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Solution of Laplace's equation in plane single-connected regions bounded by arbitrary single curves

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A method is developed to solve Laplace's equation with Dirichlet's or Neumann's conditions in plane, single-connected regions bounded by arbitrary single curves. It is based on the existence of a conformal transformation that reduces the original problem to another whose solution is known. The main advantage of the method is that it does not require the knowledge of the transformation itself, so it is applicable even when no transformation is available. The solution and its higher-order derivatives are expressed in terms of explicit quadratures easy to evaluate numerically or even analytically.

I. INTRODUCTION

Laplace's equation appears in many physical problems: gravitational or electrostatic problems; the steady, irrotational flow of incompressible fluids; the steady-state flow of heat; the steady diffusion of a solute, of neutrons, and generally, in steady diffusive processes. Consequently, several methods have been developed to solve it. There are essentially two different approaches: variable separation and Green functions (or a combination of both).¹⁻³ In addition, bidimensional problems can be treated by means of conformal mapping techniques that provide a powerful alternative method. None of these methods, of course, is always applicable and its success depends on the problem at hand. In fact, the boundary conditions and the contour itself may prevent the variable separation; moreover, Green's function or, in bidimensional problems, an appropriate conformal transformation, is generally not easy to find. On the other hand, in very complicated cases one must resort to numerical calculations.⁴⁻⁶

In this work we propose a method that allows the obtention of the general solution in bidimensional single-connected regions bounded by single closed curves. In spite of the fact that the method is based on the conformal mapping technique, the explicit knowledge of the conformal transformation is avoided. The solution is then expressed by explicit quadratures that are easy to evaluate numerically or even analytically.

In addition, the same formalism allows the direct calculation of certain magnitudes of interest as, for instance, the normal derivatives at the boundary in Dirichlet's problem, and higher-order derivatives of the unknown potential function.

II. METHOD

A. Dirichlet's conditions

In order to develop the method we use the following holomorphic functions:

$$f = \psi + i\phi, \quad (1a)$$

$$g = F + iG, \quad (1b)$$

where ψ and F are the conjugated functions of the potential ϕ , and Green's function G , corresponding to Dirichlet's conditions in the region of interest, respectively. With these definitions two equivalent expressions of the formal solution can be given as follows:

$$\phi(x', y') = \int_{\partial(R)} \phi(x, y) \frac{dg}{dZ} dZ, \quad (2)$$

from Green's formula, and

$$\phi(x', y') = \text{Im} \left\{ \frac{1}{2\pi i} \int_{\partial(R)} \frac{f(Z)}{Z - Z'} dZ \right\}, \quad (3)$$

from Cauchy's formula, where $\partial(R)$ denotes the boundary of the domain R (see Fig. 1).

We can transform the region R onto the upper half-plane R' by means of a conformal transformation $\zeta(Z)$ (see Fig 1). In the transformed domain R' the function g is given by

$$g(\zeta, \zeta', \zeta'^*) = \frac{i}{2\pi} \ln \left(\frac{\zeta - \zeta'^*}{\zeta - \zeta'} \right), \quad (4)$$

where the asterisk denotes the complex conjugate.

From Eqs. (2) and (4) the formal solution of the problem is

$$\begin{aligned} \phi[x'(\xi', \eta'), y'(\xi', \eta')] \\ = \frac{i}{2\pi} \int_{\partial(R')} \left(\frac{1}{\xi - \xi'^*} - \frac{1}{\xi - \xi'} \right) \phi[Z(\xi)] d\xi. \end{aligned} \quad (5)$$

On the other hand, by developing Eq. (3), we get,

$$\phi(x', y') = \frac{1}{2\pi} \int_{\partial(R)} \frac{\phi \cos \alpha - \psi \sin \alpha}{r} dl, \quad (6)$$

where α and r are those denoted in Fig. 1.

To obtain the explicit solution from Eq. (5) or (6), the explicit knowledge of $\zeta(Z)$ or ψ , respectively, is required. In fact, both problems reduce to the knowledge of $\zeta(Z)$, because the determination of ψ requires it. Although many conformal transformations that solve the problem are known,⁷ this is not possible for an arbitrary region R . The goal of the method presented here is to show that (and how)

