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j-Expansive Matrix-Valued Functions and Darlington Realization of Transfer-Scattering Matrices

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We obtain in this article theorems on linear fractional transformations of j-expansive matrix-valued functions wich provide a procedure to synthetize linear passive $2n$-ports. In particular, these results permit us to solve the problem of Darlington realizations of transfer-scattering matrices of linear passive $2n$-ports on the basis of the synthesis of transfer-scattering matrices of linear $4n$-ports without losses. It is a pleasure to acknowledge our debt to the remarkable paper [4] by Arov.

I. SOME KNOWN RESULTS ON MEROMORPHIC MATRIX-VALUED FUNCTIONS

A matrix A is called contractive iff $I - A^*A \geqslant 0$, where I is a unit matrix and the symbol * denotes Hermitian conjugation. Let J be a matrix for which $J^* = J$ and $J^2 = I$. A matrix A is called J-expansive iff $A^*JA - J \geqslant 0$, and J-unitary iff $A^*JA - J = 0$.

We call inner function every holomorphic function $u(z)$ ($z = re^{it}$) defined on the unit disk $D = \{z;\ |z| < 1\}$ such that $|u(z)| \leqslant 1$ ($z \in D$) and $|u(\xi)| = 1$ ($\xi = e^{it}$) a.e. on the unit circle [1].

We call outer function every function $\Phi(z)$ on D which admits a representation of the form

$$\Phi(z) = \chi \exp \frac{1}{2\Pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \ln k(t)\, dt \qquad (z \in D),$$

where $k(t) \geqslant 0$, $\ln k(t) \in L^1$ and χ is a complex number of modulus 1 [1].

Every function holomorphic and bounded in D can be written as the product of an inner function and a bounded outer function.

According to a theorem of Nevanlinna [2], the class N of functions of bounded characteristic coincides with the class of functions that can be written as the ratio of two bounded holomorphic functions ($z \in D$).

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A matrix-valued function $A(\xi)$ is said to be of the class H^{∞} if it is a.e. the boundary value of a bounded matrix-valued function $A(z)$ defined and holomorphic in D [3].

We use S to denote the class of contractive matrix-valued functions $S(z)$, defined and holomorphic in D, which satisfy condition $\|S(z)\| \leqslant 1$ $(z \in D)$ [4].

A matrix-valued function $A(z)$ is called of bounded characteristic if all its elements are functions of bounded characteristic.

A matrix-valued function $A(z)$ meromorphic in D is J-expansive if it assumes J-expansive values in each point of holomorphicity, and it is J-inner if it is J-expansive and satisfies a.e. condition $A^*(\xi)\ JA(\xi) - J = 0$. J-expansive matrix-valued functions are of bounded characteristic.

A matrix-valued function $A(\xi)$ is said to be of the class N^+ if there exists a bounded outer function $\Phi(z)$ such that $\Phi(\xi)A(\xi)$ is a matrix-valued function of the class H^∞[3].

Basic Lemma (cf. [4]). *If the matrix-valued function of order* $2n$

$$A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} \qquad (z \in D)$$

of bounded characteristic, where $\det \delta(z) \not\equiv 0$ $(z \in D)$, *satisfies conditions*:

(i) $A^*(\xi) jA(\xi) - j \geqslant 0$ *a.e. where*

$$j = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix};$$

(ii)

$$a(\xi) = \beta(\xi)\,\delta^{-1}(\xi) \in N^+; \qquad b(\xi) = \alpha(\xi) - \beta(\xi)\,\delta^{-1}(\xi)\,\gamma(\xi) \in N^+;$$

$$c(\xi) = \delta^{-1}(\xi)\,\gamma(\xi) \in N^+; \qquad d(\xi) = \delta^{-1}(\xi) \in N^+; \tag{I.1}$$

then $A^*(z) jA(z) - j \geqslant 0$ $(z \in D)$ *and* $\|a(z)\| \leqslant 1$; $\|b(z)\| \leqslant 1$; $\|c(z)\| \leqslant 1$; $\|d(z)\| \leqslant 1$.

Let $u(z)$ and $v(z)$ be inner functions. We say that a matrix-valued function $A(\xi)$ is of the class $M(u, v)$ if [3]

$$u(\xi)\,A(\xi) \in N^+; \qquad v(\xi)\,A^*(\xi) \in N^+.$$

Every function $A(\xi)$ of the class $M(u, v)$ is the boundary value of a function $A(z)$ meromorphic in D, and the same function $A(\xi)$ is the boundary value of a certain function $\tilde{A}(z)$, meromorphic in $|z| > 1$.

