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# Space of test functions for higher-order field theories

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The fundamental space $\zeta$ is defined as the set of entire analytic functions [test functions $\varphi(z)$], which are rapidly decreasing on the real axis. The variable $z$ corresponds to the complex energy plane. The conjugate or dual space $\zeta'$ is the set of continuous linear functionals (distributions) on $\zeta$. Among those distributions are the propagators, determined by the poles implied by the equations of motion and the contour of integration implied by the boundary conditions. All propagators can be represented as linear combinations of elementary (one pole) functionals. The algebra of convolution products is also determined. The Fourier transformed space $\tilde{\zeta}$ contains test functions $\tilde{\varphi}(x)$. These functions are extra-rapidly decreasing, so that the exponentially increasing solutions of higher-order equations are distributions on $\tilde{\zeta}$.

## I. INTRODUCTION

The usual quantum field theories can be considered to be mathematically sound when they are constructed from expectation values of products of field operators. An example is the Wightman functions.[1] Such a system of functions actually forms a space of distributions ($\mathscr{S}'$), defined as continuous linear functionals on the set $\mathscr{S}$ of infinitely differentiable functions which, for $|x|\to\infty$, tend to zero more rapidly than any power of $|x|^{-1}$ (rapidly decreasing functions). $\mathscr{S}'$ is the space of tempered distributions introduced by L. Schwartz.[2] The space $\mathscr{S}$ was utilized by Streater and Wightman[3] to handle and prove some important physical theorems. Similar mathematical considerations in a more general physical context were discussed by Bogolubov, Logunov, and Todorov.[4]

The space of tempered distributions appears to be favored because, among other properties, it coincides with its Fourier-transformed space. However, a new situation arises when one introduces physical theories implying higher-order equations of motion.

Let us take for example a free Lorentz-invariant higher-order equation, characterized by a polynomial in the D'Alambertian operator:

$$\left(\sum_{s=0}^{n} a_s \Box^s\right)\varphi(x)=0. \tag{1}$$

The Fourier transform of (1) is

$$\left(\sum_{s=0}^{n} a_s(-k^2)^s\right)\tilde{\varphi}(k)=0 \tag{2}$$

so that $\tilde{\varphi}(k)=0$ except when $k^2$ is one of the roots of

  

$$\left(\sum_{s=0}^{n} a_s(-k^2)^s\right)=0. \tag{3}$$

Equation (3) can be factorized as

$$a_n\prod_{s=1}^{n} (k^2-\mu_s^2)=0. \tag{4}$$

The $n$ roots of (3) are "mass parameters." For $\tilde{\varphi}(k)$ to be different from zero, we must have $k^2=\mu_s^2$. In general, $\mu_s^2$ is a complex number. An elementary exponential solution $e^{ikx}$ of Eq. (1) has a complex $k_0^2=\mathbf{k}^2+\mu_s^2$. The exponential $e^{ik_0x_0}$ is now no longer a pure plane wave, as the imaginary part of $k_0$ contributes with a factor $e^{\pm\kappa x_0}$ that blows up for $|x_0|\to\infty$. A real exponential certainly lies outside the space $\mathscr{S}'$ of tempered distributions. To enlarge this space in the sense we want, we have to take test functions that go to zero faster than any exponential of the type $e^{-a|x|}$ ($a>0$). Such is the case for the example of the spaces $\mathscr{S}_\alpha$ ($0<\alpha<1$).[5] They are formed by infinitely differentiable functions $\varphi(x)$ obeying

$$|\varphi^{(q)}|(x)\leqslant c_q e^{-a|x|^{1/\alpha}}, \tag{5}$$

where $a$ and $c_q$ are positive constants.

Any space $\mathscr{S}_\alpha$ contains all other spaces $\mathscr{S}_\beta$ for $0<\beta<\alpha$. It also contains $\mathscr{K}$, the space of infinitely differentiable functions with compact support.

All the spaces $\mathscr{S}_\alpha$ ($0<\alpha<1$) "accept" $e^{\kappa x}$ as a continuous linear functional. They also have the common feature that the Fourier transform $\tilde{\varphi}(k)$ of their fundamental functions $\varphi(x)$ belongs to $\mathscr{S}$ (on which they form a dense subset) and can be extended to the complex plane $z=k+i\kappa$ as entire analytic functions.[6] We shall see below that this is an essential property for the test functions we need.