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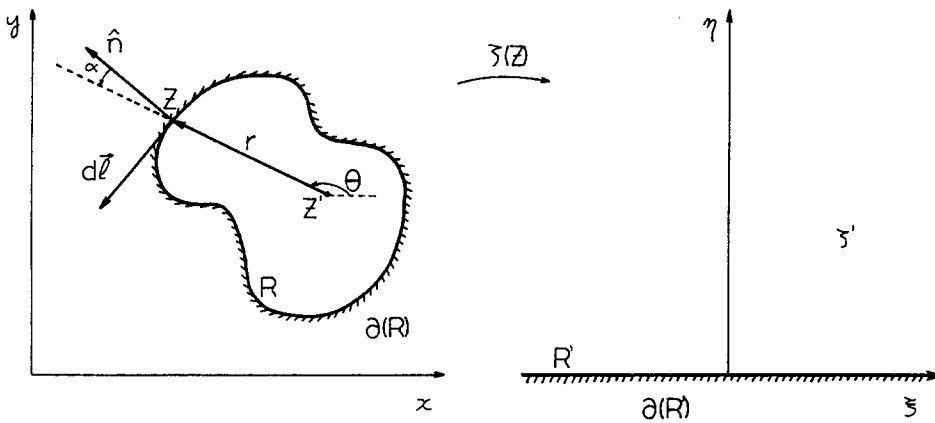


FIG. 1. The original region in Z plane and the transformed region in ζ plane.

ψ can be determined without knowing the explicit form of $\zeta(Z)$.

Noting that

$$\frac{\partial \psi}{\partial \xi'} = \frac{\partial \phi}{\partial \eta'} \quad (7)$$

and using Eq. (5), we get,

$$\frac{\partial \psi}{\partial \xi'} = \int_{-\infty}^{+\infty} h[(\xi - \xi'), \eta'] \phi(\xi) d\xi \quad (8)$$

where

$$h(u, v) = (1/\pi)(u^2 - v^2)/[u^2 + v^2]^2. \quad (9)$$

From this definition it follows that

$$\int_{-\infty}^{+\infty} h[(\xi - \xi'), \eta'] d\xi = 0, \quad (10)$$

which allows us to evaluate Eq. (8) on the boundary $\partial(R')$ as

$$\begin{aligned} \left(\frac{\partial \psi}{\partial \xi'}\right)_{\partial(R')} &= \lim_{\eta' \rightarrow 0^+} \int_{-\infty}^{+\infty} h[(\xi - \xi'), \eta'] [\phi(\xi) - \phi(\xi')] d\xi. \end{aligned} \quad (11)$$

Assuming that $\phi(\xi)$ can be developed in Taylor's series near ξ' , it can be easily proved that this limit indeed exists and is given by

$$\left(\frac{\partial \psi}{\partial \xi'}\right)_{\partial(R')} = \frac{1}{\pi} \text{pV} \int_{-\infty}^{+\infty} \frac{\phi(\xi) - \phi(\xi')}{(\xi - \xi')^2} d\xi, \quad (12)$$

where pV refers to the principal Cauchy value.

Equation (12) allows the calculation of the derivative of ψ along the boundary of the transformed region that on integration yields the function $\psi(\xi')$. Once ψ is known, the use of Eq. (6) leads to the determination of the potential ϕ at any point inside R .

As was pointed out by the referee, in many cases of interest one can avoid the intermediate step of evaluating the derivative $\partial \psi / \partial \xi'$ in order to obtain ψ . This is possible by virtue of the result derived by Sneddon⁸ that states that if ϕ is a harmonic function in the half-plane $\eta > 0$, and tends to zero as $(\xi^2 + \eta^2)^{1/2} \rightarrow \infty$ then the normal derivative of ϕ on the

boundary $\eta = 0$ is the Hilbert transform of the tangential derivative, i.e.,

$$\left(\frac{\partial \phi}{\partial \eta'}\right)_{\partial(R')} = \frac{1}{\pi} \text{pV} \int_{-\infty}^{+\infty} \frac{1}{(\xi - \xi')} \frac{\partial \phi}{\partial \xi} d\xi, \quad (13)$$

which, since $\partial \phi / \partial \eta = \partial \psi / \partial \xi$, and taking into account the properties of the Hilbert transform results in

$$\psi(\xi') = \frac{1}{\pi} \text{pV} \int_{-\infty}^{+\infty} \frac{\phi(\xi)}{(\xi - \xi')} d\xi. \quad (14)$$

For this formula to hold, however, ϕ and ψ must tend to zero as $(\xi^2 + \eta^2)^{1/2} \rightarrow \infty$. This is possible, for instance, if the potential is constant (which without loss of generality can be taken as zero) on some finite piece of the original boundary. In this case, by choosing the parametrization in such a way that $s \rightarrow \pm \infty$ corresponds to points inside the piece at constant potential, the potential at the transformed boundary will be different from zero only on a finite interval, assuring that ϕ and ψ tend to zero as $(\xi^2 + \eta^2)^{1/2} \rightarrow \infty$. Formula (14) allows then the obtention of ψ on the boundary without going through the integration of expression (12).

