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On the quantization of reducible gauge systems

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Reducible gauge theories with constraints linear in the momenta are quantized. The equivalence of the reduced phase space quantization, Dirac quantization, and BRST quantization is established, provided one transforms appropriately the Dirac wave functions under changes of representation of the constraint surface and of the reducibility functions. The ghosts of ghosts are found to play a crucial role in the equivalence proof.

I. INTRODUCTION

Gauge theories can be quantized according to at least three different methods:

(i) The reduced phase space method quantizes only the gauge invariant functions and is for that reason physically quite appealing. However, it is often not tractable because it requires the explicit finding of a complete set of gauge invariant functions.

(ii) The Dirac method realizes all the dynamical variables (gauge invariant and nongauge invariant ones) as operators in some linear space of states, and selects the physical states by means of a subsidiary condition.

(iii) The BRST method increases further the redundancy in the description of the system by introducing ghosts. The physical states are again selected by means of a subsidiary condition.

It is easy to check that the three different approaches to the quantization of gauge systems are equivalent in the case of simple constraints (see, for instance, Ref. 1). The question of their equivalence for arbitrary systems is more subtle and has attracted recently a considerable amount of interest.[1-15] Because the problem of "quantization" is inherently ambiguous (many different quantum systems possess the same $\hbar\to 0$ limit), the question of equivalence is actually ill-defined in the absence of a definite choice of quantization prescriptions. For this reason, a conclusive analysis should either exhibit correspondence rules that insure equivalence of the three quantization methods, or prove the inexistence of such rules.

The previous works on the equivalence question are all devoted to independent ("irreducible") first class constraints. The purpose of this paper is to investigate equivalence in the case of first class reducible constraints, for which some constraints are consequences of the others. The reducible case raises new problems with respect to the irreducible one. For instance, in order to get a consistent Dirac quantization, it is necessary not only that the constraints remain first-class quantum mechanically, but also that they remain dependent. Otherwise, the number of degrees of freedom in the classical and quantum theories would be different. Furthermore, in the BRST formalism, ghosts of ghost are necessary besides the standard ghosts, and it is of interest to understand their role in definite quantum models.

As the general question of equivalence is quite intricate, we restrict in this paper the analysis to reducible first class systems with constraints that are linear, homogeneous in the

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momenta. This case is already of interest since it covers p-form gauge fields and illustrate very well the crucial role played by the ghosts of ghosts. The corresponding irreducible models have been investigated in Refs. 2, 7, and 9.

In the framework of the quantization rules where the physical wave functions are taken to be densities of weight one-half in the configuration space, we show that the three methods of quantization yield the same physical spectrum, provided one transforms appropriately the Dirac wave functions under a redefinition of the constraints and of the reducibility coefficients. In order to get a consistent Dirac quantization, we also find it necessary to correct the naive Dirac operator condition by an extra term. This extra term, as well as the transformation properties of the Dirac wave functions, are quite natural from the BRST point of view. Our results generalize to the reducible case those derived by Tuynman in Ref. 9 for irreducible constraints.

Our paper is organized as follows. In the next section, we describe explicitly the models considered in this paper. We then derive the classical BRST charge that captures all the identities fulfilled by those models (Sec. III). We turn next to the quantization of the models, first along the lines of the reduced phase space method (Sec. IV), and then along those of the Dirac approach (Sec. V). We find it crucial to improve the naive Dirac quantum constraints by an appropriate term that makes them anomaly free, and we derive this term by geometric arguments. Section VI establishes the equivalence of the reduced phase space and Dirac methods by developing further the geometric interpretation. The BRST quantization and its equivalence with the other methods of quantization are given in Sec. VII. The key role played by the ghosts and ghosts of ghosts is particularly stressed since they precisely yield the anomaly cancelling term of the Dirac quantization method. Finally, Sec. VIII is devoted to concluding comments.

II. THE MODELS

The systems considered in this paper are described by n pairs of canonically conjugate variables (q^i,p_i). They are subject to m_0 bosonic constraints,

$$G_{a_0}(q^i,p_j)=0, \quad a_0=1,\ldots,m_0, \tag{2.1}$$

which we take to be linear in the momenta

$$G_{a_0}(q^i,p_j)=\xi^j_{a_0}(q^i)p_j. \tag{2.2}$$

The constraints are first class, i.e.,

$$\{G_{a_0},G_{b_0}\}=C^{c_0}_{a_0b_0}G_{c_0}, \tag{2.3}$$

where $\{,\}$ stands for the Poisson bracket in the phase space spanned by the variables (q^i,p_i). Since the constraints are linear in the momenta, the structure functions $C^{c_0}_{a_0b_0}$ can be taken to depend only on the coordinates q^i. Furthermore, the gauge transformation of a function $f(\mathbf{q})$ defined on the configuration space $\mathscr{Q}$, is

$$\delta f(\mathbf{q})=\epsilon^{a_0}\{f,G_{a_0}\}=\epsilon^{a_0}\xi^i_{a_0}\frac{\partial f}{\partial q_i}=\epsilon^{a_0}\xi_{a_0}(f), \tag{2.4}$$

and depends also only on $\mathbf{q}$. The vector fields ξ_{a_0} define the gauge transformations in the configuration space and are tangent to the gauge orbits. By inserting Eq. (2.2) in Eq. (2.3) we obtain

$$2\xi^j_{[b_0}\xi^i_{a_0],j}=\xi^i_{a_0,j}\xi^j_{b_0}-\xi^i_{b_0,j}\xi^j_{a_0}=C^{c_0}_{a_0b_0}\xi^i_{c_0}, \tag{2.5a}$$

where $_{,j}$ denotes differentiation with respect to q^j. Equation (2.5a) can be rewritten as

$$\mathcal{L}_{\xi_{a_0}}\xi_{b_0}=[\xi_{a_0},\xi_{b_0}]=-C^{c_0}_{a_0b_0}\xi_{c_0}, \tag{2.5b}$$

where [,] is the Lie bracket and $\mathcal{L}_{\xi_{a_0}}$ is the Lie derivative operator along ξ_{a_0}.

The gauge transformations generated by the constraints are said to be reducible when there exist functions $Z^{a_0}_{a_1}\not\approx 0$ such that

$$Z^{a_0}_{a_1}G_{a_0}=0. \tag{2.6}$$

Because the constraints are linear and homogeneous in the momenta, we may assume the reducibility functions $Z^{a_0}_{a_1}$ to depend only on q^i; Eq. (2.6) is then equivalent to

$$Z^{a_0}_{a_1}(\mathbf{q})\xi_{a_0}(\mathbf{q})=0,\quad a_1=1,\ldots,m_1. \tag{2.7}$$

The functions $Z^{a_0}_{a_1}$ are required to exhaust all the relations among the fields ξ_{a_0}.

It might happen that the set $\{Z^{a_0}_{a_1}\}$ is overcomplete, i.e., there exists a set of functions $Z^{a_1}_{a_2}$ such that

$$Z^{a_1}_{a_2}Z^{a_0}_{a_1}\approx 0. \tag{2.8}$$

Equation (2.8) means that $Z^{a_1}_{a_2}Z^{a_0}_{a_1}$ can be written as a combination of the constraints. Again, the functions $Z^{a_1}_{a_2}$ may be taken to depend only on $\mathbf{q}$. Since the constraints depend on the momenta but the Z's do not, the only possibility is that Eq. (2.8) is valid strongly. In general one finds a tower of reducibility equations:

$$Z^{a_k}_{a_{k+1}}Z^{a_{k-1}}_{a_k}=0,\quad a_k=1,\ldots,m_k,\quad k=1,\ldots,L-1. \tag{2.9a}$$

The tower stops with functions $Z^{a_{L-1}}_{a_L}(\mathbf{q})$ that are linearly independent,

$$\lambda^{a_L}(\mathbf{q})Z^{a_{L-1}}_{a_L}=0\Rightarrow\lambda^{a_L}=0. \tag{2.9b}$$

The theory has then order of reducibility equal to L. The number of independent gauge generators is

$$m=\sum_{k=0}^{L}(-)^k m_k. \tag{2.10}$$

It will be convenient to choose the Z's such that

$$Z^{a_{k-1}*}_{a_k}=(-)^k Z^{a_{k-1}}_{a_k}. \tag{2.11}$$

Non linearly independent gauge generators appear in a physical theory when one cannot isolate a subset of independent constraints without violating explicit covariance, locality, or global conditions. A well-known example is the case of a p-form gauge field $\mathbf{A} = (1/p!)A_{i_1 \dots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, the canonically conjugated pairs being $(A_{i_1 \dots i_p}(x), \pi^{i_1 \dots i_p}(x))$. For such a field the constraints are

$$\pi^{i_1 \dots i_p}{}_{,i_1} = 0$$

(the generalization of the Gauss law) which are not independent because of the antisymmetrization in the indices $i_1, \dots, i_p (\pi^{i_1 i_2 \dots i_p}{}_{,i_1 i_2} \equiv 0)$.

III. THE CLASSICAL BRST GENERATOR

A. Identities

The functions $\xi^i_{a_0}(\mathbf{q})$ and $Z^{a_{k-1}}_{a_k}(\mathbf{q})$ fulfill a series of identities that can be derived from Eqs. (2.5) and (2.6) by differentiation and use of the symmetry of the second partial derivatives. For instance, the Jacobi identity,

$$\{\{G_{[a_0}, G_{b_0}\}, G_{c_0]}\} = 0,$$

leads to the equation:

$$(\xi^{\ i}_{[a_0} C^{e_0}_{b_0 c_0],i} - C^{d_0}_{[a_0 b_0} C^{e_0}_{c_0] d_0}) G_{e_0} = 0.$$

For a reducible theory, this means that there exist functions $M^{a_1}_{a_0 b_0 c_0}(\mathbf{q})$ such that

$$\xi^{\ i}_{[a_0} C^{e_0}_{b_0 c_0],i} = C^{d_0}_{[a_0 b_0} C^{e_0}_{c_0] d_0} - \tfrac{2}{3} M^{a_1}_{a_0 b_0 c_0} Z^{e_0}_{a_1}, \tag{3.1}$$

the last term not being present when the theory is irreducible.

