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On the Fourier Transforms of Retarded Lorentz-Invariant Functions

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In this article we evaluate the Fourier transforms of retarded Lorentz-invariant functions (and distributions) as limits of Laplace transforms. Our method works generally for any retarded Lorentz-invariant functions $\phi(t)$ $(t \in R^n)$ which is, besides, a continuous function of slow growth. We give, among others, the Fourier transform of $G_R(t, \alpha, m^2, n)$ and $G_A(t, \alpha, m^2, n)$, which, in the particular case $\alpha = 1$, are the characteristic functions of the volume bounded by the forward and the backward sheets of the hyperboloid $u = m^2$ and by putting $\alpha = -k$ are the derivatives of k-order of the retarded and the advanced-delta on the hyperboloid $u = m^2$. We also obtain the Fourier transform of the function $W(t, \alpha, m^2, n)$ introduced by M. Riesz (*Comm. Sem. Mat. Univ. Lund* **4** (1939)). We finish by evaluating the Fourier transforms of the distributional functions $G_R(t, \alpha, m^2, n)$, $G_A(t, \alpha, m^2, n)$ and $W(t, \alpha, m^2, n)$ in their singular points.

I. INTRODUCTION

We shall evaluate the Fourier transforms of retarded Lorentz-invariant functions (and distributions) as limits of Laplace transforms. Schwartz [1, especially p. 264] has evaluated the Fourier transforms of the Marcel Riesz functions $R_\alpha(x, n)$, by evaluating their Laplace transforms (first step), and then passing to the limit (in S') for $y \to 0$, where $y \in V_- = \{y \in \mathbb{R}^n / y_0 < 0, y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}$ (second step). The method was later employed by Lavoine [2], and Vladimirov [3, 299–302]. Gelfand and Shilov [11] and Methée [14] also have evaluated the Fourier transforms of Lorentz-invariant functions but they have employed different methods.

Our method works generally for any retarded Lorentz-invariant function $\phi(t)$ $(t \in \mathbb{R}^n)$ which is, besides, a continuous function of slow growth.

We shall evaluate the Fourier transforms of the distributional functions $G_R(t, m^2, \alpha, n)$ and $G_A(t, m^2, \alpha, n)$ (formulas (I, 2; 1) and (V, 2; 1), respectively). In the particular case $\alpha = 1$, $G_R(t, m^2, \alpha, n)$ is the characteristic function of the volume bounded by the forward sheet of the hyperboloid

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$u = m^2$. (G_A is the characteristic function of the volume bounded by the backward sheet of the hyperboloid $u = m^2$.) Another particular case is obtained by putting $\alpha = -k$, $G_A(t, m^2, \alpha, n)$ is the derivative of k-order of the retarded-delta on the hyperboloid $u = m^2$.

We prove that in the particular case $n = 4$, $\alpha = 0$, our formulas coincide with the formulas due to Constantinescu [17, p. 121, formula II.55].

We shall also evaluate the Fourier transform of the function G_R (and G_A) in the singular points.

Finally, we shall obtain the Fourier transform of the function $W(t, m^2, \alpha, n)$ (formula (VI, 1; 1)) introduced by Riesz [4, p. 17] (cf. also [5, p. 89; 1, p. 179; and 6, p. 72]).

In this article we also generalize results due to Gorgé (cf. [16, 32–40]), which obtains several distributional Fourier transforms in the case $n = 4$.

I.1. *Definitions*

Let $t = (t_0, t_1, ..., t_{n-1})$ be a point of $\mathbb{R}^n$. We shall write $t_0^2 - t_1^2 - \cdots - t_{n-1}^2 = u$. By Γ_+ we designate the interior of the forward cone: $\Gamma_+ = \{t \in \mathbb{R}^n \mid t_0 > 0, u > 0\}$; and by $\bar{\Gamma}_+$ we designate its closure. Similarly, Γ_- designates the domain $\Gamma_- = \{t \in \mathbb{R}^n \mid t_0 < 0, u > 0\}$, and $\bar{\Gamma}_-$ designates its closure. We put $z = (z_0, z_1, ..., z_{n-1}) \in \mathbb{C}^n$, where $z_\nu = x_\nu + iy_\nu$, $\nu = 0, 1, 2, ..., n-1$; $\langle t, z\rangle = t_0 z_0 + t_1 z_1 + t_{n-1} z_{n-1}$; and $dt = dt_0\, dt_1 \cdots dt_{n-1}$. The tube T_- is defined by $T_- = \{z \in \mathbb{C}^n / y \in V_-\}$, where $V_- = \{y \in \mathbb{R}^n \mid y_0 < 0, y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}$. The tube T_- is defined by $T_- = \{z \in \mathbb{C}^n / y \in V_-\}$, where $V_- = \{y \in \mathbb{R}^n \mid y_0 < 0, y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}$.

Similarly, we put $T_+ = \{z \in \mathbb{C}^n / y \in V_+\}$, where $V_+ = \{y \in \mathbb{R}^n / y_0 > 0, y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0\}$.

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\phi(t)$ be a function endowed with the following properties:

(a) $\phi(t) = F(u)$,

(b) $\operatorname{supp} \phi(t) \in \bar{\Gamma}_+$,

(c) $e^{\langle t, y\rangle}\phi(t) \in L_1$ if $y \in V_-$.

We call R the family of functions $\phi(t)$ which satisfies conditions (a), (b) and (c). Similarly we call A the family of functions which satisfies conditions

(a′) $\phi(t) = F(u)$,

(b′) $\operatorname{supp} \phi(t) \in \bar{\Gamma}_-$,

(c′) $e^{\langle t, y\rangle}\phi(t) \in L_1$ if $y \in V_+$.

The Fourier transform of $\phi(t)$ is

$$[\phi]^\wedge = \int_{\mathbb{R}^n} e^{-i\langle t, x\rangle}\phi(t)\, dt, \qquad \text{(I, 1; 1)}$$

and the Laplace transform of $\phi(t)$ is

$$f(z) = L\{\phi\} = \int_{\mathbb{R}^n} e^{-i(t,z)}\phi(t)\,dt. \tag{I, 1; 2}$$

The Laplace transform of a function $\phi(t) \in \mathbb{R}$, $z \in T_-$ can be evaluated by means of the following formula (cf. [7, formula (I, 2; 1), p. 53]).

$$f(z) = L\{\phi\} = \frac{(2\Pi)^{(n-2)/2}}{\{z_1^2 + z_2^2 + \cdots + z_{n-1}^2 - z_0^2\}^{(n-2)/4}} \tag{I, 1; 3}$$

$$\times \int_0^\infty F(\lambda)\,\lambda^{(n-2)/4} K_{(n-2)/2}\{\lambda(z_1^2 + z_2^2 + \cdots + z_{n-1}^2 - z_0^2)^{1/2}\}\,d\lambda.$$

Here $K_\nu(z)$ designates the modified Bessel function of the third kind [8, vol. I, p. 371].

I.2. *The Laplace Transform of $G_R(t, \alpha, m^2, n)$*

Let m be a nonnegative number and let α be a complex parameter.
We define the n-dimensional function

$$G_R(t, \alpha, m^2, n) = \frac{(u - m^2)_+^{\alpha - 1}}{\Gamma(\alpha)}$$

$$= \frac{(u - m^2)^{\alpha - 1}}{\Gamma(\alpha)}, \qquad \text{if} \quad u - m^2 > 0 \text{ and } t > 0,$$

$$= 0, \qquad \text{if} \quad t \text{ belongs to the complementary set.}$$

The Laplace transform of $G_R(t, \alpha, m^2, n)$ is (cf. [7, formula (II, 4; 5), p. 59])

$$L\{G_R(t, \alpha, m^2, n)\} = 2^\alpha (2\Pi)^{(n-2)/2}\, m^{\alpha + (n-2)/2} \rho^{-\alpha + (2-n)/2}$$

$$\times K_{\alpha + (n-2)/2}(m\rho), \tag{I, 2; 2}$$

where we have put

$$\rho^2 = z_1^2 + z_2^2 + \cdots + z_{n-1}^2 - z_0^2. \tag{I, 2; 3}$$

Formula (I, 2; 2), which we have proved on the assumption that $\operatorname{Re}\alpha \geqslant 1$, is valid, by analytical continuation, for every complex α and $\operatorname{Im} z_0 = y_0 < 0$.

We shall evaluate the Fourier transform of $G_R(t, \alpha, m^2, n)$ by passing to the limit (in S') for $y \to 0$, where $y \in V_-$, on its Laplace transform.

That is to say, we shall consider the limit in formula (I, 2; 2) as

$$\lim_{\substack{y \to 0 \\ (y \in V_-)}} \rho^2 = \lim_{\substack{\epsilon \to 0 \\ (y_0 < 0)}} \{(x_1^2 + \cdots + x_{n-1}^2) - (x_0 + i\varepsilon y_0)^2\}, \qquad \varepsilon > 0. \tag{I, 2; 4}$$

Formula (I, 2; 4) coincides with the notation used by Schwartz [1, p. 264].

I.3. *The Fourier Transform of* $G_R(t, \alpha, m^2, n)$

We begin by subdividing the space $\mathbb{R}^n$ into four regions:

(i) the exterior of the light cone:

$$C_1 = \{x \in \mathbb{R}^n / x_0^2 - x_1^2 - \cdots - x_{n-1}^2 < 0\}; \tag{I, 3; 1}$$

(ii) the interior of the forward cone:

$$C_f = \{x \in \mathbb{R}^n / x_0^2 - x_1^2 - \cdots - x_{n-1}^2 > 0, x_0 > 0\}; \tag{I, 3; 2}$$

(iii) the interior of the backward cone:

$$C_b = \{x \in \mathbb{R}^n / x_0^2 - x_1^2 - \cdots - x_{n-1}^2 > 0, x_0 < 0\}; \tag{I, 3; 3}$$

(iv) the set of points

$$\bar{C} = \{x \in \mathbb{R} / |x_0| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}\}. \tag{I, 3; 4}$$

To evaluate the Fourier transform of $G_R(t, \alpha, m^2, n)$ we shall apply the Schwarz method on each of the four regions (i)–(iv) and then we obtain the final result, by the lineality of the Fourier transformation, by adding their respective Fourier transforms.

We begin by evaluating the Fourier transform of $G_R(t, \alpha, m^2, n)$ in C_1.

We remark that outside the light cone there are no restrictions on y_0.

Starting from formula (I, 2; 2) and passing to the limit for $z_\nu \to x_\nu$, for all $\nu = 0, 1, \ldots, n-1$, we immediately obtain

$$[G_{C_1}(t, \alpha, m^2, n)]^\wedge = 2^\alpha (2\Pi)^{(n-2)/2} m^{\alpha + (n-2)/2} \times \frac{K_{\alpha + (n-2)/2}\{m(x_1^2 + \cdots + x_{n-1}^2 - x_0^2)^{1/2}\}}{\{(x_1^2 + \cdots + x_{n-1}^2 - x_0^2)^{1/2}\}^{\alpha + (n-2)/2}}, \tag{I, 3; 5}$$

where $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 < 0$.

We shall evaluate now the Fourier transform of $G_R(t, \alpha, m^2, n)$ in the second region, it is in the interior of the forward cone.

We have, by putting in the formula (I, 2; 2),

$$\rho^2 = \lim_{\epsilon \to 0} \{x^2 + \cdots + x_{n-1}^2 - (x_0 + i\varepsilon y_0)^2\},$$

with $\varepsilon > 0$ and $y_0 < 0$,

$$[G_{C_f}(t, \alpha, m^2, n)] = 2^\alpha (2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \qquad \text{(I, 3; 6)}$$
$$\times \frac{K_{\alpha+(n-2)/2}\{me^{i(\Pi/2)}(x_0^2 - x_1^2 - \cdots - x_{n-1}^2)^{1/2}\}}{e^{i(\Pi/2)(\alpha+(n-2)/2)}\{(x_0^2 - x_1^2 - \cdots - x_{n-1}^2)^{1/2}\}^{\alpha+(n-2)/2}},$$

where $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 > 0$, $x_0 > 0$.

Now let the third region be the interior of the backward cone. Therefore, it follows that

$$[G_{C_b}(t, \alpha, m^2, n)]^{\Lambda} = 2^\alpha (2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \qquad \text{(I, 3; 7)}$$
$$\times \frac{K_{\alpha+(n-2)/2}\{me^{-i(\Pi/2)}(x_0^2 - x_1^2 - \cdots - x_{n-1}^2)^{1/2}\}}{e^{-i(\Pi/2)(\alpha+(n-2)/2)}\{(x_0^2 - x_1^2 - \cdots - x_{n-1}^2)^{1/2}\}^{\alpha+(n-2)/2}},$$

where $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 > 0$, $x_0 < 0$.

We shall evaluate the Fourier transform of $G_R(t, \alpha, m^2, n)$ in the neighborhood of $|x_0| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$.

We begin by remembering the well-known asymptotic formula, valid for $s \to 0$ (cf. formula (AIII, 2; 6) of the Appendix).

$$K_\nu(s) \sim 2^{\nu-1}\Gamma(\nu)\, s^{-\nu}. \qquad \text{(I, 3; 8)}$$

We have, taking into account formula (I, 3; 8),

$$K_{\alpha+(n-2)/2}(m\rho) \sim 2^{\alpha+(n-2)/2-1}\Gamma\left(\alpha + \frac{n-2}{2}\right)(m\rho)^{-\alpha+(n-2)/2}. \qquad \text{(I, 3; 9)}$$

By substituting (I, 3; 9) into (I, 2; 2), we obtain,

$$L[G_{\bar{C}}(t, \alpha, m^2, n)] = 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha + \frac{n-2}{2}\right)\rho^{-2(\alpha+(n-2)/2)}, \qquad \text{(I, 3; 10)}$$

for the values of $|x_0|$ in the neighborhood of $|x_0| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$.

