Orbits of non-elliptic disc automorphisms on H^p

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Motivated by the *Invariant Subspace Problem*, we describe explicitly the closed subspace H^2 generated by the limit points in the H^2 norm of the orbit of a thin Blaschke product B under composition operators C_φ induced by non-elliptic automorphisms. This description exhibits a surprising connection to model spaces. Finally, we give a constructive characterization of the C_φ-eigenfunctions in H^p for $1 \leqslant p \leqslant \infty$.

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1. Introduction and preliminaries

Given a bounded linear operator T on a separable Banach space $\mathcal{B}$, the fact that an operator has invariant subspaces may not tell you much about the operator. On the other hand, knowing that an operator has a large number of invariant closed subspaces, and, in particular, a structured family, may make it possible to obtain a lot of information about the action of the operator on $\mathcal{B}$. In this context, it is helpful to focus on the behavior of the cyclic subspaces generated by the elements of $\mathcal{B}$ under T; in other words, focusing on the subspace

$$K_x = \overline{\mathrm{span}\{T^n x : n \geqslant 0\}}^{\mathcal{B}},$$

where $x \in \mathcal{B}$. Here T^0 denotes the identity operator I and $T^n = T \circ \overset{n \text{ times}}{\cdots} \circ T$.

It turns out that knowing the cyclic subspaces of concrete operators on an infinite dimensional separable Hilbert space $\mathcal{H}$ (operators that are *universal*) would solve the long-standing open question known as the Invariant Subspace Problem. Recall that an operator U on $\mathcal{H}$ is called universal, in the sense of Rota [14], if for any bounded linear T on $\mathcal{H}$, there exist a complex constant $\lambda \neq 0$ and a closed invariant subspace $\mathcal{M}$ of U such that $U|_{\mathcal{M}}$ is similar to λT.

Note that every bounded linear operator on an infinite dimensional separable Hilbert space $\mathcal{H}$ would have a non-trivial (closed) invariant subspace $\mathcal{M}$, that is, $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \neq \mathcal{H}$, if and only if the minimal invariant subspaces of a universal operator U on $\mathcal{H}$ are just one-dimensional. In the eighties, Nordgren, Rosenthal and Wintrobe [13] proved that if φ is a hyperbolic automorphism of the unit disc and λ is in the interior of the spectrum of the composition operator C_φ acting on the classical Hardy space H^2, then $C_\varphi - \lambda I$ is a universal operator on H^2 (see also [12]). Of course, the lattice of the closed invariant subspaces of $C_\varphi - \lambda I$ coincides with that of C_φ. Thus, it is important to study the closed invariant subspaces of C_φ in H^2 and, in particular, the cyclic subspaces generated by H^2 functions.

Our discussion turns naturally to the factorization of $f \in H^p$ into its inner and outer factor. The inner factor can be factored further, into a piece carrying all of its zeros (the Blaschke factor) and one with no zeros (the singular inner factor).

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If the zero sequence of the Blaschke product is an interpolating sequence, the Blaschke product is said to be interpolating. An important subclass of the interpolating Blaschke products is the set of thin Blaschke products; recall that a Blaschke product B with zeros $\{z_n\}_{n \geqslant 1}$ is said to be thin if

$$\prod_{n \neq k} \rho(z_n, z_k) \to 1$$

as $k \to \infty$, where ρ denotes the pseudo-hyperbolic distance in the open unit disc $\mathbb{D}$. When this holds, $\{z_n\}$ is called a thin sequence. Thin Blaschke products have the closest behavior to finite Blaschke products that we can expect from infinite ones.

When φ is a non-elliptic automorphism, in [8] the first two authors exhibited Blaschke products that are cyclic for C_φ by showing that the closed linear span of the limit points of their orbits is the whole space H^2. Clearly, this forces such Blaschke products to be infinite, since the limit points of orbits of finite Blaschke products are constant functions of modulus 1. Here, we consider an arbitrary thin Blaschke product and characterize the closed linear span of limit points of its orbit, which is trivially invariant for C_φ. Concretely, we prove the following (see Theorem 2.4): Let $L_\varphi(B)$ denote the set of limit points, in the H^2 norm, of the orbit of a thin Blaschke product B under the composition operator C_φ where φ is a non-elliptic automorphism. Then the H^2-closure of the linear span of $L_\varphi(B)$ is either

$$H^2 \quad \text{or} \quad (zbH^2)^\perp,$$

where b is a Blaschke product with simple (or no) zeros that satisfies $b \circ \varphi = \gamma b$ for some $\gamma \in \partial \mathbb{D}$. We then proceed to show that the same result holds for the Hardy spaces H^p when $1 \leqslant p < \infty$. We find the appearance of model spaces surprising, and we also see from this result that a natural question follows: What are the eigenfunctions of C_φ? It is easy to see that f is an eigenfunction if and only if the same is true for its Blaschke, singular and outer factors.

In [4] Cowen studied eigenfunctions for composition operators. Later, Matache [10] characterized the singular inner eigenfunctions of C_φ in terms of the behavior of pull-back measures (see also [11] for discrete singular inner functions). Our approach provides separate characterizations for Blaschke products, singular inner functions and outer functions in H^p that are eigenfunctions. The basic idea is the same in each of the three cases, but some of the technicalities are different.

The rest of the paper is organized as follows. Section 2 is devoted to studying the orbit of thin Blaschke products. In Section 3 we characterize the eigenfunctions of composition operators induced by non-elliptic disc automorphisms. Finally, and for the sake of completeness, we end this preliminary section by recalling some basic results and notation.

1.1. Notation and basic results

Throughout this paper the open unit disc of the complex plane will be denoted by $\mathbb{D}$ and $\partial \mathbb{D}$ will stand for its boundary. We will denote the space of holomorphic functions on $\mathbb{D}$ endowed with the topology of uniform convergence on compacta by $\mathcal{H}(\mathbb{D})$.

Recall that the Hardy space H^p, $1 \leqslant p < \infty$, consists of holomorphic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_p = \left(\sup_{0 \leqslant r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is finite. The space consisting of bounded analytic functions on $\mathbb{D}$ endowed with the sup-norm will be denoted by H^∞. A classical result due to Fatou states that every Hardy function f has non-tangential limit at $e^{i\theta} \in \partial \mathbb{D}$, except possibly on a set Lebesgue measure zero (see [6], for instance). Throughout this work, $f(e^{i\theta})$ will denote the non-tangential limit of f at $e^{i\theta}$.

Recall that an automorphism φ of $\mathbb{D}$ can be expressed in the form

$$\varphi(z) = e^{i\theta} \frac{p - z}{1 - \bar{p}z} \quad (z \in \mathbb{D}),$$

where $p \in \mathbb{D}$ and $-\pi < \theta \leqslant \pi$. Recall that φ is called *hyperbolic* if $|p| > \cos(\theta/2)$ (thus, φ fixes two points on $\partial \mathbb{D}$); *parabolic* if $|p| = \cos(\theta/2)$ (so, φ fixes just one point, located on $\partial \mathbb{D}$) and *elliptic* if $|p| < \cos(\theta/2)$ (therefore, φ fixes two points in the Riemann sphere, one in $\mathbb{D}$ and the other outside $\bar{\mathbb{D}}$, see [1], for example).

Throughout this paper, the involution that interchanges 0 and w will be denoted by

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z},$$

where $z \in \mathbb{D}$. For $w \in \mathbb{D}$, we shall always denote this automorphism by φ_w. The pseudo-hyperbolic and hyperbolic metrics for $z, w \in \mathbb{D}$ are given, respectively, by
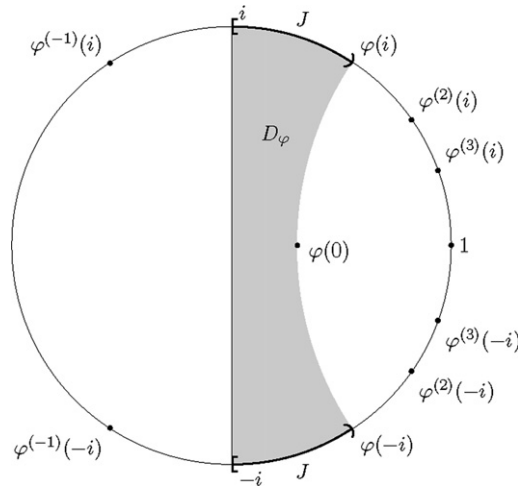
Fig. 1. A possible choice of D_φ when φ is a hyperbolic automorphism.
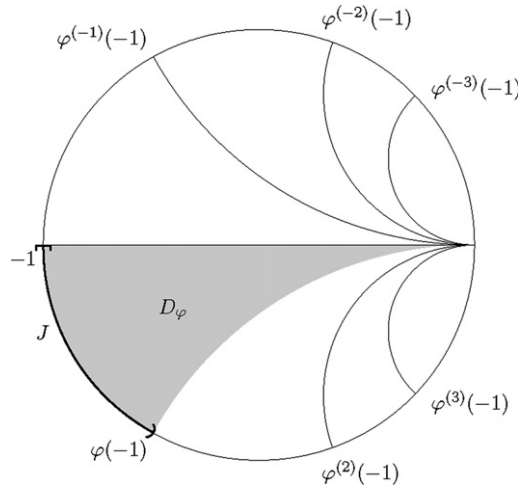

Fig. 2. A possible choice of D_φ when φ is a parabolic automorphism.

$$\rho(z, w) = \left| \varphi_w(z) \right| \quad \text{and} \quad \beta(z, w) = \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

and we will denote by $D_\rho(z, r)$ and $D_\beta(z, R)$ the respective closed balls of center z and radius r, with $0 \leqslant r < 1$ and $R \geqslant 0$.

