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Exact packing measure of central Cantor sets in the line

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ABSTRACT

In this paper we consider a class of symmetric Cantor sets in $\mathbb{R}$. Under certain separation condition we determine the exact packing measure of such a Cantor set through the computation of the lower density of the uniform probability measure supported on the set. With an additional restriction on the dimension we give also the exact centered Hausdorff measure by computing the upper density.

1. Introduction

In the study of the size of sets with Lebesgue measure zero, Hausdorff and packing dimensions and measures have been the most used tools. During the past 30 years there has been an enormous body of literature investigating Hausdorff and packing dimensions of sets (cf. [4,10]). However, the computation of the exact value of the measures is troublesome and only few results are known, most of them for Hausdorff measure.

For self-similar Cantor sets which satisfies the open set condition, the exact Hausdorff measure was computed by Marion in [9] and Ayer and Strichartz in [1], while the packing measure was obtained by Feng et al. in [6] for the classical one third Cantor set and later, Feng [5] gave the exact value for the general case. In the case of central Cantor sets (defined below), Qu et al. in [13] calculated the exact value of the Hausdorff measure. In this paper we compute the exact packing measure.

Hausdorff and packing measures are closely related to densities (see next section for definitions). In [12], the author investigated this relation. In fact, the proof in [6] relies on the lower density of the uniform measure supported on the set. For a class of non-symmetric self-similar Cantor sets in $\mathbb{R}$, the upper and lower densities of the natural weighted self-similar measure was computed by Li and Yao in [8]. In [14] Qu et al. considered central Cantor sets and, applying similar techniques, they computed the upper density under some additional hypothesis, which implies that the Hausdorff and packing dimension must coincide. In this article, we compute both – upper and lower – densities under quite general hypothesis. We do not require packing and Hausdorff dimension to coincide and – for lower density – we do not impose bounds on the packing dimension.

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2. Definitions and statements of results

In order to define the central Cantor sets we need to introduce some notations. If $k \geqslant 1$, D_k will denote the set of binary words with length k, that is,

$$D_k = \{\sigma = (\sigma_1, \dots, \sigma_k):\ \sigma_j = 0 \text{ or } 1\} = \{0,1\}^k.$$

Let $D_0 = \emptyset$ and $D = \bigcup_k D_k$. If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$ and $\tau = (\tau_1, \dots, \tau_m) \in D_m$ we define the concatenation, length and restriction by

$$\sigma\tau := (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m) \in D_{k+m},$$
$$|\sigma| := k,$$
$$\sigma|_j := (\sigma_1, \dots, \sigma_j) \in D_j \quad \text{for } j < k,$$

respectively. We also consider $\overline{D} := \{\omega = (\omega_1, \dots, \omega_k, \dots):\ \omega_i = 0 \text{ or } 1\} = \{0,1\}^{\mathbb{N}}$, with the same restriction and the concatenation defined on $D \times \overline{D}$.

Given $(r_k)_{k\geqslant 1}$ a sequence of real numbers with $0 < r_k < 1/2$, we define the collection of closed intervals $\mathcal{F} = \{I_\sigma:\ \sigma \in D\}$, called basic intervals, as follows:

(i) $I_\emptyset = [0,1]$.
(ii) For $k \geqslant 1$ and $\sigma \in D_{k-1}$, the intervals $I_{\sigma 0}$ and I_σ have the same left endpoint. $I_{\sigma 1}$ and I_σ have the same right endpoint.
(iii) $\frac{|I_{\sigma 0}|}{|I_\sigma|} = \frac{|I_{\sigma 1}|}{|I_\sigma|} = r_k$, where $|E|$ denotes the diameter of the set E.

Then, $E_k = \bigcup_{\sigma \in D_k} I_\sigma$ and $E = \bigcap_{k\geqslant 1} E_k$. The set E is called the central Cantor set associated to the ratios $(r_k)_{k\geqslant 1}$ (it is called symmetric in [14]). Central Cantor sets are nowhere dense and perfect, and they may have positive Lebesgue measure. The classical one third Cantor set is an example of central Cantor set with $r_k = 1/3$ for all $k \geqslant 1$. There is a 1–1 correspondence between points in E and words in $\overline{D}$: for every $x \in E$ there is a unique $\omega(x) := \omega \in \overline{D}$ such that $x \in I_{\omega|_k}$ for any k.