THEOREM I.1. (cf. [3, 5]). *If* $A(\xi)$ *is a nonnegative matrix-valued function of the class* $M(v, v)$, *then* $A(\xi)$ *can be factorized, i.e.,* $A(\xi) = \Phi^*(\xi)\,\Phi(\xi)$ *a.e., where* $\Phi(\xi)\,(\in M(1, v))$ *is uniquely defined by the normalization conditions* $\Phi(0) > 0$ *and* det $\Phi(z)$ *is an outer function.*

II. LINEAR FRACTIONAL TRANSFORMATIONS OF j-EXPANSIVE MATRIX-VALUED FUNCTIONS

By Darlington realization of a j-expansive matrix-valued function $T(z)$, we mean the representation of $T(z)$ as the linear fractional transformation

$$T(z) = [A(z)\,t_0 + B(z)][C(z)\,t_0 + D(z)]^{-1} \tag{II.1}$$

over a j-expansive constant matrix t_0, with a J'-inner matrix of coefficients

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

where

$$J' = \begin{pmatrix} -j & 0 \\ 0 & j \end{pmatrix}.$$

THEOREM II.1. *A j-expansive matrix-valued function* $T(z)$ *such that its boundary value function a.e.* $T(\xi)$ *is of the class* $M(u, v)$ *and satisfies a.e. one of the following conditions*:

(1) $T^*(\xi)\,jT(\xi) - j = 0$,

(2) $T^*(\xi)\,jT(\xi) - j > 0$,

can be represented as the linear fractional transformation (II.1).

Proof. (1) When $T(\xi)$ satisfies condition (1), the representation (II.1) of $T(z)$ is obtained with $t_0 = I_{2n}$ and the matrix of coefficients

$$W(z) = \begin{pmatrix} T(z) & 0 \\ 0 & I_{2n} \end{pmatrix}.$$

(2) If $T(\xi)\,(\in M(u, v))$ satisfies condition (2), there exist (cf. [3, 5]) matrix-valued functions $\theta(z)$ and $\Psi(z)$, of order $2n$, holomorphic and bounded in D, that are solutions of the factorization problems

$$\begin{aligned} T^*(\xi)\,jT(\xi) - j &= \theta^*(\xi)\,\theta(\xi) \qquad \text{a.e.}; \\ T(\xi)\,jT^*(\xi) - j &= \Psi(\xi)\,\Psi^*(\xi) \qquad \text{a.e.} \end{aligned} \tag{II.2}$$

From among the infinite set of solutions there exist uniquely defined solutions $\theta_0(z)$ and $\Psi_0(z)$ such that $\det \theta_0(z)$ and $\det \Psi_0(z)$ are outer functions, and $\theta_0(0) > 0$ and $\Psi_0(0) > 0$.

We shall obtain the representation of $T(z)$ as a linear fractional transformation (II.1) over the j-expansive constant matrix

$$t_0 = \tfrac{1}{2}(Q + J_1 + 5P),$$

where $Q = \tfrac{1}{2}(I_{2n} + j)$, $P = \tfrac{1}{2}(I_{2n} - j)$, and

$$J_1 = \begin{pmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{pmatrix};$$

and with a matrix of coefficients $W(z)$ constructed with the following blocks:

$$\begin{aligned} A(z) = {} & \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + Q\} \\ & \times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ & \times \Psi_0^{*-1}(1/\bar{z})[\sqrt{5}\,Q - 5^{-1/2}(QJ_1 + 2P)] \\ & + w^{-1}(z)\,T(z)\,\theta_0^{-1}(z)[\sqrt{5}\,QJ_1 - 5^{-1/2}(Q + 2J_1 Q)]; \\ B(z) = {} & \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + Q\} \\ & \times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ & \times \Psi_0^{*-1}(1/\bar{z})[\sqrt{5}\,P - 5^{-1/2}(2Q - PJ_1)] \\ & + w^{-1}(z)\,T(z)\,\theta_0^{-1}(z)[\sqrt{5}\,PJ_1 - 5^{-1/2}(2J_1 P - P)]; \\ C(z) = {} & \{Q[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + P\} \\ & \times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ & \times \Psi_0^{*-1}(1/\bar{z})[\sqrt{5}\,Q - 5^{-1/2}(QJ_1 + 2P)] \\ & + w^{-1}(z)\,\theta_0^{-1}(z)[\sqrt{5}\,QJ_1 - 5^{-1/2}(Q + 2J_1 Q)]; \\ D(z) = {} & \{Q[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + P\} \\ & \times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ & \times \Psi_0^{*-1}(1/\bar{z})[\sqrt{5}\,P - 5^{-1/2}(2Q - PJ_1)] \\ & + w^{-1}(z)\,\theta_0^{-1}(z)[\sqrt{5}\,PJ_1 - 5^{-1/2}(2J_1 P - P)]; \end{aligned} \tag{II.3}$$

where $w(z)$ is an inner scalar function.