Similarly, by differentiating the identity (2.7) along the orbits, one gets

$$\xi^i_{b_0}(Z^{a_0}_{a_1} \xi^j_{a_0})_{,i} = 0,$$

or, in terms of the vector fields ξ_{a_0},

$$[\mathscr{L}_{\xi_{b_0}}(Z^{a_0}_{a_1} \xi_{a_0})]^j = \xi^i_{b_0}(Z^{a_0}_{a_1} \xi^j_{a_0})_{,i} - \xi^j_{b_0,i} Z^{a_0}_{a_1} \xi^i_{a_0} = 0,$$

i.e.,

$$\xi^i_{b_0} Z^{a_0}_{a_1,i} \xi^j_{a_0} + Z^{a_0}_{a_1}(\xi^i_{b_0} \xi^j_{a_0,i} - \xi^j_{b_0,i} \xi^i_{a_0}) = 0.$$

Because of Eq. (2.5a), this is equivalent to

$$(\xi^i_{b_0} Z^{a_0}_{a_1,i} + Z^{c_0}_{a_1} C^{a_0}_{c_0 b_0}) \xi^j_{a_0} = 0.$$

The completeness of the functions $Z^{a_0}_{a_1}$ implies then the existence of functions $D^{b_1}_{b_0 a_1}(\mathbf{q})$ such that

$$\xi^i_{b_0} Z^{a_0}_{a_1,i} + Z^{c_0}_{a_1} C^{a_0}_{c_0 b_0} = D^{b_1}_{b_0 a_1} Z^{a_0}_{b_1}. \tag{3.2}$$

If one contracts this identity with $Z^{b_0}_{c_1}$, sums over b_0 and uses Eq. (2.7), one gets

$$Z^{b_0}_{c_1} Z^{c_0}_{a_1} C^{a_0}_{c_0 b_0} = Z^{b_0}_{c_1} Z^{a_0}_{b_1} D^{b_1}_{b_0 a_1}.$$

The symmetric part of this equation in a_1, c_1, reads

$$D^{b_1}_{b_0(a_1} Z^{b_0}_{c_1)} Z^{a_0}_{b_1} = 0.$$

The completeness of $Z^{a_0}_{b_1}$ implies then the existence of functions $B^{b_2}_{d_1 b_1}(\mathbf{q})$ such that

$$D^{a_1}_{b_0 c_1} Z^{b_0}_{d_1} + D^{a_1}_{b_0 b_1} Z^{b_0}_{c_1} = B^{b_2}_{d_1 c_1} Z^{a_1}_{b_2}, \tag{3.3}$$

the right-hand side of this equation being absent for reducible theories of order $L=1$.

B. The BRST generator

The identities (3.1)–(3.3) are only a few of a long list, which can be obtained by further differentiation. The most powerful way to capture all these identities is to introduce the BRST generator.[16] The identities are then contained in a unique equation, namely

$$\{\Omega,\Omega\} = 0, \tag{3.4}$$

where $\Omega = \Omega(q^i, p_j, \eta^{a_k}, \mathscr{P}_{b_k})$, the BRST generator, is a fermionic function in an extended phase space $\mathscr{E}$ including canonically conjugate pairs of ghosts $(\eta^{a_k}, \mathscr{P}_{a_k})$, $k=0,\ldots,L$, besides the original canonical variables,

$$\{\mathscr{P}_{a_k}, \eta^{b_k}\} = -\delta^{b_k}_{a_k}, \tag{3.5a}$$

$$gh(\eta^{a_k}) = -gh(\mathscr{P}_{a_k}) = k+1. \tag{3.5b}$$

In Eq. (3.4) the bracket is the Poisson bracket in $\mathscr{E}$; we remark that the nilpotency condition (3.4) is not trivial because the Poisson bracket for fermionic quantities is symmetric.

The BRST generator is unique, up to canonical transformations in $\mathscr{E}$. For reducible theories, Ω has the form[17,18]

$$\Omega = \eta^{a_0} G_{a_0} + \sum_{k=0}^{L-1} \eta^{a_{k+1}} Z^{a_k}_{a_{k+1}} \mathscr{P}_{a_k} + \text{"more,"} \tag{3.6}$$

where "more" does not contains terms of the already indicated form.

In our case the G's and the Z's are bosonic; in order that Ω be fermionic, the ghosts belonging to an even generation $(\eta^{a_0}, \mathscr{P}_{a_0}, \eta^{a_2}, \mathscr{P}_{a_2}, \ldots)$ must be fermionic, while those belonging to an odd generation must be bosonic. Due to the choice (2.11), Ω turn out to be real if

$$\eta^{a_k *} = \eta^{a_k}, \quad \mathscr{P}_{a_k}{}^* = (-)^{k+1} \mathscr{P}_{a_k}. \tag{3.7}$$

The generator Ω can be built by means of a recursive method (see Ref. 1):

$$\Omega = \sum_{p>0} \overset{(p)}{\Omega}, \tag{3.8}$$

where $\overset{(p+1)}{\Omega}$ solves the equation

$$\delta \overset{(p+1)}{\Omega} + \overset{(p)}{\Delta} = 0, \tag{3.9}$$

with

$$\overset{(p)}{\Delta} = \frac{1}{2} \sum_{k=0}^{p} \{ \overset{(p-k)}{\Omega}, \overset{(k)}{\Omega} \}_{\text{orig}} + \frac{1}{2} \sum_{k=1}^{p} \sum_{s=0}^{k-1} \{ \overset{(p-k+s+1)}{\Omega}, \overset{(k)}{\Omega} \}_{\eta^{a_s}, \mathscr{P}_{a_s}}, \tag{3.10a}$$

$$\overset{(0)}{\Omega} = \eta^{a_0} G_{a_0}. \tag{3.10b}$$

In Eq. (3.10) $\{,\}_{\text{orig}}$ is the bracket with respect to the original canonical variables, while $\{,\}_{\eta^{a_s}, \mathscr{P}_{a_s}}$ is the bracket with respect to the pair $(\eta^{a_s}, \mathscr{P}_{a_s})$. In Eq. (3.9) δ is the Koszul–Tate operator. In our case δ reads explicitly

$$\delta q^i = 0, \quad \delta p_j = 0, \tag{3.11a}$$

$$\delta \eta^{a_0} = 0, \quad \delta \mathscr{P}_{a_0} = -G_{a_0}, \tag{3.11b}$$

$$\delta \eta^{a_k} = 0, \quad \delta \mathscr{P}_{a_k} = -Z^{a_{k-1}}_{a_k} \mathscr{P}_{a_{k-1}}, \quad k = 1, \ldots, L, \tag{3.11c}$$

and is clearly nilpotent ($\delta^2 = 0$), because the reducibility Eq. (2.9a) holds strongly when the constraints are linear in the momenta [in the general case, additional terms are needed in (3.11) to achieve nilpotency].

The existence of Ω is established in Ref. 18. Its explicit form for arbitrary L is cumbersome and will not be needed here. We shall only need: (i) the crucial fact that Ω is linear in the momenta $(p_i, \mathscr{P}_{a_k})$ (Proposition 1); and (ii) the identities in Propositions 2, 3, and 4 below.

Proposition 1: In the case of constraints linear in the momenta p_j, the BRST generator Ω can be taken to be linear in the momenta $(p_j, \mathscr{P}_{a_k})$.

Proof: One has

$$\overset{(0)}{\Delta} = \tfrac{1}{2} \{ \overset{(0)}{\Omega}, \overset{(0)}{\Omega} \}_{\text{orig}} = \tfrac{1}{2} \eta^{a_0} \eta^{b_0} \{ G_{a_0}, G_{b_0} \} = \tfrac{1}{2} \eta^{a_0} \eta^{b_0} C^{c_0}_{a_0 b_0} G_{c_0} \Rightarrow \overset{(1)}{\Omega} = \tfrac{1}{2} \eta^{a_0} \eta^{b_0} C^{c_0}_{a_0 b_0} \mathscr{P}_{c_0}.$$

So let us suppose that all the $\overset{(k)}{\Omega}$'s for $k \leqslant p$ are linear in the momenta (this is true for $p=1$). Then $\overset{(p)}{\Delta}$ is linear in the momenta from (3.10). Because of the definition (3.11) for δ, we find then that $\overset{(p+1)}{\Omega}$ in Eq. (3.9) can be taken to be linear in the momenta. ◇

By expanding out $\{\Omega, \Omega\} = 0$, one finds at the lowest orders in the ghosts and their momenta the identities (2.5), (2.7), and (2.9), since $C^{c_0}_{a_0 b_0}/2$ is the coefficient of $\eta^{a_0} \eta^{b_0} \mathscr{P}_{a_0}$ in Ω, as we just got. Similarly, by calling $M^{a_1}_{a_0 b_0 c_0}/3$, the coefficient of $\eta^{a_0} \eta^{b_0} \eta^{c_0} \mathscr{P}_{a_1}$, one finds the identity (3.1). In addition one has the following.

