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Scalar concomitants of a metric and a curvature form. II

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The general form of a Lagrangian concomitant of a metric and a curvature form is found, improving substantially a previous result.

I. INTRODUCTION

In gauge field theories, Lagrangians of the form[1]

$$L = L(g_{ij};F^{\alpha}_{ij}) \tag{1}$$

are considered to obtain, through the use of variational principles, the field equations for a gauge theory. To establish the uniqueness of such equations, it is important to know the general form of Lagrangians of the type (1). Since $L/\sqrt{g}$ is a scalar, it is enough to find all scalar concomitants of such objects. In a previous paper[2] we have proved that it must be

$$L(g_{ij};F^{\alpha}_{ij}) = \sqrt{g}\, f(\phi^{\alpha\beta};\psi^{\alpha\beta}\phi^{\alpha\beta\gamma};\psi^{\alpha\beta\gamma}), \tag{2}$$

where

$$\phi^{\alpha\beta} = F^{\alpha ij}F^{\beta}{}_{ij}, \quad \psi^{\alpha\beta} = F^{\alpha ij}\, {}^*F^{\beta}{}_{ij}, \tag{3}$$

$$\phi^{\alpha\beta\gamma} = F^{\alpha i}{}_{j}F^{\beta j}{}_{h}F^{\gamma h}{}_{i}, \quad \psi^{\alpha\beta\gamma} = F^{\alpha i}{}_{j}F^{\beta j}{}_{h}\, {}^*F^{\gamma h}{}_{i}, \tag{4}$$

and where

$${}^*F^{\alpha ij} = \eta^{ijhk}F^{\alpha}{}_{hk} = (1/2\sqrt{g})\varepsilon^{ijhk}F^{\alpha}{}_{hk}.$$

Now, expression (2) is hard to handle because of the presence of the terms (4). In this paper we will prove that they can be removed from (2), obtaining an expression very similar to the one obtained in the electromagnetic case.[3] Since the latter was essential for solving the equivariant inverse problem, and so for proving the uniqueness of the Maxwell equations,[4] the result in the present paper could be useful for proving the uniqueness of the Yang–Mills equations, which will be the subject of a forthcoming paper.

II. GENERAL FORM OF THE LAGRANGIAN

Dividing or multiplying by $\sqrt{g}$ we turn densities to scalars and vice versa. So we can consider L given by (1) as a scalar. In this case, the invariance identities[5] are

$$L^{bs}g_{as} + L^{bs}_{\alpha}F^{\alpha}_{as} = 0, \tag{5}$$

where $L^{bs} = \partial L/\partial g_{bs}$, $L^{bs}_{\alpha} = \partial L/\partial F^{\alpha}_{bs}$. Since g_{as} is nonsingular, we deduce

$$L^{bs} = -L^{bt}_{\alpha}F^{\alpha}_{at}g^{as}. \tag{6}$$

Let us suppose for the moment that the Lie group G is three dimensional. Then, as we proved in Ref. 2, the skew-symmetric gauge tensor L^{bt}_{α} can be written as

$$L^{bt}_{\alpha} = a_{\alpha\beta}F^{\beta bt} + b_{\alpha\beta}\,{}^*F^{\beta bt}, \tag{7}$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are gauge invariant scalar concomitants of g_{ij} and F^{α}_{ij}. In the next step we will prove that they are symmetric in their greek indices, i.e., $a_{\alpha\beta} = a_{\beta\alpha}, b_{\alpha\beta} = b_{\beta\alpha}$. To achieve this, we replace (7) in (6) to obtain

$$L^{bs} = -a_{\alpha\beta}F^{\beta bt}F^{\alpha s}{}_{t} - b_{\alpha\beta}\,{}^*F^{\beta bt}F^{\alpha s}{}_{t}. \tag{8}$$

Taking account of the symmetry of L^{bs} in b, s, we have from (8),

$$a'_{\alpha\beta}F^{\beta bt}F^{\alpha s}{}_{t} + b_{\alpha\beta}({}^*F^{\beta bt}F^{\alpha s}{}_{t} - {}^*F^{\beta st}F^{\alpha b}{}_{t}) = 0, \tag{9}$$

where $a'_{\alpha\beta} = a_{\alpha\beta} - a_{\alpha\beta}$. Multiplying (9) by $-F^{\gamma}_{bs}$ we have

$$a'_{\alpha\beta}\phi^{\alpha\beta\gamma} + 2b_{\alpha\beta}\psi^{\gamma\alpha\beta} = 0, \tag{10}$$

where $\phi^{\alpha\beta\gamma}$ and $\psi^{\gamma\alpha\beta}$ are given by (4). It can be proved easily that $\psi^{\alpha\beta\gamma}$ is skew symmetric in all of its indices, so that (10) can be rewritten as

$$a'_{\alpha\beta}\phi^{\alpha\beta\gamma} + b'_{\alpha\beta}\psi^{\alpha\beta\gamma} = 0, \tag{11}$$

where $b'_{\alpha\beta} = b_{\alpha\beta} - b_{\beta\alpha}$.