We shall show that with our particular choice of t_0 and the matrix of coefficients $W(z)$, whose blocks are given by (II.3), relation (II.1) holds for $T(z)$.

Replacing t_0 and (II.3) in (II.1) we have

$$\begin{aligned} A(z)\,t_0 + B(z) = &\{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + Q\} \\ &\times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ &\times \Psi_0^{*-1}(1/\bar{z})\tfrac{1}{2}\sqrt{5}\,\{(Q + QJ_1 + 2P) \\ &- 5^{-1/2}[\tfrac{1}{2}(Q + 5QJ_1 + 2PJ_1 + 10P) + 2Q - PJ_1]\} \\ &+ w^{-1}(z)\,T(z)\,\theta_0^{-1}(z) \\ &\times [\tfrac{1}{2}\sqrt{5}(Q + 5QJ_1 + 2PJ_1) - 5^{-1/2}(Q + 5QJ_1 - 2J_1 Q)] \\ = &\, w^{-1}(z)\,T(z)\,\theta_0^{-1}(z)[2\sqrt{5}\,QJ_1 + (2/\sqrt{5})(Q + 2PJ_1)]. \end{aligned} \tag{II.4}$$

$$\begin{aligned} C(z)\,t_0 + D(z) = &\{Q[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] + Q\} \\ &\times \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1} \\ &\times \Psi_0{}^{*-1}(1/\bar{z})\{\tfrac{1}{2}\sqrt{5}\,(Q + QJ_1 + 2P) - 5^{-1/2} \\ &\times [\tfrac{1}{2}(Q + 5QJ_1 + 2PJ_1 + 10P) + 2Q - PJ_1]\} \\ &+ w^{-1}(z)\,\theta_0^{-1}(z)\,\tfrac{1}{2}\sqrt{5}(Q + 5QJ_1 + 2PJ_1) \\ &- \tfrac{1}{2}5^{-1/2}(Q + 5QJ_1 - 2J_1 Q) \\ = &\, w^{-1}(z)\,\theta_0^{-1}(z)[2\sqrt{5}\,QJ_1 + (2/\sqrt{5})(Q + 2PJ_1)]. \end{aligned} \tag{II.5}$$

It follows from (II.4) and (II.5) that $T(z)$ can be represented as the linear fractional transformation (II.1).

To complete the proof it remains to show that the matrix of coefficients $W(z)$ is J'-inner. Let us define now the unitary matrix

$$R \overset{\text{def}}{=} \begin{pmatrix} Q & PJ_1 \\ P & QJ_1 \end{pmatrix}, \tag{II.6}$$

verifying

$$Rj_{2n}R^* = J', \tag{II.7}$$

where

$$j_{2n} = \begin{pmatrix} -I_{2n} & 0 \\ 0 & I_{2n} \end{pmatrix};$$

and construct a matrix-valued function

$$H(z) = \begin{pmatrix} h_{11}(z) & h_{12}(z) \\ h_{21}(z) & h_{22}(z) \end{pmatrix},$$

with blocks given by the following expressions:

$$\begin{aligned} h_{11}(z) &= \{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1}\,\Psi_0^{*\,-1}(1/\bar{z}) \\ &\quad \times \sqrt{5}I_{2n} - w^{-1}(z)[QT(z) + P]\,\theta_0^{-1}(z)\,5^{-1/2}(Q - P + 2J_1); \\ h_{12}(z) &= -\{P[T^*(1/\bar{z})P + Q]^{-1}\,[T^*(1/\bar{z})Q + P] - Q\}^{-1}\,\Psi_0^{*\,-1}(1/\bar{z}) \\ &\quad \times 5^{-1/2}(Q - P + 2J_1) + w^{-1}(z)[QT(z) + P]\,\theta_0^{-1}(z)\sqrt{5}I_{2n}; \\ h_{21}(z) &= J_1\{P - Q[T^*(1/\bar{z})Q + P]^{-1}\,[T^*(1/\bar{z})P + Q]\}^{-1}\,\Psi_0^{*\,-1}(1/\bar{z}) \\ &\quad \times \sqrt{5}I_{2n} - w^{-1}(z)\,J_1[PT(z) + Q]\,\theta_0^{-1}(z)\,5^{-1/2}(Q - P + 2J_1); \\ h_{22}(z) &= -J_1\{P - Q[T^*(1/\bar{z})Q + P]^{-1}\,[T^*(1/\bar{z})P + Q]\}^{-1}\,\Psi_0{}^{*\,-1}(1/\bar{z}) \\ &\quad \times 5^{-1/2}(Q - P + 2J_1) + w^{-1}(z)\,J_1[PT(z) + Q]\,\theta_0^{-1}(z)\sqrt{5}I_{2n}. \end{aligned} \tag{II.8}$$

By virtue of (II.3), (II.6), and (II.8) it follows that

$$W(z) = RH(z)\,R^* \qquad (z \in D).$$

Hence using (II.7), we have

$$W^*(z)\,J'W(z) - J' = R[H^*(z)j_{2n}H(z) - j_{2n}]\,R^*. \tag{II.9}$$

From the above relation we conclude that the proof will be complete if we show that $H(z)$ is j_{2n}-inner. Let us multiply $H^*(z)j_{2n}H(z) - j_{2n}$, to the right and to the left by the matrix

$$t_1 = \begin{pmatrix} \sqrt{5}I_{2n} & (1/\sqrt{5})(Q - P + 2J_1) \\ (1/\sqrt{5})(Q - P + 2J_1) & \sqrt{5}I_{2n} \end{pmatrix}, \tag{II.10}$$

and introduce the matrix-valued function

$$G(z) = \begin{pmatrix} g_{11}(z) & g_{12}(z) \\ g_{11}(z) & g_{22}(z) \end{pmatrix} \stackrel{\text{def}}{=} \tfrac{1}{2}H(z)\,t_1.$$

Note that $t_1 j_{2n} t_1 = j_{2n}$. Therefore, from (II.9), (II.10), and the definition of $G(z)$, it turns out that $W(z)$ is J'-inner iff $G(z)$ is j_{2n}-inner. Let us write the blocks of $G(z)$ applying (II.8) and (II.10). We have

$$\begin{aligned}
g_{11}(z) &= \{P[T^*(1/\bar{z})P+Q]^{-1}\,[T^*(1/\bar{z})Q+P]-Q\}^{-1}\,\Psi_0^{*\,-1}(1/\bar{z});\\
g_{12}(z) &= w^{-1}(z)[QT(z)+P]\,\theta_0^{-1}(z);\\
g_{21}(z) &= J_1\{P-Q[T^*(1/\bar{z})Q+P]^{-1}\,[T^*(1/\bar{z})P+Q]\}^{-1}\,\Psi_0^{*\,-1}(1/\bar{z});\\
g_{22}(z) &= w^{-1}(z)\,J_1[PT(z)+Q]\,\theta_0^{-1}(z). \qquad \text{(II.11)}
\end{aligned}$$

From (II.11) it can be easily checked that $G(z)$ is a.e. j_{2n}-unitary. Then, to finish the proof of Theorem II.2 it is sufficient to show that the hypothesis (ii) of our basic lemma holds for $G(z)$.

Let us write, using (II.11), the matrix-valued functions

$$a(z)=g_{12}(z)\,g_{22}^{-1}(z); \qquad b(z)=g_{11}(z)-g_{12}(z)\,g_{22}^{-1}(z)\,g_{21}(z);$$
$$c(z)=g_{22}^{-1}(z)\,g_{21}(z); \qquad d(z)=g_{22}^{-1}(z).$$

After elementary calculations we get

$$\begin{aligned}
a(z) &= [QT(z)+P][PT(z)+Q]^{-1}\,J_1;\\
b(z) &= [I_{2n}-a(z)\,a^*(1/\bar{z})][PJ_1a^*(1/\bar{z})-Q]^{-1}\,\Psi_0^{*\,-1}(1/\bar{z});\\
c(z) &= w(z)\,\theta(z)[PT(z)+Q]^{-1}\\
&\quad\times\{[T(z)jT^*(1/\bar{z})-j][P-Qa^{*\,-1}(1/\bar{z})]\}^{-1}\,\Psi_0(z);\\
d(z) &= w(z)\,\theta_0(z)[PT(z)+Q]^{-1}. \qquad \text{(II.12)}
\end{aligned}$$