If φ is a non-elliptic automorphism and $n \in \mathbb{Z}$ (an integer), we denote by $\varphi^{(n)}$ the $|n|$-th iterate of φ if $n > 0$, of φ^{-1} if $n < 0$, and the identity map if $n = 0$. The action on $\mathbb{D}$ of the group $\mathcal{G}_\varphi = \{\varphi^{(n)}: n \in \mathbb{Z}\}$ leads naturally to the consideration of the quotient space $\mathbb{D}/\mathcal{G}_\varphi$, where we are identifying points $z, w \in \mathbb{D}$ such that $\varphi^{(n)}(z) = w$ for some $n \in \mathbb{Z}$.

Since the class of $z \in \mathbb{D}$ in $\mathbb{D}/\mathcal{G}_\varphi$ is the bilateral orbit $o_\varphi(z) = \{\varphi^{(n)}(z): n \in \mathbb{Z}\}$, we can represent $\mathbb{D}/\mathcal{G}_\varphi$ by any subset $D_\varphi \subset \mathbb{D}$ such that $o_\varphi(z) \cap D_\varphi$ is a singleton for every $z \in \mathbb{D}$. Figs. 1 and 2 show reasonable choices of D_φ when φ is hyperbolic and parabolic, respectively. We can transfer the quotient topology of $\mathbb{D}/\mathcal{G}_\varphi$ to D_φ, so that the one-to-one correspondence becomes a homeomorphism.

This identification allows us to think of the quotient map $P : \mathbb{D} \to \mathbb{D}/\mathcal{G}_\varphi \simeq D_\varphi$ as a map onto D_φ; that is, $P(z) = o_\varphi(z) \cap D_\varphi$, defines a continuous map from $\mathbb{D}$ onto D_φ, where D_φ has the quotient topology. So, D_φ can be identified with a subset of $\mathbb{D}$, but endowed with the topology where a base of neighborhoods of a point $z \in D_\varphi$ is given by $\{w \in D_\varphi: \inf_{n \in \mathbb{Z}} |w - \varphi^{(n)}(z)| < \varepsilon\}$, for $\varepsilon > 0$. From Figs. 1 and 2 it is not difficult to see that in both cases D_φ is homeomorphic to a two-sided truncated cylinder without the upper and lower boundaries.

2. Orbits of thin Blaschke products

We study the orbit of thin Blaschke products under composition operators induced by non-elliptic automorphisms, characterizing the closed set of its limit points (in the H^2 norm).

If φ is a self-map of $\mathbb{D}$ and f is analytic in $\mathbb{D}$, that is, $f \in \mathcal{H}(\mathbb{D})$, the orbit of f is defined by $O_\varphi(f) = \{f \circ \varphi^{(n)}: n \geqslant 0\}$. When f is a bounded analytic function, that is, $f \in H^\infty$, the orbit $O_\varphi(f)$ is precompact in the compact-open topology of $\mathcal{H}(\mathbb{D})$, so it makes sense to define the limit set

$$L_\varphi(f) = \left\{g \in H^\infty: f \circ \varphi^{(n_k)} \to g \text{ in } \mathcal{H}(\mathbb{D}) \text{ for some subsequence } \{n_k\}\right\}.$$

By the corollary that follows Proposition 2 in [2], any sup-norm bounded sequence that converges in $\mathcal{H}(\mathbb{D})$ also converges weakly in H^p for $1 < p < \infty$. Thus the points in $L_\varphi(f)$ belong to the H^p-closure of the convex hull of $O_\varphi(f)$. As we will show later, more can be said in case f is a thin Blaschke product. We proceed with a technical lemma that will be needed for our description of $L_\varphi(B)$ when B is a thin Blaschke product.

Lemma 2.1. *Let $\{z_n\}$ in $\mathbb{D}$ be a sequence such that $|z_n| \to 1$, $0 \leqslant r_k < 1$ be any sequence and $0 < \delta_k < 1$ be a sequence that tends to 1. Then there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that any sequence of points $\xi_k \in D_\rho(z_{n_k}, r_k)$ satisfies*

$$\prod_{j \neq k} \rho(\xi_j, \xi_k) > \delta_k.$$

In particular, $\{\xi_k\}$ is thin.

Proof. Let $x_j \in (0, 1)$ be a thin sequence such that $\prod_{j \neq k} \rho(x_j, x_k) > \delta_k$. It will be enough to choose the points z_{n_k} so that $\rho(D_\rho(z_{n_k}, r_k), D_\rho(z_{n_j}, r_j)) \geqslant \rho(x_k, x_j) \overset{\text{def}}{=} \alpha_{k,j}$ for all $k \neq j$ or, equivalently,

$$\beta\left(D_\beta(z_{n_k}, R_k), D_\beta(z_{n_j}, R_j)\right) \geqslant \beta_{k,j} \quad \text{for } j < k,$$

where

$$R_k = \log \frac{1 + r_k}{1 - r_k} \quad \text{and} \quad \beta_{k,j} = \log \frac{1 + \alpha_{k,j}}{1 - \alpha_{k,j}}.$$

This is quite easy to do inductively. Once $z_{n_1}, \ldots, z_{n_{k-1}}$ are chosen, simply take z_{n_k} such that $\beta(z_{n_k}, \{z_{n_1}, \ldots, z_{n_{k-1}}\}) > R_k + \sum_{j=1}^{k-1}(R_j + \beta_{k,j})$. □

The preceding lemma resembles that of Wolff and Sundberg [15, Lemma 5.4], though we do not need to control the pseudo-hyperbolic distance from our given sequence as they do. Instead, we pass to a subsequence to obtain the properties we need. Our next result provides a description of all the sets in $\mathcal{H}(\mathbb{D})$ of the form $L_\varphi(B)$, where B is a thin Blaschke product. The description contains some undetermined constants that will be irrelevant later when we take linear combinations (in the proof of Theorem 2.4).

Proposition 2.2. *Let φ be a non-elliptic automorphism. If B is a thin Blaschke product, then there exists a nonempty set $V \subseteq \partial\mathbb{D}$ such that*

$$L_\varphi(B) = \{\lambda\varphi_w: w \in E, \text{ for some } \lambda \in \partial\mathbb{D}\} \cup V, \tag{2.1}$$

where $E \subset \mathbb{D}$ is closed in $\mathbb{D}$ with $\varphi(E) = E$. Conversely, given any such set E, there is a thin product B and a set V such that $L_\varphi(B)$ is given by (2.1). Also, given B,

$$E = \left\{w \in \mathbb{D}: \exists z_j \in Z(B) \text{ and integers } m_j \to +\infty \text{ such that } \varphi^{(-m_j)}(z_j) \to w\right\}.$$

Proof. Observe in advance that for B thin, every $\mathcal{H}(\mathbb{D})$-convergent subsequence of $\{B \circ \varphi^{(n)}\}$ tends either to $\lambda\varphi_w$ or λ, for some $\lambda \in \partial\mathbb{D}$ and $w \in \mathbb{D}$. This is because if $B \circ \varphi^{(n_k)} \to f$ in $\mathcal{H}(\mathbb{D})$, there are three possible situations: in the first one, $|f(0)| = 1$ and consequently $B \circ \varphi^{(n_k)} \to \lambda \in \partial\mathbb{D}$. If $f(0) = 0$, then the definition of thin Blaschke product along with Schwarz's lemma shows that there exists $\lambda \in \partial\mathbb{D}$ such that $f(z) = \lambda z$ (see [9, Proposition 2.3], for instance). Finally, if $\xi = f(0) \neq 0$ and $|\xi| < 1$, then a computation shows that $\varphi_\xi \circ B$ is thin and $\varphi_\xi \circ B \circ \varphi^{(n_k)} \to \gamma z$ for some $\gamma \in \partial\mathbb{D}$. Thus, $B \circ \varphi^{(n_k)} \to \varphi_\xi(\gamma z) = \lambda\varphi_w(z)$, for some $\lambda \in \partial\mathbb{D}$ and $w \in \mathbb{D}$.