Proposition 2: There exist functions $D^{c_k}_{b_0 a_k}$ such that

$$\xi^i_{b_0} Z^{a_{k-1}}_{a_k}{}_{,i} + D^{a_{k-1}}_{b_0 c_{k-1}} Z^{c_{k-1}}_{a_k} = D^{c_k}_{b_0 a_k} Z^{a_{k-1}}_{c_k}, \quad k = 1, \ldots, L. \tag{3.12}$$

Proposition 3: There exist functions $B^{b_{k+1}}_{c_1 a_k}$ such that

$$D^{a_k}_{b_0c_k}Z^{b_0}_{d_1}-B^{a_k}_{d_1b_{k-1}}Z^{b_{k-1}}_{c_k}=B^{b_{k+1}}_{d_1c_k}Z^{a_k}_{b_{k+1}},\quad k=1,\ldots,L-1, \tag{3.13a}$$

$$D^{a_L}_{b_0c_L}Z^{b_0}_{d_1}=B^{a_L}_{d_1b_{L-1}}Z^{b_{L-1}}_{c_L}. \tag{3.13b}$$

Proposition 4: There exist functions $M^{d_k}_{a_0b_0c_{k-1}}$ such that

$$\xi^i_{[a_0}D^{c_k}_{b_0]a_{k,i}}=-\tfrac{1}{2}C^{e_0}_{a_0b_0}D^{c_k}_{e_0a_k}+D^{d_k}_{[a_0|a_k|}D^{c_k}_{b_0]d_k}+M^{c_k}_{a_0b_0f_{k-1}}Z^{f_{k-1}}_{a_k}+M^{f_{k+1}}_{a_0b_0a_k}Z^{c_k}_{f_{k+1}},\quad k=1,\ldots,L-1. \tag{3.14a}$$

$$\xi^i_{[a_0}D^{c_L}_{b_0]a_{L,i}}=-\tfrac{1}{2}C^{e_0}_{a_0b_0}D^{c_L}_{e_0a_L}+D^{d_L}_{[a_0|a_L|}D^{c_L}_{b_0]d_L}+M^{c_L}_{a_0b_0f_{L-1}}Z^{f_{L-1}}_{a_L}. \tag{3.14b}$$

The proof of these propositions goes as follows. Define $D^{c_0}_{a_0b_0}\equiv -C^{c_0}_{a_0b_0}$, and $B^{b_2}_{d_1c_1}/2$ to be the coefficient of $\eta^{d_1}\eta^{c_1}\mathscr{P}_{b_2}$ in Ω. Similarly, define $-D^{b_k}_{b_0a_k}$, $B^{b_{k+2}}_{d_1c_{k+1}}$ and $M^{a_{k+1}}_{a_0b_0c_k}$ ($k\neq0$) to be the respective coefficients of $\eta^{b_0}\eta^{a_k}\mathscr{P}_{b_k}$, $\eta^{d_1}\eta^{c_{k+1}}\mathscr{P}_{b_{k+2}}$ and $\eta^{a_0}\eta^{b_0}\eta^{c_k}\mathscr{P}_{a_{k+1}}$ in Ω. Then, the vanishing of the coefficients of $\eta^{a_k}\eta^{b_0}\mathscr{P}_{a_{k-1}}$, $\eta^{d_1}\eta^{c_k}\mathscr{P}_{a_k}$, and $\eta^{a_k}\eta^{b_0}\eta^{a_0}\mathscr{P}_{c_k}$ in $\{\Omega,\Omega\}$ yield respectively (3.12), (3.13), and (3.14). Note that since the ghosts η^{a_1} are bosonic, the coefficients $B^{a_2}_{c_1d_1}$ are symmetric in (c_1,d_1),

$$B^{a_2}_{c_1d_1}=B^{a_2}_{d_1c_1}. \tag{3.15}$$

Note also that the identity (3.12) reduces to the identity (3.2) for $k=1$, and (3.13) (with $B^{c_1}_{a_1b_0}=-D^{c_1}_{b_0a_1}$) becomes (3.3) for $k=1$.

It is of interest to write explicitly the BRST charge for a reducible theory of order $L=1$. One gets

$$\Omega=\eta^{a_0}G_{a_0}+\eta^{a_1}Z^{a_0}_{a_1}\mathscr{P}_{a_0}+\tfrac{1}{2}\eta^{a_0}\eta^{b_0}C^{c_0}_{a_0b_0}\mathscr{P}_{c_0}-\eta^{b_0}D^{a_1}_{b_0b_1}\eta^{b_1}\mathscr{P}_{a_1}+\tfrac{1}{3}\eta^{a_0}\eta^{b_0}\eta^{c_0}M^{a_1}_{a_0b_0c_0}\mathscr{P}_{a_1}. \tag{3.16}$$

IV. REDUCED PHASE SPACE QUANTIZATION

Because the gauge transformations are defined within the space $\mathscr{Q}$ of the q's, one can introduce the reduced configuration space $\mathscr{Y}\equiv\mathscr{Q}/\mathscr{G}$ as the quotient of the configuration space $\mathscr{Q}$ by the gauge orbits in $\mathscr{Q}$. Let y^α, $\alpha=1,\ldots,N$, be coordinates in the reduced configuration space. N is equal to n minus the number m of independent constraints. One has

$$\{y^\alpha(\mathbf{q}),G_{a_0}\}=0. \tag{4.1}$$

Let $\pi_\alpha(\mathbf{q},\mathbf{p})$ be the gauge invariant momenta conjugate to y^α,

$$\{\pi_\alpha,G_{a_0}\}\approx0,\quad \{y^\alpha,\pi_\beta\}\approx\delta^\alpha_\beta. \tag{4.2}$$

The variables y^α and π_α define a standard unconstrained system, the "reduced system" associated with the original gauge system. The reduced phase space quantization consists in quantizing this reduced system without worrying about its origins.

So, let us consider a nonconstrained system described classically by coordinates and momenta (y^α,π_α), $\alpha=1,\ldots,N$. At a given time, the quantum state of the system is given by a wave function $\varphi(y^\alpha)$ belonging to a Hilbert space. It is convenient to define the inner product in this space as:

$$(\varphi,\psi)=\int d^N y \varphi^*(y^\alpha)\psi(y^\alpha), \tag{4.3}$$

In order that the inner product (4.3) be invariant under coordinate changes, the wave functions must behave as scalar densities of weight 1/2:

$$y^\alpha \to y'^\alpha = y'^\alpha(y^\alpha) \Rightarrow \varphi(y^\alpha) \to \varphi'(y'^\alpha) = \left|\frac{\partial y^\alpha}{\partial y'^\alpha}\right|^{1/2} \varphi(y^\alpha(y'^\alpha)). \tag{4.4}$$

The product $\varphi^*\psi$ is then a density of weight 1, i.e., $d^N y \varphi^* \psi$ is a N-form in $\mathscr{Y}$. Since the integral of a N-form over a N-dimensional manifold is intrinsically defined (does not require an extra integration measure), the convention of taking the wave functions to be densities of weight 1/2 is convenient in the absence of a natural integration measure. (In practice, however, the manifold comes equipped with an integration measure ν. For instance, if it is a Riemannian manifold, one can take $\nu = g^{1/2}$. In that case, one can replace the wave functions by scalars, by redefining them as $\varphi \to \nu^{-1/2}\varphi$. Of course, this procedure also requires a redefinition of the operators in order to keep the matrix elements unchanged.)

The observables that are linear in the momenta π_α conjugate to y^α,

$$\mathbf{a}=a^\alpha(\mathbf{y})\pi_\alpha \quad \text{(classically)}, \tag{4.5}$$

possess a natural geometric interpretation since they define vector fields on the manifold of the y's. Their quantum version reads

$$\mathbf{a}=\tfrac{1}{2}(a^\alpha(\mathbf{y})\pi_\alpha+\pi_\alpha a^\alpha(\mathbf{y})), \tag{4.6}$$

and is formally Hermitian for the scalar product (4.3) whenever a^α is real. The action of $\mathbf{a}$ on a wave function yields $-i$ times its Lie derivative (as a density of weight 1/2),

$$(\mathbf{a}\varphi)(\mathbf{y}) = -i\mathscr{L}_{\mathbf{a}}\varphi \tag{4.7a}$$

$$= -i(a^\alpha \varphi_{,\alpha} + \tfrac{1}{2}a^\alpha_{,\alpha}\varphi). \tag{4.7b}$$

V. DIRAC QUANTIZATION

We now turn to the Dirac quantization, where the wave funtions are taken to depend on all the coordinates q^i and not just on the gauge invariant ones. In order to remove the unphysical degrees of freedom, one imposes on the physical states the condition

$$\hat{G}_{a_0}\psi(\mathbf{q})=0, \tag{5.1}$$

where $\hat{G}_{a_0}$ is the realization of each constraint as an operator in the space of the wave functions. Since the quantum realization of any classical function is ambiguous (factor ordering problem), we should carefully define the operator $\hat{G}_{a_0}$. Because the constraints are linear in the momenta, it is natural to take $\hat{G}_{a_0}\psi$ to be the Lie derivative of ψ along ξ_{a_0}. However, the Lie derivative of ψ is ill-defined as long as one does not give the transformation rules for ψ. So the question is: which object is ψ? In order to gain insight into this question, let us first consider the simple abelian case, with a configuration space isomorphic to R^n.

A. Abelian case

Let us thus assume that the coordinates q^i can be split as $q^i \equiv (y^\alpha, Q^A)$, in such a way that the constraints $G_{a_0} \approx 0$ are equivalent to

$$G_A \equiv P_A \approx 0, \tag{5.2}$$

P_A being the momenta conjugate to Q^A. (For an arbitrary gauge system, this can always be achieved lovally.[1]) The reduced phase space for the system is then the space of the y^α and the π_α. The discussion of Sec. IV shows that the physical wave functions depend only on y^α, i.e., are annihilated by $\hat{P}_A$. Furthermore, the scalar product (4.3) can be rewritten as

$$(\varphi,\psi) = \int d^N y d^m Q \prod_{A=1}^{m} \delta(Q^A)\varphi^*(y^\alpha)\psi(y^\alpha). \tag{5.3}$$

This expression is invariant if we transform the wave functions not only as densities of weight 1/2 under changes of the physical coordinates y^α (as we already pointed out), but also as *scalars under changes of the pure gauge coordinates* Q^A.