To evaluate the Fourier transform of $G_R(t, \alpha, m^2, n)$ in the neighborhood of $|x_0| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$, we shall consider the limit

$$\lim_{\substack{y \to 0 \\ (y<0)}} \rho^2 = \lim_{\substack{\epsilon \to 0 \\ (\epsilon > 0,\, y_0 < 0)}} \{(x_1^2 + \cdots + x_{n-1}^2 - (x_0 + iy_0\varepsilon)^2\}$$

in formula (I, 3; 10).

We get

$$[G_{\bar{C}}(t, \alpha, m^2, n)]^{\Lambda} = 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right) \qquad \text{(I, 3; 11)}$$
$$\times \lim_{\epsilon\to 0}\{x_1^2+\cdots+x_{n-1}^2-(x_0+iy_0\varepsilon)^2\}^{-\alpha+(n-2)/2}.$$

We remark that in formula (I, 3; 11) appears $\Gamma(\alpha+(n-2)/2)$, this function has simple poles if

$$\alpha+\frac{n-2}{2}=-l, \qquad l=0, 1, 2,\ldots. \qquad \text{(I, 3; 12)}$$

The distribution

$$\{x_1^2+\cdots+x_{n-1}^2-(x_0+iy_0\varepsilon)^2\}^{-\alpha+(n-2)/2} \qquad \text{(I, 3; 13)}$$

was studied by Vladimirov [3, formulas (136), p. 298, and (138), p. 299].

According to whether $2(\alpha+(n-2)/2)$ is even or odd, the distribution is of the form

$$\{x_1^2+\cdots+x_{n-1}^2-(x_0+iy_0\varepsilon)^2\}^{-(\alpha+(n-2)/2)}$$
$$=(-1)^{\alpha+(n-2)/2}\,Pf\frac{1}{\{x_1^2+\cdots+x_{n-1}^2-x_0^2\}^{\alpha+(n-2)/2}}$$
$$-i\Pi\operatorname{sgn}x_0\frac{\delta^{(\alpha+(n-2)/2)}(x_1^2+\cdots+x_{n-1}^2-x_0^2)}{(\alpha+(n-2)/2-1)!}. \qquad \text{(I, 3; 14)}$$

if $2(\alpha+(n-2)/2)$ is even, and

$$\{x_1^2+\cdots+x_{n-1}^2-(x_0-i0)^2\}^{-(\alpha+(n-2)/2)} \qquad \text{(I, 3; 15)}$$
$$=[\theta(Q)Q]^{-(\alpha+(n-2)/2)}-i(-1)^{\alpha+n/2-3/2}\operatorname{sgn}x_0[\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)},$$

if $2(\alpha+(n-2)/2)$ is odd.

Here we have put

$$(x-i0)^{\lambda}\overset{\text{def}}{=}\lim_{\epsilon\to 0}(x-i\varepsilon)^{\lambda}, \qquad \text{(I, 3; 16)}$$

$$Q=x_1^2+\cdots+x_{n-1}^2-x_0^2, \qquad \text{(I, 3; 17)}$$

$$\theta(Q)=1, \quad \text{if } Q>0,$$
$$=0, \quad \text{if } Q<0. \qquad \text{(I, 3; 18)}$$

$$\begin{aligned}\operatorname{sgn} x_0 &= 1, \quad \text{if} \quad x_0 > 0, \\ &= -1, \quad \text{if} \quad x_0 < 0.\end{aligned} \tag{I, 3; 19}$$

Pf = finite part.

We remark that the finite part of $\{x_1^2 + \cdots + x_{n-1}^2 - x_0^2\}^{-(\alpha+(n-2)/2)}$ which appears in formula (I, 3; 14) vanishes in the region $|x_0| = (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$.

Finally we have, from (I, 3; 11) and (I, 3; 14) and the previous remark, that

$$[G_{\bar{C}}(t, \alpha, m^2, n)]^{\Lambda} = 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha + \frac{n-2}{2}\right)$$

$$\times \frac{(-1)\, i\Pi \operatorname{sgn} x_0}{(\alpha + (n-2)/2 - 1)!}\, \delta(Q)^{(\alpha+(n-2)/2-1)}, \tag{I, 3; 20}$$

if $2(\alpha + (n-2)/2)$ is even.

From (I, 3; 11) and (I, 3; 15) it results

$$[G_{\bar{C}}(t, \alpha, m^2, n)] = 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha + \frac{n-2}{2}\right) \tag{I, 3; 21}$$

$$\times \{[\theta(Q)Q]^{-(\alpha+(n-2)/2)} - i(-1)^{\alpha+(n/2)-(3/2)} \operatorname{sgn} x_0[\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)}\},$$

if $2(\alpha + (n-2)/2)$ is odd.

Therefore, by adding the results (I, 3; 5), (I, 3; 6), (I, 3; 7), (I, 3; 20) or (I, 3; 21), we obtain

$$\begin{aligned}
&[G_R(t, \alpha, m^2, n)]^{\Lambda} \\
&= 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(Q)\, \frac{K_{\alpha+(n-2)/2}\{mQ^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&\quad + 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(-Q)\,\theta(x_0) \\
&\quad \times \frac{K_{\alpha+(n-2)/2}\{me^{i(\Pi/2)}(-Q)^{1/2}\}}{e^{i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\quad + 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(-Q)\,\theta(-x_0) \\
&\quad \times \frac{K_{\alpha+(n-2)/2}\{me^{-i(\Pi/2)}(-Q)^{1/2}\}}{e^{-i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\quad + \frac{2^{\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\, \Pi\Gamma(\alpha + (n-2)/2)(-1)\, i \operatorname{sgn} x_0}{(\alpha + (n-2)/2 - 1)!} \\
&\quad \times \delta(Q)^{(\alpha+(n-2)/2-1)},
\end{aligned} \tag{I, 3; 22}$$

if $2(\alpha+(n-2)/2)$ is even and $Q=x_1^2+\cdots+x_{n-1}^2-x_0^2$, and

$$
\begin{aligned}
&[G_R(t,\alpha,m^2,n)]^\Lambda \\
&\quad = 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(Q)\,\frac{K_{\alpha+(n-2)/2}\{mQ^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&\qquad + 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(-Q)\,\theta(x_0) \\
&\qquad \times \frac{K_{\alpha+(n-2)/2}\{me^{i(\Pi/2)}(-Q)^{1/2}\}}{e^{i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\qquad + 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(-Q)\,\theta(-x_0) \\
&\qquad \times \frac{K_{\alpha+(n-2)/2}\{me^{-i(\Pi/2)}(-Q)^{1/2}\}}{e^{-i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\qquad + 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right) \\
&\qquad \times \{[\theta(Q)Q]^{-(\alpha+(n-2)/2)} - i(-1)^{\alpha+(n/2)-(3/2)} \\
&\qquad \times \operatorname{sgn} x_0[\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)}\},
\end{aligned} \tag{I, 3; 23}
$$

if $2(\alpha+(n-2)/2)$ is odd and $Q=x_1^2+\cdots+x_{n-1}^2-x_0^2$.

I.4. *Equivalent Expressions of the Fourier Transform of* $G_R(t,\alpha,m^2,n)$

We shall give in this section an equivalent expression of formulas (I, 3; 22) and (I, 3; 23).

We begin by remembering that

$$\operatorname{sgn} x_0\,\delta(Q)^{(\alpha+(n-4)/2)} = \delta_{C_b}(Q)^{(\alpha+(n-4)/2)} - \delta_{C_f}(Q)^{(\alpha+(n-4)/2)}. \tag{I, 4; 1}$$

We also know that (cf. [9, p. 5, Vol. II, formulas (14)]) that

$$K_\nu(z) = \tfrac{1}{2} i\Pi e^{i(1/2)\nu\Pi} H_\nu^{(1)}(ze^{i(\Pi/2)}), \tag{I, 4; 2}$$

and

$$K_\nu(z) = -\tfrac{1}{2} i\Pi e^{-i(1/2)\nu\Pi} H_\nu^{(2)}(ze^{-i(\Pi/2)}), \tag{I, 4; 3}$$

where

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \tag{I, 4; 4}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z), \tag{I, 4; 5}$$

where

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}, \tag{I, 4; 6}$$

$$Y_\nu(z) = (\sin \nu\Pi)^{-1}[J_\nu(z) \cos \nu\Pi - J_{-\nu}(z)]. \qquad \text{(I, 4; 7)}$$

From (I, 4; 2) and (I, 4; 3), with $\nu = \alpha + (n-2)/2$ and $z = m(-Q)^{1/2}$, we have

$$K_{\alpha+(n-2)/2}\{e^{-i(\Pi/2)}m(-Q)^{1/2}\} = \tfrac{1}{2}i\Pi e^{i(\pi/2)(\alpha+(n-2)/2)}H^{(1)}_{\alpha+(n-2)/2}[m(-Q)^{1/2}] \qquad \text{(I, 4; 8)}$$

and

$$K_{\alpha+(n-2)/2}\{e^{i(\Pi/2)}m(-Q)^{1/2}\} = -\tfrac{1}{2}i\Pi e^{-i(\Pi/2)(\alpha+(n-2)/2)}H^{(2)}_{\alpha+(n-2)/2}[m(-Q)^{1/2}]. \qquad \text{(I, 4; 9)}$$

By substituting into the second and the third summands of the right-hand members of (I, 3; 22) and (I, 3; 23) the functions $K_{\alpha+(n-2)/2}\{e^{-i(\Pi/2)}m(-Q)^{1/2}\}$ and $K_{\alpha+(n-2)/2}\{e^{i(\Pi/2)}m(-Q)^{1/2}\}$ by their equivalent equations (I, 4; 8) and (I, 4; 9), we obtain

$$\begin{aligned}
[G_R(t, \alpha, m^2, n)]^\Lambda
&= 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(Q)\, \frac{K_{\alpha+(n-2)/2}\{m(Q)^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&\quad + 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}e^{-i(\Pi/2)(\alpha+(n-2)/2)} \\
&\quad \times (-\tfrac{1}{2})\, i\Pi e^{-i(\pi/2)(\alpha+(n-2)/2)} \\
&\quad \times \theta(-Q)\,\theta(x_0)\, \frac{H^{(2)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\quad + 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}e^{i(\Pi/2)(\alpha+(n-2)/2)} \cdot \tfrac{1}{2}i\Pi e^{i(\pi/2)(\alpha+(n-2)/2)} \\
&\quad \times \theta(-Q)\,\theta(-x_0)\, \frac{H^{(1)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&\quad + 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}(-i\Pi) \\
&\quad \times [\delta_{C_b}(Q)^{(\alpha+(n-4)/2)} - \delta_{C_f}(Q)^{(\alpha+(n-4)/2)}],
\end{aligned} \qquad \text{(I, 4; 10)}$$

if $2(\alpha + (n-2)/2)$ is even and $Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2$; and

$$\begin{aligned}
[G_R(t, \alpha, m^2, n)]^\Lambda
&= 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\theta(Q)\, \frac{K_{\alpha+(n-2)/2}\{m(Q)^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&\quad + \theta(-Q)\,\theta(x_0)\, 2^\alpha(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}e^{-i(\Pi/2)(\alpha+(n-2)/2)}
\end{aligned}$$

$$\times \left(-\tfrac{1}{2}\right) i\Pi e^{-i(\Pi/2)(\alpha+(n-2)/2)} \frac{H^{(2)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{(\{-Q\}^{1/2})^{\alpha+(n-2)/2}}$$

$$+ \theta(-Q)\,\theta(-x_0)\,2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} e^{i(\Pi/2)(\alpha+(n-2)/2)}$$

$$\times \tfrac{1}{2} i\Pi e^{i(\Pi/2)(\alpha+(n-2)/2)} \frac{H^{(1)}_{\alpha+(n-2)/2}(m(-Q)^{1/2})}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}}$$

$$+ 2^{\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right)$$

$$\times \{[\theta(Q)Q]^{-(\alpha+(n-2)/2)} - i(-1)^{\alpha+(n/2)-(3/2)}$$

$$\times \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)}\}, \qquad \text{(I, 4; 11)}$$

if $2(\alpha+(n-2)/2)$ is odd and $Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2$.

II. Particular Cases of Formulas (I, 4; 10) and (I, 4; 11)

II.1. *The Fourier Transform of* $G_R(t, \alpha = 0, m^2, n = 4)$

If we put $\alpha = 0$, $n = 4$ in (I, 4; 10) we get

$$[G_R(t, \alpha = 0, m^2, n = 4)]^{\Lambda} = [\delta_{C_b}(u - m^2)]^{\Lambda}$$

$$= \theta(Q)\, 2\Pi m \frac{K_1\{m(Q)^{1/2}\}}{(Q)^{1/2}}$$

$$+ \theta(-Q)\,\Pi^2 i m \left[\theta(x_0) \frac{H_1^{(2)}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}}\right.$$

$$\left. - \theta(-x_0) \frac{H_1^{(1)}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}}\right]$$

$$+ 2\Pi^2 i [\delta_{C_f}(Q) - \delta_{C_b}(Q)]. \qquad \text{(II, 1; 1)}$$

Formula (II, 1; 1) coincides with formula (5.21) in [10, p. 141].

II.2. *The Fourier Transform of* $G_R(t, \alpha = 1, m^2, n)$

If we put $\alpha = 1$ in (I, 2; 1), we obtain

$$G_R(t, \alpha = 1, m^2, n) = 1, \qquad \text{if} \quad u - m^2 > 0 \text{ and } t_0 > 0,$$

$$= 0, \qquad \text{if} \quad t \text{ belongs to the complementary set.}$$

(II, 2; 1)

Formula (II, 2; 1) defines the characteristic function of the volume bounded by the forward sheet of the hyperboloid $u = m^2$.

