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A Generalized Diffusion Equation: Radial Symmetries and Comparison Theorems

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1. INTRODUCTION

In a recent paper [2] we have considered qualitative properties of solutions of the equation

$$u_t = (k(u)\,|u_x|^{N-1}\,u_x)_x, \qquad N > 0. \tag{1.1}$$

Equation (1.1) is of course the one-dimensional (plane) version of the equation

$$u_t = \nabla \cdot (k(u)\,B), \tag{1.2}$$

where

$$B = |\nabla u|^{N-1}\,\nabla u, \qquad N > 0, \tag{1.3}$$

$k(u)\,B$ being a generalized nonlinear flux, the gradient ∇u having its usual meaning, and in three-dimensional Cartesian coordinates (x_1, x_2, x_3), $|\nabla u| \equiv (u_{x_1}^2 + u_{x_2}^2 + u_{x_3}^2)^{1/2}$.

In the present paper we wish to extend and complete some of the results presented in [2]. In particular we present in Section 5 a comparison theorem promised in [2], and in Section 4 we partially complete an existence theorem left open in that paper.

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The treatment of [2] is extended principally by the consideration of solutions of (1.2) which have radial symmetry. Thus, if u depends only on some radial variable r and t, then (1.2) can be written

$$u_t = r^{1-n}(r^{n-1}k(u)\,|u_r|^{N-1}\,u_r)_r, \tag{1.4}$$

where $n = 1$, 2, or 3 corresponds to plane, cylindrical, or spherical symmetry. First we consider similarity solutions of Eq. (1.4) in which u depends only on $\xi = |x|/t^{1/(N-1)}$ ($|x| \equiv r$). Section 2 is devoted to properties of solutions to the corresponding differential equation for $\xi > 0$. In particular, necessary and sufficient conditions for compact support of such solutions are derived in Section 2.4. These are similar to those derived in [2] for the plane case. We assume $k(u) \geqslant 0$ and $k^{1/N}(u) \in L^1$ locally. Note that these similarity solutions can be taken to refer to moving boundary problems, where $\xi = \xi_0 > 0$ specifies the moving boundary and the solution is sought in the region $\xi \geqslant \xi_0$.

Section 3 consists of existence and uniqueness theorems for such similarity solutions with $\xi_0 > 0$, $u(\xi_0) = A > 0$, $u(+\infty) = 0$. Existence results are proved for coefficients $k(s)$ which satisfy the conditions

$$k(s) \geqslant 0 \quad \text{a.e. in} \quad (0, A), \qquad k^{1/N} \in L^1(0, A),$$

and

$$\int_0^A \left(\int_x^A \left(\frac{k(s)}{s} \right)^{1/N} ds \right)^{nN/(N+1)} dx < \infty. \tag{1.5}$$

If, moreover, $k(s) > 0$ a.e. in $(0, A)$ and

$$\int_0^x k^{1/N}(s)\,ds = o(x^{(n-1-N)/nN}) \qquad \text{as} \quad x \to 0^+, \tag{1.6}$$

then there is a similarity solution $u(\xi)$ which is absolutely continuous in $\xi > \xi_0$, $u(\xi_0) = A > 0$, $u(+\infty) = 0$ such that $\xi^{n-1}k(u)\,|u'(\xi)|^{N-1}\,u'(\xi)$ coincides a.e. with an absolutely continuous function $f_{\mathrm{av}}(\xi)$, and $f_{\mathrm{av}}(\xi) \to 0$ as $\xi \to \infty$. This solution is unique (cf. Section 2.3.) We note here the particular feature that for $n > 1$, f_{av} may not tend to zero as $\xi \to \infty$.

In the Appendix exact solutions are given for the situation when $k(s) = 1$. Hence for this special case computations can be made of conditions (1.5) and (1.6). Condition (1.5) gives by direct calculation the condition $N > (n-1)/(n+1)$ and Eq. (A.6) of the Appendix shows that the average flux $\xi^{n-1}\,|u'(\xi)|^{N-1}\,u'(\xi) \equiv f_{\mathrm{av}}$ tends to zero as ξ tends to infinity in this case. On the other hand from (A.7), f_{av} clearly tends to a negative constant as ξ tends to infinity if $0 < N < (n-1)/(n+1)$. The necessary computations for $N = (n-1)/(n+1)$ are left to the reader.

In Section 4 the possibility of letting $\xi_0 \to 0^+$ is discussed. Note that this is not always a physically sensible thing to do. For example, Eq. (1.4) with $N=1$, and $n=2$ or 3 has sensible moving boundary solutions with u specified on a moving boundary, but it is not possible to specify u at $r=0$ and at $r=\infty$ and obtain a finite solution. On the other hand, when $n=1$ (the plane case) it should be possible to consider $\xi_0 \to 0+$ and so complete the existence proofs of [2]. This is done in Section 4 where existence and uniqueness is proved for the plane case.

It is also shown in Section 4 that for moving boundary problems, where ξ_0 is a priori unknown, a value of ξ_0 can be determined via a flux condition across the interface $\xi = \xi_0$.

In Section 5 comparison theorems are proved for a class of parabolic divergence structure operators including Eqs. (1.1) and (1.4).

They can be loosely described as giving conditions under which "$u \leqslant v$ on the parabolic boundary $\partial_p G$ of G implies $u \leqslant v$ in G" (cf. Section 5.1). Here we consider certain classes of generalized solutions to the equations and no smoothness hypothesis is placed on the boundary of G (although we are mainly concerned with the domain $G=(0,+\infty)\times(0,T)$). On the other hand, an integrability condition is placed on $\partial u/\partial t$ in G away from $t=0$ which has proved relevant in the case of the porous medium equation (cf. Aronson and Bénilan [1]).

Recently, Wolanski [13] has extended part of these results to solutions of nonlinear evolution equations governed by accretive operators.

In Section 6 the above-mentioned results are used to show that properties of the similarity solution of (1.4) can be carried over to solutions corresponding to more general boundary conditions.

2. Analysis of the Similarity Solution

We shall call the function $u(x,t)$, $x \in R^n$, $t \in (0,T)$ a solution to the equation

$$\nabla \cdot (k(u)\,|\nabla u|^{N-1}\,\nabla u) = u_t \tag{2.1}$$

if $u(x,t)$ admits weak derivatives locally in $L^1((R^n \backslash \{0\}) \times (0,T))$, each component of $k(u)\,|\nabla u|^{N-1}\,\nabla u$ belongs to $L^1((R^n\backslash\{0\})\times(0,T))$, and for every $\eta \in C_0^1$,

$$\iint \{k(u)\,|\nabla u|^{N-1}\,\nabla u \cdot \nabla \eta - \eta_t u\}\,dx\,dt = 0. \tag{2.2}$$

After setting $\eta = \psi(t)\,\phi(|x|/t^{1/N+1})$, $\psi \in C_0^1(0,T)$, $\phi \in C_0^1(0,\infty)$, $\xi = |x|/t^{1/(N+1)}$, we get

$$\left\{\xi^{n-1}(k(u)\,|u'|^{N-1}\,u')(\xi)+\frac{\xi^n}{N+1}\,u(\xi)\right\}$$
$$=\frac{n}{N+1}\int^{\xi} s^{n-1}u(s)\,ds+C \tag{2.3}$$

almost everywhere in $\xi>0$, and therefore $\{\xi^{n-1}(k(u)\,|u'|^{N-1}\,u')(\xi)+(\xi^n/(N+1))\,u(\xi)\}$ belongs to the Lebesgue class of a locally absolutely continuous function we shall call $h(\xi)$ (here and in what follows the prime denotes differentiation).

In turn,

$$h'(\xi)=(n/(N+1))\,\xi^{n-1}u(\xi) \qquad \text{a.e.} \tag{2.4}$$

Conversely, if $h(\xi)=\{\xi^{n-1}(k(u)\,|u'|^{N-1}\,u')(\xi)+(\xi^n/(N+1))\,u(\xi)\}$ a.e. is a locally absolutely continuous function of $\xi>0$ that satisfies (2.4) a.e., it can be shown that $u(|x|/t^{1/N+1})$ is a solution of (2.2) as stated.

From the integrability conditions it follows that $u(\xi)$ is a locally absolutely continuous function and therefore so is $\xi^{n-1}(k(u)\,|u'|^{N-1}\,u')(\xi)$ and

$$(\xi^{n-1}k(u)\,|u'|^{N-1}\,u')'+(\xi^n/(N+1))\,u'(\xi)=0 \qquad \text{a.e.} \tag{2.5}$$

We shall denote $f(\xi)$ the absolutely continuous function in the class of $(k(u)\,|u'|^{N-1}\,u')(\xi)$.

Observation. An equivalent presentation is obtained via the Kirchoff transformation $U(t)=\int_0^t k^{1/N}(s)\,ds$, $v(U(t))=t$, if $k(s)$ does not vanish a.e. in any interval (thus $U(t)$ is strictly increasing). Putting $V(\xi)=U(u(\xi))$, $\xi^{n-1}\,|V'(\xi)|^{N-1}\,V'(\xi)=\tilde{h}(\xi)$ a.e. ($\tilde{h}(\xi)$ locally absolutely continuous), it can be seen that

$$\tilde{h}'(\xi)=(n/(N+1))\,\xi^{n-1}v(V(\xi)) \qquad \text{a.e.}$$

The hypothesis of k guarantees that v is a continuous increasing function, and $v(V(\xi))=u(\xi)$ by definition. It can be shown that this hypothesis is superfluous and that an acceptable theory can be developed in which v and $u(\xi)$ have jumps, corresponding to the intervals where $k=0$ a.e. (cf. [3, 14]).

2.1.

We now assume we have a solution $u(\xi)$ in the sense just mentioned, and assume $k(s)\geqslant 0$, $k^{1/N}(s)\in L^1$ locally.

THEOREM. *The function $u(\xi)$ is monotone in $\xi>0$.*

Proof. The proof is similar to the one presented in [2]. Assuming $0 \leqslant a < b$, $u(a) = u(b) = C_1$ and $u > C_1$ in (a, b), it follows from (2.3) that

$$b^{n-1}f(b) - a^{n-1}f(a) = \frac{n}{N+1} \int_a^b s^{n-1}(u(s) - C_1)\, ds > 0.$$

From this inequality a contradiction follows (cf. [2]) recalling $0 < a < b$.