This asymmetric behavior of the wave functions under change of coordinates is undesirable since in practice, one cannot split the q^i into the y^α and the Q^A. It is thus necessary to reformulate the transformation properties of the wave functions in a manner that treats the coordinates more uniformly. To that end, we rewrite (5.3) as

$$(\varphi,\psi) = \int d^N y d^m Q \prod_{A=1}^{m} \delta(\chi^A)\,|\det\{\chi^A,G_B\}|\,\varphi^*(y^\alpha)\psi(y^\alpha), \tag{5.4}$$

where $\chi^A = 0$ define good gauge conditions. The inclusion of the "Faddeev–Popov" determinant $\det\{\chi^A,G_B\}$ makes (5.4) independent of χ^A, and equal to (5.3) (to see this, take $\chi^A = Q^A$). We then observe that under a redefinition of the Q's,

$$Q^A \to Q'^A = Q'^A(Q^A, y^\alpha), \tag{5.5}$$

the momenta P_A conjugate to Q^A transform as

$$P_A = \frac{\partial Q'^B}{\partial Q^A} P'_B. \tag{5.6}$$

Hence, the original constraints (5.2) are not identical with P'_A but differ from the constraints $G'_A \equiv P'_A \approx 0$ adapted to the Q'^A-description by a q^i-dependent linear transformation. In other words, in order to reach the description of the system in terms of the new pure gauge variables Q'^A, one must supplement the change (5.5) by a redefinition of the constraints.

Now, the scalar product (5.4) is invariant under

$$G_A \to G'_A = A_A{}^B G_B, \tag{5.7}$$

if and only if the wave functions transforms as densities of weight $-1/2$ for (5.7). Accordingly, we shall postulate that the wave functions transforms as densities of weight 1/2 for

$$q^i \to q'^i = q'^i(q^j) \tag{5.8}$$

and as densities of weight $-1/2$ for (5.7),

$$\psi(\mathbf{q}) \to \psi'(\mathbf{q}') = |\det A|^{-1/2} \left|\frac{\partial \mathbf{q}}{\partial \mathbf{q}'}\right|^{1/2} \psi(\mathbf{q}). \tag{5.9}$$

This automatically guarantees that ψ is a scalar under changes of coordinates along the gauges orbits, since the redefinition of the constraints (5.6) ($P_A \to P'_B = (\partial Q^A/\partial Q'^B)P_A$) compensates the Jacobian coming from the density weight of ψ,

$$\psi(\mathbf{y},\mathbf{Q}) \to \psi'(\mathbf{y},\mathbf{Q}') = \left|\det \frac{\partial \mathbf{Q}}{\partial \mathbf{Q}'}\right|^{-1/2} \left|\det \frac{\partial \mathbf{Q}}{\partial \mathbf{Q}'}\right|^{1/2} \psi(\mathbf{y},\mathbf{Q}).$$

The conclusion is that in order to treat uniformly the coordinates, one must require the Dirac wave functions to transform non trivially as in (5.9) under a redefinition of the constraints.

B. General case—Definition of Lie derivative of ψ

In the general case (2.1)–(2.9) of reducible constraints, not only can one "rotate" the constraints,

$$G_{a_0} \to G'_{a_0}(\mathbf{q}) = A_{a_0}{}^{b_0}(\mathbf{q}) G_{b_0}(\mathbf{q}), \tag{5.10}$$

but one can also transform the reducibility functions $Z^{a_{k-1}}_{a_k}$ as

$$Z^{a_{k-1}}_{a_k}(\mathbf{q}) \to Z'^{a_{k-1}}_{a_k}(\mathbf{q}) = A_{a_k}{}^{b_k}(\mathbf{q}) Z^{b_{k-1}}_{b_k}(\mathbf{q}) (A^{-1})_{b_{k-1}}{}^{a_{k-1}}(\mathbf{q}). \tag{5.11}$$

We shall generalize the previous transformation laws by requiring that the Dirac wave functions transform as

$$\psi(\mathbf{q}) \to \psi'(\mathbf{q}') = \prod_{k=0}^{L} |\det A_{a_k}{}^{b_k}|^{(-)^{k+1}/2} \left|\frac{\partial \mathbf{q}}{\partial \mathbf{q}'}\right|^{1/2} \psi(\mathbf{q}). \tag{5.12}$$

under (5.10), (5.11) and coordinate transformations (5.8). [Given the new and the old constraints and reducibility functions, one cannot read off uniquely the matrices $A_{a_k}{}^{b_k}$. For instance, $A_{a_0}{}^{b_0}$ is determined up to $\mu_0^{b_1} Z^{b_0}_{b_1}$. However, it is easy to convince oneself that this ambiguity in the A's does not affect ψ' in (5.12).] The law (5.12) reduces to (5.9) for irreducible constraints. Furthermore, it has a clear, intrinsic geometrical meaning.

We shall show in Sec. VI that this is the correct choice in that it yields a Dirac quantization equivalent to the reduced phase space one. In this section, we shall verify that the Dirac quantization based on (5.12) is consistent. Namely, that it leads to quantum constraints (5.1) that are still first class,

$$[\hat{G}_{a_0}, \hat{G}_{b_0}] = i\hat{C}^{c_0}_{a_0 b_0} \hat{G}_{c_0}, \tag{5.13}$$

(in that order) and that fulfill

$$\hat{Z}^{a_0}_{a_1} \hat{G}_{a_0} = 0, \tag{5.14}$$

(in that order). Equation (5.13) expresses the absence of anomalies and guarantees the compatibility of the quantum conditions $\hat{G}_{a_0}\psi = 0$. Equation (5.14) means that among $\hat{G}_{a_0}\psi = 0$, there are only m independent equations. [Since the Z's involve only the coordinates in the models discussed here, the quantum fulfillment of (2.9a) is obvious.]

To prove (5.13) and (5.14), one must compute the Lie derivative $\mathscr{L}_{\xi_{a_0}}\psi$ of the Dirac wave functions. Now, an infinitesimal diffeomorphism generated by ξ_{a_0} induces not only a linear transformation of the coordinate tangent frames, but also a redefinition (5.10), (5.11) of the constraints and of the reducibility functions. Indeed one gets from (2.5) and (3.12)

$$\mathscr{L}_{\xi_{a_0}} G_{b_0} = -C^{c_0}_{a_0 b_0} G_{c_0}, \tag{5.15a}$$

$$\xi^i_{a_0} Z^{a_{k-1}}_{b_k},_i = D^{c_k}_{a_0 b_k} Z^{a_{k-1}}_{c_k} - D^{a_{k-1}}_{a_0 c_{k-1}} Z^{c_{k-1}}_{b_k}. \tag{5.15b}$$

Therefore, the Lie derivative of ψ involves not only the term $(1/2)\xi^i_{a_{0,i}}\psi$, reflecting the weight of ψ under coordinate changes, but also terms arising from its variance under (5.10) and (5.11). Let R be the point of coordinates q^i and S the point of coordinates

$$q^i + \epsilon \xi^i_{a_0}. \tag{5.16}$$

(for fixed a_0). The Jacobian matrix of (5.16) is

$$\delta^i_j - \epsilon \xi^i_{a_{0,j}},$$

while the matrices $A_{a_k}{}^{b_k}$ induced by (5.16) are

$$A_{b_0}{}^{c_0} = \delta^{c_0}_{b_0} - \epsilon C^{c_0}_{a_0 b_0},$$

$$A_{b_k}{}^{c_k} = \delta^{c_k}_{b_k} + \epsilon D^{c_k}_{a_0 b_k}, \quad k = 1,...,L,$$

$$(G'_{b_0} = G_{b_0} + \epsilon \mathscr{L}_{\xi_{a_0}} G_{b_0}, \quad Z'^{c_{k-1}}_{b_k} = Z^{c_{k-1}}_{b_k} + \epsilon \xi^i_{a_0} Z^{c_{k-1}}_{b_k},_i).$$

Thus one gets

$$\psi_S = \psi_R + \epsilon \xi^i_{a_0} \psi_{,i},$$

$$\psi_{R\to S} = \left[1 + \frac{\epsilon}{2} \left(-\xi^i_{a_{0,i}} + C^{b_0}_{a_0 b_0} - \sum_{k=1}^{L} (-)^k D^{b_k}_{a_0 b_k} \right) \right] \psi,$$

where ψ_S is the wave function at S, ψ_R the wave function at R and $\psi_{R\to S}$ the transformed (at S) of the wave function at R under the diffeomorphism (5.16) mapping R in S. This yields finally

$$\mathscr{L}_{\xi_{a_0}} \psi = \lim_{\epsilon \to 0} \frac{\psi_S - \psi_{R\to S}}{\epsilon} = \xi^i_{a_0} \psi_{,i} + \frac{1}{2} \left(\xi^i_{a_{0,i}} - C^{b_0}_{a_0 b_0} + \sum_{k=1}^{L} (-)^k D^{b_k}_{a_0 b_k} \right) \psi. \tag{5.17}$$

The functions $C^{c_0}_{a_0 b_0}$ and $D^{c_k}_{a_0 b_k}$ are not completely defined by (2.3), (3.2), and (3.12). However, the alternating trace in (5.17) is unambiguous, so that (5.17) is well defined. (For instance, if $C^{c_0}_{a_0 b_0} \to \bar{C}^{c_0}_{a_0 b_0} = C^{c_0}_{a_0 b_0} + \mu^{c_1}_{a_0 b_0} Z^{c_0}_{c_1}$, then $D^{c_1}_{a_0 b_1} \to \bar{D}^{c_1}_{a_0 b_1} = D^{c_1}_{a_0 b_1} - \mu^{c_1}_{a_0 b_0} Z^{b_0}_{b_1}$, and $C^{c_0}_{a_0 b_0} + D^{c_1}_{a_0 b_1} \to \bar{C}^{c_0}_{a_0 b_0} + \bar{D}^{c_1}_{a_0 b_1} = C^{c_0}_{a_0 b_0} + D^{c_1}_{a_0 b_1}$.)