We shall evaluate its Fourier transform. Putting $\alpha = 1$ in formulas (I, 4; 10) and (I, 4; 11), we get

$$\begin{aligned}
&[G_R(t, \alpha = 1, m^2, n)]^\Lambda \\
&= \theta(Q)\, 2(2\Pi)^{(n-2)/2}\, m^{n/2} \frac{K_{n/2}\{m(Q)^{1/2}\}}{\{Q^{1/2}\}^{n/2}} \\
&\quad - \theta(-Q)\, \theta(x_0)(2\Pi)^{(n-2)/2}\, m^{n/2} i\Pi e^{-i(\Pi/2)n} \frac{H^{(2)}_{n/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{n/2}} \\
&\quad + \theta(-Q)\, \theta(-x_0)(2\Pi)^{(n-2)/2}\, m^{n/2} i\Pi e^{i(\Pi/2)n} \frac{H^{(1)}_{n/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{n/2}} \\
&\quad - (2\Pi)^{(n-2)/2}\, 2^{(n-2)/2} i\Pi[\delta_{C_f}(Q)^{((n-2)/2)} - \delta_{C_b}(Q)^{((n-2)/2)}], \qquad \text{(II, 2; 2)}
\end{aligned}$$

if n is even and

$$\begin{aligned}
&[G_R(t, \alpha = 1, m^2, n)]^\Lambda \\
&= \theta(Q)\, 2(2\Pi)^{(n-2)/2}\, m^{n/2} \frac{K_{n/2}\{m(Q)^{1/2}\}}{(Q^{1/2})^{n/2}} \\
&\quad - \theta(-Q)\, \theta(x_0)(2\Pi)^{(n-2)/2}\, m^{n/2} i\Pi e^{-i(\Pi/2)n} \frac{H^{(2)}_{n/2}\{m(-Q)^{1/2}\}}{\{(-Q)^{1/2}\}^{n/2}} \\
&\quad + \theta(-Q)\, \theta(-x_0)(2\Pi)^{(n-2)/2}\, m^{n/2} i\Pi e^{i(\Pi/2)n} \frac{H^{(1)}_{n/2}\{m(-Q)^{1/2}\}}{\{(-Q)^{1/2}\}^{n/2}} \\
&\quad + 2^{n/2}(2\Pi)^{(n-2)/2}\Gamma\left(\frac{n}{2}\right) \{[\theta(Q)\, Q]^{-n/2} \\
&\quad - i(-1)^{(n-3)/2} \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-n/2}\}, \qquad \text{(II, 2; 3)}
\end{aligned}$$

if n is odd.

II.3. *The Fourier Transform of* $G_R(t, \alpha = -k, m^2, n) = \delta_R^{(k)}(u - m^2)$

By putting $\alpha = -k$ in formula (I, 2; 1) and taking into account the formula (cf. [11])

$$\left\{\frac{(x - m^2)_+^{\alpha - 1}}{\Gamma(\alpha)}\right\}_{\alpha = -k} = \delta_{m^2}^{(k)}, \qquad \text{(II, 3; 1)}$$

where $k = 0, 1, 2, \ldots$, we obtain

$$G_R(t, \alpha = -k, m^2, n) = \delta_R^{(k)}(u - m^2). \qquad \text{(II, 3; 2)}$$

We shall evaluate the Fourier transform of the derivative of k-order of the delta on the hyperboloid by putting $\alpha=-k$ in formulas (I, 4; 10) and (I, 4; 11).

Therefore, we have, if $-2k+n-2$ is an even number,

$$
\begin{aligned}
&[G_R(t, \alpha=-k, m^2, n)]^\Lambda \\
&\quad = [\delta_R^{(k)}(u-m^2)]^\Lambda \\
&\quad = \theta(Q)\, 2^{-k}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2}\, \frac{K_{-k+(n-2)/2}\{m(Q)^{1/2}\}}{\{(Q)^{1/2}\}^{-k+(n-2)/2}} \\
&\qquad + \theta(-Q)\, \theta(x_0)\, 2^{-k}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2} e^{-i\Pi(-k+(n-2)/2)} \\
&\qquad \times (-\tfrac{1}{2})\, i\Pi\, \frac{H^{(2)}_{-k+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2})^{-k+(n-2)/2}} \\
&\qquad + \theta(-Q)\, \theta(-x_0)\, 2^{-k}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2} e^{i\Pi(-k+(n-2)/2)} \\
&\qquad \times \tfrac{1}{2} i\Pi\, \frac{H^{(1)}_{-k+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{-k+(n-2)/2}} \\
&\qquad + 2^{-2k+(n-4)/2}(2\Pi)^{(n-2)/2}(-i\Pi)\, \operatorname{sgn} x_0\, \delta(Q)^{(-k+(n-4)/2)}. \qquad \text{(II, 3; 3)}
\end{aligned}
$$

If $-2k+n-2$ is odd, we have

$$
\begin{aligned}
&\{G_R(t, \alpha=-k, m^2, n)\}^\Lambda \\
&\quad = [\delta_R^{(k)}(u-m^2)]^\Lambda \\
&\quad = \theta(Q)\, 2^{-k}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2}\, \frac{K_{-k+(n-2)/2}\{m(Q)^{1/2}\}}{[(Q)^{1/2}]^{-k+(n-2)/2}} \\
&\qquad - \theta(-Q)\, \theta(x_0)\, 2^{-k-1}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2} e^{-i\Pi(-k+(n-2)/2)} i\Pi \\
&\qquad \times \frac{H^{(2)}_{-k+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{-k+(n-2)/2}} \\
&\qquad + \theta(-Q)\, \theta(-x_0)\, 2^{-k-1}(2\Pi)^{(n-2)/2}\, m^{-k+(n-2)/2} e^{i\Pi(-k+(n-2)/2)} i\Pi \\
&\qquad \times \frac{H^{(1)}_{-k+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{-k+(n-2)/2}} \\
&\qquad + 2^{-2k+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(-k+\frac{n-2}{2}\right) \\
&\qquad \times [\{\theta(Q)Q\}^{-(-k+(n-2)/2)} + e^{i\Pi(-k+(n-2)/2)} \\
&\qquad \times \operatorname{sgn} x_0 \{\theta(-Q)(-Q)\}^{-(-k+(n-2)/2)}]. \qquad \text{(II, 3; 4)}
\end{aligned}
$$

Remark. Formula (II, 3; 4) requires, for its validity, that

$$-k+\frac{n-2}{2}\neq -l, \qquad l=0, 1,...; \tag{II, 3; 5}$$

and this condition always is verified because n is odd.

III. The Fourier Transform of $G_R(t, m=0, \alpha, n)=u_+^{\alpha-1}/\Gamma(\alpha)$

We shall consider the particular case of formula (I, 2; 1) when $m=0$, we get

$$\begin{aligned} G_R(t, m^2=0, \alpha, n) &= \frac{u_+^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{u^{\alpha-1}}{\Gamma(\alpha)}, \qquad \text{if} \quad u>0 \text{ and } t_0>0, \\ &= 0, \qquad\qquad \text{if} \quad t \text{ belongs to the complementary set.} \end{aligned} \tag{III, 1; 1}$$

The Laplace transform of the function defined by (II, 4; 1) is, taking into account formula (II, 4; 6) of [7, p. 13],

$$\begin{aligned} &L[G_R(t, m=0, \alpha, n)] \\ &\quad = (2\Pi)^{(n-2)/2}\, 2^{\alpha+(n-4)/2} \rho^{-2\alpha+2-n}\Gamma\left(\alpha+\frac{n-2}{2}\right), \end{aligned} \tag{III, 1; 2}$$

valid if

$$\alpha+\frac{n-2}{2}\neq -l, \qquad l=0, 1,... . \tag{III, 1; 3}$$

To evaluate the Fourier transform of $G_R(t, m=0, \alpha, n)$ we shall proceed as before, that is to say, passing to the limit (in S'), on the Laplace transform, for $y\to 0$, where $y\in V_-$.

We obtain ($\varepsilon>0$, $y_0<0$)

$$\begin{aligned} &[G_R(t, m=0, \alpha, n)]^{\Lambda} \\ &\quad = (2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right) \\ &\qquad \times \lim_{\epsilon\to 0}\{x_1^2+\cdots+x_{n-1}^2-(x_0+iy_0\varepsilon)^2\}^{(-2\alpha+2-n)/2}. \end{aligned} \tag{III, 1; 4}$$

Taking into account [3, formulas (136), p. 298, and (138), p. 299], we obtain

$$[G_R(t, m=0, \alpha, n)]^{\Lambda} = (2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right)$$
$$\times \left\{\frac{-i\Pi}{(\alpha+n/2-2)!}\operatorname{sgn} x_0\, \delta(Q)^{(\alpha+(n-4)/2)}\right.$$
$$\left. + (-1)^{\alpha+(n-2)/2}\operatorname{Pf}\frac{1}{(-Q)^{\alpha+(n-2)/2}}\right\}, \qquad \text{(III, 1; 5)}$$

if $2\alpha + n - 2$ is even, and

$$[G_R(t, m=0, \alpha, n)]^{\Lambda}$$
$$= (2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right)$$
$$\times \{[\theta(Q)Q]^{-(\alpha+n/2-3/2)-1/2} - i(-1)^{\alpha+n/2-3/2}$$
$$\times \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-(\alpha+n/2-3/2)-1/2}\}, \qquad \text{(III, 1; 6)}$$

if $2\alpha - 2 + n$ is odd.

IV. Particular Cases of the Fourier Transform of $G_R(t, m=0, \alpha, n)$

IV.1. *The Fourier Transform of $G_R(t, \alpha=1, m=0, n)$. The Characteristic Function of the Volume Bounded by the Forward Cone*

We shall consider two particular cases of formulas (III, 1; 5) and (III, 1; 6) when $\alpha = 1$ and $\alpha = -k$.

We begin by remembering that the function $G_R(t, \alpha, m^2, n)$ is, for $\alpha = 1$, $m = 0$, the characteristic function of the volume bounded by the forward cone:

$$G_R(t, \alpha=1, m=0, n) = 1, \quad \text{if} \quad u > 0,\ t_0 > 0,$$
$$= 0, \quad \text{if} \quad t \text{ belongs to the complementary set.}$$
$$\text{(IV, 1; 1)}$$

and, for $\alpha = -k$, we get

$$G_R(t, \alpha=-k, m=0, n) = \delta_R^{(k)}(u). \qquad \text{(IV, 1; 2)}$$

By putting $\alpha = 1$ in formulas (III, 1; 5) and (III, 1; 6), we obtain

$$[G_R(t, m = 0, \alpha = 1, n)]^{\Lambda}$$
$$= (2\Pi)^{(n-2)/2}\, 2^{n/2} \Gamma\left(\frac{n}{2}\right)$$
$$\times \left\{ \frac{-i\Pi}{(n/2-1)!} \operatorname{sgn} x_0 \delta(Q)^{((n/2)-1)} + (-1)^{n/2} \operatorname{Pf} \frac{1}{(-Q)^{n/2}} \right\}, \quad \text{(IV, 1; 3)}$$

if n is even, and

$$[G_R(t, m = 0, \alpha = 1, n)]^{\Lambda}$$
$$= 2^{n-1} \Pi^{n/2} \Gamma\left(\frac{n}{2}\right)$$
$$\times \{[\theta(Q)(Q)]^{-n/2} - i(-1)^{n/2-1/2} \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-n/2}\}, \quad \text{(IV, 1; 4)}$$

if n is odd.

IV.2. *The Fourier Transform of* $G_R(t, \alpha = -k, m = 0, n) = \delta_R^{(k)}(u)$

Putting $\alpha = -k$ in (III, 1; 5) and (III, 1; 6), we get

$$[G_R(t, m = 0, \alpha = -k, n)]^{\Lambda}$$
$$= [\delta_R^{(k)}(u)]^{\Lambda}$$
$$= (2\Pi)^{(n-2)/2}\, 2^{-2k+(n-4)/2} \Gamma\left(-k + \frac{n-2}{2}\right)$$
$$\times \left\{ \frac{-i\Pi}{(-k+n/2-2)!} \operatorname{sgn} x_0 \delta(Q)^{(-k+(n-4)/2)} \right.$$
$$\left. + (-1)^{-k+(n-2)/2} \operatorname{Pf} \frac{1}{(-Q)^{-k+(n-2)/2}} \right\}, \quad \text{(IV, 2; 1)}$$

where n is even and $-k + (n-2)/2 \neq -l$, $l = 0, 1, \ldots$; and

$$[G_R(t, m = 0, \alpha = -k, n)]^{\Lambda}$$
$$= [\delta_R^{(k)}(u)]^{\Lambda}$$
$$= (2\Pi)^{(n-2)/2}\, 2^{-2k+(n-4)/2} \Gamma\left(-k + \frac{n-2}{2}\right)$$
$$\times \{[\theta(Q)(Q)]^{-(-k+n/2-3/2)-1/2} - i(-1)^{-k+n/2-3/2}$$
$$\times \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-(-k+n/2-3/2)-1/2}\}, \quad \text{(IV, 2; 2)}$$

where n is odd.

Formulas (IV, 1; 5) and (IV, 1; 6), putting $k=0$ and dividing by $(2\Pi)^{n-1}$, coincide with formulas (142), p. 300, of [3].