## II. FUNDAMENTAL SPACE

We consider the fundamental space $\zeta$ as the set of all entire analytic functions $\varphi(z)$ which are rapidly decreasing on the real axis, i.e., $\varphi(z)|_{y=0}=\varphi(x)$ is a function belonging to the space $\mathscr{S}$ of Schwartz test functions. On the definition and interesting mathematical properties of the space $\zeta$, see Refs. 7–10. This space can be embedded in the space $\mathscr{S}$ of infinitely differentiable, rapidly decreasing functions. The vector space $\zeta$ can also be given a structure of countably normed space by introducing the family of norms:[11]

$$\|\varphi\|_n=\max_{|z|=n}|\varphi(z)| \quad (n\in N). \tag{6}$$

The norms are compatible and

$$\|\varphi\|_n<\|\varphi\|_{n+1}. \tag{7}$$

A scalar product in $\zeta$ is given by

$$(\psi,\varphi)=\int_{-\infty}^{+\infty} dk\,\bar{\psi}(k)\varphi(k). \tag{8}$$

The scalar product equation (8) allows the definition of a norm:

$$\|\varphi\|^2=(\varphi,\varphi). \tag{9}$$

The completion of $\zeta$ in the topology induced by the norm (9) leads to a Hilbert space $\mathscr{H}$, formed by square integrable functions ($\mathscr{H}\supset\zeta$).

The Fourier transform of the space $\mathscr{S}_\alpha(0<\alpha<1)$ is the space $\mathscr{S}^\alpha$ formed by entire analytic functions obeying the inequalities[6]

$$|x^k\varphi(x+iy)|\leq c_k e^{b|y|^{1/(1-\alpha)}} \quad (b,c_k>0). \tag{10}$$

It is evident that

$$\zeta\supset\mathscr{S}^\alpha \tag{11}$$

The Fourier transform $\mathscr{Z}$ of the space $\mathscr{K}$ is also formed by entire analytic functions, obeying (10) with $\alpha=0$. So,

$$\zeta\supset\mathscr{Z}. \tag{12}$$

The (anti)Fourier transforms $\tilde{\varphi}(x)$ of the functions $\varphi\in\zeta$ form the dual space $\tilde{\zeta}$:

$$\tilde{\varphi}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\ e^{ikx}\varphi(k) \tag{13}$$

$[\varphi(k+i\kappa)=\varphi(z)\in\zeta;\tilde{\varphi}(x)\in\tilde{\zeta}]$.

It follows from (11) and (12) that

$$\tilde{\zeta}\supset\mathscr{S}_\alpha \quad (0<\alpha<1), \tag{14}$$

$$\tilde{\zeta}\supset\mathscr{K}. \tag{15}$$

The inclusion $\zeta\supset\mathscr{Z}$ [or the Fourier-transformed (15)] is important for the examination of local properties of distributions in coordinate space ($\in\tilde{\zeta}'$). It means that among the test functions we can find those with compact support. The space of linear continuous functionals on $\zeta$ is the "conjugate" space $\zeta'$. If, as usual, we identify the conjugate $\mathscr{H}'$ with the Hilbert space $\mathscr{H}\supset\zeta$, then we have

$$\zeta'\supset\mathscr{H}\supset\zeta. \tag{16}$$

Analogously, we have the Fourier-transformed inclusions:

$$\tilde{\zeta}'\supset\mathscr{H}\supset\tilde{\zeta}. \tag{17}$$

## III. TEST FUNCTIONS AND DISTRIBUTIONS

In Sec. II we have defined four different spaces. First, we define the fundamental space $\zeta$, whose "test functions" are entire analytic and rapidly decreasing on the real axis. Second, the (anti)Fourier transformed space $\tilde{\zeta}$ is defined whose test functions are infinitely differentiable and such that multiplied by any linear exponential gives an integrable function. Third, we define the conjugate space $\zeta'$ formed by linear continuous functionals on $\zeta$, also called "distributions" on $\zeta$. Fourth, the conjugate space $\tilde{\zeta}'$ formed by distributions on $\tilde{\zeta}$ is defined.

Until now no direct connection has been established between $\zeta'$ and $\tilde{\zeta}'$. However, the Fourier transform of a distribution $f\in\tilde{\zeta}'$ is a distribution $g[=\mathscr{F}^{-1}(f)]$ defined through the relation (see Ref. 12)

$$(\mathscr{F}^{-1}(f),\mathscr{F}^{-1}(\tilde{\varphi}))=(g,\varphi)=(f,\tilde{\varphi}); \tag{18}$$

$g$ is a continuous linear functional on the fundamental space $\zeta$. In other words, $g$ is a distribution on $\zeta$, defined by (18) ($g \in \zeta'$).

