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On a Weyl-type theorem for higher-order Lagrangians

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Weyl's theorem is extended making use of the theory of concomitants to obtain a Lagrangian density for the massless bosonic fields without dimensional constants. It turns out to be quadratic in the gravitational field and encompasses all the theories that usually appear in the literature. It is shown that the gauge invariance of the Lagrangian follows from the invariance of the field equations.

I. INTRODUCTION

Weyl's theorem[1] establishes that the most general Lagrangian density that may be constructed with the metric and its derivatives up to second order which is linear in second derivatives is $a\sqrt{-g}R + b\sqrt{-g}$. This Lagrangian is the basis of general relativity (GR). But the difficulties that appear when one tries to quantize this theory have induced many researchers to correct it in different ways (increasing its order or degree, adding new bosonic or fermionic fields, and so on) to solve these problems.

The troubles of the quantization of the Einstein action are well known. It is usually believed that they are originated in the fact that if we want a dimensionless action linear in the scalar curvature, then the Lagrangian has a dimensional constant.[2,3] On the other hand, those theories that may be quantized successfully in the usual way all have dimensionless constants and come from Lagrangian densities that are quadratic in the fields.

One way to try to solve the problem is to replace the dimensional constant by fields as was first proposed by Brans and Dicke and followed, for example, by Batakis.[4] Another way is to begin with a theory without dimensional constants and then to make gravity an effective theory where the dimensional constant appears by Feynman integration over dimensional fields.[5] [A similar method is followed to obtain the Yukawa potential from the gauge theory of SU(2)×U(1) groups.] Another well-known approach is supergravity.[3] It is also known that when one uses an adequately constructed quadratic Lagrangian for gravity, at least the semiclassical theory that is obtained from it turns out to be renormalizable.[6]

Therefore for all that has been expressed, we think it is interesting to generalize somehow Weyl's theorem to study, at least, the problems of the last type of theories. Thus the purpose of this work is to obtain in a rigorous way the quadratic Lagrangians that are usually used for semiclassical gravity (or eventually more general ones: we do not ask them to be *a priori* quadratic in the Riemann tensor) using arguments as those used to deduce Weyl's theorem and the theory of concomitants which it originated, but including the hypothesis of nondimensionality of the constants which they did not use, together with the uniqueness of the Einsten tensor proved by Cartan.[7]

We also want to show the form of the Lagrangian terms for other fields coupled to the gravitational one with allowed interactions, always imposing dimensional constants not to be present. Then any of the fields could become constant, if necessary, to generate dimensional constants as in the Brans–Dicke theory. For the sake of simplicity, we only deal with bosonic massless fields in this first stage, i.e., spin-0, -1, and -2 fields, leaving the treatment of fermionic spin-$\frac{1}{2}$ and -$\frac{3}{2}$ fields for a forthcoming paper.

We also prove that, for this general Lagrangian, the gauge invariance of the field equations (a mandatory hypothesis since they have physical meaning) implies the gauge invariance of the Lagrangian itself, restricting thus severely its general form.

II. THE LAGRANGIAN DENSITY FOR THE BOSONIC FIELDS

The aim of this work is to find in a rigorous way the Lagrangian density for a theory that fulfills the conditions below.

(i) The fields involved are the metric g_{ij}, the electromagnetic potential vector A_i, and a scalar field φ. The latter will play different roles, in general the one of an ordinary field or eventually a constant.

(ii) Dimensional constants are not allowed because

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they introduce well-known problems yielding generally nonrenormalizable theories.

(iii) The Lagrangian will contain only the derivatives that appear in Eq. (1). We introduce only first derivatives of A_i and φ because we want second-order field equations for these fields. On the contrary, we allow first- and second-order derivatives of g_{ij} because naturally we want GR to be contained in our general theory. (Remember that the Hilbert action is degenerate, so the field equations are of second order even if the Lagrangian has the same order.) Besides, we do not impose any maximum degree to the gravitational field in the Lagrangian.