V

V.1. *Equivalence of $[\delta_R^{(k)}(u-m^2)]^\Lambda$, when $n=4$, $k=0$ with the Formulas Due to Lavoine and Schwartz*

Putting $\alpha=0$, $n=4$ in (II, 3; 3), we obtain

$$[\delta_R(u-m^2)]^\Lambda = \theta(Q)\, 2\Pi m \frac{K_1\{m(Q)^{1/2}\}}{(Q)^{1/2}} - 2\Pi^2 i \operatorname{sgn} x_0 \delta(Q) + \theta(-Q)\, \Pi^2 m i \left\{ \theta(x_0) \frac{H_1^{(2)}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}} - \theta(-x_0) \frac{H_1^{(1)}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}} \right\}. \qquad (\text{V}, 1; 1)$$

By substituting, in the right-hand member of (V, 1; 1), the functions $H_1^{(1)}$ and $H_1^{(2)}$ by their equivalent expressions (I, 4; 4) and (I, 4; 5), with $z=m(-Q)^{1/2}$ and remembering that

$$\theta(x_0) - \theta(-x_0) = \operatorname{sgn} x_0, \qquad (\text{V}, 1; 2)$$

and

$$\theta(x_0) + \theta(-x_0) = 1; \qquad (\text{V}, 1; 3)$$

we obtain

$$[\delta_R(u-m^2)]^\Lambda = \theta(Q)\, 2\Pi m \frac{K_1\{m(Q)^{1/2}\}}{Q^{1/2}} - 2\Pi^2 i \operatorname{sgn} x_0 \delta(Q) + \theta(-Q)\, \Pi^2 m i \operatorname{sgn} x_0 \frac{J_1\{m(-Q)^{1/2}\}}{(-Q)^{1/2}} + \theta(-Q)\, \Pi^2 m \frac{Y_1\{m(-Q)^{1/2}\}}{(-Q)^{1/2}}, \qquad (\text{V}, 1; 4)$$

where $J_\nu(z)$ and $Y_\nu(z)$ are defined by formulas (I, 4; 6) and (I, 4; 7).

Formula (V, 1; 4) is equivalent to the formula due to Lavoine [2, p. 63].

Formula (V, 1; 4) coincides, also, with the formula due to Vladimirov [3, pp. 86–88].

Formula (V, 1; 4) is equivalent to the formula (15.10), [13, p. 126].

Finally, we remark that formula (V, 1; 1) is equivalent to a formula due to Constantinescu [17, p. 121, formula (11.55)].

V.2. *The Fourier Transform of* $G_A(t, \alpha, m^2, n)$

We define the n-dimensional function

$$G_A(t, \alpha, m^2, n) = \frac{(u-m^2)_-^{\alpha-1}}{\Gamma(\alpha)}$$

$$= \frac{(u-m^2)^{\alpha-1}}{\Gamma(\alpha)}, \qquad \text{if} \quad u - m^2 > 0 \text{ and } t_0 < 0, \qquad \text{(V, 2; 1)}$$

$$= 0, \qquad \text{if} \quad t \text{ belongs to the complementary set.}$$

Here m is a real nonnegative number and α is a complex parameter.

Taking into account formula (II, 4; 5) and the final phrase of II.6 of [7], we have

$$L[G_A(t, \alpha, m^2, n)] = 2^\alpha (2\Pi)^{(n-2)/2} m^{\alpha+(n-2)/2} \rho^{-\alpha+(n-2)/2} K_{\alpha+(n-2)/2}(m\rho), \qquad \text{(V, 2; 2)}$$

where $\operatorname{Im} z_0 = y_0 > 0$.

We shall evaluate the Fourier transform of $G_A(t, \alpha, m^2, n)$ in the same manner as we evaluate the Fourier transform of $G_R(t, \alpha, m^2, n)$, in this case,

$$\lim_{\substack{y \to 0 \\ y \in V^+}} \rho^2 = \lim_{\varepsilon \to 0} \{x_1^2 + \cdots + x_{n-1}^2 - (x_0 + iy_0\varepsilon)^2\}, \qquad \text{(V, 2; 3)}$$

where $\varepsilon > 0$ and $y_0 > 0$.

Therefore, we obtain

$$[G_4(t, \alpha, m^2, n)]^\Lambda$$

$$= \theta(Q)\, 2^\alpha (2\Pi)^{(n-2)/2} m^{\alpha+(n-2)/2} \frac{K_{\alpha+(n-2)/2}\{m(Q)^{1/2}\}}{[(Q)^{1/2}]^{\alpha+(n-2)/2}}$$

$$+ \theta(-Q)\, \theta(x_0)\, 2^\alpha (2\Pi)^{(n-2)/2} m^{\alpha+(n-2)/2}$$

$$\times \frac{K_{\alpha+(n-2)/2}\{me^{-i(\Pi/2)}(-Q)^{1/2}\}}{e^{-i(\Pi/2)(\alpha+(n-2)/2)}\{(-Q)^{1/2}\}^{\alpha+(n-2)/2}}$$

$$+ \theta(-Q)\, \theta(-x_0)\, 2^\alpha (2\Pi)^{(n-2)/2} m^{\alpha+(n-2)/2}$$

$$\times \frac{K_{\alpha+(n-2)/2}\{me^{i(\Pi/2)}(-Q)^{1/2}\}}{e^{i(\Pi/2)(\alpha+(n-2)/2)}\{(-Q)^{1/2}\}^{\alpha+(n-2)/2}}$$

$$+ 2^{2\alpha+(n-4)/2} (2\Pi)^{(n-2)/2}\, i\Pi \operatorname{sgn} x_0\, \delta(Q)^{(\alpha+(n-4)/2)}, \qquad \text{(V, 2; 4)}$$

if $2(\alpha + (n-2)/2)$ is even, and

$$\begin{aligned}
&[G_A(t, \alpha, m^2, n)]^{\Lambda} \\
&= \theta(Q)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \frac{K_{\alpha+(n-2)/2}\{m(Q)^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&+ \theta(-Q)\,\theta(x_0)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \\
&\times \frac{K_{\alpha+(n-2)/2}\{me^{-i(\Pi/2)}(-Q)^{1/2}\}}{e^{-i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&+ \theta(-Q)\,\theta(-x_0)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \\
&\times \frac{K_{\alpha+(n-2)/2}\{me^{i(\Pi/2)}(-Q)^{1/2}\}}{e^{i(\Pi/2)(\alpha+(n-2)/2)}[(-Q)^{1/2}]} \\
&+ 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\Gamma\left(\alpha + \frac{n-2}{2}\right) \\
&\times \{[\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)} - e^{-i(\Pi/2)(\alpha+(n-2)/2)} \\
&\times \operatorname{sgn} x_0[\theta(Q)Q]^{-(\alpha+(n-2)/2)}\},
\end{aligned} \tag{V, 2; 5}$$

if $2(\alpha + (n-2)/2)$ is odd.

V.3. *An Equivalent Expression of the Fourier Transform of* $G_A(t, \alpha, m^2, n)$

We shall express formulas (V, 2; 4) and (V, 2; 5) in a different manner. Taking into account formulas (I, 4; 8) and (I, 4; 9), it follows that

$$\begin{aligned}
&[G_A(t, \alpha, m^2, n)]^{\Lambda} \\
&= \theta(Q)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \frac{K_{\alpha+(n-2)/2}\{m(Q)^{1/2}\}}{(Q^{1/2})^{\alpha+(n-2)/2}} \\
&+ \theta(-Q)\,\theta(x_0)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\tfrac{1}{2} i\Pi e^{i\Pi(\alpha+(n-2)/2)} \\
&\times \frac{H^{(1)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}} \\
&+ \theta(-Q)\,\theta(-x_0)\, 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}(-\tfrac{1}{2})\, i\Pi e^{-i\Pi(\alpha+(n-2)/2)} \\
&\times \frac{H^{(2)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{[(-Q)^{1/2}]^{\alpha+((n-2)/2)}} + A(\alpha, n, Q),
\end{aligned} \tag{V, 3; 1}$$

where

$$A(\alpha, n, Q) = 2^{2\alpha+(n-4)/2}(2\Pi)^{(n-2)/2}\, i\Pi \operatorname{sgn} x_0\, \delta(Q)^{(\alpha+(n-4)/2)}, \tag{V, 3; 2}$$

if $2(\alpha + (n-2)/2)$ is even, and

$$A(\alpha, n, Q) = 2^{2\alpha + (n-4)/2}(2\Pi)^{(n-2)/2}\{[\theta(-Q)(-Q)]^{-(\alpha+(n-2)/2)} - e^{-i\Pi(\alpha+(n-2)/2)} \operatorname{sgn} x_0[\theta(Q)Q]^{-(\alpha+(n-2)/2)}, \quad (V, 3; 3)$$

if $2(\alpha + (n-2)/2)$ is odd.

V.4. *Equivalence of* $[\delta_A^{(k)}(u - m^2)]^\Lambda$, *when* $n = 4$, $k = 0$ *with the Formula Due to Bogoliubov and Chirkov*

Putting $\alpha = 0$, $n = 4$ in (V, 3; 1) and (V, 3; 2), we obtain

$$[\delta_A(u-m^2)]^\Lambda = \theta(Q)\,2\Pi m\,\frac{K_1\{m(Q)^{1/2}\}}{Q^{1/2}} + \theta(-Q)\,\theta(x_0)(-i)\,\Pi^2 m\,\frac{H_1^{(1)}[m(-Q)^{1/2}]}{(-Q)^{1/2}} + \theta(-Q)\,\theta(-x_0)\,i\Pi^2 m\,\frac{H_1^{(2)}[m(-Q)^{1/2})}{(-Q)^{1/2}} + 2\Pi^2 i \operatorname{sgn} x_0 \delta(Q). \quad (V, 4; 1)$$

Formula (V, 4; 1) coincides with formula (5.20), p. 141, of [10].

By substituting the functions $H_1^{(1)}$ and $H_1^{(2)}$ by their equivalent expressions (I, 4; 4) and (I, 4; 5) and dividing both members of (V, 4; 1) by $(2\Pi)^3 i$, we get

$$\left\{\frac{1}{8\Pi^3 i}\delta_A(u-m^2)\right\}^\Lambda = -\frac{\theta(Q)}{4\Pi^2} mi\,\frac{K_1[m(Q)^{1/2}]}{Q^{1/2}} + \frac{1}{4\Pi}\operatorname{sgn} x_0 \delta(Q) - \frac{\theta(-Q)}{8\Pi}\operatorname{sgn} x_0\,\frac{J_1[m(-Q)^{1/2}]}{(-Q)^{1/2}} - \frac{\theta(-Q)}{8\Pi}\, i\,\frac{Y_1[m(-Q)^{1/2})}{(-Q)^{1/2}}. \quad (V, 4; 2)$$

Formula (V, 5; 1) is equivalent to formula (15.10), [12, p. 126].

VI

VI.1. *The Fourier Transform of a Marcel Riesz Kernel* $W(t, \alpha, m^2, n)$

We shall consider the following functions of the family R introduced by Riesz [4, p. 17] (cf. also [5, p. 89; 1, p. 179; and 6, p. 72]):

$$W(t, \alpha, m^2, n) = \frac{(m^{-2}u)^{(\alpha-n)/4}}{\Pi^{(n-2)/2} 2^{(2\alpha+n-2)/2} \Gamma(\alpha/2)} J_{(\alpha-n)/2}\{\sqrt{m^2 u}\}, \quad \text{if } t \in \Gamma_+,$$
$$= 0, \quad \text{if } t \notin \Gamma_+. \qquad \text{(VI, 1; 1)}$$

Here α is a complex parameter, m a real nonnegative number and n the dimension of the space.

$W(t, \alpha, m^2, n)$, which is an ordinary function if $\operatorname{Re} \alpha \geqslant n$, is an entire distributional function of α.

The Laplace transform of $W(t, \alpha, m^2, n)$ is, taking into account formula (II, 1; 3) of [7, p. 10],

$$L[W(t, \alpha, m^2, n)] = (\rho^2 + m^2)^{-\alpha/2}. \qquad \text{(VI, 1; 2)}$$

This formula is valid for $\operatorname{Re} \alpha > 2n - 4$ and $\operatorname{Re} \rho > 0$, this last condition effectively holds as a consequence of our assumption that $z \in T_-$. From this, $\rho^2 + m^2$ never vanishes and we conclude, by appealing to the principle of analytical in continuation, that (VI, 1; 2) is valid for every α.

We remember that

$$\rho^2 = z_1^2 + \cdots + z_{n-1}^2 - z_0^2. \qquad \text{(VI, 1; 3)}$$

To evaluate the Fourier transform of $W(t, \alpha, m^2, n)$ we proceed in a manner analogous to that of the previous paragraphs. Therefore, we obtain

$$[W(t, \alpha, m^2, n)]^\wedge = \theta(Q)(Q + m^2)^{-\alpha/2} + \theta(-Q)\,\theta(x_0)\, e^{-i\Pi(\alpha/2)}(-Q + m^2)^{-\alpha/2}$$
$$+ \theta(-Q)\,\theta(-x_0)\, e^{i\Pi(\alpha/2)}(-Q + m^2)^{-(\alpha/2)}, \qquad \text{(VI, 1; 4)}$$

where $Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2$.

VI.2. *Particular Case of* $[W(t, \alpha, m^2, n)]$, *when* $m = 0$ *and the Equivalence with a Formula Due to Schwartz*

Putting $m = 0$ in formula (VI, 1; 1), we obtain (cf. [7, formula (II, 3; 1), p. 11])

$$W(t, \alpha, m = 0, n) = R_\alpha(u) = \frac{u^{(\alpha-n)/2}}{H_n(\alpha)}, \quad \text{if } t \in \Gamma_+,$$
$$= 0, \quad \text{if } t \notin \Gamma_+. \qquad \text{(VI, 2; 1)}$$

Here we have put

$$H_n(\alpha) = \Pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha - n + 2}{2}\right). \qquad \text{(VI, 2; 2)}$$

The $R_\alpha(u)$ were introduced by Riesz [5, p. 31]. We obtain the Laplace transform of $R_\alpha(u)$ by putting $m=0$ in (VI, 1; 2), we arrive at the formula

$$L[R_\alpha(u)] = (\rho^2)^{-\alpha/2}. \qquad \text{(VI, 2; 3)}$$

We obtain the Fourier transform of $R_\alpha(u)$ immediately, putting $m=0$ in (VI, 1; 4).