For example, the functional $e^{\kappa x}/\sqrt{2\pi} \in \tilde{\zeta}'$ operates as

$$\left( \frac{e^{\kappa x}}{\sqrt{2\pi}}, \tilde{\varphi}(x) \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{\kappa x} \tilde{\varphi}(x). \tag{19}$$

However,

$$\mathscr{F}^{-1}(\tilde{\varphi}(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{-ikx} \tilde{\varphi}(x) = \varphi(k). \tag{20}$$

Equation (19) is seen to be the analytic continuation of $\varphi(k)$ to $i\kappa$, which is precisely $\varphi(z)|_{z=i\kappa}$. We can then identify $\mathscr{F}^{-1}(e^{\kappa x})$ with the continuous linear functional on $\zeta$, defined by[13]

$$(\delta(z-z_0), \varphi(z)) = \frac{1}{2\pi i} \int_L dz\, \frac{\varphi(z)}{z-z_0} = \varphi(z_0), \tag{21}$$

where $L$ is any positive loop around $z_0$.

Equation (21) is a simple example of a more general class of linear functionals on the fundamental space $\zeta$ of entire functions. If $G(z)$ is an analytic function having only a finite number of poles, and $\Gamma$ is a path (or line) on the complex $z$ plane, not touching any pole of $G(z)$, we can define the continuous linear functional:

$$(G, \varphi) = \int_\Gamma dz\, G(z) \varphi(z) \tag{22}$$

The functional defined by (22) on $\zeta$ (cf. Ref. 13) belongs to the conjugate space $\zeta'$. When $G(z)$ has a simple pole at $z_0$ and $\Gamma$ is any loop around $z_0$, then $G$ is a constant times $\delta(z-z_0)$.

In general, different $\Gamma$-curves define different functionals. However, when $G_1(z)=G_2(z)$ and $\Gamma_1$, $\Gamma_2$ have the same endpoints and can be brought into coincidence by continuous deformations, without crossing any pole of $G(z)$, then both functionals are equivalent. When a pole has to be crossed, a loop around it should be added and

$$G_1 = G_2 \pm 2\pi i c\, \delta(z-z_0) \quad (c = \text{Res } G(z)|_{z_0}). \tag{23}$$

We will be mainly interested in functionals with $G(z) = \Sigma_s c_s (z-z_s)^{-1}$.

When $\Gamma$ runs from $z=-\infty$ to $z=+\infty$, leaving all the poles of $G(z)$ below it (below $\Gamma$), we will say that the functional is of the retarded type. Similarly, when all the poles are left above $\Gamma$, we will say that the functional is of the advanced type.

It is easy to see that, according to (23), if $G(z)$ has $n$ simple poles,

$$G_{\text{ad}} = G_{\text{rt}} + 2\pi i \sum_{s=1}^{n} c_s \delta(z-z_s), \tag{24}$$

where ad (resp. rt) means advanced (resp. retarded). If no pole of $G(z)$ lies on the real axis $R$, we can define the functional $G_R$:

$$(G_R, \varphi) = \int_R dz\, G(z) \varphi(z). \tag{25}$$

We introduce the notation

$$\eta_z = \tfrac{1}{2} sg(\mathrm{Im}\, z), \tag{26}$$

where $sg(x)$ is the function sign of $x$.

Now $G_R$ can be related to $G_{\mathrm{ad}}$ and $G_{\mathrm{rt}}$:

$$G_{\mathrm{ad}} = G_R + 2\pi i \sum_{j=1}^{n} c_j \left( \frac{1}{2} - \eta_{z_j} \right) \delta(z - z_j), \tag{27}$$

$$G_{\mathrm{rt}} = G_R - 2\pi i \sum_{j=1}^{n} c_j \left( \frac{1}{2} + \eta_{z_j} \right) \delta(z - z_j). \tag{28}$$

Equations (27) and (28) imply Eq. (24).