(iv) Units: We set $c = \hbar = 1$. The action is dimensionless so to be able to construct the generating functional. Therefore

$$[S] = 1 \quad \text{and} \quad [g_{ij}] = 1.$$

Then

$$[L] = l^{-4}, \quad [\varphi] = [A_i] = l^{-1},$$

and

$$[\kappa] = l^{-1}, \quad \text{with } \kappa^2 = 16\pi G,$$

where G is the Newtonian constant.

(v) We require gauge invariance of the electromagnetic field equations.

So let L be a Lagrangian concomitant of the metric tensor, the electromagnetic potential (i.e., in a precise mathematical language a convector), a scalar field, and their derivatives up to the following order:

$$L = L(g_{ij};g_{ij,h};g_{ij,hk};A_i;A_{i,j};\varphi;\varphi_{,j}). \tag{1}$$

From condition (iii) and field dimensions (iv), by a change of scale λ in L, we will have

$$L(g_{ij};\lambda g_{ij,h};\lambda^2 g_{ij,hk};\lambda A_i;\lambda^2 A_{i,j};\lambda\varphi;\lambda^2\varphi_{,i})$$
$$= \lambda^4 L(g_{ij};g_{ij,h};g_{ij,hk};A_i;A_{i,j}\varphi;\varphi_{,i}).$$

Differentiating four times with respect to λ, making $\lambda \to 0$ and applying the replacement theorem,[8] we obtain

$$\begin{aligned} L = {} & \Lambda_1^{ijhk} R_{ijhk}\varphi^2 + \Lambda_1^{ijhks} R_{ijhk} A_s \varphi + \Lambda_2^{ijhks} R_{ijhk}\varphi_{,s} \\ & + \Lambda_1^{ijhkrs} R_{ijhk} A_r A_s + \Lambda_2^{ijhkrs} R_{ijhk} A_{r;s} + \Lambda_1 \varphi^4 \\ & + \Lambda_1^i A_i \varphi^3 + \Lambda_2^i \varphi^2 \varphi_{,i} + \Lambda_1^{ij}\varphi^2 A_i A_j + \Lambda_2^{ij} A_{i;j}\varphi^2 \\ & + \Lambda_3^{ij}\varphi\varphi_{,i} A_j + \Lambda_4^{ij}\varphi_{,i}\varphi_{,j} + \Lambda_1^{ijk}\varphi A_i A_j A_k \\ & + \Lambda_2^{ijk}\varphi A_i A_{j;k} + \Lambda_2^{ijhk} A_i A_j A_h A_k + \Lambda_3^{ijhk} A_i A_j A_{h;k} \\ & + \Lambda_4^{ijhk} A_{i;j} A_{h;k} + \Lambda_1^{ijhklmst} R_{ijhk} R_{lmst}, \end{aligned} \tag{2}$$

where $\Lambda_t^{\cdots} = \Lambda_t^{\cdots}(g_{ij})$ are tensorial densities, R_{ijhk} is the Riemann tensor, and ; stands for covariant differentiation with respect to the Christoffel symbols Γ^i_{jk}.

Recently, the $\Lambda_t^{\cdots}(g_{ij})$ have been determined for any order of the tensor.[9,10] According to those results and taking into account that $n = 4$, n being the space-time dimension, we can say that

(i) $\Lambda = \text{const}$,

(ii) $\Lambda^i = 0$,

(iii) $\Lambda^{ij} = a\sqrt{-g}\,g^{ij}$,

(iv) $\Lambda^{ijh}/\sqrt{-g}$, is a linear combination of $g^{[ij}g^{hk]}$ and ϵ^{ijhk},

where [ ] indicates the set built up from the permutation of all indexes included in [ ]. For example,

$$g^{[ij}g^{hk]} = \{g^{ij}g^{hk}, g^{ih}g^{jk}, g^{ik}g^{jh}\},$$

(v) $\Lambda^{ijh} = 0$,

(vi) $\Lambda^{ijhkl} = 0$,

(vii) $\Lambda^{ijhklm}/\sqrt{-g}$ is a linear combination of $g^{[ij}g^{hk}g^{lm]}$ and $g^{[ij}\epsilon^{hklm]}$,

(viii) $\Lambda^{ijkhlmst}/\sqrt{-g}$ is a linear combination of $g^{[ij}g^{hk}g^{lm}g^{st]}$ and $g^{[ih}g^{jk}\epsilon^{lmst]}$.

