decomposition, the representation χ carried by the vibrations appears a number of times we denote by $d_{I\chi}$, that is, $I|_H = d_{I\chi}\chi + \text{other irreps}$. Condition (41) implies that the linear combination (42) must be restricted to

$$f_\mu(\alpha) = C_k^{IMN} D_{M\mu_k}^I(\alpha), \tag{43}$$

where k runs over the d_{I_χ} representations $\chi_k = \chi$, and μ_k indicates the μ th component of irrep χ_k . This means that each representation I of G appears in the collective band a number of times equal to d_{I_χ} .

The collective band of the vibration is given by the restriction of the Hamiltonian (38) to the subspace of functions (39), which satisfy the condition (41). The first two terms in (38) are constant throughout the band, and the last one vanishes on physical states. The collective Hamiltonian reads,

$$\mathcal{H}_{\text{coll}} = \text{const} + \frac{1}{2} \mathcal{I}^{\alpha\beta} I_\alpha I_\beta - \mathcal{I}^{\alpha\beta} I_\alpha M_\beta, \tag{44}$$

where M_α stand for the restriction of the operators $(D_\alpha)_{nm} p_n q_m$ to the subspace span by the vibrational functions $\varphi_\mu(q)$, that is, $M_\alpha = \langle \varphi_\mu | (D_\alpha)_{nm} p_n q_m | \varphi_\nu \rangle$. The collective Hamiltonian is invariant under the action of G by the left, and so does not mix different representations I . Therefore, it can be diagonalized in subspaces of dimension d_{I_χ} . It is easy to check that it also commutes with the simultaneous action of H on the intrinsic states (J_i) and on the collective coordinates from the right (I_i). This ensures that it preserves the physical condition $\Phi_i = 0$ which the states ψ of Eq. (39) satisfy.

The diagonalization of $\mathcal{H}_{\text{coll}}$ in each subspace of dimension $d_{I_\chi} > 1$ must be performed case by case. However, some more information can be obtained using the H -invariance. The restricted matrices M_α no longer satisfy the G algebra, but they still satisfy,

$$[J_i, M_\alpha] = i C_{i\alpha}^\beta M_\beta, \tag{45}$$

which means that they transform under H in the same representation as J_α . By the Wigner–Eckart theorem in the group H , they are determined by Clebsch–Gordan coefficients up to as many independent constants as irreps of H are contained in this representation. The number of irreps also gives the number of independent inertia moments in $\mathcal{I}_{\alpha\beta}$. This is as far as we can go using symmetry arguments in the general case.

If G/H is a symmetric space a considerable simplification arises. In this case, there is an involutive automorphism $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$ ($\tau^2 = id_{\mathfrak{g}}$) under which the generators satisfy,¹³

$$\tau J_i \tau = J_i, \tag{46}$$

$$\tau J_\alpha \tau = -J_\alpha. \tag{47}$$

We assume that τ acts on the fields $\phi^s(\mathbf{x}, t)$ as $\tau: \phi^s(\mathbf{x}, t) \mapsto \tau^s[\phi(\mathbf{x}, t)]$ with $\tau^s[\tilde{\phi}(\mathbf{x})] = \tilde{\phi}(\mathbf{x})$. Equation (46) states that τ commutes with J_i , so a representation of H has a definite eigenvalue of $\tau(\pm 1)$, and therefore by (47) we have $M_\alpha = 0$. Another simplification is that the representation of H under which J_α transform is irreducible, so there is only one moment of inertia \mathcal{I} . The collective Hamiltonian becomes,

$$\mathcal{H}_{\text{coll}} = \frac{1}{2\mathcal{I}} I_\alpha^2 = \frac{1}{2\mathcal{I}} (I_a^2 - I_i^2), \tag{48}$$

and the d_{I_χ} states,

$$\psi_{IM_\chi}(q, \alpha) = \varphi_\mu(q) D_{M_{\mu_k}}^I(\alpha) \quad (k = 1, \dots, d_{I_\chi}), \tag{49}$$

are degenerate eigenstates, since I_a^2 and I_i^2 are Casimirs of G and H , respectively.

C. Landsman and Linden Hamiltonian

In the previous section we found the collective wavefunctions by treating the H -gauge invariance by means of the Dirac method of imposing the constraints on the states. However, the dynamical consequences of the Mackey condition (41) on the collective functions are best understood by fixing the H -gauge symmetry.