First we show that if B is a thin product, there is some $\lambda \in \partial\mathbb{D}$ that is a limit point of $\{B \circ \varphi^{(n)}\}$. Otherwise, the maximum modulus principle implies that $\sup_{n \geqslant 0} |B(\varphi^{(n)}(0))| < \alpha$ for some $\alpha < 1$. Thus, there exists $R > 0$ such that

$$\varphi^{(n)}(0) \in \{z: |B(z)| < \alpha\} \subset \bigcup_{v \in Z(B)} D_\beta(v, R)$$

for all $n \geqslant 0$. This is true because a thin product B satisfies $|B(z)| \to 1$ as $\beta(z, Z(B)) \to \infty$ (see [7, Chapter X, Lemma 1.4]). Since $\{\varphi^{(n)}(0)\}$ is an interpolating sequence (see Section 3), the number of points contained in each of these balls, $D_\beta(v, R)$, must be bounded independently of the ball, say by m. Consider only the zeros v_k of B such that $D_\beta(v_k, R)$ contains some point $\varphi^{(n_k)}(0)$. Thus, at least one of the points $\varphi^{(n_k)}(0), \varphi^{(n_k+1)}(0), \ldots, \varphi^{(n_k+m)}(0)$ must be contained in a different ball, $D_\beta(v_j, R)$, with $j \neq k$. Hence,

$$\beta(v_k, v_j) \leqslant \beta\big(v_k, \varphi^{(n_k)}(0)\big) + \sum_{l=0}^{m-1} \beta\big(\varphi^{(n_k+l)}(0), \varphi^{(n_k+l+1)}(0)\big) + R$$

$$\leqslant R + \sum_{l=0}^{m-1} \beta\big(\varphi^{(0)}(0), \varphi^{(1)}(0)\big) + R$$

$$= 2R + m\beta\big(0, \varphi(0)\big),$$

where the second inequality holds because β is φ-invariant, and this happens for every k. On the other hand, since $\{v_k\}$ is a thin sequence, $\beta(v_k, \{v_j \colon j \neq k\}) \to \infty$ as $k \to \infty$, a contradiction.

Now suppose that $w \in E = \{w \in \mathbb{D} \colon \lambda \varphi_w \in L_\varphi(B) \text{ for some } \lambda \in \partial \mathbb{D}\}$. Then there is a sequence $\{n_k\}$ such that $B \circ \varphi^{(n_k)} \to \lambda \varphi_w$. So, for any $m \in \mathbb{Z}$, $B \circ \varphi^{(n_k+m)} \to \lambda(\varphi_w \circ \varphi^{(m)})$. Now, $\varphi_w \circ \varphi^{(m)}$ vanishes at $\varphi^{(-m)}(w)$, so taking $m = -1, 1$ we see that $\varphi^{(1)}(w), \varphi^{(-1)}(w) \in E$. Thus, $\varphi(E) = E$. The fact that E is closed in $\mathbb{D}$ follows from a diagonal argument: indeed, if $w_s \in E$ is a sequence that tends to $w \in \mathbb{D}$, then for each s there exists a sequence $\{n_k(s)\}$ such that $B \circ \varphi^{(n_k(s))} \to \lambda_s \varphi_{w_s}$ for some $\lambda_s \in \partial \mathbb{D}$, when $k \to \infty$. We may assume that the sequence λ_s converges to some $\lambda_0 \in \partial \mathbb{D}$, and then we can extract a sequence from $B \circ \varphi^{(n_k(s))}$ that tends to $\lambda_0 \varphi_w$, so $w \in E$.

To prove the converse, first consider the case in which $E = \emptyset$. Since $\varphi^{(n)}(0) \to \gamma \in \partial \mathbb{D}$, the attractive fixed point of φ, for every Blaschke product B with zeros that do not accumulate at γ we have $B \circ \varphi^{(n)} \to B(\gamma) \in \partial \mathbb{D}$ in $\mathcal{H}(\mathbb{D})$.

If $E \neq \emptyset$, choose a sequence $\{\alpha_j\}$ that is dense in E and change it to $\{w_k\}_{k \geqslant 1}$, given by

$$\alpha_1, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_3, \alpha_1, \ldots,$$

so that, as sets $\{w_k\}_{k \geqslant 1} = \{\alpha_j\}_{j \geqslant 1}$, and the set of limit points in $\mathbb{D}$ of the sequence $\{w_k\}$ is E. Write $\varphi^{(n)} = \lambda_n \varphi_{z_n}$, where $|\lambda_n| = 1$ and $z_n \in \mathbb{D}$ (every automorphism can be written in this form), and observe that $|z_n| \to 1$. Now use Lemma 2.1 to choose a subsequence $\{z_{-n_k}\}$ of $\{z_{-n}\}_{n \geqslant 0}$ such that any sequence with one point in each $D_\rho(z_{-n_k}, |w_k|)$ is thin. Consider

$$B(z) = \prod_{k \geqslant 1} \gamma_k \big(\varphi_{w_k} \circ \varphi^{(-n_k)}\big)(z) = \prod_{k \geqslant 1} \gamma_k \varphi_{w_k}\big(\lambda_{-n_k} \varphi_{z_{-n_k}}(z)\big),$$

where $\gamma_k \in \partial \mathbb{D}$ are chosen so that either each factor is positive at the origin or it is z. The zeros of B are $\varphi_{z_{-n_k}}(\bar{\lambda}_{-n_k} w_k) \in D_\rho(z_{-n_k}, |w_k|)$. If $w \in E$, there is a subsequence $w_{k_j} \to w$, and

$$B \circ \varphi^{(n_{k_j})} = \varphi_{w_{k_j}} \gamma_{k_j} \prod_{k \neq k_j} \gamma_k \big(\varphi_{w_k} \circ \varphi^{(-n_k+n_{k_j})}\big).$$

Since every $\mathcal{H}(\mathbb{D})$-convergent subsequence of $\{B \circ \varphi^{(n_{k_j})}\}$ tends either to $\lambda \varphi_{z_*}$ or λ, for some $\lambda \in \partial \mathbb{D}$ and $z_* \in \mathbb{D}$, and in our case $\varphi_{w_{k_j}} \to \varphi_w$, it follows that the convergent subsequences of $\{B \circ \varphi^{(n_{k_j})}\}$ tend to automorphisms of the form $\lambda \varphi_w$, for some $\lambda \in \partial \mathbb{D}$. On the other hand, if there is $\lambda \in \partial \mathbb{D}$ such that $\lambda \varphi_w \in L_\varphi(B)$, we will show that $w \in E$. For any integer m:

$$Z\big(B \circ \varphi^{(m)}\big) = \varphi^{(-m)}\big(Z(B)\big) = \big\{\varphi^{(-m+n_k)}(w_k) \colon k \geqslant 1\big\} \subset E,$$

where the last inclusion holds because $\{w_k\}_{k \geqslant 1} \subset E$ and $\varphi(E) = E$. If $\lambda \varphi_w \in L_\varphi(B)$ for some $\lambda \in \partial \mathbb{D}$ then $0 = \inf_{m \geqslant 0} \rho(w, Z(B \circ \varphi^{(m)})) \geqslant \rho(w, E)$, and consequently $w \in E$ because E is closed.

For the last assertion of the proposition, suppose that B is a thin product with associated set E. Then $w \in E$ if and only if there exists a sequence $m_j \to +\infty$ such that $B \circ \varphi^{(m_j)} \to \lambda \varphi_w$ for some $\lambda \in \partial \mathbb{D}$. This means that $B(\varphi^{(m_j)}(w)) \to 0$, and since B is an interpolating Blaschke product, this holds if and only if $\rho(\varphi^{(m_j)}(w), Z(B)) \to 0$ (see [7, p. 395]). That is, there is a sequence $\{z_j\}$ in $Z(B)$ such that $\rho(\varphi^{(m_j)}(w), z_j) = \rho(w, \varphi^{(-m_j)}(z_j)) \to 0$, or equivalently, $w = \lim \varphi^{(-m_j)}(z_j)$. $\quad\square$

Lemma 2.3. *Let B be a thin Blaschke product. Then the H^2-limit points of $O_\varphi(B)$ are precisely the functions in $L_\varphi(B)$.*

Proof. If g is an H^2-limit point of $O_\varphi(B)$, there is a sequence $B \circ \varphi^{(n_k)} \to g$ in H^2. It is clear that $g \in H^\infty$ and bearing in mind that norm convergence in H^2 implies uniform convergence on compacta, it follows that $g \in L_\varphi(B)$.

Conversely, if $g \in L_\varphi(B)$, there is a sequence $B \circ \varphi^{(n_k)} \to g$ uniformly on compacta. This, plus the fact that $\|B \circ \varphi^{(n_k)}\|_2 = 1$, implies that $B \circ \varphi^{(n_k)} \to g$ weakly in H^2. By Proposition 2.2, we have that either $g = \lambda \varphi_w$ or $g = \lambda$, for some $\lambda \in \partial \mathbb{D}$ and $w \in \mathbb{D}$. Hence,

$$\left\| B \circ \varphi^{(n_k)} - g \right\|_2^2 = 2 - 2\operatorname{Re}\langle B \circ \varphi^{(n_k)}, g \rangle \to 2\left(1 - \|g\|^2\right) = 0 \tag{2.2}$$

as $k \to \infty$, and the lemma follows. □

Now, we are in position to state the main result of this section. If $N \subset H^2$ is a closed subspace, we denote its orthogonal complement by $N^\perp$.