So we obtain the following expression for the action:

$$\begin{aligned} S = \int L\, d^4x = \int \sqrt{-g}\{ & a_1\varphi^2 R + a_2 R A^i A_i \\ & + a_3 R^{ij} A_i A_j + a_4 R^{ij} A_{i;j} + a_5\varphi^4 + a_6\varphi^2 A^i A_i \\ & + a_7\varphi\varphi_{,i} A^i + a_8\varphi_{,i}\varphi^{,i} + a_9 A^i A^j A_{i;j} + a_{10} F_{ij} F^{ij} \\ & + a_{11} A_{i;j} A^{i;j} + a_{12} R^2 + a_{13} R_{ijhk} R^{ijhk} \\ & + a_{14} R_{ijhk} R^{ihjk} + a_{15} R_{ij} R^{ij} + a_{16}\epsilon^{ijhk} R_{ij}{}^{lm} R_{hklm} \\ & + a_{17}\epsilon^{ijhk} R_{iljm} R_{hk}{}^{lm} + \text{total divergence}\} d^4x, \end{aligned} \tag{3}$$

where R_{ij} the Ricci tensor, R the scalar curvature, and $F_{ij} = A_{i;j} - A_{j;i}$.

Let us suppose that $E(L) = \delta S/\delta\varphi$ is gauge invariant. Then $\partial E(L)/\partial A_m = 0$, which implies that

$$4a_6\varphi A_i g^{im} - a_7\Gamma^t_{th}\varphi g^{hm} + a_7\varphi g^{ht}g^{ms}g_{ts,h} = 0. \tag{4}$$

Differentiating Eq. (4), with respect to A_i, it turns out immediately that $a_6 = 0$. Then contracting Eq. (4) with g_{mk} and differentiating with respect to $g_{ab,c}$, it gives

$$a_7\varphi\{g^{bc}\delta^a{}_k - g^{ab}\delta^c{}_k + g^{ac}\delta^b{}_k\} = 0.$$

Multiplying by g_{bc} and contracting a with k, we obtain that $a_7 = 0$.

Let us suppose now that $E^r(L) = \delta S/\delta A_r$ is gauge invariant. Then $\delta E^r(L)/\partial A_m = 0$, that is,

$$\begin{aligned} & 2a_2 R g^{mr} + 2a_3 R^{mr} + 2a_6\varphi^2 g^{mr} + a_9\{A_{i;j} g^{im} g^{jr} \\ & - 2\Gamma^r_{ij} A_h g^{ih} g^{jm} - 2\Gamma^m_{ij} A_h g^{ih} g^{jr} - 2\Gamma^t_{ij} A_t g^{im} g^{jr} \\ & - \Gamma^t_{ts} A_k g^{rm} g^{sk} - \Gamma^t_{ts} A_h g^{rh} g^{sm} - A_{k;s} g^{rm} g^{sk} \\ & + A_k g^{ri} g^{mj} g^{sk} g_{ij,s} + A_h g^{ri} g^{hj} g^{sm} g_{ij,s} \\ & + A_k g^{si} g^{kj} g^{rm} g_{ij,s} + A_h g^{si} g^{mj} g^{rh} g_{ij,s}\} \\ & + a_{11}\{2\Gamma^m_{ij}\Gamma^r_{hk} g^{ih} g^{jk} + 2\Gamma^l_{ls}\Gamma^m_{hk} g^{rh} g^{sk} + 2\Gamma^m_{hk,s} g^{rh} g^{sk} \\ & - 2\Gamma^m_{hk} g^{ri} g^{hj} g^{sk} g_{ij,s} - 2\Gamma^m_{hk} g^{rh} g^{si} g^{kj} g_{ij,s}\}. \end{aligned} \tag{5}$$

Differentiating Eq. (5), with respect to $g_{ab,cd}$, taking at the point under consideration a basis, where $g_{ij} = \text{diag}(1 -1 -1 -1)$ and making $m = a = b = c = 1$, $r = d = 2$, $a_{11} = 0$ results. Taking $m = c = 2$, $r = a = 3$, and $b = d = 4$, then $a_3 = 0$. And making $d = c = 1$, $m = r = a = b = 2$, $a_2 = 0$ results.