Let us make the natural gauge choice of setting the collective coordinates related to motion on the subgroup $\alpha^i=0$. This fixes the H -gauge symmetry generated by Φ_i , i.e., $\{\alpha^i, \Phi_j\} \approx \delta_j^i$. Proceeding as before, we eliminate the pairs $(\alpha^i, \mathcal{P}_i)$ by solving \mathcal{P}_i from the equations $\Phi_i=0$. This gives

$$\mathcal{P}_i = (D_i)_{nm} p_n q_m, \tag{50}$$

where we have used that $\Theta_i^j(0) = \delta_j^i$. Since the vibrations fall into irreps of H , if we restrict to a given vibrational state classified by χ , the above equation reduces to,

$$\mathcal{P}_i = \langle \varphi_\mu | (D_i)_{nm} p_n q_m | \varphi_\nu \rangle = -i (T_i)_{\mu\nu}^\chi, \tag{51}$$

where $(T_i)_{\mu\nu}^\chi$ are the generators of H in the irrep χ . Replacing in the collective operators I_α we get,

$$I_\alpha = \Theta_\alpha^\beta(\xi) \mathcal{P}_\beta + i \Theta_\alpha^\beta(\xi) \zeta_\beta^i(\xi) (T_i)_{\mu\nu}^\chi, \tag{52}$$

where $\xi \in G/H_0$ (i.e., the coordinates $\alpha^{a=1, \dots, \dim(G/H)}$), being H_0 the identity component of H . In this gauge the collective functions f_μ become the functions $F_\mu(\xi)$ of Sec. III, and the collective Hamiltonian (44) becomes,

$$\mathcal{H}_{\text{coll}} = \text{const} - \frac{1}{2} g^{\alpha\beta} (\nabla_\alpha + A_\alpha) (\partial_\beta + A_\beta) - \frac{1}{2} g^{\alpha\beta} [(\nabla_\alpha + A_\alpha) B_\beta + B_\alpha (\partial_\beta + A_\beta)]. \tag{53}$$

The second term corresponds to Landsman–Linden Hamiltonian (28), with the metric on G/H_0 given by $g_{\alpha\beta}(\xi) = \mathcal{I}_{\gamma\delta} \zeta_\alpha^\gamma(\xi) \zeta_\beta^\delta(\xi)$, and the H_0 -connection by,

$$(A_\alpha)_{\mu\nu} = -i \zeta_\alpha^i (T_i)_{\mu\nu}^\chi. \tag{54}$$

The third term in (53) gives the coupling to an extra background field,

$$(B_\alpha)_{\mu\nu} = -i \zeta_\alpha^\beta (M_\beta)_{\mu\nu}, \tag{55}$$

which comes from the ‘‘Coriolis’’ terms in (44). In general, for real representations, this is an $SO(\dim \chi)$ connection, and is similar to the induced connections studied in Ref. 14.

The H_0 -connection ensures that Hamiltonian (53) applied to functions independent of α^i gives the same result as Hamiltonian (44) acting on functions over G which satisfy Mackey condition (40). When $H_0 \neq H$ the functions $F_\mu(\xi)$ are still restricted by condition (41) for the discrete elements of H . This restriction can be lifted including a pure gauge connection which associates a discrete element of H with each nontrivial path in $\Pi_1(G/H)$ (holonomy factors). The relation between inequivalent quantizations and holonomy factors in the path integral is discussed in Ref. 15.

V. CONCLUSIONS

We have discussed the collective bands of intrinsic states found when quantizing around a soliton solution with moduli space isomorphic to G/H . The result is that the collective band of an intrinsic vibrational state realizes an inequivalent coset space quantization given by the representation of H under which the intrinsic state transforms. The collective Hamiltonian is that of Landsman and Linden,² which describes free motion on G/H coupled to a background H -gauge field. Besides, there may be other background gauge fields coming from the Coriolis terms. The

extra degrees of freedom associated with the nontrivial quantizations are given by the intrinsic coordinates. In this way, we have given a physical example of the inequivalent quantizations studied in Refs. 1–3.