Theorem 2.4. *Let φ be a non-elliptic automorphism and B be a thin product. Write $\mathcal{H} = \overline{\operatorname{span}}\, L_\varphi(B)$ for the closure in H^2 of the linear span of $L_\varphi(B)$. Then*

$$\mathcal{H} = H^2 \quad \text{or} \quad \mathcal{H} = \left(zbH^2 \right)^\perp,$$

where b is a Blaschke product with simple (or no) zeros that satisfies $b \circ \varphi = \gamma b$ for some $\gamma \in \partial\mathbb{D}$. Conversely, if $\mathcal{H}$ is either of the above spaces, there is a thin product B such that $\overline{\operatorname{span}}\, L_\varphi(B) = \mathcal{H}$.

Proof. Since Proposition 2.2 says that $L_\varphi(B)$ contains a non-null constant and for $w \in \mathbb{D} \setminus \{0\}$,

$$\overline{w}\varphi_w(z) = \overline{w}\,\frac{w - z}{1 - \overline{w}z} = 1 - \frac{1 - |w|^2}{1 - \overline{w}z}, \tag{2.3}$$

the same proposition tells us that a subspace $\mathcal{H} \subset H^2$ has the form $\mathcal{H} = \overline{\operatorname{span}}\, L_\varphi(B)$ for B thin if and only if there is a relatively closed set $E \subset \mathbb{D}$, with $\varphi(E) = E$, such that

$$\mathcal{H} = \overline{\operatorname{span}}_{H^2}\left[\{1\} \cup \left\{ \frac{1}{1 - \overline{w}z} \colon w \in E \right\} \cup \{z \colon \text{if } 0 \in E\} \right].$$

We recall that the function $K_w(z) = (1 - \overline{w}z)^{-1}$ is the reproducing kernel in H^2 for $w \in \mathbb{D}$, meaning that $\langle f, K_w \rangle = f(w)$ for all $f \in H^2$. So, a function f is orthogonal to $\mathcal{H}$ if and only if $f(0) = 0$, $f(w) = 0$ for all $w \in E$, and $f'(0) = 0$ when $0 \in E$. That is,

$$\mathcal{H}^\perp = \left\{ f \in H^2 \colon f \in zH^2,\ f \equiv 0 \text{ on } E \text{ and } f \in z^2 H^2 \text{ if } 0 \in E \right\}.$$

There are four possibilities: $E = \emptyset$, $E \neq \emptyset$ is not a Blaschke sequence, or E is a Blaschke sequence, in which case we distinguish temporarily between the cases $0 \notin E$ and $0 \in E$. If $E = \emptyset$, then $\mathcal{H}^\perp = zH^2$. Since the zeros of a non-null function in H^2 form a Blaschke sequence, when E is not such sequence we have $\mathcal{H}^\perp = \{0\}$. If E is a Blaschke sequence, let b and b_0 be Blaschke products that satisfy $Z(b) = E$ and $Z(b_0) = E \setminus \{0\}$. If $0 \notin E$, we get $\mathcal{H}^\perp = zH^2 \cap bH^2 = zbH^2$. On the other hand, if $0 \in E$ we get $\mathcal{H}^\perp = b_0 H^2 \cap z^2 H^2 = z^2 b_0 H^2 = zbH^2$. Summing up,

1. $\mathcal{H} = H^2 \Leftrightarrow E$ is not a Blaschke sequence.
2. $\mathcal{H} = \mathbb{C} \Leftrightarrow E = \emptyset$.
3. $\mathcal{H} = (zbH^2)^\perp$, with $Z(b) = E \Leftrightarrow E$ is a Blaschke sequence, whether $0 \in E$ or not.

Finally, in the last case we have $Z(b \circ \varphi) = \varphi^{-1}(Z(b)) = \varphi^{-1}(E) = E = Z(b)$. Therefore $b \circ \varphi$ and b are Blaschke products with the same zeros, which means that $b \circ \varphi = \gamma b$ for some unimodular constant γ. □

By Theorem 2.4, the equality is attained in the inclusion $\overline{\operatorname{span}}\, L_\varphi(B) \subseteq \overline{\operatorname{span}}\, O_\varphi(B)$ if and only if $\overline{\operatorname{span}}\, L_\varphi(B) = H^2$ or $\overline{\operatorname{span}}\, L_\varphi(B) = (zbH^2)^\perp$, where b is a non-constant Blaschke product as in the theorem, and $B \in (zbH^2)^\perp$. This means that $\overline{z}\overline{b}B = \overline{z}\overline{v}$, where $v \in H^2$ is inner, or equivalently, that B divides b. Our next example shows that $\overline{\operatorname{span}}\, L_\varphi(B)$ can be much smaller than $\overline{\operatorname{span}}\, O_\varphi(B)$.

Example 2.5. There exist thin Blaschke products B such that

$$\overline{\operatorname{span}}\, O_\varphi(B) = H^2 \quad \text{and} \quad \overline{\operatorname{span}}\, L_\varphi(B) = \mathbb{C}.$$

Proof. Let $\{w_k\}$ be a sequence in D_φ such that $|w_k| \to 1$ and whose set of non-tangential accumulation points has positive measure. We show the existence of such a sequence as follows: take any maximal sequence in $\mathbb{D}$ that satisfies $\beta(w_j, w_k) \geqslant \delta > 0$ if $k \neq j$. Since the sequence is hyperbolically separated, it does not accumulate on $\mathbb{D}$. We claim that it accumulates non-tangentially at every point of $\partial\mathbb{D}$.

Otherwise, for every angle $0 < \alpha < \pi/2$, there is a point $\xi \in \partial\mathbb{D}$ such that some non-tangential sector with vertex ξ and half opening α does not contain any point w_k. If α is big enough we have that the hyperbolic distance of $r\xi$ to the sequence is larger than δ when $r < 1$ is close to 1. This contradicts the maximality of the sequence. Hence, the intersection of this sequence with D_φ satisfies the desired condition.

As we did in the proof of Proposition 2.2, we can use Lemma 2.1 to pick $n_k \to +\infty$ so that the Blaschke product

$$B(z) = \prod_{k \geqslant 1} \gamma_k \big(\varphi_{w_k} \circ \varphi^{(-n_k)}\big)(z)$$

with zeros $\varphi^{(n_k)}(w_k)$ is thin. Moreover, by Lemma 2.1 we can choose $n_k \to \infty$ so fast that B is as 'thin' as we wish, meaning that if $j \geqslant 1$ and $B_j := \prod_{k \neq j} \gamma_k(\varphi_{w_k} \circ \varphi^{(-n_k)})$, then $|B_j(\varphi^{(n_j)}(w_j))|$ is so close to 1 as we predefine (by choosing $\delta_j \to 1$ fast enough in Lemma 2.1). Furthermore, since for every $h \in H^\infty$, with $\|h\|_\infty \leqslant 1$, and $w \in \mathbb{D}$, the Schwarz–Pick inequality easily yields $\rho(|h(w)|, |w|) \leqslant |h(0)|$ (see [7, Chapter I, Corollary 1.3]), taking $h = B_j \circ \varphi^{(n_j)}$ and $w = w_j$, we can ensure that the right-hand side of

$$\rho\big(\big|B_j \circ \varphi^{(n_j)}(w_j)\big|, |w_j|\big) \leqslant \big|B_j \circ \varphi^{(n_j)}(0)\big|$$

is as close to 1 as desired. In particular, we can impose the condition

$$\frac{(1 - |B_j \circ \varphi^{(n_j)}(0)|^2)^{\frac{1}{2}}}{1 - |w_j|^2} \to 0. \tag{2.4}$$

We will check that $\overline{\mathrm{span}}\, O_\varphi(B) = H^2$. Suppose that $f \in H^2$ is orthogonal to $O_\varphi(B)$. By Proposition 2.2, $f \perp \mathbb{C}$. So, using (2.3), $f \perp (B \circ \varphi^{(n_k)})$, $f \perp \mathbb{C}$ and

$$B \circ \varphi^{(n_k)} = \big(B_k \circ \varphi^{(n_k)}\big)\gamma_k \varphi_{w_k} \quad (k \geqslant 1),$$

successively in the following chain of equalities, we get

$$\overline{\gamma_k B_k\big(\varphi^{(n_k)}(0)\big)} f(w_k) = \frac{1}{1 - |w_k|^2}\big\langle f, (1 - \overline{w}_k \varphi_{w_k})\gamma_k \big(B_k \circ \varphi^{(n_k)}\big)(0)\big\rangle$$

$$= \frac{1}{1 - |w_k|^2}\big\langle f, \overline{w}_k \big(B \circ \varphi^{(n_k)}\big) + (1 - \overline{w}_k \varphi_{w_k})\gamma_k \big(B_k \circ \varphi^{(n_k)}\big)(0)\big\rangle$$

$$= \Big\langle f, \frac{\overline{w}_k}{1 - |w_k|^2}\big(B \circ \varphi^{(n_k)}\big) - \frac{\gamma_k \overline{w}_k}{1 - |w_k|^2}\varphi_{w_k}\big(B_k \circ \varphi^{(n_k)}\big)(0)\Big\rangle$$

$$= \frac{w_k}{1 - |w_k|^2}\big\langle f, \gamma_k \varphi_{w_k}\big[\big(B_k \circ \varphi^{(n_k)}\big) - \big(B_k \circ \varphi^{(n_k)}\big)(0)\big]\big\rangle.$$