Differentiating Eq. (5) with respect to $A_{a,b}$ and taking into account that $a_2 = a_3 = a_{11} = 0$,

$$a_9(g^{am}g^{br} - g^{rm}g^{ab}) = 0$$

results. Letting $a = b = 1$, $r = m = 2$, we get $a_9 = 0$.

Let us finally suppose that $E^{ij}(L) = \delta S/\delta g_{ij}$ is gauge invariant, then $\partial E^{ij}(L)/\partial A_m = 0$. It is clear due to what we have just seen that this is the same as

$$\frac{\partial E^{ij}(L_4)}{\partial A_m} = 0, \quad \text{where } L_4 = a_4\sqrt{-g}R^{ij}A_{i;j}.$$

As L_4 is linear in $g_{ij,hk}$, fourth-order derivatives do not appear in $E^{ij}(L_4)$. Differentiating $\partial E^{ij}(L_4)/\partial A_m$ with respect to $g_{kt,hsc}$ and then contracting with $g_{st}g_{ah}g_{ck}$, it results, after a long computation, that $a_4 = 0$. So we have proved the following.

Theorem: If L is a Lagrangian density of the form

$$L = L(g_{ij};g_{ij,h};g_{ij,hk};A_i;A_{i,j};\varphi;\varphi_{,i}),$$

which satisfies hypotheses (i)–(v), then gauge invariance of the associated Euler–Lagrange equations implies gauge invariance of the Lagrangian and it becomes

$$\begin{aligned} L = {} & b_1\sqrt{-g}R\varphi^2 + b_2\sqrt{-g}\varphi^4 + b_3\sqrt{-g}\varphi_{,i}\varphi_{,j}g^{ij} \\ & + b_4\sqrt{-g}F_{ij}F^{ij} + b_5\sqrt{-g}R^2 + b_6\sqrt{-g}R_{ijhk}R^{ijhk} \\ & + b_7\sqrt{-g}R_{ij}R^{ij}. \end{aligned} \tag{6}$$

Note that $b_5\sqrt{-g}R^2$, $b_6\sqrt{-g}R_{ijhk}R^{ijhk}$, and $b_7\sqrt{-g}R_{ij}R^{ij}$ terms may be related through the Gauss–Bonnet theorem.

III. SOME CONSIDERATIONS ON THE CONSTRUCTED LAGRANGIAN

Now we can reobtain several well-known classical theories from the Lagrangian of Eq. (6) choosing different values for the constants.

(a) General relativity: General relativity is a particular case of Eq. (6) avoiding dimensional constants as in the Brans–Dicke theory[11] whose Lagrangian may be written

$$L = -\sqrt{-g}\{-\varphi^2R + 4wg^{ij}\varphi_{,i}\varphi_{,j}\},$$

w being a numerical constant. This is obtained from our Eq. (6) taking $b_1 = -\frac{1}{2}$, $b_3 = 4w$, and all other coefficients equal to zero. Finally, the Einstein Lagrangian is obtained setting $\varphi^2 = \kappa^{-2}$.

(b) Maxwell–Einstein: The electromagnetic field is minimally coupled to the gravitational field replacing all partial derivatives which appear in its formulation in Minkowskian space-time by covariant ones. Maxwell electromagnetism plus relativity can be obtained from Eq. (6) making $b_1 = -\frac{1}{2}$; $b_4 = 2\pi/137$; all other $b_i = 0$ and thinking of φ^2 as κ^{-2}.