As a consequence of this theorem, and of the assumption of local (Lebesgue) integrability of $k^{1/N}$, it follows that $\int_0^{u(\xi)} k^{1/N}(s)\, ds$ is locally absolutely continuous and

$$\left(\int_0^{u(\xi)} k^{1/N}(s)\, ds\right)' = k^{1/N}(u(\xi))\, u'(\xi), \qquad \text{a.e.} \quad \xi.$$

As we are interested in boundary conditions of the type $u = A > 0$ at $\xi = \xi_0 \geqslant 0$, $\lim_{\xi \to +\infty} u = 0$, we shall assume throughout that $u(\xi)$ is a monotone decreasing function of ξ. From (2.3) we obtain

$$\begin{aligned} 0 \geqslant \xi^{n-1}f(\xi) &= C_0 + \frac{1}{N+1} \int_{\xi_0}^{\xi} ns^{n-1}u(s)\, ds - \frac{\xi^n}{N+1} u(\xi) \\ &\geqslant C_0 + \frac{1}{N+1} \int_{\xi_0}^{\xi} ns^{n-1}(u(s) - u(\xi))\, ds - \xi_0^n u(\xi_0) \\ &\geqslant C_0 - (\xi_0^n/(N+1))\, u(\xi_0) = C_{\xi_0}. \end{aligned} \tag{2.6}$$

We now take into account the definition of $h(\xi)$ and the equation $h'(\xi) = (n/(N+1))\, \xi^{n-1}u(\xi)$ to obtain

$$h'(\xi) - (n/\xi)\, h(\xi) = -(n/\xi)\, (\xi^{n-1}f(\xi)) \tag{2.7}$$

and upon integration,

$$h(\xi) = n\xi^n \int_\xi^\infty \frac{f(s)}{s^2}\, ds, \qquad \xi \geqslant \xi_0 > 0. \tag{2.8}$$

Now if $f(\xi_1) = 0$, $\xi_1 > \xi_0$, $0 \leqslant (\xi_1^n/(N+1))\, u(\xi_1) \leqslant n\xi_1^n \int_{\xi_1}^\infty (f(s)/s^2)\, ds \leqslant 0$, whence $u(\xi_1) = 0$ and $u(\xi) = 0$ in $[\xi_1, \infty)$ by monotonicity. We observe that $f(\xi_1) = 0$ wherever u remains constant in an open interval. We have therefore proved the following theorem (cf. also Bouillet *et al.* [3] for $N = 1$ and a slightly different operator).

THEOREM. *A solution $u(\xi)$ as stated, such that $u(\xi_0) = A > 0$, $\xi_0 \geqslant 0$, $\lim_{\xi \to \infty} u(\xi) = 0$, is strictly monotone decreasing at those $\xi > 0$ with $u(\xi) > 0$.*

2.2.

We point out here that $\xi^{n-1}f(\xi)$ is proportional to the flux average through the surface $|x| = \xi \cdot t^{1/N+1}$ for fixed $\xi > 0$, $t > 0$. It will be of interest to have conditions under which $f_{av} \equiv \xi^{n-1}f(\xi)$ tends to zero as $\xi \to +\infty$: from (2.5) follows the existence of $\lim_{\xi\to\infty} \xi^{n-1}f(\xi) = f_\infty \leqslant 0$. From (2.8), it follows by L'Hospital's rule that

$$h(\xi) \sim \xi^{n-1}f(\xi) = f_{av}(\xi) \to f_\infty$$

and

$$\xi^n u(\xi) \to 0 \qquad \text{as} \quad \xi \to \infty.$$

This is also a consequence of (2.3) and the theorem in Section 2.1.

It will be sufficient to find conditions for $h(\xi) \to 0$ as $\xi \to \infty$; these can be obtained by applying Jensen's inequality—for $N \leqslant 1$—and bound (2.6) for $f(\xi)$ to $\int_\xi^\infty |f(s)|\, s^{-2}\, ds$ to give

$$|h(\xi)| \leqslant C \cdot \xi^{n-1-N} \left(\int_0^{\sigma/\xi^n} k^{1/N}(s)\, ds \right)^N, \qquad \text{where} \quad \xi^n u(\xi) \leqslant \sigma.$$

For $N > 1$, and suitable $p > 1$, Holder's inequality applied to

$$\int_\xi^\infty (-f(s))^{1/Nq} (-f(s))^{(N-1)/Nq}\, s^{-2}\, ds, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

coupled with (2.6) gives

$$|h(\xi)| \leqslant C\xi^{(n-1-N)/qN} \left(\int_0^{\sigma/\xi^n} k^{1/N}(s)\, ds \right)^{1/q}.$$

We can put these two bounds together in a sufficient condition for $h(\xi)$ to tend to zero as $\xi \to \infty$, namely,

$$\int_0^x k^{1/N}(s)\, ds = o(x^{(n-1-N)/nN}) \qquad \text{as} \quad x \to 0^+. \tag{2.9}$$

If k is bounded near $x = 0$, this condition is satisfied for $N \geqslant (n-1)/(n+1)$ (cf. in this connection Cases (ii) and (iii) of the Appendix).

2.3. *Uniqueness in the Case $f_{av} \to f_\infty = 0$*

Let $u(\xi)$, $\tilde{u}(\xi)$ be two solutions of the same boundary value problem, taking values $A > 0$ at ξ_0 and limit zero as $\xi \to \infty$. Consider the case in which $f_\infty = 0$ for all solutions (cf. Section 2.2). The proof in [2, Section 2.1] needs only minor modifications, namely, [2, formula (2.7)] needs to be replaced with

$$b^{n-1}(\tilde{f}(b)-f(b))-a^{n-1}(\tilde{f}(a)-f(a))=\frac{n}{N+1}\int_a^b s^{n-1}(\tilde{u}(s)-u(s))\,ds, \tag{2.10}$$

where the assumption $\tilde{u}(\xi)>u(\xi)$ for $0\leqslant a<\xi<b$, with equality at the endpoints, leads to contradiction. The remaining possibility (cf. [2, Eq. (2.8)]) now reads

$$0\leqslant\frac{n}{N+1}\int_{\xi_0}^{\infty} s^{n-1}(\tilde{u}(s)-u(s))\,ds=-\xi_0^{n-1}(\tilde{f}(\xi_0)-f(\xi_0))\leqslant 0$$

and implies $\tilde{u}\equiv u$ (here $\tilde{f}(\xi)=(k(\tilde{u})\,|\tilde{u}'|^{N-1}\,\tilde{u}')(\xi)$). Note that in [2] the term $-(\tilde{f}(0+)-f(0+))$ on the right-hand side of the formula preceding (2.7) was inadvertently omitted. We summarize these results in

THEOREM. *Assume $k(s)\geqslant 0$ is such that $k^{1/N}$ is locally Lebesgue integrable, $N>0$, $n=1,2,\ldots$, $\xi_0\geqslant 0$, $A>0$.*

Then there is at most one $u(\xi)$ such that $u(\xi)$ is absolutely continuous, $f(\xi)=(k(u)\,|u'|^{N-1}\,u')(\xi)$ is (essentially) absolutely continuous, $u(\xi_0)=A$, $\lim_{\xi\to\infty} u=0$, $\lim_{\xi\to\infty}\xi^{n-1}f(\xi)=\lim_{\xi\to\infty}f_{\mathrm{av}}=0$, and

$$(\xi^{n-1}k(u)\,|u'|^{N-1}\,u')'\,(\xi)=-(1/(N+1))\,\xi^n u'(\xi) \qquad \text{a.e. in} \quad \xi>\xi_0.$$

This function $u(\xi)$ is monotone decreasing (and strictly so where $u(\xi)>0$), and therefore $\int^{u(\xi)} k^{1/N}(s)\,ds$ is an absolutely continuous function such that $(\int^{u(\xi)} k^{1/N}(s)\,ds)'=k^{1/N}(u(\xi))\,u'(\xi)$ is essentially continuous and $f(\xi)=\operatorname{sgn}(k^{1/N}(u(\xi))\,u'(\xi))\,|k^{1/N}(u(\xi))\,u'(\xi)|^N=-\,|k^{1/N}(u(\xi))\,u'(\xi)|^N$ a.e. Furthermore, if (2.9) holds as $x\to 0+$, then the condition $\lim_{\xi\to\infty}\xi^{n-1}f(\xi)=0$ is a consequence of the remaining hypotheses.

2.4. *Compact Support*

We again reproduce the argument of [2]. Let $\xi^{n-1}f(\xi)=f_{\mathrm{av}}\to f_\infty$ as $\xi\to\infty$, $f_\infty\leqslant 0$. Integrating (2.5) we have

$$f_\infty-\xi^{n-1}f(\xi)=-\frac{1}{N+1}\int_\xi^\infty s^n u'(s)\,ds,$$

whence

$$\xi^{n-1}k(u)(-u')^N\geqslant\frac{\xi^n}{N+1}\int_\xi^\infty(-u'(s))\,ds-f_\infty=\frac{\xi^n u(\xi)}{N+1}-f_\infty.$$

Therefore

$$k(u)(-u')^N \geqslant \frac{\xi u(\xi)}{N+1} - \frac{f_\infty}{\xi^{n-1}}$$

and, at every ξ with $u(\xi) > 0$,

$$\left(\frac{k(u)}{u}\right)^{1/N} (-u'(\xi)) \geqslant \left(\frac{\xi}{N+1} - \frac{f_\infty}{u(\xi)\,\xi^{n-1}}\right)^{1/N}$$

and

$$\int_{u(\xi)}^{A} \left(\frac{k(s)}{s}\right)^{1/N} ds = \int_{\xi_0}^{\xi} \left(\frac{k(u)}{u}\right)^{1/N} (-u'(s))\,ds$$
$$\geqslant \int_{\xi_0}^{\xi} \left(\frac{\xi}{N+1} - \frac{f_\infty}{u(\xi)\,\xi^{n-1}}\right)^{1/N} d\xi \geqslant \int_{\xi_0}^{\xi} \left(\frac{s}{N+1}\right)^{1/N} ds.$$

Hence $\int_0^A (k(s)/s)^{1/N}\,ds < \infty$ implies that the values ξ at which $u(\xi) > 0$ are bounded, i.e., $\operatorname{supp} u = [\xi_0, a]$, $0 \leqslant \xi_0 < a < \infty$. This obviously implies $f_\infty = 0$.

This condition is also necessary, for if $u(\xi) = 0$ and $f_{\mathrm{av}} = 0$ for $\xi \geqslant a$, one easily obtains

$$\int_{u(\xi)}^{u(\xi_0)} \left(\frac{k(s)}{s}\right)^{1/N} ds \leqslant \left(\frac{a^n}{N+1}\right)^{1/N} \int_{\xi_0}^{\xi} s^{(1-n)/N}\,ds,$$
$$\xi < a, \quad u(\xi) \to 0 \quad \text{as} \quad \xi \to a-.$$

Note. We must assume here $\xi_0 > 0$. This is of no consequence to the proof, as it is enough to show the integrability of $(k(s)/s)^{1/N}$ near $s = 0$, due to the assumed integrability of $k^{1/N}(s)$.

3. Existence

We shall discuss the case $\xi_0 > 0$, $\xi^{n-1} f(\xi) = f_{\mathrm{av}}(\xi) \to 0$ as $\xi \to \infty$, $k(s) > 0$ a.e.

3.1. *Construction of the Inverse Function System* (*cf.* [2, *Section* 2.3])

We recall (cf. definition of solution to (2.1)) that $(k(u)\,|u'|^{N-1}\,u')(\xi)$ is well defined and finite for a.e. ξ, and coincides with the absolutely continuous function $f(\xi) < 0$ at almost every ξ such that $u(\xi) > 0$ (cf. Section 2.1). It follows that $-\infty < u'(\xi) < 0$ a.e. wherever $u(\xi) > 0$.