Consequently

$$\big|B_k\big(\varphi^{(n_k)}(0)\big)\big|\big|f(w_k)\big| \leqslant \frac{|w_k|}{1 - |w_k|^2}\|f\|_2\big\|\big(B_k \circ \varphi^{(n_k)}\big) - \big(B_k \circ \varphi^{(n_k)}\big)(0)\big\|_2$$

$$= \frac{|w_k|}{1 - |w_k|^2}\|f\|_2\big(1 - \big|\big(B_k \circ \varphi^{(n_k)}\big)(0)\big|^2\big)^{\frac{1}{2}} \to 0$$

by (2.4); that is, $f(w_k) \to 0$. Since the sequence $\{w_k\}$ accumulates non-tangentially on a set of positive measure in $\partial\mathbb{D}$, the non-tangential limits of f must vanish on a set of positive measure, implying that $f = 0$. This proves our claim.

The equality $\overline{\mathrm{span}}\, L_\varphi(B) = \mathbb{C}$ will follow from the last assertion of Proposition 2.2 if we show that the set E associated with B is empty. So, suppose that $w \in E$. Since the zeros of B are $\varphi^{(n_k)}(w_k)$, the proposition says that there are integers $m_j, k_j \to +\infty$ such that $\varphi^{(-m_j)}(\varphi^{(n_{k_j})}(w_{k_j})) \to w$. Applying the quotient map $P : \mathbb{D} \to D_\varphi$, we obtain

$$\lim w_{k_j} = \lim P\big(\varphi^{(-m_j + n_{k_j})}(w_{k_j})\big) = P(w) \in D_\varphi,$$

where the limit is taken in the D_φ topology. This is not possible, because $|w_{k_j}| \to 1$. $\quad\square$

2.1. Generalization to Hardy spaces

The proof of Theorem 2.4 is more natural and transparent in the context of the Hilbert space H^2, but with very minor modifications we can see that both Lemma 2.3 and Theorem 2.4 are valid for H^p when $1 \leqslant p < \infty$. We recall that the composition operator C_φ is bounded on H^p for $p \geqslant 1$ (see [5, p. 121]). To see that the lemma holds simply replace (2.2) by

$$\big\|B \circ \varphi^{(n_k)} - g\big\|_p^p \leqslant \big\|B \circ \varphi^{(n_k)} - g\big\|_{2(p-1)}^{p-1}\big\|B \circ \varphi^{(n_k)} - g\big\|_2 \leqslant 2^{p-1}\big\|B \circ \varphi^{(n_k)} - g\big\|_2,$$

which by (2.2) tends to 0 as $k \to \infty$, where the first inequality is the Cauchy–Schwarz inequality and the second holds because $|B \circ \varphi^{(n_k)} - g| \leqslant 2$.

Let $1 < p < \infty$ and $N \subset H^p$ be a subspace. The annihilator of N is

$$N^{\perp} = \left\{ f \in H^q \colon \langle h, f \rangle = 0 \text{ for all } h \in N \right\},$$

where $1/p + 1/q = 1$. Mimicking the proof of Theorem 2.4 we see that if φ is a non-elliptic automorphism and B is a thin product, a space $\mathcal{H} \subset H^p$ has the form $\mathcal{H} = \overline{\operatorname{span}}_{H^p} L_{\varphi}(B)$ if and only if it is closed and $\mathcal{H}^{\perp} = \{0\} \subset H^q$ or $\mathcal{H}^{\perp} = zbH^q$, where b is a Blaschke product as in Theorem 2.4. This is the same as saying that

$$\mathcal{H} = H^p \quad \text{or} \quad \mathcal{H} = \left(zbH^q\right)^{\perp} = H^p \cap b\overline{H}^p,$$

where the bar means complex conjugation. This is the H^p version of Theorem 2.4 for $1 < p < \infty$. The result for H^1 requires a different argument. Since the first of the following spaces

$$\operatorname{span} L_{\varphi}(B) \subset \overline{\operatorname{span}}_{H^2} L_{\varphi}(B) \subset \overline{\operatorname{span}}_{H^1} L_{\varphi}(B) := \mathcal{H}$$

is dense in the latter, so is the one in middle, which by Theorem 2.4, is either H^2 or $H^2 \cap b\overline{H}^2$, with both cases occurring. In the first case $\mathcal{H} = H^1$ and in the second, $\mathcal{H}$ is the H^1-norm closure of $H^2 \cap b\overline{H}^2$, which by Lemma 5.8.14 of [3] is $H^1 \cap b\overline{H}^1$. This gives Theorem 2.4 for H^1.

3. A constructive characterization of eigenvectors

While our work in this section is related to those in [4, Section 4], our approach to the problem is quite different. The function $h(z) = i\frac{1+z}{1-z}$ maps $\mathbb{D}$ onto $\mathbb{C}_+ = \{v \in \mathbb{C} \colon \operatorname{Im} v > 0\}$, with $h^{-1}(v) = \frac{v-i}{v+i}$. If φ is a hyperbolic automorphism of $\mathbb{D}$, by conjugating C_{φ} with an invertible operator, we can assume that its fixed points are -1 and 1, where 1 is the attractive point. Thus, $\varphi = h^{-1} \circ \tilde{\varphi} \circ h$, where $\tilde{\varphi} \colon \mathbb{C}_+ \to \mathbb{C}_+$ is $\tilde{\varphi}(w) = \alpha w$, with $\alpha > 1$ (see [1], for instance). A straightforward calculation shows that

$$\varphi(z) = -\varphi_{\frac{1-\alpha}{1+\alpha}}(z) \quad \text{and} \quad \varphi^{(n)}(z) = -\varphi_{\frac{1-\alpha^n}{1+\alpha^n}}(z), \quad \forall n \in \mathbb{Z}. \tag{3.5}$$

The same argument with $\varphi \colon \mathbb{D} \to \mathbb{D}$ parabolic, where we now assume that its fixed point is 1, and therefore $\tilde{\varphi}(w) = w + t$, with $t \in \mathbb{R} \setminus \{0\}$, shows that

$$\varphi(z) = \left(\frac{t-2i}{t+2i}\right)\varphi_{\frac{t}{t-2i}}(z) \quad \text{and} \quad \varphi^{(n)}(z) = \left(\frac{nt-2i}{nt+2i}\right)\varphi_{\frac{nt}{nt-2i}}(z), \quad \forall n \in \mathbb{Z}. \tag{3.6}$$

We can further assume that $t > 0$, since otherwise the treatment is analogous.

From the above expressions for $\tilde{\varphi}$, it is easy to see in both cases that if $w_0 \in \mathbb{C}_+$ then $\{\tilde{\varphi}^{(n)}(w_0) \colon n \in \mathbb{Z}\}$ is an interpolating sequence for $H^{\infty}(\mathbb{C}_+)$ (see [7, Chapter VII]), so $\{\varphi^{(n)}(z_0) \colon n \in \mathbb{Z}\}$ is an interpolating sequence for $H^{\infty}(\mathbb{D})$ for any fixed $z_0 \in \mathbb{D}$.

3.1. Blaschke product eigenvectors

It is clear that a Blaschke product b satisfies $b \circ \varphi = \gamma b$ for some $\gamma \in \partial\mathbb{D}$ if and only if b and $b \circ \varphi$ have the same zeros. This means that for every zero w of b, $\{\varphi^{(n)}(w) \colon n \in \mathbb{Z}\}$ are zeros of b, each one with the same multiplicity as w. Hence, b is solely determined by the sequence of its zeros in D_{φ}. So, a characterization of the Blaschke products that are eigenvectors is tantamount to a characterization of the Blaschke sequences $\{z_k\}_{k \geqslant 1}$ in D_{φ} such that $\{\varphi^{(n)}(z_k) \colon k \geqslant 1, \; n \in \mathbb{Z}\}$ is also a Blaschke sequence. We have to distinguish between the hyperbolic and the parabolic case.