We shall prove that the boundary value functions a.e. of the matrix-valued functions given by relations (II.12) are of the class N^+. Let us note that the j-expansivity of $T(z)$ implies that $a(z)$ is a contractive matrix-valued function ($z\in D$), i.e., $a(z)\in S$ and $a(\xi)\in N^+$. Since $T(\xi)\in M(u,v)$, then $v(\xi)\,T^*(\xi)\,(\in N^+)$ is a.e. the boundary value function of the holomorphic matrix-valued function $v(z)\,T^*(1/\bar{z})$ ($z\in D$) (cf. [3]). Therefore, $T^*(1/\bar{z})$ is a matrix-valued function of bounded characteristic in D and, by virtue of [3, Theorem 3.1] and of the relation between $a(z)$ and $T(z)$, the result is that $\Psi_0^*(\xi)$ and $a^*(\xi)$ are a.e. boundary value functions of the matrix-valued functions $a^*(1/\bar{z})$ and $\Psi_0^*(1/\bar{z})$ of bounded characteristic in D.

The above conclusions allow us to affirm that the functions $b(\xi)$, $c(\xi)$, and $d(\xi)$, defined by relations (I.1), are a.e. boundary value functions of the matrix-valued functions $b(z)$, $c(z)$, and $d(z)$, of bounded charateristic in D, given by relations (II.12).

Consider now the factorization problems

$$I_{2n}-a^*(\xi)\,a(\xi)=\phi^*(\xi)\,\phi(\xi) \qquad \text{a.e.};$$
$$I_{2n}-a(\xi)\,a^*(\xi)=\eta(\xi)\,\eta^*(\xi) \qquad \text{a.e.}$$

Using the expression $a(z)$ in terms of $T(z)$ we have:

$$I_{2n} - a^*(\xi)\, a(\xi) = J_1 [PT(\xi) + Q]^{*-1} [T^*(\xi) jT(\xi) - j][PT(\xi) + Q]^{-1} J_1;$$
$$T(\xi) jT^*(\xi) - j = [a(\xi) J_1 P - Q]^{-1} [I_{2n} - a(\xi)\, a^*(\xi)][a(\xi) J_1 P - Q]^{*-1}.$$

From (II.1) and the preceding relations we conclude that a.e. $d(\xi) = w(\xi)\,\phi(\xi)$ and $b(\xi) = \eta(\xi)$. Recalling that $\phi(z)$ and $\eta(z)$ are bounded holomorphic matrix-valued functions ($z \in D$) (cf. [3–5]) it follows that $d(\xi)$ and $b(\xi)$ are functions of the class N^+.

Finally we are going to show that $c(\xi) \in N^+$. Since we know that $\Psi_0(z)$ and $d(z)$ are holomorphic matrix-valued functions ($z \in D$), if we choose the inner scalar function $w(z)$ to be the common denominator of the matrix-valued function $\{[T(z) jT^*(1/\bar{z}) - j][P - Qa^{*-1}(1/\bar{z})]\}^{-1}$ of bounded characteristic in D, then $c(z)$ is an holomorphic matrix-valued function ($z \in D$) and $c(\xi) \in N^+$. We have proved that $G(z)$ is j_{2n}-expansive ($z \in D$) and hence $W(z)$ is J'-inner.

This completes the proof of Theorem II.1.

A similar theorem for J_1-expansive matrix-valued functions in $\operatorname{Re} p > 0$ was stated without proof in the paper [6] by Efimov and Potapov.

THEOREM II.2. *Let $T(z)$ be a matrix-valued function of order $2n$, which can be represented as a linear fractional transformation* (II.1) *over a j-expansive constant matrix t, with a J'-inner matrix of coefficients*

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix};$$

then, $T(z)$ is a j-expansive matrix-valued function and $T(\xi) \in M(u, v)$.