Therefore the function inverse to $u(\xi)$, $\xi(x)$ such that $\xi(u(\eta)) = \eta$ provided $u(\eta) > 0$, is locally absolutely continuous.

Now $u(\xi(x)) = x$ if $x \in (0, A]$ and so u is an inverse function to $\xi(x)$ which is locally absolutely continuous by hypotheses on weak derivability. With the argument used before it follows that $\xi'(x) < 0$ a.e. in $(0, A)$. Changing to the new unknown in Eq. (2.5) gives the following problem:

> To find $\xi(x) \geqslant 0$, monotone decreasing in $(0, A]$, locally absolutely continuous, and $F(x) \leqslant 0$, locally absolutely continuous in $(0, A]$ such that

$$\xi'(x) = -\left(\frac{k(x)}{-F(x)}\right)^{1/N} \quad \text{a.e.}, \qquad (\xi^{n-1}(x)\,F(x))' = -\frac{\xi^n(x)}{N+1} \quad \text{a.e.},$$

$$\xi(A) = \xi_0 > 0, \qquad (\xi^{n-1}F)(0^+) = 0. \tag{3.1}$$

Observation. This setting hints at the impossibility of existence of an absolutely continuous solution $u(\xi)$ in the case where $k(s)$ vanishes on a set of positive measure (cf. the observation following Eq. (2.5)). We shall assume $k(s) > 0$ a.e., but the existence proof will give a solution to (3.1) for any $k(s) \geqslant 0$ such that $k^{1/N}(s) \in L^1(0, A)$—and a corresponding generalized solution to the problem (3.2) below whose discussion is beyond our present purposes (cf. [3, 14]).

Conversely, assuming $k(s) > 0$ a.e. from a solution $\xi(x)$, $F(x)$ to (3.1), a solution $u(\xi)$ to

$$(\xi^{n-1}k(u)\,|u'(\xi)|^{N-1}\,u'(\xi))' = -\frac{\xi^n}{N+1}\,u'(\xi) \qquad \text{a.e. in} \quad (\xi_0, \infty),$$

$$u(\xi_0) = A > 0, \qquad \lim_{\xi \to \infty} u(\xi) = 0, \tag{3.2}$$

is found which has the regularity properties required: $u(\xi)$ is the absolutely continuous function inverse to $\xi(x)$ in $\xi_0 \leqslant \xi < \sup\{\xi(x) : x \in (0, A)\}$ (or $+\infty$ if $\xi(x)$ is not bounded), $u(\xi) = 0$ for $\xi > \sup\{\xi(x) : x \in (0, \mathrm{A})\}$; $F(x) = -k(x)(-\xi'(x))^{-N}$ a.e. is such that $\xi^{n-1}(x)\,F(x) = -\xi^{n-1}(x)\,k(x)(-\xi'(x))^{-N}$ is also absolutely continuous essentially and monotone strictly decreasing in $(0, A)$. The latter is Lipschitz continuous away from $x = 0$ and therefore $\xi^{n-1}(u(\xi))\,F(u(\xi)) = -\xi^{n-1}k(u(\xi))(-u'(\xi)^N$ is absolutely continuous as a function of $\xi > 0$. Equation (3.2) is now easily verified.

3.2

Upon integration of (3.1) it follows that the problem stated in Section 3.1 is equivalent to the integral equation

$$\xi(x) = \xi_0 + \int_x^A \left(\frac{(N+1)\,k(s)\,\xi^{n-1}(s)}{\int_0^s \xi^n(t)\,dt}\right)^{1/N} ds, \tag{3.3}$$

which is meaningful provided $\xi^n(t)$ is integrable near $t=0$. Putting $(T\xi)(x)$ for the right-hand side above, the integral equation takes the form $\xi = T\xi$.

The operator T acts on the set of functions $\xi(x)$, monotone decreasing in $(0, A)$, $\xi(A) = \xi_0 > 0$. For these functions the inequality (easily proved)

$$\xi^{-n}(x) \int_0^x \xi^n(s)\, ds \geqslant x, \qquad x \in (0, A), \tag{3.4}$$

implies

$$(T\xi)(x) \leqslant \xi_0 + \xi_0^{-1/N} \int_x^A \left(\frac{(N+1)\, k(s)}{s}\right)^{1/N} ds \equiv K(x), \tag{3.5}$$

say.

We shall make the provisional assumption that $K^n(x)$ be integrable in $(0, A)$, which essentially means

$$\int_0^A \left(\int_x^A \left(\frac{k(s)}{s}\right)^{1/N} ds\right)^n dx < \infty. \tag{3.6}$$

Now we have a lower bound $>\xi_0$ for $T\xi$

$$(T\xi)(x) \geqslant \xi_0 + \xi_0^{(n-1)/N} \int_x^A \left(\frac{(N+1)\, k(s)}{\int_0^s K^n(t)\, dt}\right)^{1/N} ds \equiv H(x), \tag{3.7}$$

and so we define

$$E = \{\xi(x) : \xi \text{ is monotone decreasing, } H(x) \leqslant \xi(x) \leqslant K(x) \text{ in } (0, A]\}.$$

It is clear from (3.3) and (3.5) that if, say, $0 < x < z \leqslant A$, and $\xi(x) \in E$, then

$$|(T\xi)(x) - (T\xi)(z)| \leqslant \xi_0^{-1/N} \int_x^z \left(\frac{(N+1)\, k(s)}{s}\right)^{1/N} ds,$$

whence it follows that $T(E)$ is an equi-uniformly continuous family of functions in $[\delta, A]$ for every $\delta > 0$. We distinguish two cases.

Case (a): $\int_0^A (k(s)/s)^{1/N}\, ds < \infty$. In this case, (3.6) and (3.7) are immediate and T maps the convex set E into itself in a compact manner (obviously one can put $\delta = 0$ above). We shall show its continuity in norm $\|\cdot\|_\infty$. Assume $\xi(x), \eta(x) \in E$; for every $x \in (0, A)$,

$$|(T\xi)(x) - (T\eta)(x)| \leqslant \int_x^A ((N+1)\, k(s))^{1/N} \left| \left(\xi(s) \cdot \xi^{-n}(s) \int_0^s \xi^n\, dt\right)^{-1/N} - \left(\eta(s) \cdot \eta^{-n}(s) \int_0^s \eta^n\, dt\right)^{-1/N} \right| ds.$$

Here the denominators are $\geqslant \xi_0 s \geqslant \xi_0 x$. Thus

$$\|(T\xi)-(T\eta)\|_\infty$$
$$\leqslant 2\xi_0^{-1/N}\int_0^\delta \left(\frac{(N+1)\,k(s)}{s}\right)^{1/N} ds + \int_\delta^A ((N+1)\,k(s))^{1/N}$$
$$\times \left| \left(\xi(s)\cdot\xi^{-n}(s)\int_0^s \xi^n\,dt\right)^{-1/N} - \left(\eta(s)\cdot\eta^{-n}(s)\int_0^s \eta^n\,dt\right)^{-1/N} \right| ds.$$

Select first $\delta > 0$ so that the first term is $<\varepsilon/2$; the parentheses in the second term being continuous functions in $[\delta, A]$, the right-hand side can be made $<\varepsilon$ if $\|\xi-\eta\|_\infty$ is sufficiently small.

The existence of a fixed point $\xi = T\xi \in E$ is ensured by Schauder's theorem.

Case (b). $\int_0^A (k(s)/s)^{1/N}\,ds = \infty$. For every integer $m \geqslant 1/A$ define

$$\begin{aligned} (T_m\xi)(x) &= (T\xi)(x), && \text{if} \quad 1/m \leqslant x \leqslant A, \\ &= (T\xi)(1/m), && \text{if} \quad 0 \leqslant x \leqslant 1/m. \end{aligned}$$

Here T_m is the operator corresponding to (3.3) with $k(s)$ replaced by $\tilde{k}(s) = k(s)$ if $1/m \leqslant s \leqslant A$, $\tilde{k}(s) = 0$ otherwise (cf. Section 3.1, Observation).

The reasoning in Case (a) applies to T_m, furnishing the existence of a fixed point $\xi_m(x)$ to T_m for each m. Fix $m_1 > 1/A$. For $m \geqslant m_1$, $\{\xi_m(x)\}$ is a uniformly bounded equi-uniformly continuous set of functions on $[1/m_1, A]$ (cf. Case (a)) from which we select a uniformly convergent subsequence on $[1/m_1, A]$. To the functions in this subsequence a similar argument applies on $[1/m_2, A]$ for certain $m_2 > m_1$, thus yielding a convergent subsequence on $[1/m_2, A]$. By continuing this way and employing a diagonal selection we finally obtain a subsequence $\xi_\nu(x)$, converging pointwise in $(0, A]$ to a function $\xi(x)$, the convergence being uniform on compacts of $(0, A]$.

In order to pass to the limit in (3.3) we need an integrable majorant to the sequence $\{\xi_\nu^n\}$. We already have $K(x)$, but it will not be sufficient for it leaves out some $k = \text{const}$ cases. Taking into account the fact that $\xi_\nu(x)$ satisfies (3.3) for $1/\nu \leqslant x \leqslant A$, we differentiate to give

$$\begin{aligned} \xi_\nu'(x) &= -\left(\frac{(N+1)\,k(x)\cdot\xi_\nu^{n-1}(x)}{\int_0^x \xi_\nu^n(t)\,dt}\right)^{1/N} \\ &\geqslant -\left(\frac{(N+1)\,k(x)}{x}\right)^{1/N}\cdot\frac{1}{\xi_\nu^{1/N}(x)}, \qquad 1/\nu \leqslant x \leqslant A. \end{aligned}$$

Integrating this differential inequality with $\xi_\nu(A) = \xi_0$ gives

$$\xi_\nu(x) \leqslant \left\{ \xi_0^{(N+1)/N} + \left(\frac{N+1}{N}\right) \int_x^A \left(\frac{(N+1)\,k(s)}{s}\right)^{1/N} ds \right\}^{N/(N+1)} \tag{3.8}$$

for $1/\nu \leqslant x \leqslant A$. But the fixed point $\xi_\nu(x)$ to the operator T_ν equals the constant $\xi_\nu(1/\nu)$ for $x < 1/\nu$—being the limit of constants in this interval. Therefore (3.8) is valid over the whole region $(0, A]$. As the right-hand side of (3.8) is independent of ν, (3.8) gives an integrable bound for $\{\xi_\nu^n(x)\}$ provided

$$\int_0^A \left\{ \int_x^A \left(\frac{k(s)}{s}\right)^{1/N} ds \right\}^{nN/N+1} dx < \infty. \tag{3.9}$$

It remains to show that $\xi(x) = (T\xi)(x)$ on $(0, A]$. Let $0 < x \leqslant A$ and $1/\nu_0 < x$. Clearly,

$$\xi_\nu(x) = \xi_0 + \int_x^A \left(\frac{(N+1)\,k(s)\cdot \xi_\nu^{n-1}(s)}{\int_0^s \xi_\nu^n(t)\,dt}\right)^{1/N} dt.$$

The uniform convergence of ξ_ν to ξ on $[1/\nu_0, A]$, the pointwise convergence on $(0, A]$, and (3.9) yield the result.