From now on we will adopt the notation $[G(z)]_\Gamma$ for the functional defined by Eq. (22). For example, we will write

$$\delta(z - z_0) = \frac{1}{2\pi i} \frac{1}{[z - z_0]_{L_0}} \quad (L_0 = \text{loop around } z_0). \tag{29}$$

For a single pole, (24), (27), and (28) give

$$\frac{1}{[z - z_0]_{\mathrm{ad}}} = \frac{1}{[z - z_0]_{\mathrm{rt}}} + \frac{1}{[z - z_0]_{L_0}}, \tag{30}$$

$$\frac{1}{[z - z_0]_{\mathrm{ad}}} = \frac{1}{[z - z_0]_R} + \frac{(\frac{1}{2} - \eta_{z_0})}{[z - z_0]_{L_0}}, \tag{31}$$

$$\frac{1}{[z - z_0]_{\mathrm{rt}}} = \frac{1}{[z - z_0]_R} - \frac{(\frac{1}{2} + \eta_{z_0})}{[z - z_0]_{L_0}}. \tag{32}$$

## IV. PROPAGATORS AS FUNCTIONALS ON $\zeta$

In general, a propagator is a Green function for the equation of motion of a field theory. When the field evolution is given by the usual Klein–Gordon equation

$$(\Box + m^2)\varphi(x) = j(x).$$

We first look for a function $G(x)$ obeying

$$(\Box + m^2) G(x) = \delta(x) \tag{33}$$

under prescribed boundary conditions. Such a solution is most easily found by Fourier transforming (33):

$$(k^2 - m^2) G(k) = 1, \tag{34}$$

$$G(k) = \frac{1}{k^2 - m^2} = \frac{1}{k_0^2 - \omega^2}. \tag{35}$$

Of course, Eq. (35) only determines $G(k)$ for $k_0^2 \neq \omega^2$. A prescription must be given to handle the poles at $k_0 = \pm\omega$. This prescription is related to the boundary conditions to be imposed on the physical solutions of Eq. (4). Classically, $G(x)$ is determined by the choice of the retarded Green function:

$$G_{\text{Class}} = \frac{1}{[k_0^2 - \omega^2]_{\text{rt}}}. \tag{36}$$

For quantum field theories we impose Feynman's conditions, namely, positive energies should be propagated by a retarded solution and negative energies by an advanced one. This fixes the Feynman path $\Gamma \equiv F$, which runs below the pole at $k_0 = -\omega$ and above the pole at $k_0 = +\omega$:

$$G_{\text{Quan}} = \frac{1}{[k_0^2 - \omega^2]_F}. \tag{37}$$

The path $F$ can be taken along the real $k_0$ axis by the well-known procedure of giving the mass a small negative imaginary part:

$$G_F = \frac{1}{[k_0^2 - \omega^2]_R}, \tag{38}$$

where now $\omega^2 = \mathbf{k}^2 + m^2 - i\epsilon$.

For the higher-order equation (1) we can proceed in a similar way. The equation for the Fourier-transformed Green function can be written [cf. Eq. (4)]

$$\prod_{s=1}^{n} (k^2 - \mu_s^2) G(k) = 1, \tag{39}$$

from which we obtain

$$G(k) = \prod_{s=1}^{n} \frac{1}{k_0^2 - \omega_s^2}. \tag{40}$$

It is a matter of algebraic manipulation to write (40) into the form

$$G(k) = \sum_{s=1}^{n} \frac{b_s}{k_0^2 - \omega_s^2}, \tag{41}$$

where $b_s$ are appropriate $\omega_s$-dependent complex numbers. Again, for the complete determination of $G(k)$, it is necessary to take into account the boundary conditions. It is outside the scope of the present paper to give the physical considerations needed to choose the correct $\Gamma$ for $G(k)$. For quantum field theories the choice depends on the roots $(\mu_s^2)$ of the characteristic equation (3). When a root is real and positive, Feynman's path must be chosen. When the root is real and negative (see Ref. 14) or when it is complex (see Ref. 15), $\Gamma$ must be half-advanced and half-retarded. See Refs. 14–17 for the reasons behind those choices (which are related to unitarity of the $S$-matrix).

The propagator for the higher-order equation (1) is the functional

$$G(k) = \sum_{s=1}^{n} \frac{b_s}{[k_0^2 - \omega_s^2]_{\Gamma_s}}, \tag{42}$$

where $\Gamma_s$ is Feynman's path when $\mu_s^2>0$ and $\Gamma_s=\frac{1}{2}\mathrm{ad}+\frac{1}{2}\mathrm{rt}$ for all other terms. These propagators are particular cases of "tempered ultrahyperfunctions" (see Ref. 10).