Some of the consequences for the collective quantization of a soliton with moduli space G/H are the following. The states in the collective band of a vibration carrying the representation χ of H are classified by the irreducible representations I of the symmetry group G . Each irrep I appears in the band as many times as χ is contained in its decomposition into irreps of H . This also gives to lowest order the dimension of the collective Hamiltonian matrix within this subspace, from which the energies of the states are obtained. Determining the collective bands is analogous to calculating the rotational spectra of polyatomic molecules,¹⁶ and our results can be understood as a generalization of this problem to a general symmetry group.

This work may be of interest for obtaining the spins and isospins bands of the ground state of multiskyrmions^{17,18} and their excited states, which have been found for topological charges $B = 2$ and $B = 4$ by Barnes *et al.*^{19,20} The symmetry group of the Skyrme model is the direct product of the spatial and isospacial rotations and the combined parity (without considering spatial translations) $G = \text{SO}(3)_S \times \text{SO}(3)_I \times P$. As described in Ref. 21, the symmetry group of a Skyrmeion $H \subset G$ is given by pairs $(h, D(h))$, where h is an element of the spatial group $\text{O}(3)_S$, and $D(h)$ an element of the isospin group $\text{O}(3)_I$. The mapping $D: h \rightarrow D(h)$ is a three-dimensional real representation of H , and $\det h = \det D(h)$. For example, for the $B = 2$ case $H = D_\infty$ (not considering parity). The ground state is in the non trivial one-dimensional representation Σ^- if the Skyrmeion is quantized as a fermion. Therefore, the lowest allowed state of the band are $(I = 1, S = 0)$ or $(I = 0, S = 1)$, since they are the lowest irreps of G containing Σ^- in its decomposition. Similarly, the other states of the band can be obtained. The $B = 2$ case has already been considered in Ref. 22, but we expect the more systematic treatment presented here will be useful in more complicated situations, as for $B > 2$, where the subgroup H is discrete.²³

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¹G. W. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York, 1969).

²N. P. Landsman and N. Linden, Nucl. Phys. B **365**, 121 (1991).

³D. McMullan and I. Tsutsui, Ann. Phys. **237**, 269 (1995), hep-th/9308027.

⁴E. Witten, Nucl. Phys. B **223**, 422 (1983); **223**, 433 (1983).

⁵T. H. R. Skyrme, Proc. R. Soc. London **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

⁶J. P. Garrahan, M. Kruczenski, and D. R. Bes, Phys. Rev. D **53**, 7176 (1996), hep-th/9603041.

⁷A. Hosoya and K. Kikkawa, Nucl. Phys. B **101**, 271 (1975).

⁸D. R. Bes and J. Kurchan, *The Treatment of Collective Coordinates in Many-Body Systems, Lecture Notes in Physics* (World Scientific, Singapore, 1990), Vol. 34.

⁹J. Alfaro and P. H. Damgaard, Ann. Phys. **202**, 398 (1990).

¹⁰H. J. W. Müller-Kirsten and J.-z. Zhang, Phys. Rev. D **50**, 6531 (1994); Phys. Lett. B **339**, 65 (1994).

¹¹M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).

¹²A. Perelomov, *Generalized Coherent States and Their Applications* (Springer-Verlag, Berlin, 1986).

¹³A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (World Scientific, Singapore, 1986).

¹⁴P. Maraner, J. Phys. A **28**, 2939 (1995), hep-th/9409080.

¹⁵S. Tanimura and I. Tsutsui, Ann. Phys. **258**, 137 (1997), hep-th/9609089.

¹⁶L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Nonrelativistic Theory* (Pergamon, Oxford, 1965).

¹⁷E. Braaten, S. Townsend, and L. Carson, Phys. Lett. B **391**, 150 (1990).

¹⁸R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **79**, 363 (1997), hep-th/9702081.

¹⁹C. Barnes, W. K. Baskerville, and N. Turok, Phys. Lett. B **411**, 180 (1997), hep-th/9704028.

²⁰C. Barnes, W. K. Baskerville, and N. Turok, Phys. Rev. Lett. **79**, 367 (1997), hep-th/9704012.

²¹R. A. Leese and N. S. Manton, Nucl. Phys. A **572**, 575 (1994).

²²R. A. Leese, N. S. Manton, and B. J. Schroers, Nucl. Phys. **442**, 228 (1995), hep-th/9502405.

²³P. Irwin, "Zero mode quantization of multi-Skyrmions," hep-th/9804142.