THEOREM. *Let $k(s) \geqslant 0$ a.e. in $(0, A)$, $k^{1/N} \in L^1(0, A)$. Then*

$$\int_0^A \left\{ \int_x^A \left(\frac{k(s)}{s}\right)^{1/N} ds \right\}^{nN/(N+1)} dx < \infty \tag{3.9}$$

is a sufficient condition for the existence of a solution to the inverse function system (3.1).

If, moreover, $k(s) > 0$ a.e. in $(0, A)$ and

$$\int_0^x k^{1/N}(s)\,ds = o(x^{(n-1-N)/nN}) \qquad as \quad x \to 0^+, \tag{2.9}$$

then there is a solution $u(\xi)$ to problem (3.2) *which is absolutely continuous in $\xi > \xi_0$ such that $\xi^{n-1}k(u)\,|u'(\xi)|^{N-1}\,u'(\xi)$ coincides a.e. with an absolutely continuous function $f_{\mathrm{av}}(\xi)$, and $f_{\mathrm{av}}(\xi) \to 0$ as $\xi \to \infty$. This solution is unique (cf. Section* 2.3).

Remark. Both conditions (2.9) and (3.9) are met if $k(s)$ is bounded near zero and $N > (n-1)/(n+1)$ (cf. Appendix).

Sufficient conditions for (3.9) are given by the following bounds:

(a) If $nN/(N+1) \leqslant 1$ $(N \leqslant 1/(n-1))$,

$$\int_0^A \left(\int_x^A \left(\frac{k(s)}{s}\right)^{1/N} ds\right)^{nN/(N+1)} dx \leqslant A^{((N+1)-nN)/(N+1)} \left\{\int_0^A \left(\frac{k(s)}{s}\right)^{1/N} s\, ds\right\}^{nN/(N+1)}. \tag{3.10}$$

(b) If $nN/(N+1) \geqslant 1$ $(N \geqslant 1/(n-1)$,

$$\int_0^A \left(\int_x^A \left(\frac{k(s)}{s}\right)^{1/N} ds\right)^{nN/(N+1)} dx \leqslant \left\{\int_0^A \left(\frac{k(s)}{s}\right)^{1/N} s^{(N+1)/nN}\, ds\right\}^{nN/(N+1)}.$$

(Condition (a) can be obtained by use of Jensen's inequality; (b) is a straightforward application of Minkowski's integral inequality to the left-hand side).

It is perhaps worth noting another integral equation formulation that can be deduced formally from (3.2). This can be written

$$(\xi(x))^{(N+1)/N} = \xi_0^{(N+1)/N} + \frac{N+1}{N} \int_x^A ((N+1)\,k(s))^{1/N} \cdot (J(s))^{-1/N}\, ds,$$

where

$$J(s) = s + n \int_0^s ((N+1)\,k(t))^{1/N} \times (\xi(t))^{-2+(N-1)/N} \left\{\frac{1}{\xi^n(t)} \int_0^t \xi^n(r)\, dr\right\}^{(N-1)/N} dt, \tag{3.11}$$

and reduces to [2, Eq. (2.37)] when $\xi_0 = 0$ and $n = 1$. However, we have been unable to prove that solutions of (3.11) imply solutions of (3.2) in general, except in the case of $N \geqslant 1$.

4. The Problem $\xi_0 \to 0+$: Completion of the Existence Proof in [2]

Put $\xi_0 = \alpha > 0$ for convenience in problem (3.1). We shall discuss the behavior of the solutions $\xi(x) = \xi_\alpha(x)$ to this problem as $\alpha \to 0^+$. Assuming throughout that $N > n-1$, and introducing the new dependent variables $\zeta_\alpha(x) = (\xi_\alpha(x))^{(1-n+N)/N}$, $F_1(x) = \xi^{n-1}(x) F(x)$, problem (3.1) becomes

$$\zeta'_\alpha(x) = -\left(\frac{1-n+N}{N}\right)\left(\frac{k(x)}{-F_1(x)}\right)^{1/N},$$

$$F'_1(x) = -\frac{\zeta_\alpha^{nN/(1-n+N)}}{N+1}, \qquad \text{a.e. } x,$$

$$\zeta_\alpha(A) = \alpha^{(1-n+N)/N}, \qquad F_1(0^+) = 0. \tag{4.1}$$

Here $F_1(x)$ is actually the flux average f_{av} as a function of concentration.

This system is equivalent to the integral equation

$$\zeta_\alpha(x) = \alpha^{(1-n+N)/N} + \frac{(1-n+N)}{N}\int_x^A \left(\frac{(N+1)\,k(s)}{\int_0^s \zeta_\alpha^{nN/(N+1-n)}\,dt}\right)^{1/N} ds. \tag{4.2}$$

LEMMA. *Let $0 \leqslant \alpha \leqslant \beta$. Then the corresponding solutions ζ_α, ζ_β of* (4.1) *satisfy $\zeta_\alpha(x) \leqslant \zeta_\beta(x)$, $0 < x \leqslant A$.*

Proof. Assume this is not the case. We can only expect two possibilities.

Case (1). $\zeta_\alpha(x) > \zeta_\beta(x)$ in (a, b), $\zeta_\alpha(a) = \zeta_\beta(a)$, $\zeta_\alpha(b) = \zeta_\beta(b)$, $0 < a < b \leqslant A$. Then

$$\zeta_\alpha(x) - \zeta_\alpha(a) = -\left(\frac{1-n+N}{N}\right)\int_a^x \left(\frac{k(s)}{-F_{1\alpha}(s)}\right)^{1/N} ds,$$

and we have a similar relation for ζ_β. Subtracting gives

$$\begin{aligned} 0 &< \zeta_\alpha(x) - \zeta_\beta(x) \\ &= \left(\frac{1-n+N}{N}\right)\int_a^x k^{1/N}(s)\left\{\frac{1}{(-F_{1\beta}(s))^{1/N}} - \frac{1}{(-F_{1\alpha}(s))^{1/N}}\right\} ds \end{aligned}$$

for $a < x < b$, with equality at $x = b$. However,

$$-F'_{1\beta}(x) = \frac{(\zeta_\beta(x))^{nN/(1-n+N)}}{N+1} < \frac{(\zeta_\alpha(x))^{nN/(1-n+N)}}{N+1} = F'_{1\alpha}(x)$$

and therefore $F_{1\alpha}$ and $F_{1\beta}$ (being absolutely continuous) cannot touch each other in more than one point. As $-F_{1\beta} \leqslant -F_{1\alpha}$ near a, the integral will be >0 at b, a contradiction.

Case (2). $\zeta_\alpha(x) > \zeta_\beta(x)$ in $(0, b)$, $b \leqslant A$, $\zeta_\alpha(b) = \zeta_\beta(b)$. Now

$$\zeta_\alpha(b) - \zeta_\alpha(x) = -\frac{1-n+N}{N}\int_x^b \left(\frac{k(s)}{-F_\alpha(s)}\right)^{1/N} ds,$$

and we have a similar relation for ζ_β. Hence

$$0 > \zeta_\beta(x) - \zeta_\alpha(x)$$
$$= \frac{1-n+N}{N} \int_x^b k^{1/N}(s) \left\{ \frac{1}{(-F_{1\beta}(s))^{1/N}} - \frac{1}{(-F_{1\alpha}(s))^{1/N}} \right\} ds$$

and again $-F'_{1\beta} < -F'_{1\alpha}$. But $F_{1\alpha}(0^+) = F_{1\beta}(0^+) = 0$, whence the contradiction $-F_{1\beta} < -F_{1\alpha}$ near b. The proof is complete.

Now fix $\beta > 0$, and let $0 < \alpha < \beta$. It is clear that

$$\zeta_\alpha(x) \geqslant \alpha^{(1-n+N)/N} + \left(\frac{1-n+N}{N}\right) \int_x^A \left(\frac{(N+1)\,k(s)}{\int_0^s \zeta_\beta^{nN/(N+1-n)}\,dt}\right)^{1/N} ds,$$

whence

$$\zeta_\alpha(x) > \frac{\int_x^A \{(N+1)\,k(s)\}^{1/N}\,ds}{(\int_0^A \zeta_\beta^{nN/(N+1-n)}\,dt)^{1/N}} > 0, \qquad 0 < \alpha < \beta.$$

(Recall we assume $k(s) > 0$ a.e.). This lower bound is independent of α, $0 < \alpha < \beta$. Therefore as $\alpha \to 0^+$, $\zeta_\beta(x) \geqslant \zeta_\alpha(x) \to \zeta(x) >$ bound above for every $x \in (0, A)$ (and correspondingly, $\xi_\alpha(x) \to \xi(x) > 0$ on $(0, A)$).

By passing to the limit as $\xi_0 \equiv \alpha \to 0^+$ in Section 3.3 we have, by the bounded convergence theorem,

$$\xi(x) = \int_x^A \left(\frac{(N+1)\,k(s)\,\xi^{n-1}(s)}{\int_0^s \xi^n(t)\,dt}\right)^{1/N} ds.$$

This integral equation is equivalent to the following problem

$$\xi'(x) = -\left(\frac{k(x)}{-F(x)}\right)^{1/N} \qquad \text{a.e.,} \qquad \xi(A) = 0.$$
$$(\xi^{n-1}(x)\,F(x))' = -\frac{\xi^n(x)}{(N+1)} \qquad \text{a.e.,} \qquad (\xi^{n-1}F)(0^+) = 0, \quad (4.3)$$

$\xi(x) \geqslant 0$, $F(x) \leqslant 0$, both locally absolutely continuous in $(0, A)$, and this in turn is the inverse function system corresponding to our problem (cf. (3.2)) with $\xi_0 = 0$.

Observe that $(\xi^{n-1}F)(A) = (-1/N+1) \int_0^A \xi^n(s)\,ds < 0$. Now if $n > 1$, $\xi^{n-1}(x) \to 0$ as $x \to A^-$, and therefore $F(x)$, locally absolutely continuous in $(0, A)$, tends to $-\infty$ as $x \to A$.

Thus the proof of uniqueness in 2.3 is not complete when $\xi_0 = 0$.

On the other hand, the lemma above is also a proof of uniqueness of the solution $\xi(x)$, $F(x) = -(1/(N+1)\,\xi^{n-1}(x)) \int_0^x \xi^n(s)\,ds$ to the inverse function system (4.3).

The physically relevant case for boundary condition $u(0) = A > 0$ is that of planar symmetry $n = 1$. This case was left infinished in [2], but the proof above was announced. We state now the result.