In order to evaluate Eq. (12) or, when possible, Eq. (14), only the values of the potential ϕ on the boundary are required, so that all we need to know is the transformation $\zeta(Z)$ restricted to that boundary. This can be easily done if the boundary of the original region R is given in parametric form. In fact, if the boundary is given by

$$x = X(s), \quad (15a)$$

$$y = Y(s), \quad (15b)$$

where s is a real parameter which, when ranging from $-\infty$ to ∞ counterclockwise, describes the full contour $\partial(R)$, the restricted transformation is

$$\zeta(\xi) = X(\xi) + iY(\xi). \quad (16)$$

In this manner, the potential on the boundary is given as a function of ξ by

$$\phi(\xi) = \phi[X(\xi), Y(\xi)]. \quad (17)$$

In order to evaluate Eq. (6), the explicit form of the terms entering in the integrand can be expressed in terms of known functions as

$$\begin{aligned} & \frac{\phi \cos \alpha dl}{r} \\ &= \phi[X(\xi), Y(\xi)] \\ & \times \frac{[X(\xi) - x'](dY/d\xi) - [Y(\xi) - y'](dX/d\xi)}{[X(\xi) - x']^2 + [Y(\xi) - y']^2} \\ & \times d\xi, \end{aligned} \quad (18a)$$

$$\begin{aligned} & \frac{\psi \sin \alpha dl}{r} \\ &= \psi[X(\xi), Y(\xi)] \\ & \times \frac{[Y(\xi) - y'](dY/d\xi) + [X(\xi) - x'](dX/d\xi)}{[X(\xi) - x']^2 + [Y(\xi) - y']^2} \\ & \times d\xi. \end{aligned} \quad (18b)$$

As can be seen, we have avoided the use of the explicit conformal transformation, which is the major difficulty in actual calculations.

B. Neumann's conditions

Using the definitions given by Eqs. (1), G being now Green's function for Neumann's conditions, the formal solution is given by

$$\phi(x', y') = \text{Im} \left\{ \int_{\partial(R)} g \frac{df}{dZ} dZ \right\}. \quad (19)$$

We can transform the original region R onto the upper half-plane R' as in Dirichlet's case by means of a conformal transformation $\zeta(Z)$. In R' the function g is now given by $g(\zeta, \zeta', \zeta'^*) = -K\zeta + (i/2\pi) \ln[(\zeta - \zeta')(\zeta - \zeta'^*)]$,

$$(20)$$

where K is a real constant.

We can then obtain the derivative of the potential ϕ along the contour $\partial(R')$ from Eq. (19). A straightforward calculation leads to

$$\left(\frac{\partial \phi}{\partial \xi'} \right)_{\partial(R')} = -\frac{1}{\pi} pV \int_{-\infty}^{+\infty} \frac{1}{\xi - \xi'} \frac{\partial \phi}{\partial \eta} d\xi, \quad (21)$$

whose integrand is known through the relationship

$$\left(\frac{\partial \phi}{\partial \eta} \right)_{\partial(R')} d\xi = - \left(\frac{\partial \phi}{\partial n} \right)_{\partial(R')} dl. \quad (22)$$

On the other hand, since f is holomorphic,

$$\frac{\partial \psi}{\partial \xi} = \frac{\partial \phi}{\partial \eta}, \quad (23)$$

we can obtain the values of ϕ and ψ on the boundary $\partial(R')$ by integration of Eqs. (21) and (23). The potential $\phi(x', y')$ can then be obtained from Eqs. (6) and (18).

It is interesting to note that the "energy" U associated to the potential ϕ :

$$U = \iint_R |\nabla \phi|^2 dx dy, \quad (24)$$

and expressed in terms of the magnitudes evaluated at the boundary as

$$U = \int_{\partial(R)} \phi \frac{\partial \phi}{\partial n} dl = - \int_{\partial(R')} \phi \frac{\partial \phi}{\partial \eta} d\xi, \quad (25)$$

can be easily evaluated from Eq. (12) [or Eq. (14) if possible] in Dirichlet's case, and from Eqs. (21) and (22) in Neumann's case.