It follows that

$$[R_\alpha(u)]^\Lambda = \theta(Q)\, Q^{-\alpha/2} + \theta(-Q)\,\theta(x_0)\, e^{-i\Pi(\alpha/2)}(-Q)^{-\alpha/2} + \theta(-Q)\,\theta(-x_0)\, e^{i\Pi(\alpha/2)}(-Q)^{-\alpha/2}. \qquad \text{(VI, 2; 4)}$$

Formula (VI, 2; 4) coincides with formula (VII, 7; 8) [1, p. 264].

Remark. Formulas (VI, 1; 4) and (VI, 2; 4) must be interpreted in different ways according to whether α is, or is not, an exceptional value.

In this section we obtain the formulas in the case that α is not an exceptional value. The particular case when α is an exceptional value will be studied in Sections XI–XIII.

VII

VII.1. *The Fourier Transform of* $G(t, \alpha, m^2, n) = G_R(t, \alpha, m^2, n) + G_A(t, \alpha, m^2, n)$

We shall define the function $G(t, \alpha, m^2, n)$ by

$$G(t, \alpha, m^2, n) \overset{\text{def}}{=} G_R(t, \alpha, m^2, n) + G_A(t, \alpha, m^2, n), \qquad \text{(VII, 1; 1)}$$

where G_R and G_A are defined by formulas (I, 2; 1) and (V, 2; 1), respectively.

Its Fourier transform will be evaluated by adding the Fourier transforms of G_R and G_A.

Therefore we have, taking into account (I, 4; 10) ((I, 4; 11)) and (V, 3; 1) ((V, 3; 2)), the formula

$$[G(t, \alpha, m^2, n)]^\Lambda = A + B + C, \qquad \text{(VII, 1; 2)}$$

where

$$A = \theta(Q)\, 2^{\alpha+1}(2\Pi)^{(n-2)/2} \frac{m^{\alpha+(n-2)/2}}{(Q^{1/2})^{\alpha+(n-2)/2}} K_{\alpha+(n-2)/2}\{mQ^{1/2}\}, \qquad \text{(VII, 1; 3)}$$

$$B = \theta(-Q)\, 2^{\alpha-1}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2}\Pi i e^{i\Pi(\alpha+(n-2)/2)} \times \frac{H^{(1)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}}, \qquad \text{(VII, 1; 4)}$$

$$C = -\theta(-Q)\, 2^{\alpha-1}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} i\Pi e^{-i\Pi(\alpha+(n-2)/2)} \times \frac{H^{(2)}_{\alpha+(n-2)/2}\{m(-Q)^{1/2}\}}{(-Q)^{1/2}}. \quad \text{(VII, 1; 5)}$$

Remembering that the following formulas are valid (cf. formulas (9), p. 4, [9, vol. II; 11, 289–290]):

$$H^{(1)}_{-(\alpha+(n-2)/2)}[m(-Q)^{1/2}] = e^{i(\alpha+(n-2)/2)\Pi} H^{(1)}_{\alpha+(n-2)/2}[m(-Q)^{1/2}], \quad \text{(VII, 1; 6)}$$

$$H^{(2)}_{-(\alpha+(n-2)/2)}[m(-Q)^{1/2}] = e^{-i(\alpha+(n-2)/2)\Pi} H^{(2)}_{\alpha+(n-2)/2}[m(-Q)^{1/2}], \quad \text{(VII, 1; 7)}$$

$$\frac{K_{\alpha+(n-2)/2}[m(Q-i0)^{1/2}]}{(Q-i0)^{(1/2)(\alpha+(n-2)/2)}} = \frac{K_{\alpha+(n-2)/2}(mQ^{1/2})}{(Q^{1/2})^{\alpha+(n-2)/2}} + \frac{\Pi}{2} i \frac{H^{(1)}_{-(\alpha+(n-2)/2)}(m(-Q)^{1/2})}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}}, \quad \text{(VII, 1; 8)}$$

$$\frac{K_{\alpha+(n-2)/2}[m(Q+i0)^{1/2}]}{(Q+i0)^{(1/2)(\alpha+(n-2)/2)}} = \frac{K_{\alpha+(n-2)/2}(mQ^{1/2})}{(Q^{1/2})^{\alpha+(n-2)/2}} - \frac{\Pi}{2} i \frac{H^{(2)}_{-(\alpha+(n-2)/2)}(m(-Q)^{1/2})}{[(-Q)^{1/2}]^{\alpha+(n-2)/2}}. \quad \text{(VII, 1; 9)}$$

Taking into account formulas (VII, 1; 6)–(VII, 1; 9) we finally obtain

$$[G_R(t, \alpha, m^2, n)]^{\Lambda} \quad \text{(VII, 1; 10)}$$
$$= 2^{\alpha}(2\Pi)^{(n-2)/2}\, m^{\alpha+(n-2)/2} \times \left\{ \frac{K_{\alpha+(n-2)/2}\{m(Q-i0)^{1/2}\}}{(Q-i0)^{(1/2)(\alpha+(n-2)/2)}} + \frac{K_{\alpha+(n-2)/2}\{m(Q+i0)^{1/2}\}}{(Q+i0)^{(1/2)(\alpha+(n-2)/2)}} \right\}.$$

VII.2. *The Particular Cases of* $[G(t, \alpha, m^2, n)]$ *when* $\alpha = 0$ *and* $m = 0$. *The Equivalence between the* $[\delta(u - m^2)]^{\Lambda}$ *due to Gelfand and Our Formula*

Putting $\alpha = 0$ in formula (VII, 1; 10) we obtain

$$[G(t, \alpha = 0, m^2, n)]^{\Lambda} = [\delta(u - m^2)]^{\Lambda} = 2^{n/2-1} m^{n/2-1} \Pi^{n/2-1} \times \left\{ \frac{K_{n/2-1}\{m(Q-i0)^{1/2}\}}{[(Q-i0)^{1/2}]^{n/2-1}} + \frac{K_{n/2-1}\{m(Q+i0)^{1/2}\}}{[(Q+i0)^{1/2}]^{n/2-1}} \right\}. \quad \text{(VII, 2; 1)}$$

Formula (VII, 2; 1) coincides with formula (7), p. 294, of [11].

Remarks. (1) Putting $m = 0$ in formula (VII, 2; 1) we obtain

$$[G(t, \alpha, m = 0, n)]^{\Lambda} = \left(\frac{u^{\alpha - 1}}{\Gamma(\alpha)}\right)^{\Lambda}. \qquad \text{(VII, 2; 2)}$$

(2) We can also obtain the Fourier transform of $G_R(t, \alpha, m = 0, n)$ by putting, directly, $m = 0$ in the Laplace transform of $G_R(t, \alpha, m, n)$.

VIII

VIII.1. *The Equivalence between the $[\delta_R^{(k)}(u)]^{\Lambda}$ Due to Methée and Our Formula, when n Is Even*

In this section we shall prove the equivalence between the Fourier transform of $\delta^{(k)}(u)$ (when k is a regular or a singular point) due to Methée [14, p. 156] and our formulas (IV, 2; 1) and (IV, 2; 2).

Methée [14, p. 156, formula (5.5)] proves that, for $k \neq , I = \{(n-2)/2, n/2, (n+2)/2, \ldots, (n-2)/2 + h, h = 0, 1, \ldots; n$ even$\}$ and n *even* the following formulas are valid:

$$(H_+^k)^{\Lambda} = (-1)^{(n-2)/2 + k}\, \nu_2(n, k)\, \sigma^{2 - n + 2k}, \qquad \text{(VIII, 1; 1)}$$

$$(H_-^k)^{\Lambda} = -i 2^{n - 2 - 2k} \Pi^{n/2} H_-^{(n-4)/2 - k}, \qquad \text{(VIII, 1; 2)}$$

where

$$\nu_2(n, k) = \Pi^{(n-2)/2} 2^{n - 2 - 2k} \Gamma\left(\frac{n-2}{2} - k\right), \qquad \text{(VIII, 1; 3)}$$

$$H_{\pm}^k = H^k \pm \bar{H}^k, \qquad \text{(VIII, 1; 4)}$$

$$H^k = \delta^{(k)}(\Gamma_+) = \delta_+^{(k)}, \qquad \text{(VIII, 1; 5)}$$

$$\bar{H}^k = \delta^{(k)}(\Gamma_-) = \delta_-^{(k)}, \qquad \text{(VIII, 1; 6)}$$

$$\Gamma_+ = \{x \in \mathbb{R}^n / u = x_0^2 - x_1^2 - \cdots - x_{n-1}^2 = 0, x_0 > 0\} \qquad \text{(forward cone)},$$

$$\Gamma_- = \{x \in \mathbb{R}^n / u = x_0^2 - x_1^2 - \cdots - x_{n-1}^2 = 0, x_0 < 0\} \qquad \text{(backward cone)}$$

$$\sigma^{2m} = P_+^m + P_-^m, \qquad \text{(VIII, 1; 7)}$$

if m is even, and

$$\sigma^{2m} = P_+^m - P_-^m, \qquad \text{(VIII, 1; 8)}$$

if m is odd, where

$$P = u = x_0^2 - x_1^2 - \cdots - x_{n-1}^2. \qquad \text{(VIII, 1; 9)}$$

From (VIII, 1; 4) we have

$$H^k = \tfrac{1}{2}\{H_+ + H_-\}. \qquad \text{(VIII, 1; 10)}$$

From (VIII, 1; 1), (VIII, 1; 2) and (VIII, 1; 10) we obtain, if *n is even* and $k \neq$ [illegible];

$$[H^k]^\Lambda = \{\tfrac{1}{2}(-1)^{(n-2)/2+k}\, \nu_2(n,k)\, \sigma^{2-n+2k} - i2^{n-2-2k}\Pi^{n/2} H_-^{(n-4)/2-k}\}. \qquad \text{(VIII, 1; 11)}$$

Taking into account (VIII, 1; 3), it must be $-k + (n-2)/2 \neq -l$, $k, l = 0, 1, 2, \ldots$, and also, $2(-k + (n-2)/2)$ even, which implies n even.

Formula (VIII, 1; 11) can be explicitly written

$$\begin{aligned}[\delta_R^{(k)}(u)]^\Lambda = \frac{1}{2} &\left\{(-1)^{(n-2)/2+k}\, \Pi^{(n-2)/2} 2^{n-2-k} \Gamma\left(\frac{n-2}{2} - k\right)\right\} \\ &\times [P_+^{(2-n+2k)/2} + P_-^{(2-n+2k)/2}] \\ &- i2^{n-2-2k}\Pi^{n/2}[\delta_+^{((n-4)/2-k)} - \delta_-^{((n-4)/2-k)}]. \end{aligned} \qquad \text{(VIII, 1; 12)}$$

Formula (VIII, 1; 12), due to Methée, which expresses the derivative of k order of the delta on the cone, when n is even, coincides (taking into account (I, 4; 1) and (VIII, 1; 7)) with our formula (IV, 2; 1).

VIII.2. *The Equivalence between the* $[\delta_R^{(k)}(u)]^\Lambda$ *Due to Methée and Our Formula, when n is Odd*

Here, we shall prove the equivalence between the formula due to Methée [14, p. 156, formula (5.5)] and our formula (IV, 2; 2) which expresses the Fourier transform of the derivative of k-order of the delta on the cone, when n is odd.

Methée's formula says

$$[H^k]^\Lambda = \tfrac{1}{2}\{\nu_2(n,k)[\mathscr{S}^{2-n+2k} + i(-1)^{(n-1)/2+k} S_-^{2-n+2k}]\}, \qquad \text{(VIII, 2; 1)}$$

where

$$\nu_2(n,k) = \Pi^{(n-2)/2} 2^{n-2-2k} \Gamma\left(\frac{n-2}{2} - k\right), \qquad \text{(VIII, 2; 2)}$$

$\mathscr{S}^p$ coincides with the distribution Q_-^λ (cf. [11, p. 269, formula (47)]):

$$\mathscr{S}^{2-n+2k} = Q_-^{(2-n+2k)/2}, \qquad \text{(VIII, 2; 3)}$$

$$Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2, \qquad \text{(VIII, 2; 4)}$$

$$S_-^{2-n+2k} = S^{2-n+2k} - \overline{S}^{2-n+2k} \qquad \text{(VIII, 2; 5)}$$

$$= \theta(-Q)\,\theta(x_0)(-Q)^{(2-n+2k)/2} - \theta(-Q)\,\theta(-x_0)(-Q)^{(2-n+2k)/2}. \qquad \text{(VIII, 2; 6)}$$

With our notation formula (VIII, 2; 1) says

$$[\delta_R^{(k)}(u)]^\Lambda = \Pi^{(n-2)/2} 2^{n-3-2k} \Gamma\left(\frac{n-2}{2} - k\right)$$
$$\times \{\theta(Q)\, Q^{(2-n+2k)/2} + i(-1)^{(n-1)/2+k}[\theta(-Q)\,\theta(x_0)(-Q)^{(-n+2)/2+2k}$$
$$- \theta(-Q)\,\theta(-x_0)(-Q)^{(-n+2)/2+2k}]\}, \qquad \text{(VIII, 2; 7)}$$

when n *is odd* and $k \neq I' = \{(n-2)/2 + h,\ h = 0, 1, \ldots\}$.

Taking into account that $\Theta(x_0) - \Theta(-x_0) = \operatorname{sgn} x_0$, the coincidence between the formula (VIII, 2; 7) due to Methée and our formula (IV, 2; 2), is evident.