THEOREM. *Let $N > 0$, $k(s) > 0$ a.e. in $(0, A)$, $k^{1/N}(s) \in L^1(0, A)$. Assume (3.9) for $n = 1$. Then there is a (unique) solution to the system (cf. [2, Eq. (215)])*

$$\xi'(x) = -\left(\frac{k(x)}{-F(x)}\right)^{1/N} \qquad \text{a.e.,}$$

$$F'(x) = -\frac{1}{N+1}\,\xi(x) \qquad \text{a.e.}$$

$$\xi(A) = 0, \qquad F(0) = 0, \tag{4.4}$$

and a fortiori a (unique) solution to the problem (cf. [2, Section 2.1, Theorem 2])

> $u(\xi)$, $f(\xi) = (k(u)\,|u'|^{N-1}\,u')(\xi)$ *essentially absolutely continuous locally in* $(0, \infty)$, $u(0) = A$, $\lim_{\xi \to \infty} u(\xi) = 0$,
>
> $(k(u)\,|u'|^{N-1}\,u')'\,(\xi) = -(\xi/(N+1))\,u'(\xi)$ *a.e.* in $(0, \infty)$.

Furthermore, $\lim_{\xi \to \infty} f(\xi) = 0$ ((2.9) *is obviously satisfied when* $n = 1$).

4.1

We shall now consider the problem of determining $\xi_0 = \alpha$ from a moving boundary condition of the form

$$f(\xi_0) = -\frac{1}{N+1}\,L\xi_0, \qquad L > 0 \tag{4.5}$$

or

$$f_{\text{av}}(\xi_0) = \xi_0^{n-1} f(\xi_0) = -(1/(N+1))\,L\xi_0^n.$$

In terms of the inverse function system this is

$$\alpha^{n-1} F_\alpha(A) = -(L/(N+1))\,\alpha^n.$$

Putting ξ_α, F_α for the solutions of inverse system (3.1) with $\xi_0 = \alpha$, we find

$$(\xi_\alpha^{n-1} F_\alpha)(A) = -\frac{1}{N+1} \int_0^A \left\{ \alpha^{(N+1-n)/N} + \left(\frac{N+1-n}{N} \right) \int_x^A \left(\frac{(N+1)\, k(s)}{\int_0^s \xi_\alpha^n \, dt} \right)^{1/N} ds \right\}^{nN/(N+1-n)} dx$$
$$= -\frac{1}{N+1} \int_0^A \xi_\alpha^n \, dx.$$

Recall that $\xi_\alpha \searrow \xi(x)$ as $\alpha \to 0^+$, where $\xi(x) > 0$ in $(0, A)$. It follows that

$$(\xi_\alpha^{n-1} F_\alpha)(A) \searrow -\frac{1}{N+1} \int_0^A \xi^n(x)\, dx < 0, \qquad \alpha \to 0^+,$$

$$(\xi_\alpha^{n-1} F_\alpha)(A) = -\frac{\alpha^n}{N+1} \int_0^A \left\{ 1 + \frac{N+1-n}{N \alpha^{(N+1-n)/n}} \times \int_x^A \left(\frac{(N+1)\, k(s)}{\int_0^s \xi_\alpha^n \, dt} \right)^{1/N} ds \right\}^{nN/(N+1-n)} dx$$
$$\sim -A \frac{\alpha^n}{N+1} \qquad \text{as} \quad \alpha \to +\infty.$$

Therefore, if $L > A$ and $N > n - 1$, there is at least one value $\alpha = \xi_0$ satisfying (4.5).

5. Comparison Theorems

Section 5.1 includes the main hypotheses and the statement of the main result. In Section 5.2 we sketch the main idea of the proof. The complete demonstration of the theorem in 5.1 is contained in Section 5.3.

Generalizations of this result are included in Section 5.4. We point out that the theorem in 5.4(b) extends the comparison result to operators of the form $\partial u/\partial t = (\partial/\partial x)\, A(x, t, u, D(u)(\partial u/\partial x))$, $D(u) \geqslant 0$, which are our main concern. We also mention in this connection Theorems 5.4(d) and (e) for unbounded domains G, and Theorem 5.4(f).

Section 5.5 includes a discussion of the hypotheses made in Section 5.1, together with a comment of other comparison results available in the literature.

5.1

We shall present a comparison result for solutions—in a generalized sense—of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} A \left(x, t, u, \frac{\partial u}{\partial x}\right) \tag{5.1}$$

in a domain G, under the following assumptions: G is a domain in (x, t) space, $x \in R^1$, and

$$\begin{aligned} &\text{projection of } G \text{ on } t\text{-axis} = (0, T), \qquad T > 0, \\ &\text{projection of } G \text{ on } x\text{-axis} = \Omega; \end{aligned} \tag{5.2}$$

Ω is an interval; to be more specific we shall assume Ω to be contained in $\{x > 0\}$. (5.3)

The function $A(x, t, u, p)$ is continuous in its four arguments, while $p \to A(\cdot, \cdot, \cdot, p)$ is strictly increasing—as a function of p, other arguments held fixed. (5.4)

We shall establish comparison results for solutions $u(x, t)$ within the class $\mathscr{H}$ of functions such that

$u(x, t)$, $(\partial u/\partial x)(x, t)$, $(\partial u/\partial t)(x, t)$ (weak derivatives), belong to $L^1(G \cap \{(x, t) : \delta < t < T\})$, for every $\delta > 0$. (5.5i)

$X_G(x, t) \cdot u(x, t)$ has a limit $u_0(x)$ in $L^1(\Omega)$ as $t \to 0^+$ (i.e., $\|(X_G u)(\cdot, t) - u_0(\cdot)\|_{L^1(\Omega)} \to 0$). (5.5ii)

$A(x, t, u(x, t), (\partial u/\partial x)(x, t)) \in L^1_{\text{loc}}(G)$. (5.5iii)

Here $X_G(x, t)$ is the characteristic function of G.

Equation (5.1) is meant to be satisfied in a weak sense as follows: for every $\phi \in C_0^\infty(G)$.

$$\int_G u \frac{\partial \phi}{\partial t} \, dx \, dt = \int_G A \left(x, t, u \frac{\partial u}{\partial x}\right) \cdot \frac{\partial \phi}{\partial x} \, dx \, dt. \tag{5.6}$$

Assumption (5.5i) immediately gives

$$\frac{\partial}{\partial x} A \left(x, t, u, \frac{\partial u}{\partial x}\right) \in L^1(G \cap \{(x, t) : \delta < t < T\}), \tag{5.7}$$

and (5.1) is satisfied a.e. in G.

Remark. Under the conditions above, if $u \in \mathscr{H}$, then $A(x, t, u(x, t), (\partial u/\partial x)(x, t))$ belongs to the Lebesgue class of a function absoluteiy continuous in x for almost every $t \in (0, T)$. The function $u(x, t)$ is itself in the Lebesgue class of a function absolutely continuous in almost every line parallel to an axis.

Let $u, v \in \mathscr{H}$. We define the expression "$u \leqslant v$ on $\partial_p G$" as follows:

a.e. $\bar{t} \in (0, T)$, if $(\bar{x}, \bar{t}) \in \partial G$, $\liminf_{x \to \bar{x}} (v - u)(x, \bar{t}) \geqslant 0$; (5.8i)

a.e. $\bar{x} \in \Omega$, if $(\bar{x}, \bar{t}) \in \partial G$, $\liminf_{t \to \bar{t}+} (v - u)(\bar{x}, t) \geqslant 0$ (if $G = \Omega \times (0, T]$, it is enough to assume $u_0(x) \leqslant v_0(x)$ a.e. (cf. (5.5ii)), (5.8ii)

the limits in both (5.8i) and (5.8ii) need only hold along an open line segment contained in G of which $(\bar{x}, \bar{t})$ is an endpoint.

We shall refer to those points $(\bar{x}, \bar{t}) \in \partial G$ as belonging to the parabolic boundary $\partial_p G$ of G (this concept differs from the one in Walter [11]).

Our main result is

THEOREM. *Let u, $v \in \mathscr{H}$ be solutions of Eq.* (5.1) *in a bounded domain G. Then $u \leqslant v$ on $\partial_p G$ implies $u(x, t) \leqslant v(x, t)$ a.e. in G.*

The proof of this result is included in Sections 5.2 and 5.3. Some generalizations needed for the diffusion problem are contained in Section 5.4.

5.2

In order to clarify the main idea of the proof, assume first that both u and v are classical solutions of the equation

$$\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t}$$

(that is, Eq. (2.1) with $N = 1$, $n = 1$), continuous with continuous derivatives up to the boundary $\partial_p G$, $G = \Omega \times (0, T)$ and $u \leqslant v$ on $\partial_p G$ in the usual sense.

Assuming $D = \{(x, t) \in G: u(x, t) > v(x, t), t \leqslant \tau\}$ nonvoid, and integrating the equation there gives

$$\begin{aligned}\int dx \int X_D(x, t) \frac{\partial}{\partial t}(v - u)\, dt \\ = \int dt \int X_D(x, t) \frac{\partial}{\partial x}\{k(v)\, v_x - k(u)\, u_x\}\, dx.\end{aligned}$$

Take the left-hand side. For each fixed x, $X_D(x, t) = 1$ in a union (at most denumerable) of segments of the form

$$(t_0, t^0) \cup \bigcup_j (t_j, t^j) \cup (t_m, \tau),$$

where the point $(x, t_0) \in \partial_p G$ unless (t_0, t^0) is empty; (x, t_j), (x, t^j), (x, t_m) are interior points to G that belong to ∂D. It clearly follows that $u = v$ at those points, while $u(x, \tau) > v(x, \tau)$ for some $\tau \in (0, T)$ and x in an open set of the line if $D \neq \phi$ is to hold.

Therefore the left-hand side above is

$$\leqslant \int dx \left(\sum_{j=0} \int_{t_j}^{t^j} + \int_{t_m}^{\tau} \right) X_D(x, t) \frac{\partial}{\partial t}(v - u)\, dt$$

$$= \int (v - u)(x, \tau)\, X_D(x, \tau)\, dx < 0.$$

On the other hand, it is not difficult to see (with similar arguments) that the right-hand side of the equation is $\geqslant 0$, thus producing a contradiction that shows $D = \phi$. (This proof is essentially included by Douglas *et al.* [5], who study elliptic operators in divergence form of more general nature than our Eq. (5.1), while assuming smoothness of the solutions considered.)

5.3. *Proof of Theorem* 5.1.