Proof. Let us write the expression (II.1) in the form

$$\begin{pmatrix} T(z) \\ I_{2n} \end{pmatrix} = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \begin{pmatrix} t \\ I_{2n} \end{pmatrix} [C(z)t + D(z)]^{-1}.$$

Then,

$$\begin{aligned} T^*(z) jT(z) - j &= [C(z)t + D(z)]^{*-1} \begin{pmatrix} t \\ I_{2n} \end{pmatrix}^* \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}^* \\ &\quad \times \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \begin{pmatrix} t \\ I_{2n} \end{pmatrix} [C(z)t + D(z)]^{-1}. \end{aligned}$$

Since $W(z)$ is J'-inner ($z \in D$) and t is j-expansive, it follows that

$$T^*(z)\,jT(z) - j \geqslant [C(z)t + D(z)]^{*-1} \times (t^*jt - j)[C(z)t + D(z)]^{-1} \geqslant 0.$$

Then $T(z)$ is a j-expansive matrix-valued function ($z \in D$) and, consequently, it is of bounded characteristic and there exists an inner function $u(z)$ such that $u(\xi)\,T(\xi) \in N^+$.

Consider now the J'-inner matrix-valued function $W(z)$. From the equation

$$W^*(\xi)\,J'\,W(\xi) - J' = 0 \qquad \text{a.e.},$$

we can define the matrix-valued function

$$\tilde{W}(z) = J'\,W^*(1/\bar{z})\,J',$$

of bounded characteristic in $|z| > 1$. Therefore, $T(\xi)$ is the limiting value a.e. of functions $T(z)$, of bounded characteristic in D, and $\tilde{T}(z)$, of bounded characteristic in $|z| > 1$. Using [3, Theorem 2.8] we can conclude that $T(\xi) \in M(u, v)$. This completes the proof of Theorem II.2.

III. Real Realizations

For physical applications we are interested in real functions, i.e., functions $T(z)$ such that $\overline{T(\bar{z})} = T(z)$, where the bar stands for complex conjugation.

We call real realization of a j-expansive matrix-valued function $T(z)$ the representation of $T(z)$ as a linear fractional transformation (II.1) over a j-expansive constant real matrix t, with a real matrix of coefficients $W(z)$.

THEOREM III.1. *Let $T(z)$ be a real j-expansive matrix-valued function such that $T(\xi) \in M(u, v)$ and one of the following conditions hold a.e.:*

(1) $T^*(\xi)\,jT(\xi) - j = 0$,

(2) $T^*(\xi)\,jT(\xi) - j > 0$,

then there exists a real realization of $T(z)$.

The proof of this theorem is essentially similar to that of [4, Theorem 6.1]..

The linear fractional transformation $z = (p+1)/(p-1)$ $(p = x + iy)$, transforms the unit disk D on the right half plane $\operatorname{Re} p > 0$. Then, it is obvious that theorems analogous to Theorems II.1, II.2, and III.1 hold mutatis mutandis for j-expansive matrix-valued functions in the right half plane $\operatorname{Re} p > 0$.

The transfer-scattering matrix $T(p)$ of a linear passive $2n$-port is a real rational matrix-valued function, of order $2n$, j-expansive in the right half plane $\operatorname{Re} p > 0$, i.e.,

$$\text{(i)}\quad T^*(p)jT(p) - j \geqslant 0 \qquad \text{in} \quad \operatorname{Re} p > 0;$$

$$\text{(ii)}\quad \overline{T(\bar{p})} = T(p) \qquad \text{in} \quad \operatorname{Re} p > 0.$$

For a $2n$-port without losses, $T(p)$ satisfies the additional condition,

$$\text{(iii)}\quad T^*(p)jT(p) - j = 0 \qquad \text{in} \quad \operatorname{Re} p = 0.$$

In view of results obtained in [6] for real rational matrix-valued functions, J_1-expansive in the right half plane $\operatorname{Re} p > 0$ (chain matrices), it is easy to conclude that a $2n$-port, whose transfer-scattering matrix $T(p)$ is known, can be constructed by closing with quadrupoles the output $2n$-ports of a $4n$-port without losses. The transfer-scattering matrix $T(p)$ is given by the formula

$$T(p) = [A(z)t + B(p)][C(p)t + D(p)]^{-1}, \tag{III.1}$$

where t is the transfer-scattering matrix of the system formed by the n quadrupoles. The matrix of coefficients $W(p)$ of the linear fractional transformation (III.1), whose blocks are defined by formulas equivalent to (II.3), can be expressed in terms of the transfer-scattering matrix $H(p)$ of the $4n$-port without losses, by the relation $W(p) = RH(p)R^*$, where R is defined by (II.6).

We want to point out that we have obtained a method to synthesize a passive $2n$-port, with a fixed transfer-scattering matrix, on the basis of the synthesis of a $4n$-port without losses.

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