VIII.3. *The Equivalence between the Fourier Transform of a Power of the Cone Due to Methée and Our Formula*

In this section we shall prove the equivalence between the formula due to Methée [14, p. 162, formula (7.7)] and our formula (III, 1; 6), which express the Fourier transform of a power of the cone.

We write, as does Methée,

$$S^p = u^{p/2}, \qquad \text{(VIII, 3; 1)}$$

$u > 0$, $x_0 > 0$, p positive integer.

In our notation this is

$$S^p = \Gamma\left(\frac{p+2}{2}\right) G_R\left(t, \frac{p+2}{2}, m = 0, n\right). \qquad \text{(VIII, 3; 2)}$$

By multiplying the both members of (III, 1; 6) by $\Gamma((p+2)/2)$ and putting $\alpha = (p+2)/2$, it results

$$[S^p]^\Lambda = \left[\Gamma\left(\frac{p+2}{2}\right) G_R\left(t, \frac{p+2}{2}, m = 0, n\right)\right]^\Lambda \qquad \text{(VIII, 3; 3)}$$
$$= 2^{n+p-1} \Pi^{(n-2)/2} \Gamma\left(\frac{p+n}{2}\right) \Gamma\left(\frac{p+2}{2}\right)$$
$$\times [\theta(Q)\, Q^{-p/2-n/2} - i(-1)^{p/2+n/2-1/2} \operatorname{sgn} x_0\, \theta(-Q)(-Q)^{-p/2-n/2}],$$

where $p + n$ is odd.

Formula (VIII, 3; 3) coincides with formula (7.7), of [14, p. 162] due to Methée, which says:

$$[S^p]^\wedge = \frac{1}{2}\lambda(n,p)\Big\{-iS_-^{-n-p}\sin(n+p)\frac{\Pi}{2}$$

$$+ S_+^{-n-p}\cos(n+p)\frac{\Pi}{2} + \mathscr{S}^{-n-p}\Big\}, \qquad \text{(VIII, 3; 4)}$$

where

$$\lambda(n,p) = \Pi^{(n-2)/2}2^{n+p}\Gamma\left(\frac{n+p}{2}\right)\Gamma\left(\frac{p+2}{2}\right),$$

$$S_+^k = S^k + \bar{S}^k = \theta(-Q)(-Q)^{k/2}, \qquad \text{(VIII, 3; 5)}$$

where S^k and $\bar{S}^k$ are defined by (VIII, 2; 6) and $\mathscr{S}^p$ by (VIII, 2; 3).

IX

IX.1. *The Fourier Transform of $\delta_R^{(k)}(u)$, in the Singular Points ($k = (n-2)/2 + h$, $h = 0, 1, \ldots$)*

In this section we shall evaluate the Fourier transform of $\delta_R^{(k)}(u)$, when $k \in \mathscr{N} = \{(n-2)/2 + h, h = 0, 1, \ldots\}$.

We start by writing formula (IV, 2; 1):

$$[\delta_R^{(k)}(u)]^\wedge = (2\Pi)^{(n-2)/2}\, 2^{-2k+(n-4)/2}(-i\Pi)\operatorname{sgn} x_0\,\delta(Q)^{(-k+(n-4)/2)}$$

$$+ (-1)(2\Pi)^{(n-2)/2}\, 2^{-2k+(n-4)/2}$$

$$\times\Gamma\left(-k+\frac{n-2}{2}\right)\operatorname{Pf}\frac{1}{Q^{-k+(n-2)/2}}, \qquad \text{(IX, 1; 1)}$$

where $Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2$. This formula is valid when $2(-k + (n-2)/2)$ is even, which implies n even, and $-k + (n-2)/2 \neq -h$, $h = 0, 1, \ldots$.

We shall evaluate (IX, 1; 1) in the singular points, that is to say when $k \in \mathscr{N}$:

$$\boxed{k = h + \frac{n-2}{2}, \qquad h = 0, 1, \ldots.} \qquad \text{(IX, 1; 2)}$$

We know that

$$\left.\frac{Q^{\alpha-1}}{\Gamma(\alpha)}\right|_{\alpha=-l} = \delta(Q)^{(l)}. \qquad \text{(IX, 1; 3)}$$

Then, we have

$$\delta(Q)^{(-k+(n-4)/2)} = \delta(Q)^{(-1-h)} = \frac{Q^h}{\Gamma(h+1)}. \qquad \text{(IX, 1; 4)}$$

Now, we shall study the second summand of the right-hand member of (IX, 1; 1) when $k \in \mathscr{I}$.

To do this we shall first evaluate $\delta_R^{(k)}(u - m^2)$ when $k = (n-2)/2 + h$, $h = 0, 1, \ldots$ and then we shall pass to the limit for $m \to 0$.

We begin by considering the following formula due to Gelfand[1] (cf. [11, p. 294, formula (6)])

$$[\delta^{(t-1)}(m^2 + Q)]^{\wedge} = (-1)^{t+1}\, 2^{n/2-t} \Pi^{n/2-t} m^{n/2-t} \qquad \text{(IX, 1; 5)}$$
$$\times \left\{ \frac{K_{n/2-t}[m(Q - i0)^{1/2}]}{(Q - i0)^{1/2(n/2-t)}} + \frac{K_{n/2-t}[m(Q + i0)^{1/2}]}{(Q + i0)^{1/2(n/2-t)}} \right\}.$$

Remembering that $K_\mu = K_{-\mu}$ and taking into account formula defining $K_\nu(z)$ [9, p. 9, formula (37)], we have

$$(-1)^{t+1}\, 2^{n/2-t} \Pi^{n/2-1} m^{n/2-t} \frac{K_{n/2-t}[m(Q - i0)^{1/2}]}{(Q - i0)^{1/2(n/2-t)}}$$
$$= \frac{(-1)^{t+1}\, 2^{n/2-t} \Pi^{n/2-t} m^{n/2-t}}{(Q - i0)^{1/2(n/2-t)}} \left[(-1)^{t-n/2+1} I_{t-n/2}[m(Q - i0)^{1/2}]\right]$$
$$\times \log\left[\tfrac{1}{2}(m(Q - i0)^{1/2})\right]$$
$$+ \tfrac{1}{2} \sum_{p=0}^{t-n/2-1} (-1)^p \left[\tfrac{1}{2}(m(Q - i0)^{1/2})\right]^{2p-t+n/2} \frac{(t - n/2 - p - 1)!}{p!}$$
$$+ \tfrac{1}{2}(-1)^{t-n/2} \sum_{p=0}^{\infty} \left[\tfrac{1}{2}(m(Q - i0)^{1/2})\right]^{t-n/2+2p}$$
$$\times \left\{ \frac{[\psi(t - n/2 + p + 1) + \psi(p + 1)]}{p!(t - n/2 + p)!} \right\}, \qquad \text{(IX, 1; 6)}$$

where

$$I_\nu(z) = \sum_{s=0}^{\infty} \frac{(z/2)^{2s+\nu}}{s!\,\Gamma(s + \nu + 1)}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$.

[1] We remark that Gelfand defines $[f]^{\wedge} = \int_{\mathbb{R}^n} e^{i(x,y)} f(x)\, d(x)$ while Methée and the author both define $[f]^{\wedge} = \int_{\mathbb{R}^n} e^{-i(x,y)} f(x)\, dx$.

We remember that

$$\log\left[\frac{m(Q-i0)^{1/2}}{2}\right]=\log m-\log 2+\tfrac{1}{2}\log(Q-i0)$$
$$=\log m-\log 2+\tfrac{1}{2}\{\log|Q|-i\Pi H(-Q)\},$$

where H is the Heaviside function.

Then the first summand of the right-hand member of (IX, 1; 6) results

$$(-1)^{t+1}\,2^{n/2-t}\Pi^{n/2-t}m^{n/2-t}(-1)^{t-n/2+1}$$
$$\times\frac{I_{t-n/2}[m(Q-i0)^{1/2}]\log((1/2)\,m(Q-i0)^{1/2}]}{(Q-i0)^{1/2(n/2-it)}}$$
$$=(-1)^{n/2}\,2^{n/2-t}\Pi^{n/2-1}\sum_{p=0}^{\infty}\frac{(1/2)^{2p+t-n/2}\,m^{2p}(Q-i0)^{p+t-n/2}}{p!\,\Gamma(p+t-n/2+1)}$$
$$\times\{\log m-\log 2+\tfrac{1}{2}[\log|Q|-i\Pi H(-Q)]\}. \quad \text{(IX, 1; 7)}$$

Taking into account that in our case is $t-1=k=(n-2)/2+h$, and by passing to the limit for $m\to 0$ in (IX, 1; 7) we, finally, obtain

$$\lim_{m\to 0}(-1)^{n/2}\,2^{n/2-t}\Pi^{n/2-1}\frac{I_{t-n/2}[m(Q-i0)^{1/2}]\log[(1/2)\,m(Q-i0)^{1/2}]}{(Q-i0)^{1/2(n/2-t)}}$$
$$=(-1)^{n/2}\frac{\Pi^{(n-2)/2}}{h!}2^{-2h}Q^{h}[-\log 2+\tfrac{1}{2}\log Q-\tfrac{1}{2}i\Pi H(-Q)]. \quad \text{(IX.1; 8)}$$

Considering the third summand of the right-hand member of (IX, 1; 6) when $t=n/2+h$ one obtains

$$\lim_{m\to 0}(-1)^{t+1}\,2^{n/2-t}\Pi^{n/2-1}m^{n/2-t}\tfrac{1}{2}(-1)^{t-n/2}$$
$$\times\sum_{p=0}^{\infty}[\tfrac{1}{2}(m(Q-i0)^{1/2}]^{t-n/2+2p}\frac{\{\psi(t-(n/2)+p+1)+\psi(p+1)\}}{p!(t-(n/2)+p)!}$$
$$=(-1)^{-(n/2)+1}\,2^{-1-2h}Q^{h}\Pi^{(n-2)/2}\left[\frac{\psi(h+1)+\psi(1)}{h!}\right]. \quad \text{(IX, 1; 9)}$$

Proceeding in the same manner for the second summand of (IX, 1; 5), we have

$$\lim_{m\to 0} (-1)^{n/2} 2^{n/2-t} \Pi^{n/2-1} \frac{I_{t-n/2}[m(Q+i0)^{1/2}] \log[(1/2)m(Q+i0)^{1/2}]}{(Q+i0)^{(1/2)(n/2-t)}}$$

$$= \frac{(-1)^{n/2}}{h!} \Pi^{n/2-1} 2^{-2h} Q^h [-\log 2 + \tfrac{1}{2}\log Q + \tfrac{1}{2} i\Pi H(-Q)], \tag{IX, 1; 10}$$

and

$$\lim_{m\to 0} (-1)^{t+1} 2^{n/2-t} \Pi^{n/2-1} m^{n/2-t} \tfrac{1}{2} (-1)^{t-n/2}$$

$$\times \sum_{p=0}^{\infty} [\tfrac{1}{2}(m(Q+i0)^{1/2})]^{t-n/2+2p} \frac{[\psi(t-n/2+p+1)+\psi(p+1)]}{p!\,(t-n/2+p)!}$$

$$= (-1)^{-n/2+1} 2^{-1-2h} Q^h \Pi^{(n-2)/2} \frac{[\psi(h+1)+\psi(1)]}{h!}. \tag{IX, 1; 11}$$

Adding (IX, 1; 8), (IX, 1; 9), (IX, 1; 10) and (IX, 1; 11) we finally obtain

$$\lim_{m\to 0} [\delta_R^{(k)}(m^2+Q)]^\Lambda = \frac{\Pi^{(n-2)/2}}{2^{1-2h}\Gamma(h+1)} (-1)^{n/2-1} \tag{IX, 1; 12}$$

$$\times \{Q^h[\psi(h+1)+\psi(1)+2\log 2 - \log|Q|]\}.$$

Therefore, we have

$$[\delta_R^{(k)}(u)]^\Lambda = -i(2\Pi)^{(n-2)/2} 2^{-n/2-2h} \Pi \operatorname{sgn} x_0 \frac{Q}{\Gamma(h+1)} \tag{IX, 1; 13}$$

$$+ \frac{\Pi^{(n-2)/2}}{2^{1-2h}\Gamma(h+1)} [Q^h[\psi(h+1)+\psi(1)+2\log 2 - \log|Q|]].$$

IX.2. *The Equivalence between the Fourier Transform of $\delta_R^{(k)}(u)$ in the Singular Points $k = (n-2)/2 + h$, $h = 0, 1, \ldots$, and Our Formula*

We start by writing the formula due to Methée [14, p. 159]:

$$[\delta_R(u)^{((n-2)/2+h)}] = \frac{1}{2} \left\{ \frac{-i\Pi^{n/2}}{4^h\Gamma(h+1)} S_-^{2h} + \frac{\Pi^{(n-2)/2}}{4^h\Gamma(h+1)} \right.$$

$$\left. \times [\sigma^{2h}(\psi(h+1)+\psi(1)+\log 4) - N^{2h+1}] \right\}, \tag{IX, 2; 1}$$

where

$$\psi(z) = \frac{d\log\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{IX, 2; 2}$$

$$N^{2m,k} = |u|^m \log^k u \qquad \text{if} \quad x \in \Omega_1 u \Omega_3,$$
$$= u^m \log^k |u| \qquad \text{if} \quad x \in \Omega_2. \qquad (\text{IX}, 2; 3)$$

In (IX, 2; 3) $\Omega_1 = \Gamma_+$, $\Omega_3 = \Gamma_-$ and $\Omega_2 = \mathscr{C}(\Gamma_+ \cup \Gamma_-)$.

$$\sigma^{2m} = P_+^m + P_-^m,$$

if m is even, and (IX, 2; 4)

$$\sigma^{2m} = P_+^m - P_-^m,$$

if m is odd.

$$P = u = x_0^2 - x_1^2 - \cdots - x_{n-1}^2$$

$$P_+^\lambda = P^\lambda \text{ if } P \geqslant 0, \qquad P_-^\lambda = 0 \qquad \text{if } P > 0,$$
$$= 0 \quad \text{if } P < 0 \qquad \text{and} \qquad = (-P)^\lambda \text{ if } P \leqslant 0. \qquad (\text{IX}, 2; 5)$$

We know that

$$S_-^{2h} = S^{2h} - \bar{S}^{2h}$$
$$= \theta(x_0)\,\theta(Q)\,Q^h - \theta(-x_0)\,\theta(Q)\,Q^h \qquad (\text{IX}, 2; 6)$$
$$= Q^h \operatorname{sgn} x_0.$$

Therefore, taking into account formulas (IX, 1; 4), (IX, 1; 12) and (IX, 2; 6) we prove, immediately, the equivalence between the Methée formula (IX, 2; 1) and our formula (IX, 1; 13).