Let $u, v \in \mathscr{H}$ and $u \leqslant v$ on $\partial_p G$. Put $D = \{(x, t) \in G : u(x, t) > v(x, t)\}$, $\tau \in (0, T)$, $0 < \delta < \tau$. We have

$$\iint_{D \cap \{\delta < t < \tau\}} \frac{\partial}{\partial t} (v - u)\, dx\, dt$$

$$= \iint_{D \cap \{\delta < t < \tau\}} \frac{\partial}{\partial x} \left\{ A\left(x, t, v, \frac{\partial v}{\partial x}\right) - A\left(x, t, u, \frac{\partial u}{\partial x}\right) \right\} dx\, dt.$$

Applying Fubini's theorem (cf. (5.5)) and (5.7) gives

$$\int_\Omega dx \int_\delta^\tau X_D(x, t) \frac{\partial}{\partial t} (v - u)\, dt$$

$$= \int_\delta^\tau dt \int_\Omega X_D(x, t) \frac{\partial}{\partial x} \left\{ A\left(x, t, v, \frac{\partial v}{\partial x}\right) - A\left(x, t, u, \frac{\partial u}{\partial x}\right) \right\} dx. \tag{5.9}$$

Consider the left-hand side (l.h.s.): for a.e. x

$$\int_\delta^\tau X_D(x, t) \frac{\partial}{\partial t} (v(x, t) - u(x, t))\, dt$$

$$= \left(\int_\delta^{t^0} + \sum_j \int_{t_j}^{t^j} + \int_{t_m}^{\tau} \right) \frac{\partial}{\partial t} (v(x, t) - u(x, t))\, dt. \tag{5.10}$$

This is due to the fact that a.e. x, $X_D(v-u)$ is absolutely continuous in $t \in (\delta, \tau)$ and $G \cap \{\delta < t < \tau\}$ is an open set of the plane; therefore the set $\{t: (x, t) \in D, \delta < t < \tau\}$ is an open set of the segment (δ, τ)—for a.e. x in the projection of D—and admits a representation

$$(\delta, t^0) \cup \bigcup_j (t_j, t^j) \cup (t_m, \tau).$$

Here (δ, t^0), (t_m, τ) may be void for certain x, the union is at most denumerable, and $u(x, t^0) = v(x, t^0)$, $u(x, t^j) = v(x, t^j)$, $u(x, t_j) = v(x, t_j)$, $u(x, t_m) = v(x, t_m)$ if all the points belong in G, while equality also follows from definition (5.8) if (x, t_m), (x, t_j) lie on $\partial_p G$. If $(x, t^j) \in \partial_p G$ is the right endpoint of a vertical line segment contained in D, we only know that $\lim_{t \to t^j-}(v-u) \leqslant 0$ (the limit exists due to integrability of $(\partial/\partial t)(v-u)(x, t)$, except for the values of x in a set of measure zero (cf. (5.5i))). At $t = \delta$ and $t = \tau$, the values of $v-u$ may be negative on a set of values of x.

Performing the integrations indicated in (5.10) we obtain, a.e. x,

$$\int_\delta^\tau X_D(x, t) \frac{\partial}{\partial t}(v-u)\, dt \leqslant (X_D \cdot (v-u))(x, \tau) - (X_D \cdot (v-u))(x, \delta),$$

whence in (5.9),

$$\text{l.h.s.} \leqslant \int_\Omega X_D(x, \tau)(v-u)(x, \tau)\, dx - \int_\Omega X_D(x, \delta)(v-u)(x, \delta)\, dx. \quad (5.11)$$

The first integral is nonpositive. The second integral tends to 0 due to (5.5ii) and the pointwise convergence to zero of $X_D(x, \delta)(v-u)(x, \delta)$ as $\delta \to 0^+$ (cf. (5.8ii)).

We study now the right-hand side (r.h.s.) of (5.9). Fix t, $\delta < t < \tau$.

For a.e. t in these conditions, $\{A(x, t, v(x, t), (\partial/\partial x)\, v(x, t)) - A(x, t, u(x, t), (\partial/\partial x)\, u(x, t))\}$ and $(v-u)(x, t)$ are absolutely continuous as functions of x, and therefore $D \cap \{(x, t): x \in \Omega\} = U_i(x_i, x^i)$, the union is at most denumerable, and $u = v$ at those endpoints that belong to G, with inequality as side limit if they fall on $\partial_p G$. Hence, a.e. $t \in (\delta, \tau)$,

$$\int_\Omega X_D(x, t) \frac{\partial}{\partial x} \{A(x, t, v, (\partial v/\partial x)) - A(x, t, u, (\partial u/\partial x))\}\, dx$$
$$= \sum_i [A(x, t, v, (\partial v/\partial x) - A(x, t, u, (\partial u/\partial x)]_{x_i}^{x^i}. \quad (5.12)$$

Assume now we have conditions under which the last sum is $\geqslant 0$ for a.e.

$t \in (\delta, \tau)$, $\delta > 0$. Clearly $0 \leqslant$ r.h.s. ($=$ l.h.s., of course), and therefore (cf. (5.11)), $0 \geqslant \int_\Omega X_D(x, \tau)(v-u)(x, \tau)\, dx \geqslant$ r.h.s. $+ \int_\Omega X_D(x, \delta)(v-u)(x, \delta)\, dx \geqslant -\varepsilon$, $\varepsilon > 0$ arbitrarily small as $\delta \to 0^+$. It follows that $\int_\Omega X_D(x, \tau)(v-u)(x, \tau)\, dx = 0$, which implies $X_D(x, \tau) = 0$ a.e. x. But $\tau \in (0, T)$ is arbitrary and D is a measurable subset of G, whence meas $D = 0$, thus showing $u(x, t) \leqslant v(x, t)$ a.e. in G.

Returning to the claim made above, we observe that for a.e. (fixed) t, if $u \in \mathscr{H}$, then $A(x, t, u(x, t), (\partial u/\partial x)(x, t)) = Z(x, t)$, locally absolutely continuous as a function of x. Now $A(x, t, u, p) = Z$, considered as an equation for p, admits a unique continuous solution $p = H(x, t, u, Z)$ (cf. (5.4), the continuity is a consequence of the uniqueness—which is obvious—and the closed graph theorem). That is, after replacement, $(\partial u/\partial x)(x, t) = H(x, t, u(x, t), Z(x, t))$ and therefore $(\partial u/\partial x)(x, t)$ is (locally at least) continuous in x (a.e. t).

Consider, for instance, the term

$$T(x_i) = A(x, t, v, \partial v/\partial x) - A(x, t, u, \partial u/\partial x)|_{x = x_i}:$$

$$\begin{aligned} 0 \leqslant D_+(u-v)(x_i) &= \liminf_{x \to x_i +} \frac{u(x, t) - v(x, t)}{x - x_i} \\ &= \liminf_{x \to x_i +} \left(\int_{x_i}^{x} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) dx \Big/ (x - x_i) \right). \end{aligned}$$

Thus $(\partial u/\partial x) - (\partial v/\partial x)$, continuous in x (a.e. t), cannot have a negative limit at $x = x_i$. It follows that $(\partial v/\partial x)(x_i, t) \leqslant (\partial u/\partial x)(x_i, t)$ and $T(x_i) \leqslant 0$, a.e. $t \in (\delta, \tau)$.

The case at x^i is treated in an analogous way. This completes the proof.

5.4

The result in Sections 5.1–5.3 can be improved in several ways:

(a) Differential inequalities may be considered;

(b) THEOREM. *Theorem* 5.1 *is valid for operators* $A(x, t, u, D(u)(\partial u/\partial x))$, $A(x, t, u, p) \in \mathscr{H}$ *as before*, $D(s) \geqslant 0$, $D(s)$ *locally integrable Lebesgue, provided it is assumed that the functions* u *of* $\mathscr{H}$ *are such that* $\int^{u(x,t)} D(s)\, ds$ *is absolutely continuous in* x *for a.e.* t *(this is the case if, e.g., D is measurable bounded or it is assumed that* $D(u)(\partial u/\partial x) \in L^1$; *we observe that the resulting operator may not be strictly increasing in the variable* $\partial u/\partial x$).

Proof. It is enough to show, say, that $(D(v)(\partial v/\partial x))(x_i, t) \leqslant (D(u)(\partial u/\partial x))(x_i, t)$. With the argument employed before, $(D(u)(\partial u/\partial x) - D(v)$

$(\partial v/\partial x))(x, t)$ is a continuous function for a.e. t, which coincides a.e. x with $(\partial/\partial x)\int_{v(x,t)}^{u(x,t)} D(s)\,ds$. Now again,

$$0 \leqslant D_+\left(\int_{v(x,t)}^{u(x,t)} D(s)\,ds\right)(x_i, t)$$

$$= \liminf_{x\to x_i+}(x-x_i)^{-1}\int_{v(x,t)}^{u(x,t)} D(s)\,ds$$

$$= \liminf_{x\to x_i+}(x-x_i)^{-1}\int_{x_i}^{x}\left\{D(u)\frac{\partial u}{\partial x} - D(v)\frac{\partial v}{\partial x}\right\} ds;$$

therefore the integrand cannot have a negative limit at x_i.

(c) If G is not bounded (i.e., if $\Omega = \operatorname{proj}_x G$ is not bounded), then it is clear that in the proof of the theorem we must consider the occurrence of intervals $(x_i, +\infty)$. The theorem is clearly true if

(i) The inclusion $\{(x, t): x \geqslant \text{certain } \alpha > 0\} \subset G$ takes place for t in a subset of $(0, T)$ of zero measure.

(ii) For both u and v in $\mathscr{K}$, $A(x, t, u(x, t), (\partial u/\partial x)(x, t)) \to 0$ as $x \to \infty$ a.e. $t \in (0, T)$.

Conditions (5.5) are, however, very strong. They allow for the following variant of 5.4(c, ii).

(d) THEOREM. *Under the assumptions in Section* 5.4(b), *the comparison theorem is valid in unbounded G provided $A = A(x, t, p)$, i.e., A is independent of u.*

Proof. It will be enough to show that

$$B(v, u) = A\left(x, t, D(v)\frac{\partial v}{\partial x}(x, t)\right) - A\left(x, t, D(u)\frac{\partial u}{\partial x}(x, t)\right)$$

tends to a nonnegative limit as $x \to +\infty$, for a.e. t such that $\{(x, t): x > \text{some } x(t)\} \subset D = \{(x, t) \in G : u > v\}$.

Assume the contrary, and let us select $P \subset (0, T)$, where $\lim_{x\to\infty} B(v, u)(x, t) < 0$, $\lim_{x\to\infty} u(x, t)$, $\lim_{x\to\infty} v(x, t)$ exist and $\{(x, t): x > \text{some } x(t)\} \subset D$ (the existence of these limits a.e. t in P is a consequence of (5.5), (5.7) (cf. also Remark), and Fubini's theorem: the absolute continuity of $u(x, t)$, say, in the x variable gives $u(x, t) - u(x(t), t) = \int_{x(t)}^{x} (\partial/\partial x)\, u(s, t)\,ds$ a.e. t, and $(\partial u/\partial x)$ is integrable in $(x(t), \infty)$ a.e. $t \in P$).

We have $A(x, t, D(v)(\partial u/\partial x)(x, t)) < A(x, t, D(u)(\partial u/\partial x)(x, t)$ for $x >$ some $x_1(t) \geqslant x(t)$, $t \in P$, whence we obtain $D(v)\, v_x < D(u)\, u_x$, i.e.,

$$\frac{\partial}{\partial x}\int_v^u D(s)\, ds > 0 \qquad \text{for} \quad x > x_1(t),$$

$$0 < \lim_{x\to\infty} \int_{x_1(t)}^{x} \frac{\partial}{\partial x}\int_v^u D(s)\, ds\, dx$$

$$= \lim_{x\to\infty} \int_{v(x,t)}^{u(x,t)} D\, ds - \int_{v(x(t),t)}^{u(x(t),t)} D\, ds;$$

this readily implies

$$\lim_{x\to\infty} u(x, t) > \lim_{x\to\infty} v(x, t),$$

that is, $(u - v) \geqslant n(t) > 0$ in certain $(x_2(t), +\infty)$, $t \in P$. But $|u - v|$ is integrable in $G \cap \{t > \delta\}$ for every $\delta > 0$ (cf. (5.5)). Applying Fubini's theorem once more we find meas $P = 0$.