We can now use the identity

$$\frac{1}{k_0^2-\omega^2}=\frac{1}{2\omega}\left(\frac{1}{k_0-\omega}-\frac{1}{k_0+\omega}\right)$$

to write Eq. (42) as a linear combination of "elementary" functionals of the form

$$G_j=\frac{1}{[z-\omega_j]_{\Gamma_j}}. \tag{43}$$

Furthermore, with (30)–(32) we can choose $\Gamma_j$ to coincide (at will) with rt, ad, or $R$, if we add to (43) a corresponding loop term, which also has the form (43) with $\Gamma_j\equiv L_j$.

## V. CONVOLUTION PRODUCTS

To solve the field equations with sources, the only known general method is the perturbative solution. The method leads to Feynman diagrams in which each propagator is represented by an internal line. The external lines represent the incoming and outgoing particles of the scattering process. The simplest diagrams (tree diagrams) give the lowest approximation. The next-to-lowest order corresponds to diagrams with an internal loop, implying an integration over an internal momentum variable. A second-order loop contains two lines and the integration is a "convolution product" of the involved propagators.

For two elementary functionals (43), the convolution product is given by[18]

$$(G_1*G_2,\varphi)=\int_{\Gamma_1}dz_1\int_{\Gamma_2}dz_2\,\frac{\varphi(z_1+z_2)}{(z_1-\omega_1)(z_2-\omega_2)}. \tag{44}$$

When $\Gamma_1=L_1$ and $\Gamma_2=L_2$, a double application of Cauchy's theorem gives [cf. (21)]

$$\delta(z-\omega_1)*\delta(z-\omega_2)=\delta(z-\omega_1-\omega_2). \tag{45}$$

Let us now take $\Gamma_1=\mathrm{ad}$ and $G_2=\delta(z-\omega_2)$. Equation (44) gives

$$(G_{\mathrm{ad}}*\delta(z-\omega_2),\varphi)=\int_{\mathrm{ad}}dz_1\,\frac{\varphi(z_1+\omega_2)}{(z_1-\omega_1)}. \tag{46}$$

A change of variable $z_1\to\xi-\omega_2$ (taking into account that ad goes below the pole at $z_1=\omega_1$, i.e., below the pole at $\xi=\omega_1+\omega_2$) leads to

$$\frac{1}{[z-\omega_1]_{\mathrm{ad}}}*\delta(z-\omega_2)=\frac{1}{[z-\omega_1-\omega_2]_{\mathrm{ad}}}. \tag{47}$$

Analogously,

$$\frac{1}{[z-\omega_1]_{\mathrm{rt}}}*(z-\omega_2)=\frac{1}{[z-\omega_1-\omega_2]_{\mathrm{rt}}}. \tag{48}$$

Thus, under convolution, the $\delta$-functional behaves as a translation operator for another $\delta$-functional and for an advanced or retarded functional.

If we want the convolution (44) with $\Gamma_1=R$ and $\Gamma_2=R$, we write

$$(G_R * G_R, \varphi) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \frac{\varphi(x+y)}{(x-\omega_1)(y-\omega_2)} \tag{49}$$

$$= \int_{-\infty}^{+\infty} dt \varphi(t) \int_{-\infty}^{+\infty} dx (x-\omega_1)^{-1}(t-x-\omega_2)^{-1} \tag{50}$$

using the formula[19]

$$\int_{-\infty}^{+\infty} dx \, \frac{1}{(x-a)(x-b)} = 2\pi i \, \frac{\eta_a - \eta_b}{a-b}.$$

We obtain

$$(G_R * G_R, \varphi) = 2\pi i \int_{-\infty}^{+\infty} dt \, \frac{\varphi(t)(\eta_{\omega_1} - \eta_{t-\omega_2})}{t-\omega_1-\omega_2} \tag{51}$$

$(\eta_{t-\omega_2} = \eta_{-\omega_2} = -\eta_{\omega_2})$, which means that

$$\frac{1}{[z-\omega_1]_R} * \frac{1}{[z-\omega_2]_R} = 2\pi i \, \frac{(\eta_{\omega_1} + \eta_{\omega_2})}{[z-\omega_1-\omega_2]_R}. \tag{52}$$