Remark. We observe that in the case that $2(-k + (n-2)/2)$ is odd, which implies n odd, the Fourier transform of $\delta_R^{(k)}(u)$ has no singular points because the argument of $\Gamma(z)$, which appears in the second summand of the right-hand member of (IX, 1; 1), never is a negative integer or zero.

X. The Fourier Transform of $G_R(t, \alpha = -k, m^2, n)$

In this paragraph we shall study the Fourier transform of $G_R(t, \alpha, m^2, n)$, when $\alpha = -k$, it is $[\delta_R^{(k)}(u - m^2)]^\Lambda$.

We begin with the case n even. Taking into account the formula (I, 3; 22), we must study

$$\frac{K_{-k+(n-2)/2}[mi(\pm Q)^{1/2}]}{(\pm Q)^{1/2(-k+(n-2)/2)}}$$

and $\delta(Q)^{(-k+(n-4)/2)}$. We write, by definition, $\theta(\pm Q)(\pm Q)^\lambda = (Q\pm)^\lambda$.

From the definitory formula of $K_\nu(z)$ [9, p. 9, formula (37)] and remembering (cf. [11, p. 255]) formulas (15) and (15′) that $(Q\pm)^\lambda$ has two kinds of singularities, for $\lambda=-1,-2,\ldots,-l$ and $\lambda=-n/2,-n/2-1,\ldots,-n/2-l$, we must interpret $(\pm Q)^{-r}$, $r=1,\ldots,n/2-k-1$, $k\in Z^+$, as the finite part of $Q_\pm^\lambda$, for $\lambda=-r$, $r=1,\ldots,n/2-k-1$, $k\in Z^+$. This finite part is evaluated in the Appendix, paragraph V of [19]. On the other hand, $\delta(Q)^{(-k+(n-4)/2)}$ has no singularities. This is a consequence of the fact that $\delta(Q)^{(\alpha)}$ has no singularities when $\alpha\geqslant(n-2)/2$ [11, p. 250] and in our case is $\alpha=-k+(n-4)/2$, $k=0,1,\ldots$.

Now we shall consider the case n odd.

In this case

$$\frac{K_{-k+(n-2)/2}\{mi(\pm Q)^{1/2}\}}{(\pm Q)^{1/2(-k+(n-2)/2)}} \quad \text{and} \quad (\pm Q)^{-(-k+(n-3)/2)-1/2}$$

do not have singularities.

Therefore, we can conclude that, in both cases, n even or odd,

$$[G_R(t,\alpha=-k,m^2,n)]^\Lambda=[\delta_R^{(k)}(u-m^2)]^\Lambda \tag{X, 1; 1}$$

does not have singularities.

We remark that, taking into account formulas (I, 3; 22) and (V, 2; 4), when $\alpha=-k$, n even, and (I, 3; 23) and (V, 2; 5) when $\alpha=-k$, n odd, it results that

$$[G_R(t,\alpha=-k,m^2,n)+G_A(t,\alpha=-k,m^2,n)]^\Lambda=[\delta^{(k)}(u-m^2)]^\Lambda$$

has no singular points and our result coincides with the formula due to Gelfand [11, p. 294, formula (6) and (7)].

XI

XI.1. *The Fourier Transform of the Marcel Riesz Kernel $W(t,\alpha,m^2,n)$ in the Singular Points*

We shall evaluate in this section the Fourier transform of the Riesz function (cf. (VI, 1; 1) when α is an exceptional value).

We start with the particular case of formula (VI, 1; 1), when $m=0$ (formula (VI, 2; 1)). We repeat here, by commodity, formula (VI, 2; 1):

$$\begin{aligned} W(t,\alpha,m=0,n)=R_\alpha(u)&=\frac{u^{(\alpha-n)/2}}{H_n(\alpha)} && \text{if} \quad t\in\Gamma_+,\\ &=0 && \text{if} \quad t\notin\Gamma_+. \end{aligned} \tag{XI, 1; 1}$$

In this formula

$$H_n(\alpha) = \Pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha-n+2}{2}\right).$$

The Laplace transform of $R_\alpha(u)$ is, taking into account the formula (II, 3; 3), p. 11, of [7],

$$L[W(t, \alpha, m=0, n)] = L\left[\frac{u^{(\alpha-n)/2}}{H_n(\alpha)}\right] = (\rho^2)^{-\alpha/2}. \qquad \text{(XI, 1; 1)}$$

We remember that

$$\rho^2 = z_1^2 + \cdots + z_{n-1}^2 - z_0^2. \qquad \text{(XI, 1; 2)}$$

Applying our method for evaluating the Fourier transforms by evaluating their Laplace transforms and then passing to the limit (in S') for $y \to 0$, where $y \in V_-$, we have

$$[W(t, \alpha, m=0, n)]^\wedge = \lim_{y\to 0} (\rho^2)^{-\alpha/2}, \qquad \text{(XI, 1; 3)}$$

where $y \in V_-$, it is, $y \in \mathbb{R}^n$, $y_0 = \operatorname{Im} z_0 < 0$, $y_0^2 - y_1^2 - \cdots - y_{n-1}^2 > 0$.
From (XI, 1; 3) and (I, 2; 4), with $\varepsilon y_0 = -\varepsilon$, we have

$$[W(t, \alpha, m=0, n)]^\wedge = \{x_1^2 + \cdots + x_{n-1}^2 - (x_0 - i0)^2\}^{-\alpha/2}. \qquad \text{(XI, 1; 4)}$$

The distribution $\{x_1^2 + \cdots + x_{n-1}^2 - (x_0 - i0)^2\}^{-\alpha/2}$ was studied by Vladimirov [3, p. 298, formula (136), p. 299, formula (138)], accordingly as α is even or odd), and is given by the following formulas

$$\{x_1^2 + \cdots + x_{n-1}^2 - (x_0 - i0)^2\}^{-\alpha/2} = \frac{-i\Pi \operatorname{sgn} x_0 \delta(Q)^{(k-1)}}{(k-1)!} + (-1)^k \operatorname{Pf} \frac{1}{Q^k}, \qquad \text{(XI, 1; 5)}$$

if $\alpha = 2k$, $k = 0, 1, \ldots$ and $Q = x_0^2 - x_1^2 - \cdots - x_{n-1}^2$;

$$\{x_1^2 + \cdots + x_{n-1}^2 - (x_0 - i0)^2\}^{-\alpha/2} = [\theta(Q) Q]^{-k-1/2} - i(-1)^k \operatorname{sgn} x_0 [\theta(-Q)(-Q)]^{-k-1/2}, \qquad \text{(XI, 1; 6)}$$

if $\alpha = 2k+1$, $k = 0, 1, \ldots$.

XI.2. *The Fourier Transform of* $W(t, \alpha, m \neq 0, n)$

We shall study now the Fourier transform of $W(t, \alpha, m, n)$, when $m \neq 0$.

We repeat the definitory formula of $W(t, \alpha, m, n)$ (cf. formula (VI, 1; 1)):

$$W(t, \alpha, m, n) = \frac{(m^{-2}u)^{(\alpha-n)/4}}{\Pi^{(n-2)/2}2^{(2\alpha+n-2)/2}\Gamma(\alpha/2)} J_{(\alpha-n)/2}\{\sqrt{m^2u}\}, \quad \text{if } t \in \Gamma_+.$$
$$= 0, \quad \text{if } t \notin \Gamma_+. \quad \text{(XI, 2; 1)}$$

In this formula α is a complex parameter, m a real nonnegative number and n the dimension of the space.

The Laplace transform of $W(t, \alpha, m^2, n)$ is, taking into account the formula (II, 1; 3), p. 10, of [7],

$$L[W(t, \alpha, m^2, n)] = (\rho^2 + m^2)^{-\alpha/2}. \quad \text{(XI, 2; 2)}$$

Therefore, as always,

$$[W(t, \alpha, m^2, n)]^\Lambda = \lim_{\epsilon \to 0} \{m^2 + x_1^2 + \cdots + x_{n-1}^2 - (x_0 + i\varepsilon y_0)^2\}^{-\alpha/2}, \quad \text{(XI, 2; 3)}$$

where $y_0 < 0$.

When $\operatorname{Re} \alpha \leqslant 0$, we obtain

$$[W(t, \alpha, m^2, n)]^\Lambda = (m^2 + Q)^{-\alpha/2}, \quad \text{(IX, 2; 3)}$$

when $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 < 0$.

Here is

$$Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2. \quad \text{(XI, 2; 4)}$$

This is a consequence of the fact that we evaluate the Fourier transform in the exterior of the forward cone and in this region there are no prescriptions over y.

Now, we shall consider the interior of the forward cone: $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \geqslant 0$ and $x_0 \geqslant 0$.

From (XI, 2; 3) results

$$\begin{aligned}[W(t, \alpha, m^2, n)]^\Lambda &= \{m^2 + x_1^2 + x_1^2 + \cdots + x_{n-1}^2 - (x_0 - i0)^2\}^{-\alpha/2} \\ &= (m^2 + Q + i0)^{-\alpha/2},\end{aligned} \quad \text{(XI, 2; 5)}$$

when $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \geqslant 0$ and $x_0 \geqslant 0$.

Taking into account the formulas on pp. 565 and 566, of [15], we have ($\lambda \in \mathbb{C}$)

$$(m^2 + Q \pm i0)^\lambda = (m^2 + Q)_+^\lambda + e^{\pm i\Pi\lambda}(m^2 + Q)_-^\lambda, \quad \text{(XI, 2; 6)}$$

where

$$(m^2+Q)_+^\lambda = (m^2+Q)^\lambda, \quad \text{if} \quad m^2+Q \geqslant 0, \\ = 0, \quad \text{if} \quad m^2+Q<0. \qquad \text{(XI, 2; 7)}$$

and

$$(m^2+Q)_-^\lambda = 0, \quad \text{if} \quad m^2+Q \geqslant 0, \\ = [-(m^2+Q)]^\lambda, \quad \text{if} \quad m^2+Q<0. \qquad \text{(XI, 2; 8)}$$

Then, formula (XI, 2; 5) can be written, equivalently,

$$[W(t,\alpha,m^2,n)]^\Lambda = (m^2+Q)_+^{-\alpha/2} + e^{-i\Pi(\alpha/2)}(m^2+Q)_-^{\alpha}, \quad \text{(XI, 2; 9)}$$

when $Q \leqslant 0$ and $x_0 > 0$.

Let now, $x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \geqslant 0$ and $x_0 \leqslant 0$ be the interior of the backward cone.

From (XI, 2; 3) and (XI, 2; 6) one obtains

$$[W(t,\alpha,m^2,n)]^\Lambda = (m^2+Q-i0)^{-\alpha/2} \\ = (m^2+Q)_+^{-\alpha/2} + e^{i(\Pi\alpha/2)}(m^2+Q)_-^{-\alpha/2}, \qquad \text{(XI, 2; 10)}$$

when $Q \leqslant 0$ and $x_0 < 0$.

Finally, summarizing the results (XI, 2; 3), (XI, 2; 9) and (XI, 2; 10), we have

$$[W(t,\alpha,m^2,n)]^\Lambda = \theta(Q)(m^2+Q)^{-\alpha/2} \qquad \text{(XI, 2; 11)} \\ + \theta(-Q)\,\theta(x_0)\{(m^2+Q)_+^{-\alpha/2} + e^{-i(\Pi\alpha/2)}(m^2+Q)_-^{-\alpha/2}\} \\ + \theta(-Q)\,\theta(-x_0)\{(m^2+Q)_+^{-\alpha/2} + e^{i(\Pi\alpha/2)}(m^2+Q)_-^{-\alpha/2}\},$$

where $\operatorname{Re}\alpha \leqslant 0$ and $Q = x_1^2 + \cdots + x_{n-1}^2 - x_0^2$.

XI

XI.3. *The Fourier Transform of $W(t,\alpha,m\neq 0,n)$, in the Singular Points*

We consider now the particular case of (XI, 2; 3) when $\alpha = 2k$, $k = 1, 2, \ldots$.

We have from (XI, 2; 3)

$$[W(t,\alpha,m^2,n)]^\Lambda = \lim_{\epsilon\to 0}\{m^2 + x_1^2 + \cdots + x_{n-1}^2 - (x_0 + i\varepsilon y_0)^2\}^{-k}. \quad \text{(XI, 3; 1)}$$

When $x_0 > 0$, we have

$$[W(t, \alpha, m^2, n)]^\Lambda = (m^2 + Q + i0)^{-k} \qquad \text{(XI, 3; 2)}$$

while, when $x_0 < 0$, is

$$[W(t, \alpha, m^2, n)]^\Lambda = (m^2 + Q - i0)^{-k}. \qquad \text{(XI, 3; 3)}$$

Taking into account formula (1.6), p. 565, of [15] that says

$$(m^2 + Q \pm i0)^{-k} = (m^2 + Q)^{-k} \mp \frac{(-1)^{k-1}\, i\Pi}{(k-1)!}\, \delta^{(k-1)}(m^2 + Q). \qquad \text{(XI, 3; 4)}$$

Finally, from (XI, 3; 2), (XI, 3; 3) and (XI, 3; 4), we have

$$W[(t, \alpha = 2k, m^2, n)] = (m^2 + Q)^{-k} + \frac{(-1)^k\, i\Pi}{(k-1)!} \operatorname{sgn} x_0 \delta^{(k-1)}(m^2 + Q). \qquad \text{(XI, 3; 5)}$$

We remark that formula (XI, 2; 11) is also valid when $\alpha = 2k + 1$, $k = 0, 1, \ldots$.