Theorem 3.1. *Let $\{z_k\}_{k \geqslant 1}$ be Blaschke sequence in D_{φ}. Then $\{\varphi^{(n)}(z_k) \colon k \geqslant 1, \; n \in \mathbb{Z}\}$ is a Blaschke sequence*

(1) *always when φ is hyperbolic,*
(2) *if and only if $\{z_k\}$ stays outside of some horocycle tangent to $\partial\mathbb{D}$ at the fixed point of φ, when φ is parabolic.*

Proof. Geometrically, it will be more convenient to look at things in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$. Hence, the Blaschke condition for the sequence $z_k = x_k + iy_k$ is $\sum \frac{y_k}{1+|z_k|^2} < \infty$. To prove (1) we can assume that φ is a hyperbolic automorphism that fixes 0 and ∞, and therefore has the form $\varphi(w) = \alpha w$ with $1 \neq \alpha > 0$. We can also assume that $\alpha > 1$, since the proof is the same in both cases. In this case, we can take $D_{\varphi} = \{z \in \mathbb{C}_+ \colon 1 \leqslant |z| < \alpha\}$, and we must show that if $\{z_k\}_{k \geqslant 1}$ in D_{φ} is a Blaschke sequence then so is $\{\alpha^n z_k \colon k \geqslant 1, \; n \in \mathbb{Z}\}$. Hence,

$$\sum_{k\geqslant 1,\, n\in\mathbb{Z}} \frac{\alpha^n y_k}{1+|\alpha^n z_k|^2} \leqslant \sum_{n\geqslant 0}\sum_{k\geqslant 1} \frac{\alpha^n y_k}{\alpha^{2n}|z_k|^2} + \sum_{n<0}\sum_{k\geqslant 1} \alpha^n y_k$$

$$\leqslant \left[\sum_{n\geqslant 0}\frac{1}{\alpha^n} + \sum_{n<0}\alpha^n\right]\sum_{k\geqslant 0} y_k \quad (\text{since } |z_k|\geqslant 1)$$

$$\leqslant \left[\frac{\alpha+1}{\alpha-1}\right]\sum_{k\geqslant 0}\frac{y_k(1+\alpha^2)}{1+|z_k|^2} < \infty \quad (\text{since } |z_k|<\alpha).$$

To prove (2) we can assume that φ is a parabolic automorphism that fixes ∞, and therefore has the form $\varphi(w)=w+t$, where $t\in\mathbb{R}$, and we can also assume that $t>0$, since the proof is the same in both cases.

In the case at hand, we take $D_\varphi = \{z\in\mathbb{C}_+ \colon 0\leqslant \operatorname{Re} z < t\}$, and show that given a Blaschke sequence $\{z_k\}_{k\geqslant 1}$ in D_φ, $\{z_k + nt \colon k\geqslant 1,\ n\in\mathbb{Z}\}$ is a Blaschke sequence if and only if the sequence $\{y_k\}$ is bounded.

First suppose that there is some constant $C>0$ such that $y_k\leqslant C$ for all $k\geqslant 1$. Since $|x_k+nt|\geqslant |n|t-x_k\geqslant (|n|-1)t$, we have that $(x_k+nt)^2\geqslant (|n|-1)^2 t^2$ for all $n\neq 0$. Consequently, if $n\in\mathbb{Z}\setminus\{0\}$,

$$\frac{1+|z_k|^2}{1+|z_k+nt|^2} \leqslant \frac{1+t^2+C^2}{1+(|n|-1)^2 t^2},$$

leading to

$$\sum_{k\geqslant 1,\, n\in\mathbb{Z}\setminus\{0\}} \frac{y_k}{1+|z_k+nt|^2} = \sum_{k\geqslant 1,\, n\in\mathbb{Z}\setminus\{0\}} \frac{y_k}{1+|z_k|^2}\frac{1+|z_k|^2}{1+|z_k+nt|^2}$$

$$\leqslant \sum_{k\geqslant 1}\frac{y_k}{1+|z_k|^2}\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1+t^2+C^2}{1+(|n|-1)^2 t^2},$$

which is convergent. Obviously, the sum is also bounded for $n=0$.

If the sequence $\{y_k\}$ is not bounded, fix an arbitrary $y_k\geqslant t+1$ and consider all the values of $n\in\mathbb{Z}$ such that $-\frac{y_k}{t}\leqslant -[\frac{y_k}{t}]\leqslant n\leqslant [\frac{y_k}{t}]\leqslant \frac{y_k}{t}$, where $[a]$ denotes the largest integer $\leqslant a$. Since $0\leqslant x_k<t$, we have

$$-y_k\leqslant x_k-y_k\leqslant x_k+nt\leqslant x_k+y_k\leqslant t+y_k\leqslant 2y_k,$$

and consequently $(x_k+nt)^2\leqslant (2y_k)^2$. Thus,

$$\sum_{-[\frac{y_k}{t}]\leqslant n\leqslant [\frac{y_k}{t}]} \frac{y_k}{1+|z_k+nt|^2} \geqslant \frac{1}{y_k}\sum_{-[\frac{y_k}{t}]\leqslant n\leqslant [\frac{y_k}{t}]}\frac{y_k^2}{1+5y_k^2}$$

$$\overset{(y_k\geqslant 1)}{\geqslant} \frac{1}{y_k}\sum_{-[\frac{y_k}{t}]\leqslant n\leqslant [\frac{y_k}{t}]}\frac{1}{6}$$

$$= \frac{1}{6y_k}\left(2\left[\frac{y_k}{t}\right]+1\right)$$

$$\geqslant \frac{1}{6y_k}\left(\frac{y_k}{t}\right) = \frac{1}{6t}.$$

Consequently,

$$\sum_{k\geqslant 1,\, n\in\mathbb{Z}} \frac{y_k}{1+|z_k+nt|^2} \geqslant \sum_{y_k\geqslant t+1}\sum_{-[\frac{y_k}{t}]\leqslant n\leqslant [\frac{y_k}{t}]}\frac{y_k}{1+|z_k+nt|^2} \geqslant \sum_{y_k\geqslant t+1}\frac{1}{6t},$$

which is infinite if there are infinitely many $y_k\geqslant t+1$. □

3.2. Outer eigenfunctions

If φ is given by (3.5) (the hyperbolic case) consider the circular intervals $J=(\varphi(i),i]\cup[-i,\varphi(-i))$ of $\partial\mathbb{D}$, and if φ is given by (3.6) (the parabolic case) consider $J=[-1,\varphi(-1))$ (see Figs. 1 and 2).

Lemma 3.2. *If φ is given by* (3.5) *then*

$$\frac{1}{4}\alpha^{-|n|} \leqslant \left|\varphi^{(n)'}(w)\right| \leqslant (\alpha+1)^2\alpha^{-|n|}, \quad \forall w\in J,\ \forall n\in\mathbb{Z}. \tag{3.7}$$

If φ is given by (3.6) *then there is a constant $c(t) > 0$ such that*

$$\frac{1}{n^2 t^2 + 4} \leqslant |\varphi^{(n)'}(w)| \leqslant \frac{c(t)}{n^2 t^2 + 4}, \quad \forall w \in J, \ \forall n \in \mathbb{Z}. \tag{3.8}$$

Proof. If φ comes from (3.5),

$$|\varphi^{(n)'}(w)| = \frac{1 - |\frac{1-\alpha^n}{1+\alpha^n}|^2}{|1 - \frac{1-\alpha^n}{1+\alpha^n} w|^2} = \frac{\frac{4\alpha^n}{(1+\alpha^n)^2}}{|1 - \frac{1-\alpha^n}{1+\alpha^n} w|^2}.$$

When $w \in J$ and $n \in \mathbb{Z}$, $\frac{1-\alpha^n}{1+\alpha^n} w$ remains in the angular sector $\{re^{i\theta}: 0 \leqslant r \leqslant 1, \ \arg\varphi(i) < \theta < \arg\varphi(-i)\}$. Since $\mathbb{D} \cap \partial D_\varphi$ is orthogonal to $\partial\mathbb{D}$ (see Fig. 1), it follows that

$$\frac{2^2}{(\alpha+1)^2} = |1 - \varphi(0)|^2 \leqslant \left| 1 - \frac{1-\alpha^n}{1+\alpha^n} w \right|^2 \leqslant 2^2.$$

For $n \in \mathbb{Z}$,

$$\alpha^{-|n|} \leqslant \frac{4\alpha^n}{(1+\alpha^n)^2} \leqslant 4\alpha^{-|n|},$$

so combining the above inequalities we obtain the desired result. If φ comes from (3.6),

$$|\varphi^{(n)'}(w)| = \frac{1 - |\frac{nt}{nt-2i}|^2}{|1 - \frac{nt}{nt-2i} w|^2} = \frac{\frac{4}{n^2 t^2 + 4}}{|1 - \frac{nt}{nt-2i} w|^2}.$$

When $w \in [-1, \varphi(-1))$, Fig. 2 shows that

$$
\begin{aligned}
2 \geqslant \left| \overline{w} - \frac{nt}{nt-2i} \right| &\geqslant |\varphi(-1) - 1| - \left| 1 - \frac{nt}{nt-2i} \right| \\
&= 2\left[\frac{1}{(t^2+1)^{1/2}} - \frac{1}{(n^2 t^2 + 4)^{1/2}} \right] \\
&\overset{\text{if } n \neq 0}{\geqslant} 2\left[\frac{1}{(t^2+1)^{1/2}} - \frac{1}{(t^2+4)^{1/2}} \right].
\end{aligned}
$$

The claim follows for $n \neq 0$ by inserting the above inequalities in the expression of $|\varphi^{(n)'}(w)|$. Since $|\varphi^{(0)'}(w)| = 1$, it also follows for $n = 0$ by taking $c(t) > 4$. $\quad\square$

Lemma 3.3. *Let $1 \leqslant p < \infty$ and φ be a non-elliptic automorphism. If $f_0 \geqslant 0$ belongs to $L^p(J)$ and $\log f_0 \in L^1(J)$ with respect to the linear Lebesgue measure $|dz|$, then for $\lambda > 0$,*

$$f(z) := \sum_{n \in \mathbb{Z}} \chi_{\varphi^{(n)}(J)}(z) \lambda^n f_0\big(\varphi^{(-n)}(z)\big)$$

is in $L^p(\partial\mathbb{D})$ and $\log f \in L^1(\partial\mathbb{D})$ if and only if

(1) $\frac{1}{\sqrt[p]{\alpha}} < \lambda < \sqrt[p]{\alpha}$ *in the hyperbolic case* (*i.e.: when φ is given by* (3.5)),
(2) $\lambda = 1$ *in the parabolic case* (*i.e.: when φ is given by* (3.6)).