C. General case—Consistency of Dirac quantization

In view of the above discussion, we take the operator $\hat{G}_{a_0}$ in (5.1) to be $-i\mathscr{L}_{\xi_{a_0}}$, i.e.,

$$\hat{G}_{a_0}\psi \equiv -i\mathscr{L}_{\xi_{a_0}}\psi = 0, \tag{5.18}$$

or

$$i\hat{G}_{a_0} = i\hat{\xi}^i_{a_0}\hat{p}_i + \frac{1}{2}\left(\hat{\xi}^i_{a_0,i} - \hat{C}^{b_0}_{a_0b_0} + \sum_{k=1}^{L}(-)^k \hat{D}^{b_k}_{a_0b_k}\right). \tag{5.19}$$

Because the second term in the right-hand side of (5.19) is multiplied by $\hbar$ (set equal to one here), one can view it as arising from an ordering ambiguity. [For any operator $\hat{A}$, one has $\hbar\hat{A} = \hbar\hat{A}\hat{1} = -i\hat{A}(\hat{q}\hat{p} - \hat{p}\hat{q})$. So, one can always view $\hbar\hat{A}$ as arising from an ordering ambiguity in the classically equal to zero observable $0 = -iA(pq - qp)$.] In the limit $\hbar \to 0$, $\hat{G}_{a_0}$ goes over into $\xi^i_{a_0}p_i$ and so, possesses the correct classical limit.

To verify the consistency of the Dirac quantization, one must check (5.13) and (5.14). This is direct because the Lie derivative (5.17) has the following crucial properties:

$$\text{(i) } \mathscr{L}_{\xi_{a_0}}\psi \text{ transforms as } \psi. \tag{5.20}$$

$$\text{(ii) } \mathscr{L}_{\mu\xi_{a_0}}\psi = \mu\mathscr{L}_{\xi_{a_0}}\psi \text{ for any } \mu(\mathbf{q}). \tag{5.21}$$

$$\text{(iii) } [\mathscr{L}_{\xi_{a_0}}, \mathscr{L}_{\xi_{b_0}}]\psi = \mathscr{L}_{[\xi_{a_0},\xi_{b_0}]}\psi. \tag{5.22}$$

The property (5.20) follows from the definition (5.17) of the Lie derivative since in $\mathscr{L}_{\xi_{a_0}}\psi$, one takes the difference of two objects that transform in the same manner at S. It can be checked straightforwardly. The property (5.21) follows from the fact that ψ is in essence a scalar under changes of coordinates along the gauge orbits and can be verified by using Propositions 2 and 3 above. Finally the property (5.22) reflects the fact that ψ provides a representation of the diffeomorphism group, i.e., $\psi_{R\to S_2} = (\psi_{R\to S_1})_{\to S_2}$. It can be established by using Proposition 4. We leave the details of the calculations to the reader.

From (5.21), one gets

$$Z^{a_0}_{a_1}\hat{G}_{a_0}\psi = -iZ^{a_0}_{a_1}\mathscr{L}_{\xi_{a_0}}\psi = -i\mathscr{L}_{Z^{a_0}_{a_1}\xi_{a_0}}\psi = 0,$$

and from (5.22) and (5.21),

$$[\hat{G}_{a_0}, \hat{G}_{b_0}]\psi = -[\mathscr{L}_{\xi_{a_0}}, \mathscr{L}_{\xi_{b_0}}]\psi = -\mathscr{L}_{[\xi_{a_0},\xi_{b_0}]}\psi = \mathscr{L}_{C^{c_0}_{a_0b_0}\xi_{c_0}}\psi = C^{c_0}_{a_0b_0}\mathscr{L}_{\xi_{c_0}}\psi = iC^{c_0}_{a_0b_0}\hat{G}_{c_0}.$$

This proves (5.13) and (5.14). [Because of (5.21), one can interpret $\mathscr{L}_{\xi_{a_0}}$ as a kind of covariant derivative. This derivative has flat connection due to (5.22).] In addition, the property (5.20) guarantees the covariance of the Dirac conditions under changes of coordinates and redefinitions of the constraints and of the reducibility functions.

D. Observables linear in the momenta

The quantum definition of an observable $A(\mathbf{q},\mathbf{p})$ linear in the momenta

$$A = a^i(\mathbf{q})p_i, \tag{5.23}$$

$$\{A,G_{a_0}\}\approx 0, \tag{5.24}$$

can be done along the same geometrical lines. Indeed, it follows from (5.23) and (5.24) that

$$\mathscr{L}_\alpha \xi_{b_0} = -X^{c_0}_{b_0}\xi_{c_0}, \tag{5.25}$$

and thus also

$$\mathscr{L}_\alpha Z^{a_{k-1}}_{a_k} = X^{c_k}_{a_k} Z^{a_{k-1}}_{c_k} - X^{a_{k-1}}_{c_{k-1}} Z^{c_{k-1}}_{a_k}, \tag{5.26}$$

where α is the vector field on $\mathscr{Q}$ defined by

$$\alpha = a^i(\mathbf{q})\,\frac{\partial}{\partial q^i}. \tag{5.27}$$

The functions $X^{b_k}_{a_k}(\mathbf{q})$ are subject to indentities similar to that fulfilled by $D^{b_k}_{a_0a_k}$. By repeating the steps leading to (5.17), one gets, for any vector field α

$$\mathscr{L}_\alpha \psi = a^i\psi_{,i} + \frac{1}{2}\left(a^i_{,i} - X^{b_0}_{b_0} + \sum_{k=1}^{L}(-)^k X^{b_k}_{b_k}\right)\psi. \tag{5.28}$$

The group property of diffeomorphisms implies

$$[\mathscr{L}_\alpha, \mathscr{L}_\beta]\psi = \mathscr{L}_{[\alpha,\beta]}\psi \tag{5.29}$$

and

$$[\mathscr{L}_\alpha, \mathscr{L}_{\xi_{a_0}}]\psi = \mathscr{L}_{[\alpha,\xi_{a_0}]}\psi = -X^{b_0}_{a_0}\mathscr{L}_{\xi_{b_0}}\psi, \tag{5.30}$$

[from (5.25) and (5.21)]. Note that $\mathscr{L}_{\mu\alpha}\psi \neq \mu\mathscr{L}_\alpha\psi$ in general (unless $\alpha = \alpha^{a_0}\xi_{a_0}$).

One can take for the quantum operator $\hat{A}$ associated with A minus i times the Lie derivative along α,

$$\hat{A}\psi = -i\mathscr{L}_\alpha\psi. \tag{5.31}$$

Because of (5.30), $\hat{A}$ maps Dirac states on Dirac states so that (5.31) is consistent.

E. Conclusions

In this section, we have shown that the transformation law (5.12) for the Dirac wave functions leads to a quantization procedure that is consistent. The absence of anomaly in the algebra of the quantum constraints would not have been achieved had we taken $\psi(\mathbf{q})$ to be merely a density of weight one-half in $\mathscr{Q}$, without weight for the redefinitions of the constraints. Indeed for densities of weight 1/2, one does not have $\mathscr{L}_{\mu\xi_{a_0}}\psi = \mu\mathscr{L}_{\xi_{a_0}}\psi$ (unless $\xi^i_{a_0}\partial_i\mu = 0$) and thus, (5.13) and (5.14) would fail. The extra terms in (5.17) containing the structure functions are therefore crucial.

The quantization procedure allows also for a geometrical consistent definition of the observables that are linear in the momenta. The resulting expression for $\hat{A}$ is not formally Hermitian in the scalar product $\int d\mathbf{q}\,\chi^*(\mathbf{q})\psi(\mathbf{q})$. However, this is no harm because $\int d\mathbf{q}\,\chi^*(\mathbf{q})\psi(\mathbf{q})$ is not the physical scalar product. We shall derive the correct physical scalar product and prove its equivalence with the reduced phase space quantization scalar product in the next section. This requires a better understanding of the transformation law (5.12).

VI. EQUIVALENCE OF THE REDUCED PHASE SPACE AND DIRAC QUANTIZATIONS

A. Equivalence of physical spectrum

The reduced configuration space $\mathcal{Q}/\mathcal{G}$ is the quotient of the configuration space $\mathcal{Q}$ by the gauge orbits. There is a natural map $\pi:\mathcal{Q}\to\mathcal{Q}/\mathcal{G}$ that maps any point of $\mathcal{Q}$ on its equivalence class in $\mathcal{Q}/\mathcal{G}$. If the wave functions of the reduced phase space quantization were scalars in $\mathcal{Q}/\mathcal{G}$, they would induce by pull-back scalars in $\mathcal{Q}$ that are constants along the gauge orbits:

$$f\in C^\infty(\mathcal{Q}/\mathcal{G})\to\pi^*f=f\circ\pi\in C^\infty(\mathcal{Q}),\quad \partial_{\xi_{a_0}}(\pi^*f)=0.$$

However, the wave functions of the reduced phase space quantization are not scalars in $\mathcal{Q}/\mathcal{G}$. Rather, they are densities of weight 1/2. We show here that they induce objects on $\mathcal{Q}$ with the transformation law (5.12) which have, furthermore, zero Lie derivative (5.17) along the gauge generators ξ_{a_0}.

Consider the vectors $\partial/\partial y^\alpha$ tangent to the coordinate lines (y^α) in a local chart of $\mathcal{Q}/\mathcal{G}$. Let $(\partial/\partial y^\alpha)^R\equiv\mathbf{Y}_\alpha$ be vector fields in $\mathcal{Q}$ that project down to $\partial/\partial y^\alpha$. At each point of $\mathcal{Q}$, the vectors $\mathbf{Y}_\alpha$ and ξ_{a_0} provide an overcomplete set of vectors, which is a basis if and only if the constraints are independent. Among the ξ_{a_0}, one can choose locally m independent vectors ξ_A, which, together, with the Y_α, form a basis of the tangent space to $\mathcal{Q}$. Let us expand the vectors $\partial/\partial q^i$ in terms of (Y_α,ξ_A),

$$\frac{\partial}{\partial q^i}=\mu_i^\alpha(\mathbf{q})\mathbf{Y}_\alpha+\mu_i^A(\mathbf{q})\xi_A, \tag{6.1}$$

and let us define

$$\mu(\mathbf{q})\equiv|\det(\mu_i^\alpha,\mu_i^A)|, \tag{6.2}$$

Even though the vectors $\mathbf{Y}_\alpha$ are not unique $(\mathbf{Y}_\alpha\to\mathbf{Y}_\alpha+k_\alpha^A\xi_A)$, the determinant is unambiguous since the ambiguity in $\mathbf{Y}_\alpha$ simply modifies the rows μ_i^α by linear combinations of the rows μ_i^A.