C. Determination of higher-order derivatives of ϕ

We have developed a method that allows the evaluation of ϕ from the knowledge of the function f at the boundary in Secs. II A and II B. From this function we can also evaluate the higher-order derivatives of the potential ϕ . In fact, by using the integral Cauchy formulas and taking real and imaginary parts, the order- p derivatives of ϕ can be written in terms of

$$\begin{aligned} \text{Re} \left\{ \frac{d^p f}{dz^p} \right\} &= \text{Re} \left\{ \frac{p!}{2\pi i} \int_{\partial(R)} \frac{f(Z)}{(Z - Z')^{p+1}} dZ \right\} \\ &= \frac{p!}{2\pi} \int_{\partial(R)} \frac{\phi \sin \chi_p + \psi \cos \chi_p}{r^{p+1}} dl, \end{aligned} \quad (26a)$$

$$\begin{aligned} \text{Im} \left\{ \frac{d^p f}{dz^p} \right\} &= \text{Im} \left\{ \frac{p!}{2\pi i} \int_{\partial(R)} \frac{f(Z)}{(Z - Z')^{p+1}} dZ \right\} \\ &= \frac{p!}{2\pi} \int_{\partial(R)} \frac{\phi \cos \chi_p - \psi \sin \chi_p}{r^{p+1}} dl, \end{aligned} \quad (26b)$$

where χ_p is given in terms of the angles denoted in Fig. 1 by $\chi_p = \alpha + p\theta$.

As an example, developing the integrands of Eqs. (26) as in Sec. II A we get the analogous of Eqs. (18) for the case $p = 1$:

$$\cos \chi_1 = \frac{[(X(\xi) - x')^2 - (Y(\xi) - y')^2](dY/d\xi) - 2(X(\xi) - x')(Y(\xi) - y')(dX/d\xi)}{r^2}, \quad (27a)$$

$$\sin \chi_1 = \frac{2(X(\xi) - x')(Y(\xi) - y')(dY/d\xi) + [(X(\xi) - x')^2 - (Y(\xi) - y')^2](dX/d\xi)}{r^2}. \quad (27b)$$

The higher-order derivatives can be easily obtained in the same way. The advantage of this method is that derivatives of any order can be obtained from the same data as the required for the potential.

III. APPLICATIONS

The method outlined in Sec. II A is specially suited for situations in which the potential ϕ is piecewise constant, because in those cases the evaluation of Eq. (12) and its subse-

quent integration to obtain ψ can be performed analytically. So, the general results are

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{\partial(R')} = \frac{1}{\pi} \sum_{j=0}^N (\phi_{j+1} - \phi_{i+1}) \left[\frac{1 - \delta_{j,0}}{\xi_j - \xi} - \frac{1 - \delta_{j,N}}{\xi_{j+1} - \xi} \right], \quad (28)$$

$$(\psi)_{\partial(R')} = \frac{1}{\pi} \sum_{j=0}^N \{ (\phi_{j+1} - \phi_{i+1}) [(1 - \delta_{j,N}) \times \ln|\xi_{j+1} - \xi| - (1 - \delta_{j,0}) \ln|\xi_j - \xi|] \}, \quad (29)$$

where the notations of Fig. 2 was used and $\xi_i < \xi < \xi_{i+1}$. We mention in passing that formula (29) could be obtained from the direct evaluation of Eq. (14).

A case of practical interest is given by the electrostatic potential generated by conducting boundaries. In this situation the inducted surface charge density per unit length, $\sigma = \partial\phi/\partial n$, is given by

$$\sigma[X(\xi), Y(\xi)] = - \left(\frac{\partial\psi}{\partial\xi}\right)_{\partial(R')} / \sqrt{\left(\frac{dX}{d\xi}\right)^2 + \left(\frac{dY}{d\xi}\right)^2}. \quad (30)$$

Let us consider as an example the problem of a square with one side at a potential assumed to be one, and the rest of the boundary at zero potential. The length of each side is normalized to one. We immediately get by using Eqs. (28) and (30):

$$\sigma = - \frac{1}{\pi} \frac{1/t^2 + 1/(4-t)^2}{1/(4-t) - 1/t + \frac{1}{2}}. \quad (31)$$

To obtain this formula we have used the following parametric representation of the boundary:

$$\begin{aligned} X(t) = t, & \quad Y(t) = 0, & \text{if } 0 < t < 1, \\ X(t) = 1, & \quad Y(t) = t - 1, & \text{if } 1 < t < 2, \\ X(t) = 3 - t, & \quad Y(t) = 1, & \text{if } 2 < t < 3, \\ X(t) = 0, & \quad Y(t) = 4 - t, & \text{if } 3 < t < 4, \end{aligned}$$

with the potential: 1 if $0 < t < 1$; 0 if $1 < t < 4$; and with the relation $s = 1/(4-t) - 1/t$ between t and the s parameter appearing in Eqs. (15).