Furthermore, $f \circ \varphi = \lambda f$ on $\partial\mathbb{D}$ in both cases.

Proof. Clearly f is defined almost everywhere on $\partial\mathbb{D}$ so that $f|_J = f_0$ and $(f \circ \varphi^{(n)})(z) = \lambda^n f(z)$ for all $n \in \mathbb{Z}$ (see Figs. 1 and 2). Thus,

$$
\begin{aligned}
\int_{\varphi^{(n)}(J)} f(z)^p \, |dz| &= \int_J f\big(\varphi^{(n)}(w)\big)^p |\varphi^{(n)'}(w)| \, |dw| \\
&= \int_J \lambda^{pn} f(w)^p |\varphi^{(n)'}(w)| \, |dw|.
\end{aligned}
\tag{3.9}
$$

In the hyperbolic case we insert the inequalities (3.7) in the above expression to obtain a constant $C(\alpha) > 0$ such that

$$\frac{1}{4}\frac{\lambda^{pn}}{\alpha^{|n|}}\int\limits_J f(w)^p\,|dw| \leqslant \int\limits_{\varphi^{(n)}(J)} f(z)^p\,|dz| \leqslant C(\alpha)\frac{\lambda^{pn}}{\alpha^{|n|}}\int\limits_J f(w)^p\,|dw|.$$

Summing over $n \in \mathbb{Z}$,

$$\frac{1}{4}\sum_{n\in\mathbb{Z}}\frac{\lambda^{pn}}{\alpha^{|n|}}\int\limits_J f^p \leqslant \int\limits_{\partial\mathbb{D}} f^p \leqslant C(\alpha)\sum_{n\in\mathbb{Z}}\frac{\lambda^{pn}}{\alpha^{|n|}}\int\limits_J f^p.$$

For the convergence of the sum it is necessary and sufficient that $\lambda^p/\alpha < 1$ and $\lambda^p\alpha > 1$; that is, $\frac{1}{\sqrt[p]{\alpha}} < \lambda < \sqrt[p]{\alpha}$. The same idea, with f^p replaced by $\log f$, gives

$$\frac{1}{4}\sum_{n\in\mathbb{Z}}\frac{1}{\alpha^{|n|}}\int\limits_J (\log f + n\log\lambda) \leqslant \int\limits_{\partial\mathbb{D}} \log f \leqslant C(\alpha)\sum_{n\in\mathbb{Z}}\frac{1}{\alpha^{|n|}}\int\limits_J (\log f + n\log\lambda),$$

which converges because $\alpha > 1$.

In the parabolic case we insert the inequalities (3.8) in (3.9) to obtain a constant $c(t) > 0$ such that

$$\frac{\lambda^{pn}}{n^2t^2+4}\int\limits_J f(w)^p|dw| \leqslant \int\limits_{\varphi^{(n)}(J)} f(z)^p|dz| \leqslant c(t)\frac{\lambda^{pn}}{n^2t^2+4}\int\limits_J f(w)^p|dw|.$$

Summing over $n \in \mathbb{Z}$,

$$\sum_{n\in\mathbb{Z}}\frac{\lambda^{pn}}{n^2t^2+4}\int\limits_J f^p \leqslant \int\limits_{\partial\mathbb{D}} f^p \leqslant c(t)\sum_{n\in\mathbb{Z}}\frac{\lambda^{pn}}{n^2t^2+4}\int\limits_J f^p.$$

The sum converges if and only if $\lambda = 1$. Finally, observing that for $\lambda = 1$ we have $\log(f \circ \varphi) = \log f$, the same proof, with f^p replaced by $\log f$, gives

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2t^2+4}\int\limits_J \log f \leqslant \int\limits_{\partial\mathbb{D}} \log f \leqslant c(t)\sum_{n\in\mathbb{Z}}\frac{1}{n^2t^2+4}\int\limits_J \log f,$$

which converges. $\square$

Theorem 3.4. *Let $1 \leqslant p < \infty$ and φ be a non-elliptic automorphism. There is an outer function*

$$F(z) = \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log f(e^{i\theta})\frac{d\theta}{2\pi}\right) \in H^p$$

such that $F \circ \varphi = \gamma\lambda F$ for $\lambda > 0$ and some $\gamma \in \partial\mathbb{D}$ if and only if

(1) $\frac{1}{\sqrt[p]{\alpha}} < \lambda < \sqrt[p]{\alpha}$ *in the hyperbolic case* (*where φ is given by* (3.5)),
(2) $\lambda = 1$ *in the parabolic case,*

and f is given by Lemma 3.3. Moreover, γ depends on φ and f_0 but it is independent of λ in both cases.

Proof. Since $|F(\xi)| = f(\xi)$ for almost every $\xi \in \partial\mathbb{D}$, we also have $|F(\varphi(\xi))| = f(\varphi(\xi)) = \lambda f(\xi)$, where the last equality comes from the lemma. That means that the outer functions $F \circ \varphi$ and λF have the same modulus on $\partial\mathbb{D}$, and consequently they differ by a multiplicative constant of modulus one (see [7, Chapter II, Theorem 4.6]). To complete the proof, notice that

$$\gamma\lambda = \frac{F(\varphi(0))}{F(0)} = \exp\left(\int\limits_{-\pi}^{\pi}\left[\frac{e^{i\theta}+\varphi(0)}{e^{i\theta}-\varphi(0)}-1\right]\log f(e^{i\theta})\frac{d\theta}{2\pi}\right)$$

$$= \exp\left(\int\limits_{\partial\mathbb{D}}\frac{2\varphi(0)}{e^{i\theta}-\varphi(0)}\log f(e^{i\theta})\frac{|d(e^{i\theta})|}{2\pi}\right).$$

So, if we partition $\partial\mathbb{D} = \bigcup_{n\in\mathbb{Z}}\varphi^{(n)}(J)$ and change variables (as in (3.9)), we obtain

$$\gamma = \exp\left(i \int_{\partial \mathbb{D}} \operatorname{Im} \frac{2\varphi(0)}{e^{i\theta} - \varphi(0)} \log f\left(e^{i\theta}\right) \frac{|d(e^{i\theta})|}{2\pi} \right)$$

$$= \exp\left(i2\varphi(0) \int_J \sum_{n \in \mathbb{Z}} \operatorname{Im} \frac{|\varphi^{(n)'}(e^{i\theta})|}{\varphi^{(n)}(e^{i\theta}) - \varphi(0)} \left[\log f_0\left(e^{i\theta}\right) + n \log \lambda \right] \frac{|d(e^{i\theta})|}{2\pi} \right)$$

$$= \exp\left(i2\varphi(0) \int_J \log f_0\left(e^{i\theta}\right) \sum_{n \in \mathbb{Z}} \operatorname{Im} \frac{|\varphi^{(n)'}(e^{i\theta})|}{\varphi^{(n)}(e^{i\theta}) - \varphi(0)} \frac{|d(e^{i\theta})|}{2\pi} \right),$$

where the last equality holds in the hyperbolic case because J is symmetric with respect to the real axis, $\varphi(0) \in \mathbb{R}$ and $\varphi^{(n)}(e^{-i\theta}) = \overline{\varphi^{(n)}(e^{i\theta})}$, and in the parabolic case because $\lambda = 1$. $\square$

Remark 3.5. The case $p = \infty$ is easier. If a bounded function is an eigenvector corresponding to an eigenvalue a, then $|a| = 1$. On the other hand, if $f_0 \in L^{\infty}(J)$, $\log f_0 \in L^1(J)$ and $\lambda = 1$ in Lemma 3.3, it is clear that the function f of the lemma is bounded and the same proof shows that $\log f \in L^1(\partial \mathbb{D})$. Thus, Theorem 3.4 holds for $p = \infty$ with $\lambda = 1$ in both the hyperbolic and parabolic case.