Proposition 5: The determinant $\mu(\mathbf{q})$ transforms as

$$\mu'(\mathbf{q}'(\mathbf{q}))=\det\frac{\partial y'^\alpha}{\partial y^\alpha}\det\frac{\partial q^i}{\partial q'^i}(\det A_A{}^B)^{-1}\mu(\mathbf{q}) \tag{6.3}$$

under the transformations

$$q^i\to q'^i=q'^i(q^j), \tag{6.4}$$

$$y^\alpha\to y'^\alpha=y'^\alpha(y^\beta), \tag{6.5}$$

$$\xi_A\to\xi'_A=A_A{}^B\xi_B. \tag{6.6}$$

Proof: This simply follows from standard properties of determinants.

Corollary: Let $\varphi(\mathbf{y})$ be a density of weight one-half in the reduced configuration space, and $\pi:q^i\to y^\alpha=y^\alpha(\mathbf{q})$ the projection from $\mathcal{Q}$ to $\mathcal{Q}/\mathcal{G}$. Then, the function $\psi(\mathbf{q})=|\mu(\mathbf{q})|^{1/2}\varphi(y^\alpha(\mathbf{q}))$ is a density of weight 1/2 in $\mathbf{q}$-space that transforms with $|\det A_A{}^B|^{-1/2}$ under $\xi_A\to\xi'_A=A_A{}^B\xi_B$.

Proof: Obvious.

If the vectors ξ_{a_0} are independent (i.e., $m_0=m$ and $\xi_{a_0}\equiv\xi_A$) the analysis is done. But if they are dependent, this is not the whole story. In that case, the dimension m of the tangent space $T_{\mathscr{G}}$ to the orbits is given by Eq. (2.10), which can be written as

$$m=m_0-(m_1-(m_2-(...-m_L)...)),$$

so strongly suggesting that $T_{\mathscr{G}}$, should be regarded as a multiple quotient space.

Let us begin by considering the simplest case $L=1$. In this case we regard $T_{\mathscr{G}}$, at each point $\mathbf{q}\in\mathscr{Q}$, as a quotient space V_0/V_1, where dim $V_0=m_0$ and dim $V_1=m_1$; then $m=m_0-m_1$ in agreement with Eq. (2.10). We define the space V_0 as a vector space generated by m_0 linearly independent vectors $\{\Xi_{a_0}\}$, $a_0=1,...,m_0$. In V_0, the $\{\Xi_{a_0}\}$ forms a basis. We demand that in the quotient V_0/V_1, the vectors Ξ_{a_0} are mapped on the vectors ξ_{a_0},

$$\Xi_{a_0}|_{T_{\mathscr{G}}}=\xi_{a_0}. \tag{6.7}$$

This is the case if we take V_1 to be the space generated by the vectors $\Xi_{a_1}\equiv Z^{a_0}_{a_1}\Xi_{a_0}$, $a_1=1,...,m_1$. Indeed, these vectors are mapped on zero,

$$\Xi_{a_1}|_{T_{\mathscr{G}}}\equiv Z^{a_0}_{a_1}\Xi_{a_0}|_{T_{\mathscr{G}}}=Z^{a_0}_{a_1}\xi_{a_0}=0, \tag{6.8}$$

as they should. Since the order of reducibility is $L=1$, then Eq. (2.9b) tell us that $\{\Xi_{a_1}\}$ is a set of m_1 linearly independent vectors. So we can replace the basis $\{\Xi_{a_0}\}$ by $\{\Xi_A,\Xi_{a_1}\}$, where the Ξ_A's are the m vectors associated with the basis $\{\xi_A\}$ of $T_{\mathscr{G}}$ via Eq. (6.7). Therefore it is clear that $T_{\mathscr{G}}$ can be regarded as V_0/V_1, where the equivalence relation is such that the vectors $\Xi_{a_1}\in V_1$ are identified with zero. One has

$$\xi_{a_0}=\mu^A_{a_0}(\mathbf{q})\Xi_A+\mu^{a_1}_{a_0}(\mathbf{q})\Xi_{a_1}. \tag{6.9}$$

Define

$$\mu_0(\mathbf{q})\equiv|\det(\mu^A_{a_0},\mu^{a_1}_{a_0})|. \tag{6.10}$$

Again $\mu_0(\mathbf{q})$ does not depend on how one chooses to lift ξ_A.

Proposition 6: The determinant $\mu_0(\mathbf{q})$ transforms as

$$\mu_0\to\mu_0'=|\det A_A{}^B||\det A_{a_1}{}^{b_1}||\det A_{a_0}{}^{b_0}|\mu_0, \tag{6.11}$$

under a redefinition

$$\xi_A\to\xi_A'=A_A{}^B\xi_B, \tag{6.12}$$

$$\Xi_{a_0}\to\Xi_{a_0}'=A_{a_0}{}^{b_0}\Xi_{b_0}, \tag{6.13}$$

$$\Xi_{a_1}\to\Xi_{a_1}'=A_{a_1}{}^{b_1}\Xi_{b_1} \tag{6.14}$$

[which are equivalent to the transformation laws (5.7), (5.10), and (5.11) for $k=1$].

Proof: Obvious.

Corollary: The function $\psi(\mathbf{q})=\mu^{1/2}\mu_0^{-1/2}\varphi(y^\alpha(\mathbf{q}))$ is a density of weight 1/2 in $\mathscr{Q}$ that transforms with $|\det A_{a_1}{}^{b_1}|^{1/2}|\det A_{a_0}{}^{b_0}|^{-1/2}$ under redefinitions of the constraints and of the Ξ_{a_1}. In particular, it does not depend on the choice of the intermediate vectors Ξ_A.

This corollary is a direct consequence of Propositions 5 and 6, and of the fact that a redefinition of the constraints G_{a_0} yields a redefinition of the vectors ξ_{a_0}.

For $L=1$ the analysis is done. If the $Z^{a_0}_{a_1}$ are, however, not independent, one should keep going and regard the vector space V_1 itself as a quotient, etc..., until one reaches the last reducibility stage. For $L>1$ the set $\{\Xi_{a_1}\}$ is not linearly independent, so no longer we define V_1 to be the space that they generate. Rather, we denote that subspace of V_0 by W_0. Let $\{\Xi_{A_0}\}$ be a basis for W_0; then Eq. (6.9) now reads,

$$\Xi_{a_0}=\mu^A_{a_0}\Xi_A+\mu^{A_0}_{a_0}\Xi_{A_0}. \tag{6.15}$$

The relations (2.9) among the Z's mean

$$\dim W_0=m_1-(m_2-(m_3-(.......-m_L)...))=m_0-m,$$

which suggests again to regard W_0 as a multiple quotient space. So, define a vector space V_1, $\dim V_1=m_1$, with basis $\{\mathcal{N}_{a_1}\}$, and consider the subspace $W_1\subset V_1$ generated by the vectors

$$\mathcal{N}_{a_2}\equiv Z^{a_1}_{a_2}\mathcal{N}_{a_1},\quad a_2=1,...,m_2. \tag{6.16}$$

These are not linearly independent if $L>2$. The dimension of W_1 is, according to Eq. (2.9b), $\dim W_1=m_2-(m_3-(m_4.....-m_L)...)=m_1-m_0+m$. Let $\{\mathcal{N}_{A_1}\}$ be a basis for W_1. The quotient space V_1/W_1 can be identified with W_0, and the image of $\mathcal{N}_{a_1}$ in the mapping $V_1\rightarrow W_0$ (which we denote by $\mathcal{N}_{a_1}|_{W_0}$) can be identified with Ξ_{a_1} since one has

$$\mathcal{N}_{a_2}|_{W_0}=Z^{a_1}_{a_2}\mathcal{N}_{a_1}|_{W_0}=Z^{a_1}_{a_2}\Xi_{a_1}=Z^{a_1}_{a_2}Z^{a_0}_{a_1}\Xi_{a_0}=0.$$

Then a basis for V_1 is $\{\mathcal{N}_{A_0},\mathcal{N}_{A_1}\}$, where $\mathcal{N}_{A_0}$ is any vector projecting to Ξ_{A_0} in the mapping $V_1\rightarrow W_0$. The $\mathcal{N}_{a_1}$'s can be expanded in this basis:

$$\mathcal{N}_{a_1}=\mu^{A_0}_{a_1}\mathcal{N}_{A_0}+\mu^{A_1}_{a_1}\mathcal{N}_{A_1}. \tag{6.17}$$

By using the same argument for W_1 and so on, we will obtain

$$T\mathcal{G}=V_0/(V_1/(V_2/...V_L)...).$$

One can define

$$\mu\equiv|\det(\mu^\alpha_i,\mu^A_i)|, \tag{6.18a}$$

$$\mu_0\equiv|\det(\mu^A_{a_0},\mu^{A_0}_{a_0})|, \tag{6.18b}$$

$$\mu_1\equiv|\det(\mu^{A_0}_{a_1},\mu^{A_1}_{a_1})|, \tag{6.18c}$$

$$\vdots$$

$$\mu_{L-1}\equiv|\det(\mu^{A_{L-2}}_{a_{L-1}},\mu^{a_L}_{a_{L-1}})|, \tag{6.18d}$$

where the μ's are the coefficients that appear in the equations generalizing (6.17). These determinants do not depend on how one lifts the basis vectors of W_k to V_{k+1}. One has the following.