(e) There is yet another variant of Theorems in 5.1 and 5.4(b) as applied to unbounded domains G.

THEOREM. *The comparison result remains valid if hypotheses* (5.5) *are replaced with the following ones on the functions u, v to be compared:*

$u(x, t)$, $\partial u(x, t)/\partial x$, $\partial u(x, t)/\partial t$, $v(x, t)$, $\partial v(x, t)/\partial x$, $\partial v(x, t)/\partial t$ (weak derivatives) belong to $L^1(G \cap \{(x, t)\colon\ 0 < x \leqslant m, \delta < t < T\})$, for every $m > 0$, $\delta > 0$; (5.5i′)

There exist sequences $K_i < M_i$, K_i, $M_i \to +\infty$, $i \to \infty$ for which

$$\int_0^T \frac{1}{M_i - K_i}\int_{K_i}^{M_i} \left| A\left(x, t, v, \frac{\partial v}{\partial x}\right) - A\left(x, t, u, \frac{\partial u}{\partial x}\right)\right| X_G(x, t)\, dx\, dt$$

tends to zero, and moreover

$$\|X_G(\cdot, t)\cdot(v - u)(\cdot, t)\|_{L^1(0,m)} \to 0$$

$$\text{as} \quad t \to 0^+, \qquad \text{for every} \quad m > 0. \qquad (5.5\text{ii}')$$

Alternatively, the outer integral above can have lower limit $\delta > 0$, for any δ, if $\|X_G(\cdot, t)(v - u)(\cdot, t)\|_{L^1(\Omega)} \to 0$, as in (5.5ii);

$A(x, t, u, \partial u/\partial x)$, $A(x, t, v, \partial v/\partial x)$ belong to $L^1_{\mathrm{loc}}(G)$. (5.5iii′)

Proof. Multiplying Eq. (5.1) by $\rho_i(x)$, a piecewise linear function that equals one in $(0, K_i)$ and equals zero in (M_i, ∞), and applying Fubini's theorem and the considerations in Section 5.3, easily gives, for $0 < \delta < \tau < T$,

$$\int_\Omega X_D(x,\tau)(u-v)(x,\tau)\,\rho_i'(x)\,dx$$

$$\leqslant \int_\Omega X_D(x,\delta)(u-v)(x,\delta)\,\rho_i(x)\,dx$$

$$+\int_\delta^\tau \frac{1}{M_i-K_i}\int_{K_i}^{M_i} \left| A\left(x,t,v,\frac{\partial v}{\partial x}\right) - A\left(x,t,u\frac{\partial u}{\partial x}\right)\right| X_D(x,t)\,dx\,dt,$$

whence $X_D(x,\tau)=0$ a.e. follows for every τ, by application of (5.5ii′).

(f) The function $u = \text{const}$ is not integrable, in general. However, it satisfies Eq. (5.1) if $A(x,t,u,0)=0$, and comparison with any $u \in \mathscr{H}$ is possible.

THEOREM. *Solutions $u \in \mathscr{H}$ of Eq.* (5.1) *with* $A(x,t,u,0)=0$ *satisfy maximum and minimum principles.*

COROLLARY. *The functions $u \in \mathscr{H}$ are bounded by their suprema and infima on $\partial_p G$ (cf.* (5.8)).

(g) For the case $G \subset R^{n+1}$, $x \in R^n$, our main theorem admits a straightforward extension under the following assumptions:

> $A(x,t,u,u_x)$ is a vector function whose components $A_i = A_i(x,t,u,\partial u/\partial x_i)$ (A_i depends only on $\partial u/\partial x_i$) satisfy hypotheses (5.4) (similarly for the case in Section 5.4(b), where $A_i = A_i(x,t,u,D_i(u)\,\partial u/\partial x_i)$).

However, we must assume now

$$\frac{\partial}{\partial x_i} A_i\left(x,t,u,D_i(u)\frac{\partial u}{\partial x_i}\right) \in L^1(G \cap \{(x,t):\delta < t < T\}), \qquad \text{for every} \quad \delta > 0. \tag{5.13}$$

We need only note that Fubini's theorem allows us to study each variable separately.

5.5

We shall end this section with a few comments.

Conditions (5.5) are similar to those valid for the Cauchy problem for the porous medium equation $\partial u/\partial t = \Delta(u^m)$; here $A(x, t, u, \nabla u) = mu^{m-1}\nabla u$ (cf. Aronson and Bénilan [1]). These authors employ the initial condition $\lim_{t\to 0+} \|u(t, \cdot) - u_0(\cdot)\|_{L^1(R^n)} = 0$, which compares with our (5.5ii).

For the case $k_i = \text{const}$, the operator

$$\nabla \cdot A(x, t, u, \nabla u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(k_i(u) \left| \frac{\partial u}{\partial x_i} \right|^{N_i - 1} \frac{\partial u}{\partial x_i} \right)\right.$$

—a sort of generalized Laplacean—has monotocity properties that permit the use of very mild integrability assumptions (cf. Lions [7], Brézis [4]). A comparison result for

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - \frac{\partial u}{\partial t}, \qquad p \geqslant 2$$

—the operator includes multivalued terms—is found in Nagai [8]. Here trace spaces are used and smooth domains considered.

We recall the need of condition (5.13) for the operators above in many space variables. They are otherwise amenable to our methods; for $A = (A_i)$,

$$A = \left| k_i^{1/N_i}(u) \frac{\partial u}{\partial x_i} \right|^{N_i - 1} \left(k_i^{1/N_i}(u) \frac{\partial u}{\partial x_i} \right) = |p_i|^{N_i - 1} p_i$$

is an increasing function of p_i, if $N_i > 0$, $k_i \geqslant 0$, and $k_i^{1/N_i} \in L^1_{\text{loc}}$ are imposed.

It should be noted that boundary conditions of mixed type can also be studied with similar arguments as long as the flux $A(x, t, u, \nabla u)$ is a datum only at points of $\partial_p G$ which cannot be attained as downward limits in t.

As an example of operators and comparison results that fall within our framework we mention the Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(u) \frac{\partial u}{\partial x} \right) - b(u) \frac{\partial u}{\partial x},$$

describing motion under gravity and capillarity of a fluid in an unsaturated porous medium (cf. Gilding [6]). In fact, even the conclusion of the Theorem in Section 5.4(d) applies to this operator if it is assumed that $b(u) \geqslant 0$.

Theorem 5.4(f) gives results like those in [10, Section 5, Theorem] for our divergence structure operator and solutions in $\mathscr{H}$. Note that we can handle discontinuous boundary conditions.

We shall state the theorem for the domain $G = \{(x, t) : x > 0,\ 0 < t < T\}$ assuming the definitions in Redheffer and Walter, [10, Sects. 4, 5].

THEOREM. *Let $u \in \mathscr{H}$. If S_0 is a saw on $\{(x, t): t = T, x > 0\}$ and if $\varepsilon > 0$, then there exists a saw S of the same order and type on $\partial_p G$ such that*

$$u(T, x_i) \leqslant \limsup_{\partial_p G} u(\tau(s_i), \xi(s_i)) + \varepsilon \qquad \text{for an } H\text{-point},$$

$$u(T, x_i) \geqslant \liminf_{\partial_p G} u(\tau(s_i), \xi(s_i)) - \varepsilon \qquad \text{for an } L\text{-point}.$$

Here $((\tau(s_i), \xi(s_i)) \in \partial_p G$ and the limits are as described in (5.8).

Remark. The sharp result of [10, Theorem, p. 64] relies on the fact that $\sup_G u = \sup\{u(x, t): (x, t) \in \partial_p G\} = u(\bar{x}, \bar{t})$ for certain $(\bar{x}, \bar{t}) \in \partial_p G$, under the assumption that $u(\bar{x}, \bar{t})$ is continuous in $\bar{G}$).

6. Compact Support Behaviour of Solutions of General Boundary Value Problems

6.1

We return now to generalized diffusion equation (2.1), i.e.,

$$\nabla \cdot (k(u) \, |\nabla u|^{N-1} \, \nabla u) = \partial u/\partial t.$$

We observe that, except for the case of a single space variable ($\nabla = \partial/\partial x$), this operator does not satisfy condition $A_i = A_i(x, t, u, \partial u/\partial x_i)$. Therefore we could only apply our comparison theorem to solutions of (2.1) that are radial functions $u = u(r, t)$, for which the equation becomes

$$\left(\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1}\right) \left(k(u) \left| \frac{\partial u}{\partial r} \right|^{N-1} \frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}. \tag{6.1}$$

(We are clearly thinking in a polar (cylindrical or spherical) coordinate system, whose Jacobian is to be taken into account in the integration.)

We shall discuss now the case $n = 1$, $r = x$. Putting $A(x, t, u, p) = |p|^{N-1} p$ and $p = k^{1/N}(u) \cdot (\partial u/\partial x)$, we see that all considerations made in Section 5 are applicable to

$$\frac{\partial}{\partial x} (k(u) \, |u_x|^{N-1} \, u_x) = \frac{\partial u}{\partial t} \tag{6.2}$$

provided $u(x, t)$ belongs to the corresponding class $\mathscr{H}$ and $k(s) \geqslant 0$, $k^{1/N}(s) \in L^1_{\text{loc}}$ (and set $D(u) = k^{1/N}(u)$ in Section 5.4(b)).

THEOREM. *The problem $u \in \mathscr{H}$ and $\int^{u(x,t)} k^{1/N}(s) \, ds$ is absolutely* *continuous in x for a.e.* $t \in (0, T)$, $(\partial/\partial x)(k(u)\,|u_x|^{N-1}\,u_x) = \partial u/\partial t$ *a.e. in* $G = \{(x, t)\colon x > 0, 0 < t < T\}$,

$$\lim_{x \to 0+} u(x, t) = A > 0, \qquad a.e. \quad t \in (0, T),$$

$$\lim_{t \to 0+} u(x, t) = 0, \qquad a.e. \quad x > 0,$$

has at most a unique solution $u(x, t) \in \mathscr{H}$. *This solution is monotone decreasing in x. Furthermore, if the assumptions in Theorem, Section* 4, *are fulfilled, then* $u(x, t) = u(x/t^{1/(N+1)})$.

The uniqueness is an obvious consequence of the comparison theorem, and is valid for more general boundary conditions, attained in the a.e. sense described in (5.8). The monotonicity (proved independently for the similarity solution, cf. Section 2.1) is also a general result, valid in the case of operator (6.2) for discontinuous boundary data (cf. Section 5.5 and Redheffer and Walter [10]).

Finally, it is easily verified that the solution $u = u(x/t^{1/(N+1)}) \in \mathscr{H}$.

6.2

In this section we shall assume the hypotheses of Section 4, Theorem.