(c) Scalar field Lagrangians: It is widely accepted that the Lagrangian for the massless scalar self-interacting field in curved space-time is

$$L = \xi\varphi^2\sqrt{-g}R + \sqrt{-g}g^{ij}\varphi_{,i}\varphi_{,j} + \sqrt{-g}\lambda\varphi^4.$$

So we must take in Eq. (6) $b_1 = \xi$, $b_2 = \lambda$, $b_3 = 1$, with $\xi = 0$ or $\frac{1}{6}$ for minimal or conformal coupling, respectively, and all other coefficients $b_i = 0$. With two different scalar fields we could also add the $\kappa^{-2}\sqrt{-g}\,R$ term to obtain GR coupled to the matter scalar field.

(d) Quadratic Lagrangian for the gravitational field: In spite of the fact that predictions based on Einstein's relativity are in large agreement with experience, it is necessary to form a generalization for high energies—high curvatures—which could circumvent its quantum difficulties. Quantum field theories in curved space-time take into account the quantum properties of matter, including the eventual perturbations of the metric due to back reaction, but they do not quantize the background geometry which remains as a classical external field. The results obtained by this procedure—although incomplete—would eventually represent some semiclassical limit of a more general theory where the space-time geometry would be quantized too. We believe this limit describes the physical phenomena in those regions that verify $\max(|R^{ij}{}_{hk}|) < c^3/16\pi Gh \equiv l_p^{-2}$, where l_p is the Planck longitude, $l_p \sim 10^{-32}$ cm.

The pure gravitational terms of the Lagrangian of Eq. (6)—with the identification of $b_1\varphi^2$ with κ^{-2} and $b_2\varphi^4$ with Λ, Λ the cosmological constant—are those generally used by the quantum field theory in curved space-time. Its action for gravitation is

$$S_g = \int \sqrt{-g}\{(R/2\kappa^2 - 2\Lambda) + \alpha R^2 + \beta R_{ij}R^{ij}\}d^4x$$

obtained by taking $b_1 = 1$, $b_2 = -2$, $b_3 = \alpha$, $b_7 = \beta$ in our Eq. (6) and making use of the Gauss–Bonnet theorem.

Besides its quantum interest, the quadratic Lagrangian theories may give information about classical features of space-time for high curvatures. For example, we can consider the theory with Lagrangian $L = \{(R/2\kappa^2 - 2\Lambda) + \alpha R^2\}\sqrt{-g}$ that can be obtained from our Eq. (6) making $b_1 = 1$, $b_2 = -2$, $b_5 = \alpha$, and all other $b_i = 0$. These kinds of theories are studied in Ref. 12 where it is shown that black holes have no hair as it is in ordinary GR.

Our Lagrangian density also contains the action which Strominger shows to have non-negative energy for asymtotically flat limits[13] if we choose $F_{ij} = 0$, $b_1\varphi^2 = 1$, $b_5 = \frac{1}{2}\beta^2$, and all other $b_i = 0$.

IV. CONCLUSIONS

We have extended Weyl's theorem, making use of the theory of concomitants that comes from it, to obtain a Lagrangian density for the massless bosonic fields, without dimensional constants. It turns out quadratic in the gravitational field even if this order was not prescribed from the outset. This circumvents, at least at the semiclassical level, the quantum difficulties of the Einstein Lagrangian that comes from the original Weyl theorem. We have also limited the possible couplings among the bosonic matter fields and among them and the metric, choosing only Lagrangians with no dimensional constants.

Moreover, we have shown that it is not necessary to ask the Lagrangian to be gauge invariant because its invariance follows from the invariance of the field equations. This also excludes interactions which would be allowed if we only took into account tensorial concomitants and dimensional analysis.

The Lagrangian density thus obtained encompasses all theories that usually appear in the literature as we have shown in Sec. III.

In a forthcoming paper this work will be extended to include fermionic fields, making use of spinorial concomitants, allowing us to compare the results not only with GR but also with supergravity theories.

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