Similarly, by multiplying (9) by $-{}^*F^{\gamma}{}_{bs}$ we obtain

$$a'_{\alpha\beta}\psi^{\alpha\beta\gamma} - b'_{\alpha\beta}\phi^{\alpha\beta\gamma} = 0. \tag{12}$$

Now, since $\dim G = 3$, and from the skew symmetry of $\phi^{\alpha\beta\gamma}$ and $\psi^{\alpha\beta\gamma}$, we have

$$\phi^{\alpha\beta\gamma} = \lambda\varepsilon^{\alpha\beta\gamma}, \quad \psi^{\alpha\beta\gamma} = \mu\varepsilon^{\alpha\beta\gamma}, \tag{13}$$

where λ and μ are scalar concomitants of g_{ij} and F^{α}_{ij}. Replacing (13) in (11) and (12) gives us

$$\varepsilon^{\alpha\beta\gamma}[a'_{\alpha\beta}\lambda + b'_{\alpha\beta}\mu] = 0, \tag{14}$$

and

$$\varepsilon^{\alpha\beta\gamma}[a'_{\alpha\beta}\mu - b'_{\alpha\beta}\lambda] = 0. \tag{15}$$

Taking $\gamma = 3$ in (14) and (15) it follows that

$$\lambda a'_{12} + \mu b'_{12} = 0, \tag{16}$$

and

$$\mu a'_{12} - \lambda b'_{12} = 0. \tag{17}$$

Since the determinant of the system given by (16) and (17) is $-(\lambda^2 + \mu^2) \neq 0$, we have

$$a'_{12} = b'_{12} = 0. \tag{18}$$

Similarly, taking $\gamma = 2$ and $\gamma = 1$ in (14) and (15) it follows that

$$a'_{31} = b'_{31} = a'_{23} = b'_{23} = 0. \tag{19}$$

From (18) and (19) we obtain the claimed symmetry of $a_{\alpha\beta}$ and $b_{\alpha\beta}$.

Now, we are in position to prove that L depends only on $\phi^{\alpha\beta}$ and $\psi^{\alpha\beta}$. To obtain this let us suppose

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$$\phi^{\alpha\beta} = \text{const}, \quad \psi^{\alpha\beta} = \text{const}. \tag{20}$$

Then

$$0 = d\phi^{\alpha\beta} = \frac{\partial\phi^{\alpha\beta}}{\partial g_{ij}} dg_{ij} + \frac{\partial\phi^{\alpha\beta}}{\partial F^{\gamma}_{ij}} dF^{\gamma}_{ij}$$

$$= -[F^{\alpha i}{}_k F^{\beta kj} + F^{\alpha j}{}_k F^{\beta ki}] dg_{ij} + F^{\beta ji} dF^{\alpha}_{ij} + F^{\alpha ji} dF^{\beta}_{ij}. \tag{21}$$

Similarly

$$0 = d\psi^{\alpha\beta} = -\tfrac{1}{2} g^{ij}\psi^{\alpha\beta} dg_{ij} + {}^*F^{\beta ji} dF^{\alpha}_{ij} + {}^*F^{\alpha ji} dF^{\beta}_{ij}. \tag{22}$$

Then

$$\begin{aligned}
dL &= L^{bs} dg_{bs} + L^{bt}_{\alpha} dF^{\alpha}_{bt} = L^{bt}_{\alpha}(-F^{\alpha}_{at} g^{as} dg_{bs} + dF^{\alpha}_{bt}) \\
&= (a_{\alpha\beta} F^{\beta bt} + b_{\alpha\beta} {}^*F^{\beta bt})(-F^{\alpha}_{at} g^{as} dg_{bs} + dF^{\alpha}_{bt}) \\
&= -\tfrac{1}{2} a_{\alpha\beta}(F^{\beta bt} F^{\alpha s}{}_t + F^{\alpha bt} F^{\beta s}{}_t) dg_{bs} \\
&\quad + \tfrac{1}{2} a_{\alpha\beta}(F^{\beta bt} dF^{\alpha}_{bt} + F^{\alpha bt} dF^{\beta}_{bt}) \\
&\quad - \tfrac{1}{2} b_{\alpha\beta}({}^*F^{\beta bt} F^{\alpha s}{}_t + {}^*F^{\alpha bt} F^{\beta s}{}_t) dg_{bs} \\
&\quad + \tfrac{1}{2} b_{\alpha\beta}({}^*F^{\beta bt} dF^{\alpha}_{bt} + {}^*F^{\alpha bt} dF^{\beta}_{bt}) \\
&= \tfrac{1}{2} a_{\alpha\beta}[-(F^{\beta i}{}_k F^{\alpha jk} + F^{\alpha i}{}_k F^{\beta jk}) dg_{ij} \\
&\quad + F^{\beta ij} dF^{\alpha}_{ij} + F^{\alpha ij} dF^{\beta}_{ij}] \\
&\quad + \tfrac{1}{2} b_{\alpha\beta}[-({}^*F^{\beta i}{}_k F^{\alpha jk} + {}^*F^{\alpha i}{}_k F^{\beta jk}) dg_{ij} \\
&\quad + {}^*F^{\beta ij} dF^{\alpha}_{ij} + {}^*F^{\alpha ij} dF^{\beta}_{ij}].
\end{aligned} \tag{23}$$