Proposition 7: Let $\varphi(y^\alpha)$ be a density of weight one-half on the reduced configuration space $\mathscr{Q}/\mathscr{G}$, and let $\psi(\mathbf{q})$ be defined through

$$\psi(\mathbf{q})=\mu^{1/2}\mu_0^{-1/2}\mu_1^{1/2}\cdots\mu_{L-1}^{(-1)^L/2}\varphi(y^\alpha(\mathbf{q})). \tag{6.19}$$

Then ψ (q) transforms as in (5.12) and fulfills

$$\mathscr{L}_{\xi_{a_0}}\psi(\mathbf{q})=0. \tag{6.20}$$

Proof: The first part is obvious, so let us prove only (6.20). It is enough to check (6.20) in a particular coordinate system and with a particular choice of the constraints and of the reducibility functions. We take the q^i coordinates to split into gauge invariant coordinates y^α and pure gauge invariant Q^A, as in Sec. IV, $q^i\equiv(y^\alpha,Q^A)$. The constraints can then be taken to be

$$G_{a_0}=(G_A,G_{\bar{a}_0}),$$

with $G_A\equiv P_A$, $G_{\bar{a}_0}\equiv0$. Similarly, the reducibility functions can be taken to be zero or one.[18,1] With that choice, the Lie derivative of ψ reduces to

$$\mathscr{L}_{\xi_A}\psi\equiv\frac{\partial\psi}{\partial Q^A}, \tag{6.21a}$$

$$\mathscr{L}_{\xi_{\bar{a}_0}}\psi\equiv0. \tag{6.21b}$$

Furthermore, the vectors ξ_A, ξ_{a_0}, Ξ_{a_1}, $\mathscr{N}_{a_2}$, etc., can be taken in such a way that the determinants μ, μ_0, $\mu_1,\ldots,\mu_{L-1}$, are all equal to one. Hence $\psi(\mathbf{q})=\psi(y^\alpha)$ does not depend on Q^A, establishing (6.20) [$\Leftrightarrow$(6.21)]. This proves Proposition 7.

Conversely, let $\psi(\mathbf{q})$ be an object that transforms as in (5.12) and that fulfills (6.20). Then $\psi(\mathbf{q})|\mu|^{-1/2}|\mu_0|^{1/2}\ldots|\mu_{L-1}|^{(-)^{L+1}/2}$ depends only on y^α and defines a density of weight 1/2 on the reduced configuration space.

We thus conclude that a density of weight 1/2 in $\mathscr{Q}/\mathscr{G}$ induces naturally a Dirac state as defined in Sec. V and vice versa. The Dirac and reduced phase-space quantizations give the same spectrum of physical states.

B. Observables

Similarly, the action of the observables that are linear in the momenta are equivalent in both quantization methods. Namely, if ψ is a reduced phase space state and ψ_D the corresponding Dirac state, and if α is a vector field in $\mathscr{Q}/\mathscr{G}$ and α_D a vector field that project down to α (fulfilling accordingly $[\alpha_D,\xi_{a_0}]\sim\xi_{b_0}$), then $(\mathscr{L}_\alpha\psi)_D=\mathscr{L}_{\alpha_D}\psi_D$. This simply follows from the group property of mappings and the commutativity of the following

$$\begin{array}{ccc}\mathscr{Q} & \xrightarrow{\gamma^t_D} & \mathscr{Q}\\ \pi\downarrow & & \downarrow\pi\\ \mathscr{Q}/\mathscr{G} & \xrightarrow{\gamma^t} & \mathscr{Q}/\mathscr{G},\end{array} \tag{6.22}$$

in which γ^t and γ_D^t are elements of the one-parameter group of diffeomorphisms generated respectively by α and α_D. The equivalence of the linear observables was already pointed out by Tuynman[9] in the irreducible case.

C. Scalar product

Finally, we turn to the scalar product. One defines the scalar product of two Dirac states as

$$\int d\mathbf{q}\, \psi^*(\mathbf{q})\varphi(\mathbf{q}) \prod_{A=1}^{m} \delta(\chi^A(\mathbf{q}))|\det\{G_A,\chi^B\}||\mu_0||\mu_1|^{-1}|\mu_2|\cdots|\mu_{L-1}|^{(-)^{L+1}}, \quad (6.23)$$

where $\chi^A=0$ are gauge conditions and G_A a subset of irreducible constraint functions. [If there is no such subset that is globally defined, one must introduce a partition of unity and generalize (6.23) in the standard manner.] The expression (6.23) is: (i) invariant under redefinitions of the constraints and of the reducibility functions; this is because one has included the factors $|\mu_0||\mu_1|^{-1}|\mu_2|\cdots|\mu_{L-1}|^{(-)^{L+1}}$ besides the usual Fadeev–Popov determinant; and (ii) invariant under changes of coordinates. By choosing the coordinates and the constraints as in the proof of the Proposition 7, one can rewrite (6.23) as

$$\int \psi^*(y^\alpha)\varphi(y^\alpha), \quad (6.24)$$

thereby proving the equivalence of the scalar products in the Dirac and reduced phase space quantizations.

VII. BRST QUANTIZATION

A. BRST charge

In the standard BRST method for quantizing a constrained system, the classical BRST generator is realized as an Hermitian operator on the Hilbert space of the functions depending on the original variables q^i and the ghosts η^{a_k}. We say that the theory is free from BRST anomalies, if a realization $\hat{\Omega}=\hat{\Omega}^\dagger$ can be found such that the classical property (3.4) becomes

$$[\hat{\Omega},\hat{\Omega}]=0,$$

i.e.,

$$\hat{\Omega}^2=0 \quad (7.1)$$

(remember that the graded commutator is symmetric for fermionic quantities).

As it was already proved in Sec. III, the BRST generator is linear in the original momenta and the ghost momenta, when the constraints are linear in the momenta. So it has the generic structure

$$\Omega=\sum_{k=-1}^{L} \Omega^{a_k}\mathscr{P}_{a_k}, \quad \mathscr{P}_{a_{-1}}\equiv p_i, \quad (7.2)$$

with

$$\Omega^{a_k*}=(-)^{k+1}\Omega^{a_k}, \quad (7.3)$$

since Ω is real. We will prove that the Hermitian ordering

$$\hat{\Omega}=\tfrac{1}{2}(\hat{\Omega}_R+\hat{\Omega}_L), \tag{7.4a}$$

$$\hat{\Omega}_R\equiv \sum_{k=-1}^{L} \Omega^{a_k}\hat{\mathscr{P}}_{a_k}, \tag{7.4b}$$

$$\hat{\Omega}_L\equiv \sum_{k=-1}^{L} \hat{\mathscr{P}}_{a_k}\Omega^{a_k}, \tag{7.4c}$$

leads to a theory free from BRST anomalies. In fact,

$$\hat{\Omega}=\hat{\Omega}_R-\frac{i}{2}\sum_{k=-1}^{L}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{a_k}, \tag{7.5}$$

so that

$$[\hat{\Omega},\hat{\Omega}]=[\hat{\Omega}_R,\hat{\Omega}_R]-i\left[\sum_{k=-1}^{L}\Omega^{a_k}\hat{\mathscr{P}}_{a_k},\sum_{j=-1}^{L}\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{b_j}\right]. \tag{7.6}$$

The first term in the right-hand side of (7.6) is zero (it is the same calculation as in the classical case). Hence one gets

$$[\hat{\Omega},\hat{\Omega}]=-i\sum_{k,j=-1}^{L}\Omega^{a_k}\frac{\partial^{2(\text{left})}}{\partial\eta^{a_k}\,\partial\eta^{b_j}}\Omega^{b_j}. \tag{7.7}$$

Now the odd vector field Ω^{a_k}, defined on the configuration space of the q's and the η's, has vanishing Lie bracket with itself,

$$\sum_{k,j=-1}^{L}\Omega^{a_k}\frac{\partial^{(\text{left})}\Omega^{b_j}}{\partial\eta^{a_k}}=0. \tag{7.8}$$

This is just the expression of the classical nilpotency of Ω, $\{\Omega,\Omega\}=0$.

Proposition 8: Let Ω^{a_k} be an odd vector field that has vanishing Lie bracket with itself. Then

$$\sum_{k,j=-1}^{L}\Omega^{a_k}\frac{\partial^{2(\text{left})}}{\partial\eta^{a_k}\,\partial\eta^{b_j}}\Omega^{b_j}=0. \tag{7.9}$$

Proof: By differentiating (7.8) with respect to η^{b_j},

$$\begin{aligned}
0&=\sum_{k,j=-1}^{L}(-)^j\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\left(\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}\right)\\
&=\sum_{k,j=-1}^{L}(-)^j\left(\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}+(-)^{k(j+1)}\Omega^{a_k}\frac{\partial^{2(\text{left})}}{\partial\eta^{b_j}\,\partial\eta^{a_k}}\Omega^{b_j}\right)\\
&=\sum_{k,j=-1}^{L}(-)^j\left(\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}+(-)^{k(j+1)}\Omega^{a_k}\frac{\partial^{2(\text{left})}}{\partial\eta^{a_k}\,\partial\eta^{b_j}}\Omega^{b_j}(-)^{(k+1)(j+1)}\right)\\
&=\sum_{k,j=-1}^{L}\left((-)^j\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}+\Omega^{a_k}\frac{\partial^{2(\text{left})}}{\partial\eta^{a_k}\,\partial\eta^{b_j}}\Omega^{b_j}\right).
\end{aligned}$$

But it is easy to prove that the first term in the right-hand side of this equation is zero by itself:

$$\sum_{k,j=-1}^{L}(-)^j\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}=\sum_{k,j=-1}^{L}(-)^j(-)^{j+k+1}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}$$

$$=-\sum_{k,j=-1}^{L}(-)^k\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}$$

$$=-\sum_{k,j=-1}^{L}(-)^j\frac{\partial^{(\text{left})}}{\partial\eta^{b_j}}\Omega^{a_k}\frac{\partial^{(\text{left})}}{\partial\eta^{a_k}}\Omega^{b_j}.$$

This proves Proposition 8, and as a corollary, the nilpotency of the quantum $\hat{\Omega}$ [Eq. (7.1)].