XII. The Fourier Transform of $G_R(t, \alpha > 0, m^2, n)$ in the Singular Points

We shall study the Fourier transform of $G_R(t, \alpha, m^2, n)$, when $\alpha > 0$, $m \neq 0$, n even or odd, in the singular points.

We start by studying $[G_R(t, \alpha > 0, m^2, n)]^\Lambda$, when n is even.

From (I, 3; 22) we must study the terms of the form

$$\frac{K_{\alpha + (n-2)/2}[m(\pm Q)^{1/2}]}{(\pm Q)^{(1/2)(\alpha + (n-2)/2)}} \quad \text{and} \quad \delta(Q)^{(\alpha + (n-4)/2)}.$$

An explicit and detailed study of

$$\frac{K_{\alpha + (n-2)/2}\{m(\pm Q)^{1/2}\}}{(\pm Q)^{1/2}}$$

in their singular points appears in the Appendix, A.IV of [19]. On the other hand, we know that $\delta(Q)^{(\alpha + (n-4)/2)}$ (cf. [11, Chap. III, Sect. 2.1]) has no singularities if $\alpha < 1$. When $\alpha \geqslant 1$ we interpret $\delta(Q)^{(\alpha + (n-4)/2)}$ as follows:

$$\langle \delta(Q)^{(\alpha + (n-4)/2)}, \phi \rangle = \frac{(-1)^{\alpha + n/2}}{4} (u_+^{-\alpha}, \psi(u)), \qquad \text{(XII, 1; 1)}$$

where

$$\psi(r, x_n) = \int \phi \, d\Omega^{(p)} \, dx_n, \tag{XII, 1; 2}$$

$$u_+^{-\alpha} = u^{-\alpha}, \quad \text{if} \quad u > 0,$$
$$= 0, \quad \text{if} \quad u < 0. \tag{XII, 1; 3}$$

$d\Omega^{(p)}$ is the element of area of the p-dimensional sphere.

We can regularize $u_+^{-\alpha}$ for $\alpha \neq n$, n positive integer, by analytical continuation of $u_+^{-\alpha}$, $\operatorname{Re} \alpha < 1$ and for $\alpha = n$, u_+^{-n} is the constant term in the Laurent development of $u_+^{-\alpha}$ in the neighborhood of $\alpha = n$.

When n is odd, taking into account formula (I, 3; 23) we must study the terms of the form

$$\frac{K_{\alpha+(n-2)/2}\{m(\pm Q)^{1/2}\}}{(\pm Q)^{(1/2)(\alpha+(n-2)/2)}}$$

and $(\pm Q)^{-1/2-(\alpha+(n-3)/2)}$.

For the study of

$$\frac{K_{\alpha+(n-2)/2}\{m(\pm Q)^{1/2}\}}{(\pm Q)^{(1/2)(\alpha+(n-2)/2)}}$$

see A.IV of the Appendix of [19].

We know that [11, p. 255, formulas (15) and (15′)] $(\pm Q)^{\lambda}$ has two kinds of singularities when

$$\lambda = -1, -2, \ldots, -k, \ldots,$$
$$\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \ldots, -\frac{n}{2} - k, \ldots. \tag{XII, 1; 4}$$

In our case is

$$\lambda = -\alpha - \frac{n}{2} + 1, \qquad n \text{ odd}. \tag{XII, 1; 5}$$

Therefore if

$$\alpha = r + \tfrac{1}{2}, \qquad r = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{XII, 1; 6}$$

or,

$$\alpha = s, \qquad s = 1, 2, \ldots, \tag{XII, 1; 7}$$

$(\pm Q)^{-\alpha-n/2+1}$ has singularities and in this case, we must interpret $(\pm Q)^{-\alpha-n/2+1}$ as the finite part of $(\pm Q)^{\lambda}$, for $\lambda=-\alpha-n/2+1$. The explicit evaluation of this finite part appears in A.V. of the Appendix of [19], where we write, by definition, $\theta(\pm Q)(\pm Q)^{\lambda}=(Q\pm)^{\lambda}$.

XIII

XIII.1. *The Fourier Transform of $G_R(t, \alpha, m=0, n)$ in the Singular Points, when $2\alpha-2+n$ Is Even*

From (III, 1; 5) we have, when $2\alpha-2+n$ is even,

$$[G_R(t, m=0, \alpha, n)]^{\wedge} \tag{XIII, 1; 1}$$
$$= (2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}(-i\Pi)\operatorname{sgn} x_0\,\delta(Q)^{(\alpha+(n-4)/2)}$$
$$+ (2\Pi)^{(n-2)/2}\, 2^{\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right)(-1)^{\alpha+(n-2)/2}\operatorname{Pf}\frac{1}{Q^{\alpha+(n-2)/2}}.$$

In the first summand of the right-hand member appears $\delta(Q)^{(\alpha+(n-4)/2)}$. When $\alpha\geqslant 1$ we must substitute $\delta(Q)^{(\alpha+(n-4)/2)}$ by its "regularized" expression (see paragraph XII.1).

When $\alpha+(n-2)/2=-r$, $r=0, 1, \ldots$, $\Gamma(z)$ has simple poles while Q^r has no singular points. To regularize the second summand of the right-hand member of XIII.1 we write, as usual,

$$\lim_{\lambda\to -r}\frac{d}{d\lambda}\left\{(2\Pi)^{(n-2)/2}\, 2^{2(\lambda-(n-2)/2)+(n-4)/2}(\lambda+r)(-1)^{\lambda}\,\Gamma(\lambda)\operatorname{Pf}\frac{1}{Q^{\lambda}}\right\}$$
$$=\Pi^{(n-2)/2}2^{-2r+(n-4)/2}(-1)^r\,Q^r\operatorname*{Pf}_{\lambda=-r}\Gamma(\lambda)$$
$$+\Pi^{(n-2)/2}2^{-2r+(n-4)/2}(-1)^r\operatorname*{res}_{\lambda=-r}\Gamma(\lambda)\operatorname*{Pf}_{\lambda=-r}[Q^{-\lambda}], \tag{XIII, 1; 2}$$

where

$$\lambda=\alpha+\frac{n-2}{2}. \tag{XIII, 1; 3}$$

The second summand of the right-hand member of (XIII, 1; 2) vanishes because Q^r has no singular points.

The explicit evaluation of the finite part and the residue of $\Gamma(\lambda)$, for $\lambda=-r$, appears in A.I of the Appendix of [19].

XIII.2. *The Fourier Transform of $G_R(t, \alpha, m = 0, n)$ in the Singular Points, when $2\alpha - 2 + n$ Is Odd*

Now, we shall consider formula (III, 1; 6). We have, for $2\alpha - 2 + n$ odd,

$$\begin{aligned}[G_R(t, m = 0, \alpha, n)]^{\wedge} = {} & (2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha + \frac{n-2}{2}\right) \\ & \times [\theta(Q)\, Q^{-(\alpha+(n-3)/2)-1/2} - i(-1)^{\alpha+n/2-3/2} \\ & \times \operatorname{sgn} x_0\, \theta(-Q)(-Q)^{-(\alpha+(n-3)/2)-1/2}]. \qquad \text{(XIII, 2; 1)}\end{aligned}$$

When $\alpha = -h + 1 - n/2$ the function $\Gamma(\alpha + n/2 - 1)$ has simple poles and when $\alpha = r$, $r = 1, 2, \ldots$ or $\alpha = s - n/2$, $s = 1, 2, \ldots$; n odd, the distribution $Q_{\pm}^{-\alpha-n/2+1}$ has singularities. We observe, therefore, that double poles never exist in (XIII, 2; 1).

We know that Q_+^{λ} has the same set of singularities of Q_-^{λ} (it is sufficient to interchange the roles of p and q of the quadratic form Q). Therefore we shall only regularize, by the usual method, the first summand of the right-hand member of (XIII, 2; 1).

For

$$\alpha = -h - \frac{n}{2} + 1, \qquad h = 0, 1, 2, \ldots, n \text{ odd}, \qquad \text{(XIII, 2; 2)}$$

we have

$$\begin{aligned}\underset{\alpha=-n/2-h+1}{\mathrm{Pf}} & \left[(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n}{2}-1\right) Q_+^{-\alpha-n/2+1}\right] \\ = {} & \lim_{\alpha\to-h-n/2+1} \frac{d}{d\alpha}\left\{\left(\alpha+h+\frac{n}{2}-1\right)(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\right. \\ & \left.\times\,\Gamma\left(\alpha+\frac{n}{2}-1\right) Q_+^{-\alpha-n/2+1}\right\} \\ = {} & 2^{-2h}\Pi^{(n-2)/2} Q_+^h \underset{\alpha=-h-n/2+1}{\mathrm{Pf}} \Gamma\left(\alpha+\frac{n}{2}-1\right) \\ & + 2^{-2h}\Pi^{(n-2)/2} \underset{\alpha=-h-n/2+1}{\mathrm{Res}} \Gamma\left(\alpha+\frac{n}{2}-1\right) \\ & \times \left[\frac{d}{d\alpha} Q_+^{-(\alpha+n/2-1)}\right]_{\alpha=-h-n/2+1}. \qquad \text{(XIII, 2; 3)}\end{aligned}$$

The explicit evaluation of the finite part and residue of $\Gamma(z)$ appears in A.I of the Appendix of [19]. Taking into account formulas (A.I, 2; 3) and (A.I, 1; 6), we, finally, obtain,

$$\underset{\alpha=-n/2-h+1}{\mathrm{Pf}}\left[(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n}{2}-1\right) Q_+^{-\alpha-n/2+1}\right]$$

$$= 2^{-2h}\Pi^{(n-2)/2}\,\frac{(-1)^h}{h!}\,[\psi(1+h)\, Q_+^h - \log Q Q_+^h], \qquad \text{(XIII, 2; 4)}$$

where $\psi(1+n) = 1 + 1/2 + \cdots + 1/n - \mathscr{C}$, with $\mathscr{C}$ Euler constant.

For

$$\alpha = k+1, \qquad k = 0, 1,\ldots, \qquad \text{(XIII, 2; 5)}$$

we have

$$\underset{\alpha=k+1}{\mathrm{Pf}}\left[(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma\left(\alpha+\frac{n-2}{2}\right) Q_+^{-(\alpha+n/2-1)}\right]$$

$$= \lim_{\alpha\to k+1}\frac{d}{d\alpha}\left\{(\alpha-k-1)(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\right.$$

$$\left.\times\,\Gamma\left(\alpha+\frac{n-2}{2}\right) Q_+^{-(\alpha+n/2-1)}\right\}$$

$$= (2\Pi)^{(n-2)/2}\, 2^{2k+n/2}\left\{\Gamma\left(k+\frac{n}{2}\right)\underset{\alpha=k+1}{\mathrm{Pf}}\, Q_+^{-\alpha-n/2+1}\right.$$

$$\left.+\,\Gamma'\left(k+\frac{n}{2}\right)\underset{\alpha=k+1}{\mathrm{Res}}\, Q_+^{-\alpha-n/2+1}\right\}. \qquad \text{(XIII, 2; 6)}$$

The explicit value of the finite part and the residue of Q_+^λ, for $\lambda = -k$, appears in A.V and A.VI of the Appendix of [19].

For

$$\alpha = k+1-\frac{n}{2}, \qquad k = 1, 2,\ldots, \qquad n \text{ odd}, \qquad \text{(XIII, 2; 7)}$$

we write, as always,

$$\underset{\alpha=k+1-n/2}{\mathrm{Pf}}\left[(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\Gamma(\alpha+(n-2)/2)\, Q_+^{-\alpha-n/2+1}\right]$$

$$= \lim_{\alpha\to k+1-n/2}\frac{d}{d\alpha}\left\{\left(\alpha-k-1+\frac{n}{2}\right)(2\Pi)^{(n-2)/2}\, 2^{2\alpha+(n-4)/2}\right.$$

$$\left.\times\,\Gamma\left(\alpha+\frac{n-2}{2}\right) Q_+^{-\alpha-n/2+1}\right\}$$

$$= (2\Pi)^{(n-2)/2}\, 2^{2k-n/2}\{\Gamma(k)|_{\alpha=k+1-n/2}\underset{\alpha=k+1-n/2}{\mathrm{Pf}}\, Q_+^{-\alpha-n/2+1}$$

$$+\,\Gamma'(k)\underset{\alpha=k+1-n/2}{\mathrm{Res}}\, Q_+^{-\alpha-n/2+1}\}. \qquad \text{(XIII, 2; 8)}$$

Remark. We obtain the same result putting $m=0$ in the regularized formula of $[G_R(t, m \neq 0, \alpha, n)]^{\Lambda}$ or regularizing $[G_R(t, m=0, \alpha, n)]^{\Lambda}$ in the singular points.

Starting by $[G_R(t, m \neq 0, \alpha, n)]$ we must express $K_\nu(z)$ in a neighborhood at the origin (see the asymptotic development of $K_\nu(z)$, cf. A.III of the Appendix) of [19], instead of formulas (I, 3; 22) and (I, 3; 23) and this process is equivalent to regularizing directly $[G(t, m=0, \alpha, n)]^{\Lambda}$.

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