Now, the first term in the right-hand side of (23) is zero because of (21), and the second term is zero because of (22) and the identity

$${}^*F^{\beta i}{}_k F^{\alpha jk} + {}^*F^{\alpha i}{}_k F^{\beta jk} = -\tfrac{1}{2} g^{ij}\psi^{\alpha\beta},$$

which is easy to prove.

In summary, we have obtained that L is a constant when $\phi^{\alpha\beta}$ and $\psi^{\alpha\beta}$ are also constant. Then, when the Lie group is three dimensional we have proved that there is a function f of real variables such that

$$L = f(\phi^{\alpha\beta}; \psi^{\alpha\beta}). \tag{24}$$

Let us suppose now that $\dim G > 3$. Then we have the result (2) from Ref. 2. But, for each α, β, γ fixed, $\phi^{\alpha\beta\gamma}$ and $\psi^{\alpha\beta\gamma}$ are scalar concomitants of a metric tensor and three skew-symmetric tensors, namely, $F^{\alpha}_{ij}, F^{\beta}_{ij}, F^{\gamma}_{ij}$, and so they can be written in the form (24). So the result is also true when $\dim G > 3$.

If $\dim G = 2$, i.e.,

$$L = L(g_{ij}; F^1_{ij}; F^2_{ij}),$$

let F^3_{ij} be an auxiliary and arbitrary skew-symmetric tensor. We can write

$$L = L(g_{ij}; F^1_{ij}; F^2_{ij}; F^3_{ij}),$$

with $\partial L / \partial F^3_{ij} = 0$. Then from (24)

$$L = f(\phi^{\alpha\beta}; \psi^{\alpha\beta}) \quad (1 \leqslant \alpha, \beta \leqslant 3). \tag{25}$$

Differentiating (25) with respect to F^3_{ij} we have

$$0 = \frac{\partial f}{\partial\phi^{13}} F^{1ji} + \frac{\partial f}{\partial\phi^{23}} F^{2ji} + 2\frac{\partial f}{\partial\phi^{33}} F^{3ji} + \frac{\partial f}{\partial\psi^{13}} {}^*F^{1ji} + \frac{\partial f}{\partial\psi^{23}} {}^*F^{2ji} + 2\frac{\partial f}{\partial\psi^{33}} {}^*F^{3ji}.$$

Since $F^1, F^2, F^3, {}^*F^1, {}^*F^2, {}^*F^3$ are linearly independent in a dense subset of the set of the concomitance variables, then

$$\frac{\partial f}{\partial\phi^{\alpha 3}} = \frac{\partial f}{\partial\psi^{\alpha 3}} = 0 \quad (1 \leqslant \alpha \leqslant 3)$$

and so

$$L = h(\phi^{\alpha\beta}; \psi^{\alpha\beta}) \quad (1 \leqslant \alpha, \beta \leqslant 2).$$

In summary, taking account that (2) and (7) are valid in a dense subset of the set of the concomitance variables, we have proved the following theorem.

Theorem: If $L = L(g_{ij}; F^{\alpha}_{ij})$ is a scalar density, then there is a function f of real variables such that

$$L = \sqrt{g} f(\phi^{\alpha\beta}; \psi^{\alpha\beta}),$$

in a dense subset of the set of the concomitance variables, where $\phi^{\alpha\beta}, \psi^{\alpha\beta}$ are shown in Eq. (3).

Remark: The result in the theorem is not true in general for the whole set of concomitance variables. Otherwise, every scalar density would be an even function of $F^1, \ldots, F^r$, and $\phi^{\alpha\beta\gamma}$ is a counterexample.

[1] As for any gauge field theory, we are working in a manifold M endowed with the metric tensor g_{ij} and with a G-principal fiber bundle P with base space M. The F^{α}_{ij} are the coefficients of $\sigma^* F$ in some basis of the Lie algebra LG of G, where $\sigma: U \subset M \to P$ is a local section and F is the curvature form. See S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley-Interscience, New York, 1963), Vol. 1.

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