Now, using (45), (47), (48), and (52) and the relations (30)–(32), it is possible to obtain the complete convolution algebra of the elementary functionals (43). We will simply quote the following equations:

$$\frac{1}{[z-\omega_1]_{\mathrm{ad}}} * \frac{1}{[z-\omega_2]_{\mathrm{ad}}} = \frac{2\pi i}{[z-\omega_1-\omega_2]_{\mathrm{ad}}}, \tag{53}$$

$$\frac{1}{[z-\omega_1]_{\mathrm{rt}}} * \frac{1}{[z-\omega_2]_{\mathrm{rt}}} = -\frac{2\pi i}{[z-\omega_1-\omega_2]_{\mathrm{rt}}}, \tag{54}$$

$$\frac{1}{[z-\omega_1]_{\mathrm{ad}}} * \frac{1}{[z-\omega_2]_{\mathrm{rt}}} = 0. \tag{55}$$

Finally, if we define the $W$-functional as

$$W = \frac{1}{2} \frac{1}{[z-\omega]_{\mathrm{ad}}} + \frac{1}{2} \frac{1}{[z-\omega]_{\mathrm{rt}}}, \tag{56}$$

we obtain [cf. (53), (54), and (30)]

$$W * W = -\pi^2 \delta(z-\omega_1-\omega_2). \tag{57}$$

Equation (57) is interesting. The functional (56) is a generalization of Cauchy's principal value Green function to complex values of the pole. The convolution with $W$ is then a generalization of the Hilbert transform.[20] Equation (57) says that the inverse of a $W(\omega)$-transformation is a $W(-\omega)$-transformation,

$$f \to f' = W(\omega) * f, \tag{58}$$

$$f' \to f'' = W(-\omega) * f' = W(-\omega) * W(\omega) * f = -\pi^2 \delta * f, \tag{59}$$

$$f''=-\pi^2 f. \tag{60}$$

## VI. DISCUSSION

When a field theory implies higher-order equations of motion, the elementary solutions $e^{ikx}$ in general contain complex exponents. An exponentially increasing function lies outside the space of tempered distributions $\mathscr{S}'$, which at most accepts functions of "power growth" as linear continuous functionals. The needed extension in the space of distributions is most easily described by taking the space $\zeta$ of entire analytic functions rapidly decreasing on the real axis of the complex energy plane. The (anti)Fourier dual space $\tilde{\zeta}$ is formed by "extra-rapidly decreasing" functions which, multiplied by any increasing exponential (of order one), gives integrable functions. In this way, complex exponentials (or hyperbolic functions) and their linear combinations are continuous linear functionals $\in \tilde{\zeta}'$. We can also find in $\tilde{\zeta}$ all the infinitely differentiable functions with compact support. This inclusion makes it possible to define local properties of distributions, particularly the principle of microcausality.

Likewise, the Fourier transforms of these functions are well defined as functionals on $\zeta$. The propagators of higher-order field theories are represented by functionals characterized by the poles implied by the equations of motion, and by the associated contour of integration on the complex energy plane implied by the boundary conditions, imposed on the solutions for physical reasons.

All propagators can be represented as linear combinations of elementary (one pole) functionals whose convolution algebra can readily be determined.

For the usual second-order case (normal Klein–Gordon equation), the propagator is Feynman's functional associated with the real axis of the energy plane or, at most, with a contour that runs along the real axis with infinitesimal deviation to avoid the poles. Any subsequent operation with Feynman propagators can be handled in the space of tempered distributions. For higher-order equations the situation changes; the poles of the propagators are well outside the real axis. The contour of integration must in general "move" freely all over the complex plane, avoiding the poles in an appropriate way. The test functions must not hinder or obstruct those integrations with unphysical singularities. It is then most reasonable that the fundamental space $\zeta$ shall contain only entire analytic functions.

The elementary, single-pole functionals obey a simple convolution algebra. The complex $\delta$-functional is a translation operator for another $\delta$-functional and for the advanced and retarded (or $W$) Green functions. The convolution product of two advanced (resp. retarded) functionals gives another advanced (resp. retarded) functional. The convolution of an advanced times a retarded functional gives zero. These results are expected in view of the fact that the Fourier transform of an advanced (resp. retarded) function is zero for $t<0$ (resp. $t>0$). It is also interesting that the convolution of two $W$-functionals gives a complex $\delta$-functional. On one hand, this is related to properties of the Hilbert transform.[20] On the other hand, it is related to the unitarity of the physical scattering matrix.[17]

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