THEOREM. *Let* $u(x, t) \in \mathscr{H}$ *be a solution of*

$$\frac{\partial}{\partial x}(k(u))\,|u_x|^{N-1}\,u_x) = u_t \qquad a.e.\ in \quad G = \{(x, t)\colon x > 0, 0 < t \leqslant T\},$$

$$\lim_{t \to 0+} u(x, t) = u_1(t) \qquad a.e. \quad t \in (0, T),$$

$$\lim_{t \to 0+} u(x, t) = u_0(x) \qquad a.e. \quad x > 0,$$

such that $\int^{u(x,t)} k^{1/N}(s)\,ds$ *be absolutely continuous as a function of x for almost every* $t \in (0, T)$.

We assume that $u(x)$ *has compact support,* $0 \leqslant u_0(x) \leqslant M$, $0 \leqslant u_1(t) \leqslant M$, $k(s) > 0$ *a.e., and assume also that* $k^{1/N}(s)$ *is Lebesgue integrable in the range of the solution* $u(x, t)$.

Then $\int_0^M (k(s)/s)^{1/N}\,ds < \infty$ *is a necessary and sufficient condition for* $u(x, t)$ *to have compact support as a function of x for* $0 < t \leqslant T$.

Note. A similar result for the equation $(k(u)\,u_x)_x = u_t$ was established by Peletier [9] assuming differentiability of $k(s)$ and a weak formulation of the problem above, and assuming also that $u_1(t)$ is bounded away from zero in $(0, T)$. His proof is based on a construction of classical solutions that approximate u and on a standard maximum principle for smooth solutions of the equations. More recently, Nagai [8] has shown compact support behaviour in $R^n \times (0, \infty)$ for solutions of differential inclusions involving

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{N-1} \frac{\partial u}{\partial x_i}\right).$$

We also cite in this connection Díaz and Herrero [12].

Proof of the Theorem. We assume that the integral is finite and let $w(\xi)$, $\xi = x/(t+\alpha)^{1/N+1}$ be a solution of the similarity problem $(k(w)\,|w'|^{N-1}\,w')' = (-1/(N+1))\,\xi w'$, $w(0) = M + \varepsilon$, $w(\infty) = 0$, with $\varepsilon > 0$ and $\alpha > 0$ chosen so that $w(x/\alpha^{1/N+1}) \geqslant u_0(x)$ for every x. (We may assume that $k(s)$ is defined for $s \in (0, M+\varepsilon)$, or else put $k = 1$ in $(\sup u, M+\varepsilon)$).

It follows that $w(x/(t+\alpha)^{1/N+1})$ is a solution of the partial differential equation whose data on $\partial_p G$ majorise that of $u(x, t)$. Therefore

$$0 \leqslant u(x,t) \leqslant w(x/(t+\alpha)^{1/N+1}) \qquad \text{in} \quad G.$$

This proves that $u(x, t)$ has compact support in x, due to the property of $w(\xi)$ (cf. Section 2.4).

Suppose now that $u(x, t)$ has compact support in G as a function of x, and $u_0(x) \geqslant 0$ is not zero a.e. There is an $x_0 > 0$ such that $u(x_0, t)$ is absolutely continuous locally for $t > 0$, $\lim_{x \to x_0} u(x,t) = u(x_0, t)$ a.e. t. $\lim_{t \to 0+} u(x_0, t) = u_0(x_0) > 0$, and hence $u(x_0, t) \geqslant \delta > 0$ for $0 < t < \gamma$. Put $\xi = (x - x_0)/t^{1/N+1}$, $x > x_0$, and let $v(\xi)$ be the solution of the similarity problem with $v(0) = \delta$, $v(+\infty) = 0$. Then $v((x-x_0)/t^{1/N+1})$ is a solution to the partial differential equation in $H = \{(x,t)\colon x > x_0,\ 0 < t \leqslant \gamma\}$ whose data on $\partial_p H$ are majorised by those of $u(x, t)$. Therefore $0 \leqslant v((x-x_0)/t^{1/N+1}) \leqslant u(x,t)$ and thus v has compact support as a function of x. Clearly $v(\xi)$ has also compact support and therefore $\int_0^\delta (k(s)/s)^{1/N}\,ds < \infty$. The integrability of $k^{1/N}(s)$ in (δ, M) completes the proof. The situation where $\lim_{x \to 0+} u(x, t_0) > 0$, $t_0 > 0$ can be treated in a similar way.

Remark. It is not difficult to see that the argument in this theorem can be extended to prove compact support behavior of solutions to a Cauchy problem in $(-\infty, +\infty)$.

6.3

In order to obtain the compact support result in the more general hypotheses for existence made in Section 3.2 for the similarity solution, we must be able to compare the solution $u(x, t)$ with similarity solutions $v(\xi)$ such that $v(\xi_0) = \text{const}$, $\xi_0 > 0$. Reverting to the (r, t) space, this represents a boundary $r = \xi_0 t^{1/N+1}$, $\xi_0 > 0$. We recall that in many dimensions the case

$r=0$ has little meaning, so we may assume that the boundary of G lies a positive distance apart from $r=0$ and that $\xi_0 < r/t^{1/N+1}$ in G.

The modifications needed to extend the proof of the theorem of Section 6.2 to arbitrary G or the case of a radial solution $u(r, t)$ are not essential, the basic feature being that bounded data had bounded support on $\partial_p G$. We shall include, instead, a consideration of the following result that is interesting in itself, for it points out the possibility of weakening—for certain operators A—the integrability assumption (5.5i). (Cf. in this connection the comment at the end of Section 5.5).

THEOREM. *For the operator* $(\partial/\partial x)(k(u)\,|u_x|^{N-1}\,u_x)=\partial u/\partial t$, *when a similarity solution is available, maximum principle* 5.4(f) *holds if* $u(x,t)\in\mathscr{H}$ *is assumed bounded in G, and condition* (5.5i) *is replaced with*

$$u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \in L^1(G\cap\{(x,t): \delta < t < T, x < m\}),$$

for every $\delta > 0$, $m > 0$ (*cf. Section* 5.4(e)).

Proof. We recall from the proof of the comparison theorem that all integrability assumptions are actually made on the set $D\cap\{0<\delta<t<\tau\}$, where $D=\{(x,t)\in G: v<u\}$.

Assume $|u(x,t)|\leqslant M$ in G, $u\leqslant$ const U ($<M$) on $\partial_p G$. We shall show that $u\leqslant U$ on G. Let $v=v(x/(t+\alpha)^{1/N+1})$ be a solution of the similarity equation with $v(0)=U$, $v(+\infty)=M+1$, $\varepsilon>0$ given,

$$U\leqslant v(\xi)\leqslant U+\varepsilon \qquad \text{in} \quad 0\leqslant\xi\leqslant\xi_1.$$

Now the set $D=\{(x,t)\in G: u>v\}$ is bounded due to the choice of $v(\infty)=M+1>u$. Therefore it must be empty and $u(x,t)\leqslant v(x/(t+\alpha)^{1/N+1})$. Now $\alpha>0$ is to be determined yet, and we observe that $x\leqslant\xi_1(t+\alpha)^{1/N+1}$ sweeps the region G as $\alpha\to+\infty$.

APPENDIX

We consider here the solution of (2.5) when $k(u)=1$, i.e., we solve

$$(\xi^{n-1}\,|u'|^{N-1}\,u')' + \xi^n u'/(N+1) = 0, \qquad N>0, \tag{A.1}$$

subject to the conditions

$$u=A>0, \qquad \xi=\xi_0, \qquad u\to 0 \quad \text{as} \quad \xi\to\infty. \tag{A.2}$$

A first integral of (A.1) can be written

$$u' = -\xi^{(1-n)/N} \left\{ C - \frac{(N-1)\,\xi^{1+n+(1-n)/N}}{(N+1+Nn-n)(N+1)} \right\}^{1/(N-1)}, \tag{A.3}$$

where C is an arbitrary constant to be determined from boundary conditions (A.2). It is necessary to consider certain cases separately.

Case (i): $N > 1$. In this case $\int_0^1 (1/u)^{1/N}\, du < \infty$ and hence we expect $u = 0$ for $\xi \geqslant a$, where a is to be determined. Equation (A.3) and the condition $u' = 0$ at $\xi = a$ give

$$C = \frac{(N-1)\,a^{1+n+(1-n)/N}}{(N+1+Nn-n)(N+1)} \tag{A.4}$$

and an integration of (A.3) gives

$$u = \int_{\xi}^{a} \xi^{(1-n)/N} \left\{ C - \frac{(N-1)\,\xi^{1+n+(1-n)/N}}{(N+1+Nn-n)(N+1)} \right\}^{1/(N-1)} d\xi, \tag{A.5}$$

where a is to determined from the boundary condition $u = A$ at $\xi = \xi_0$. If in addition $N > n-1$, the integral for u will exist even if $\xi_0 = 0$ and so we can allow $\xi_0 = 0$ for such cases. The condition $u = A$ at $\xi = \xi_0$ for fixed ξ_0 can clearly be satisfied for any A by changing the corresponding value of a.

Case (ii): $1 > N > (n-1)/(n+1)$. From (A.3) and the inequality $1 > N > (n-1)/(n+1)$ we deduce that

$$u' \sim -\xi^{2/(N-1)} \qquad \text{as} \quad \xi \to \infty.$$

Integrating (A.3) now gives

$$u = \int_{\xi}^{\infty} \xi^{(1-n)/N} \left\{ C + \frac{(1-N)\,\xi^{1+n+(1+n)/N}}{(N+1)(N+1+Nn-n)} \right\}^{1/(N-1)} d\xi. \tag{A.6}$$

We cannot let $\xi \to 0$ because the integral would not then exist if $n \geqslant 2$. However, if $u = A$ when $\xi = \xi_0 > 0$, then (A.6) gives an equation to determine C in terms of ξ_0. Note that (A.6) is a decreasing function of C for fixed ξ_0 and hence u at $\xi = \xi_0$ can take any value from infinity to zero as C ranges from $(-(1-N)\,\xi_0^{1+n+(1-n)/N})/((N+1)(N+1+Nn-n))$ to infinite. The case $n = 1$, $1 > N > 0$ is similar (cf. [2]).

Case (iii): $0 < N < (n-1)/(n+1)$. Now $u' \sim \xi^{(1-n)/N}$ as $\xi \to \infty$ and

$$u = \int_{\xi}^{\infty} \xi^{(1-n)/N} \left\{ C - \frac{(1-N)\,\xi^{1+n+(1-n)/N}}{(N+1)(n-Nn-N-1)} \right\}^{1/(N-1)} d\xi. \tag{A.7}$$

In order that the integral should exist for $\xi > \xi_0 > 0$ we require

$$C > \frac{(1-N)\,\xi_0^{1+n+(1-n)/N}}{(N+1)(n-1-N-Nn)}. \tag{A.8}$$

However, for a fixed value of ξ_0 we can satisfy the boundary condition $u = A$ at $\xi = \xi_0$ for any A since equality in (A.8) would correspond to A infinite and from (A.7) u at $\xi = \xi_0$ decreases as C increases, and tends to zero as C tends to infinity.

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