It is important to point out that, even in this simple case, the obtention of σ from the expression of ϕ calculated by

means of the usual method of variable separation, is not possible due to the fact that the arising series is not derivable, a "blemish" commonly encountered in these methods when the potential on the boundary is discontinuous.

As a second example let us consider a circle of radius one with boundary conditions of the Neumann type given by: $\partial\phi/\partial\rho = \cos(\theta)$ on $\rho = 1$; where (ρ, θ) are polar coordinates. Using Eq. (23), the expression of ψ on the boundary is immediately found to be: $\psi = -\sin(\theta) + \text{const.}$ With the transformation $\xi = -\cot(\theta/2)$, Eq. (21) leads to

$$\left[\frac{\partial\phi}{\partial\theta'}\right]_{\rho=1} = -\frac{1}{2\pi} \csc^2(\theta'/2) pV \times \int_0^{2\pi} \frac{\cos(\theta)}{\cot(\theta/2) - \cot(\theta'/2)} d\theta. \quad (32)$$

The integration can be analytically performed and the result is simply: $\partial\phi/\partial\theta' = -\sin(\theta')$ on the boundary that on integration yields: $\phi = \cos(\theta') + \text{const.}$ With these expressions of ϕ and ψ , Eq. (6) leads on integration to: $\phi = \rho' \times \cos(\theta') + \text{const.}$, which agrees with the result obtained by classical methods.

These simple examples show how the proposed method can be used. Analytical solutions, however, are not always possible and in more involved cases one must resort to numerical evaluation of the integrals appearing in Eqs. (6) and (11) or (21). We wish to point out here that this numerical procedure is fast and easy to perform even in very complicated problems.

IV. FINAL REMARKS AND CONCLUSIONS

We have developed a method that allows to obtain the potential satisfying Laplace's equation with Dirichlet's or Neumann's conditions on a plane region bounded by an arbitrary curve.

The potential and its derivatives of any order are expressed by explicit quadratures which allows further developments. The great advantage of the method presented here is that the explicit form of the conformal transformation is not required. Therefore, the potential in very complicated regions, for which no conformal transformations are available, can be directly computed from the boundary conditions alone.

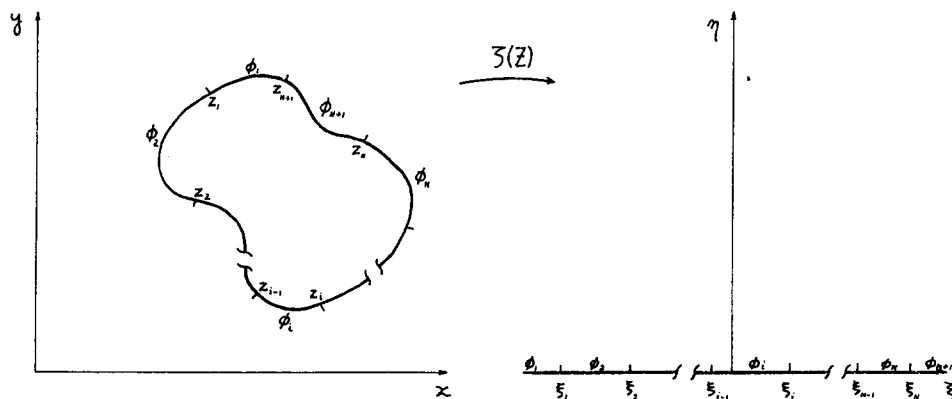


FIG. 2. Sketch of a Dirichlet's problem with a piecewise constant potential at the boundary (Z and ζ planes).

From the numerical point of view, the method is extremely low memory consuming and it allows the evaluation of the potential function ϕ with any precision avoiding the use of grids or numerical transformations.

Moreover, special cases of practical interest as, for instance, the electrostatic potential generated by conducting boundaries or the temperature distribution in regions whose boundary is composed by several constant temperature sections, etc.; admits solutions given by only a quadrature easy to evaluate even in very complicated regions. As a bonus, some magnitudes of interest (viz. the induced charge or the heat flow through the boundary) are obtained analytically.

Finally, for both Neumann's and Dirichlet's conditions, the total energy associated to the potential field can be straightforwardly computed from the formalism itself.

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