3.3. Singular inner eigenvectors

For μ a positive, finite, singular measure on $\partial \mathbb{D}$, its associated singular inner function is

$$S_{\mu}(z) = \exp\left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu\left(e^{i\theta}\right) \right).$$

The following lemma was proved by Matache in [10] as a means to characterize the singular inner eigenfunctions in terms of pull-back measures. We give a different proof.

Lemma 3.6. Let φ be any automorphism and ν be a (finite positive) singular measure on $\partial \mathbb{D}$. Then there exists a unique singular measure μ that satisfies $|S_{\nu}(\varphi^{-1}(z))| = |S_{\mu}(z)|$ for $z \in \mathbb{D}$, and it is given by

$$\mu(E) = \int_{\varphi^{-1}(E)} |\varphi'| \, d\nu \tag{3.10}$$

for Borel sets $E \subset \partial \mathbb{D}$.

Proof. The existence holds because $S_{\nu}(\varphi^{-1}(z))$ is a singular inner function, so there must exist a singular measure μ and $\lambda \in \partial \mathbb{D}$ such that $S_{\nu}(\varphi^{-1}(z)) = \lambda S_{\mu}(z)$. The uniqueness follows because two singular inner functions with the same modulus have the same associated measure (see [7, p. 70]). To prove the last statement consider first the case $\nu = \delta_{\xi}$, the Dirac measure concentrated at a point $\xi \in \partial \mathbb{D}$. If μ_{ξ} is the measure that satisfies $|S_{\delta_{\xi}}(\varphi^{-1}(z))| = |S_{\mu_{\xi}}(z)|$, the first function extends continuously to $\partial \mathbb{D} \setminus \{\varphi(\xi)\}$ as the constant 1, and thus the same holds for $|S_{\mu_{\xi}}(z)|$, which means that $\mu_{\xi} = c\delta_{\varphi(\xi)}$ for some constant $c > 0$. Moreover,

$$c = \mu_{\xi}(\mathbb{D}) = -\log|S_{\mu_{\xi}}(0)| = -\log|S_{\delta_{\xi}}(\varphi^{-1}(0))| = \frac{1 - |\varphi^{-1}(0)|^2}{|\xi - \varphi^{-1}(0)|^2} = |\varphi'(\xi)|.$$

So, $\mu_{\xi} = |\varphi'(\xi)|\delta_{\varphi(\xi)}$, which is the measure defined by (3.10) for $\nu = \delta_{\xi}$. For an arbitrary singular measure ν, write $\nu = \int \delta_{\xi} \, d\nu(\xi)$ and consider the singular measure $\mu := \int \mu_{\xi} \, d\nu(\xi)$, where the integrals converge weak-$*$ in the space of finite Borel measures. Since μ_{ξ} is the measure that satisfies (3.10) for δ_{ξ}, then μ is the measure that satisfies (3.10) for ν. Moreover, since the map $\sigma \mapsto P_z(\sigma) = -\log|S_{\sigma}(z)|$ (the Poisson integral of σ) is linear,

$$-\log|S_{\nu}(\varphi^{-1}(z))| = P_{\varphi^{-1}(z)}(\nu) = \int P_{\varphi^{-1}(z)}(\delta_{\xi}) \, d\nu(\xi) = \int P_z(\mu_{\xi}) \, d\nu(\xi) = P_z(\mu) = -\log|S_{\mu}(z)|,$$

where we can take the integral of measures outside of the Poisson integral because the kernel of P_w is continuous on $\partial \mathbb{D}$ for every $w \in \mathbb{D}$, and the equality in the middle is proved above. $\square$

In order to present a statement that is as clear as possible, we allow the possibility of $\nu \equiv 0$ as a singular measure in the next corollary, and we interpret this to mean that $S_{\nu} \equiv 1$.

Corollary 3.7. Let φ be a non-elliptic automorphism and ν be a non-negative measure supported on the fixed points of φ. Then $|S_{\nu}(\varphi^{-1}(z))| = |S_{\nu}(z)|$ for $z \in \mathbb{D}$ if and only if

(1) $\nu \equiv 0$ in the hyperbolic case,
(2) $\nu = a\delta_1$ in the parabolic case, where δ_1 is the Dirac measure at 1 and $a \geqslant 0$.

Proof. By (3.10), $|S_\nu(\varphi^{-1}(z))| = |S_\nu(z)|$ if and only if $\nu(\{\xi\}) = |\varphi'(\xi)|\nu(\{\xi\})$ for any fixed point ξ of φ. So, $\xi \in \{-1, 1\}$ when φ is hyperbolic, and since $|\varphi'(1)| = \alpha^{-1}$ and $|\varphi'(-1)| = \alpha$, where $\alpha \neq 1$, then $\nu(\{\xi\}) = 0$. If φ is parabolic, we have that $\xi = 1$ and $|\varphi'(1)| = 1$, which means that $\nu(\{1\})$ can be any non-negative number. $\quad\square$

Now we are in position to state the result concerning singular inner functions.

Theorem 3.8. Let φ be a non-elliptic automorphism and $J \subset \partial\mathbb{D}$ be the set associated with φ by the paragraph that precedes Lemma 3.2. If ν_0 is a singular measure on $\partial\mathbb{D}$ with mass concentrated in J, write

$$d\nu := \sum_{n \in \mathbb{Z}} d\nu_n, \quad \text{where } \nu_n(G) = \int_{\varphi^{(-n)}(G)} \big|\varphi^{(n)'}(e^{i\theta})\big| \, d\nu_0(e^{i\theta}). \tag{3.11}$$

Then S_ν is a singular inner function that satisfies $S_\nu(\varphi(z)) = \gamma S_\nu(z)$ for some $\gamma \in \partial\mathbb{D}$. If φ is parabolic, the same holds for $S_{\nu + a\delta_1}$ for any $a \geqslant 0$. Conversely, every singular inner eigenvector S_μ has this form, with $\nu_0 = \chi_J \mu$ and, if φ is parabolic, $a = \mu(\{1\})$.

Proof. First we prove that ν is a finite measure. Observe that the whole mass of ν_n is concentrated in $\varphi^{(n)}(J)$ and that these sets are pairwise disjoint. Hence,

$$\nu(\partial\mathbb{D}) = \sum_{n \in \mathbb{Z}} \nu_n(\partial\mathbb{D}) = \int_{\partial\mathbb{D}} \sum_{n \in \mathbb{Z}} \big|\varphi^{(n)'}(e^{i\theta})\big| \, d\nu_0(e^{i\theta}) = \int_J \sum_{n \in \mathbb{Z}} \big|\varphi^{(n)'}(e^{i\theta})\big| \, d\nu_0(e^{i\theta}).$$

It follows from Lemma 3.2 that the above quantity is finite.

By (3.10), $|S_{\nu_0}(\varphi^{(-n)}(z))| = |S_{\nu_n}(z)|$, and consequently,

$$\big|S_{\nu_{n-1}}(z)\big| = \big|S_{\nu_0}\big(\varphi^{(-[n-1])}(z)\big)\big| = \big|S_{\nu_0}\big(\varphi^{(-n)}\big(\varphi(z)\big)\big)\big| = \big|S_{\nu_n}\big(\varphi(z)\big)\big|.$$

Therefore

$$\log\big|S_\nu(z)\big| = \sum_{n \in \mathbb{Z}} \log\big|S_{\nu_{n-1}}(z)\big| = \sum_{n \in \mathbb{Z}} \log\big|S_{\nu_n}\big(\varphi(z)\big)\big| = \log\big|S_\nu\big(\varphi(z)\big)\big|,$$

which means that $S_\nu(\varphi(z))/S_\nu(z)$ is some constant of modulus 1. Corollary 3.7 says that we can add to ν an atom at 1 when φ is parabolic.

To prove the converse, suppose that S_μ is a singular inner eigenvector and write $\nu_n = \chi_{\varphi^{(n)}(J)}\mu$. If we fix $n \in \mathbb{Z}$, for any Borel set $G \subset \varphi^{(n)}(J)$, the equality $|S_\mu(\varphi^{-(n)}(z))| = |S_\mu(z)|$ together with (3.10) yields

$$\nu_n(G) = \mu(G) = \int_{\varphi^{(-n)}(G)} \big|\varphi^{(n)'}(e^{i\theta})\big| \, d\mu(e^{i\theta}) = \int_{\varphi^{(-n)}(G)} \big|\varphi^{(n)'}(e^{i\theta})\big| \, d\nu_0(e^{i\theta}).$$

Thus, $\mu = \nu + \mu_\infty$, where ν is given by (3.11), with $\nu_0 = \chi_J \mu$ and μ_∞ supported on the fixed points of φ. But since S_ν and S_μ are both eigenvectors, the same holds for S_{μ_∞}. The theorem now follows from Corollary 3.7. $\quad\square$

We can summarize this discussion by saying that if $1 \leqslant p < \infty$, and $h = FBS \in H^p$, where F is outer, B is a Blaschke product and S is a singular inner function, then h is an eigenvector of C_φ if and only if F, B and S are respectively given by Theorems 3.4, 3.1 and 3.8. The same holds for $p = \infty$, with F given by Remark 3.5.

Acknowledgments

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