Let us now expand explicitly (7.5). One finds (recall that $-D^{b_k}_{a_0a_k}$ is the coefficient of $\eta^{a_0}\eta^{a_k}\mathscr{P}_{b_k}$ in Ω, see proof of Proposition 2 in Sec. III),

$$\hat{\Omega}=\hat{\eta}^{a_0}\hat{G}_{a_0}+\sum_{k=0}^{L}\hat{\Omega}^{a_k}\hat{\mathscr{P}}_{a_k}, \tag{7.10}$$

with the same operator $\hat{G}_{a_0}$ *as in the Dirac quantization method* [Eq. (5.19)]. That is, $\hat{G}_{a_0}$ *is the coefficient of* η^{a_0} *in the* $\eta-\mathscr{P}$ *ordering of the Hermitian BRST charge* (7.4). In this view, the term $\xi^i_{a_0,i}$ comes from the reordering of the original degrees of freedom $[(1/2)(\xi^i_{a_0}p_i+p_i\xi^i_{a_0}=\xi^i_{a_0}p_i-(i/2)\xi^i_{a_0,i}]$, the term $C^{b_0}_{a_0b_0}$ comes from the reordering of the ghosts degrees of freedom $(\eta^{a_0},\mathscr{P}_{a_0})$, while the terms $D^{b_k}_{a_0a_k}$ comes from the reordering of the ghosts of ghosts $(\eta^{a_k},\mathscr{P}_{a_k})$. *Thus each generation of ghosts contributes to the Dirac constraint operators* $\hat{G}_{a_0}$.

The consistency of the Dirac quantization scheme—i.e., no anomaly in (5.13) and fulfillment of (5.14)—can also be viewed as a direct consequence of the absence of BRST anomaly. Indeed, the nilpotency of (7.1) implies straightforwardly (5.13) and (5.14) as one can see by examining the first terms of $\hat{\Omega}^2$. One can thus say that the absence of anomaly in the algebra of the Dirac constraints follows from the inclusion in $\hat{G}_{a_0}$ of the ghost contribution as well as of the contribution from the ghosts of ghosts. It should be noted in that respect that one could have achieved the classical nilpotency condition $\{\Omega,\Omega\}=0$ without the ghosts of ghosts. But then, one would not have found $[\hat{\Omega},\hat{\Omega}]=0$ quantum mechanically, since the $D^{b_k}_{a_0b_k}$ -terms in $\hat{G}_{a_0}$ are essential. (For a different approach to the BRST quantization of reducible systems with linear constraints, see Ref. 19.)

We leave to the reader to check that similar considerations apply to the observables that are linear in the momenta, which, in the BRST quantization scheme, fulfill $[\hat{A},\hat{\Omega}]=0$, see, e.g., Ref. 1.

B. BRST physical states

In the BRST method, the physical states are annihilated by the BRST charge,

$$\hat{\Omega}\psi=0. \tag{7.11}$$

The link with the Dirac method as developed in the previous sections, is obtained by demanding, in addition, that the physical states be annihilated by the ghost momenta

$$\hat{\mathscr{P}}_{a_k}\psi=0,\quad k=0,\ldots,L. \tag{7.12a}$$

Then the physical wave functions $\psi(q^i,\eta^{a_k})$ do not depend on the ghosts,

$$\psi=\psi(\mathbf{q}). \tag{7.12b}$$

When (7.12) is inserted in (7.11), one gets

$$\hat{\eta}^{a_0}\hat{G}_{a_0}\psi=0, \tag{7.13a}$$

i.e.,

$$\hat{G}_{a_0}\psi=0, \tag{7.13b}$$

which are exactly the Dirac physical state conditions. Hence the BRST physical states fulfilling (7.11) are exactly the same as the Dirac physical states.

Since we have adopted a Hermitian ordering for $\hat{\Omega}$, adapted to the formal scalar product $\int dq\, d\eta\, \psi^*(q,\eta)\chi(q,\eta)$, we shall require the BRST wave functions to transform as superdensities of weight 1/2 under changes of coordinates in the configuration space of the q^i and the ghosts,

$$q^i,\eta^{a_k}\rightarrow q'^i,\eta'^{a_k}, \tag{7.14a}$$

$$\psi\rightarrow\psi'=\psi\left|\operatorname{sdet}\frac{\partial(q,\eta)}{\partial(q',\eta')}\right|^{1/2}. \tag{7.14b}$$

This leaves the integral $\int dq\, d\eta\, \psi^*\chi$ invariant. For the states (7.11), *the rule (7.14) exactly yields the transformation law (5.12) for the Dirac states.* Indeed, the redefinition $G'_{a_0} = A_{a_0}{}^{b_0}G_{b_0}$ induces the transformation $\eta^{a_0} = A_{a_0}{}^{b_0}\eta'^{b_0}$ of the ghosts, in order to leave $G_{a_0}\eta^{a_0}$ invariant, $G_{a_0}\eta^{a_0} = G'_{a_0}\eta'^{a_0}$. Similarly, the redefinitions of the reducibility functions are equivalent to a transformation of the higher order ghosts. Hence, the transformation law (5.12) has a very direct explanation in terms of the BRST quantization.

We again leave it to the reader to check that the BRST observables that are linear in the momenta reproduce correctly the Lie derivative when acting on the states (7.11) and (7.13).

C. BRST inner product

To complete the proof of equivalence of the BRST method with the Dirac method, it remains to discuss the scalar product.

Now, if one computes the integral $\int dq\, d\eta\, \psi^*(q,\eta)\chi(q,\eta)$ for the BRST physical states (7.11), (7.12), one obtains an ill-defined result. The way out is to introduce a so-called "nonminimal sector" (i.e., further variables that do not change the physics) and to regularize $\int dq\, d\eta\, \psi^*(q,\eta)\chi(q,\eta)$ by inserting the operator $\exp-[\hat{K},\hat{\Omega}]$ between ψ^* and χ, and integrating also over the nonminimal variables, with the natural measure $dq\, d\eta\, d$ (nonminimal variables). Here, $\hat{K}$ is a ghost minus one operator depending on all the variables and chosen so that the integral $\int dq\, d\eta\, d$ (nonminimal) $\psi^*(q,\eta)\exp[\hat{K},\hat{\Omega}]\chi(q,\eta)$ is well defined. The operator $\exp-[\hat{K},\hat{\Omega}]$ is (formally) equivalent to the unit operator between physical states because $\hat{\Omega}$ is (formally) Hermitian with that natural measure. When this regularization is appropriately carried through, one finds that the BRST scalar product coincides with the Dirac scalar product for the states obeying (7.11) and (7.12).

This result is derived in detail in Ref. 1 for the irreducible case (Chap. 14). It is easy to see, by using the invariance of the scalar product under changes of coordinates $q^i,\eta^{a_k}\rightarrow q'^i,\eta'^{a_k}$, that the same result applies to the reducible case as well.

VIII. CONCLUSIONS

In this paper we have established the equivalence of the reduced phase space, Dirac and BRST quantization methods for reducible gauge systems described by constraints linear in the momenta. We have shwon that densities of weight one-half in the reduced configuration space define densities of weight one-half in the original configuration space, which have a nontrivial weight under redefinitions of the constraints and of the reducibility functions. Because of this extra variance, the Lie derivative of the Dirac wave functions contains extra terms besides those characteristic of ordinary density of weight one-half. These terms guarantee the absence of anomalies in the Dirac quantization scheme, as well as the reducibility of the quantum constraints. Finally, we have given a BRST interpretation of the Dirac analysis. In particular, we have shown that the extra anomaly cancelling terms in the quantum constraints could be thought of as arising from the ghosts and the ghosts of ghosts, which play thus a fundamental role.

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[1] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton U.P., Princeton, NJ, 1992).
[2] K. V. Kuchař, Phys. Rev. D **34**, 3031, 3044 (1986).
[3] M. J. Gotay, J. Math. Phys. **27**, 2051 (1986).
[4] A. O. Barvinsky and V. N. Ponomariov, Phys. Lett. B **167**, 289 (1986).
[5] B. Kostant and S. Sternberg, Ann. Phys. (NY) **176**, 49 (1987).
[6] K. V. Kuchař and C. G. Torre, J. Math. Phys. **30**, 1769 (1989).
[7] D. Mc Mullan and J. Paterson, J. Math. Phys. **30**, 477, 487 (1989).
[8] C. Duval, J. Elhadad, and G. M. Tuynman, Commun. Math. Phys. **126**, 535 (1990).
[9] G. M. Tuynman, J. Math. Phys. **31**, 83 (1990).
[10] P. Hajiček and K. V. Kuchař, Phys. Rev. D **41**, 1091 (1990).
[11] A. O. Barvinsky, Phys. Lett. B **241**, 210 (1990).
[12] R. Loll, Phys. Rev. D **41**, 3785 (1990).
[13] G. Kunstatter, Class. Quant. Grav. **9**, 1469 (1992).
[14] A. O. Barvinsky and V. Krykhtin, "Dirac and BFV quantization methods in the one-loop approximation: Closure of the quantum constraint algebra and the conserved inner product," preprint, Alberta Thy-7-92 (1992).
[15] G. M. Tuynman, Lett. Math. Phys. **21**, 205 (1991).
[16] M. Henneaux, Phys. Rep. **126**, 1 (1985).
[17] I. A. Batalin and E. S. Fradkin, Phys. Lett. B **122**, 157 (1983); Rev. Nuovo Cimento **9**, 1 (1986).
[18] J. Fisch, M. Henneaux, J. D. Stasheff, and C. Teitelboim, Commun. Math. Phys. **120**, 379 (1989).
[19] G. M. Tuynman, "Geometric quantization of the BRST charge," MSRI preprint 06124-89 (1989).