We need to introduce more notation. For $\sigma \in D_k$ we denote by s_k the length of I_σ and the length of the gap between the intervals $I_{\sigma 0}$ and $I_{\sigma 1}$ will be denoted by y_{k+1}. With this notation,

$$s_k = r_1 \cdots r_k, \qquad s_{k-1} = 2s_k + y_k \quad \text{and} \quad y_k = (1 - 2r_k) r_1 \cdots r_{k-1}.$$

Let $\mathcal{H}^s$ and $\mathcal{P}^s$ denote the s-dimensional Hausdorff and packing measures, respectively (see [4,10] for definitions and properties of these measures and corresponding dimensions). The asymptotic behavior of the sequence $(2^n s_n^s)_n$ is related to $\mathcal{H}^s(E)$ and $\mathcal{P}^s(E)$. In fact, there are finite and positive constants c_1, c_2, c_3 and c_4 such that

$$c_1 \liminf_{n\to\infty} 2^n s_n^s \leqslant \mathcal{H}^s(E) \leqslant c_2 \liminf_{n\to\infty} 2^n s_n^s \tag{1}$$

and

$$c_3 \limsup_{n\to\infty} 2^n s_n^t \leqslant \mathcal{P}^t(E) \leqslant c_4 \limsup_{n\to\infty} 2^n s_n^t. \tag{2}$$

Equivalence (1) was shown by Besicovitch and Taylor [2] while (2) was established in [7], Theorem 4.2, replacing packing measure by packing premeasure; then, an application of the mass distribution principle implies (2); see [3], Theorem 3.5. Both papers assume that the lengths of the removed gaps are decreasing, but if the Cantor set is central, an inspection of the proof of that theorems shows that this hypothesis is not necessary.

In particular, the Hausdorff and packing dimensions of E are given by

$$\dim_H E = \liminf_{n\to\infty} \frac{\log 2^n}{|\log s_n|} \quad \text{and} \quad \dim_P E = \limsup_{n\to\infty} \frac{\log 2^n}{|\log s_n|}, \tag{3}$$

respectively, and this values may not coincide.

In [13], Qu et al. established the following result.

Theorem. *(See [13].) If the sequence $(y_k)_{k\geqslant 1}$ of gaps lengths is decreasing, then*

$$\mathcal{H}^s(E) = \liminf_{n\to\infty} 2^n s_n^s. \tag{4}$$

In fact, their result is for homogeneous Cantor sets, which are a wider class of symmetric Cantor sets.

Our goal is to give the exact value of the packing measure of a central Cantor set E. We will require the following separation condition:

$$\text{there exists} \quad \beta < \frac{1}{2} \quad \text{such that} \quad r_k \leqslant \beta \quad \text{for all } k \text{ large enough.} \tag{5}$$

Our main result is the following.

Theorem 1. *Let E be a central Cantor set for which* (5) *holds. Then*

$$\mathcal{P}^t(E) = 2^t \limsup_{n\to\infty} 2^n (s_n + y_n)^t. \tag{6}$$

Remark 2. If $\mathcal{P}^t(E) = 0$ *or* ∞, then (6) holds in view of (2) and because

$$2^n s_n^t < 2^n (s_n + y_n)^t < 2 \cdot 2^{n-1} s_{n-1}^t, \tag{7}$$

the last inequality is because $s_{n-1} = 2s_n + y_n$.

Remark 3. We note further that Meinershagen [11] compute the packing measure of a class of Cantor sets that includes central Cantor sets. When restricted to this subclass, the hypothesis assumed on that paper implies that Hausdorff and packing dimensions must agree and it must be smaller than $\log 2/\log(5/2)$.

We emphasize that condition (5) is quite general, since it does not require that the dimensions match nor impose bounds on the packing dimension. In fact, given $\beta < 1/2$, by (3) we have $\dim_P E \leqslant \log 2/|\log \beta|$.

The proof of Theorem 1, which is given in Section 3, relies on the computation of the lower density of a natural measure. Given $t > 0$ and ν a measure on $\mathbb{R}$, the lower t-density of ν at $x \in \mathbb{R}$ is defined by

$$\Theta_*^t(\nu, x) := \liminf_{r\to 0} \frac{\nu(B(x,r))}{(2r)^t},$$

where $B(x,r)$ is the closed ball centered at x with radio r. The upper density $\Theta^{*t}(\nu, x)$ is defined analogously by taking lim sup instead of lim inf. There is one natural measure supported on E that we will denote by μ_E and is the only probability measure satisfying that $\mu_E(I_\sigma) = 2^{-|\sigma|}$. For central Cantor sets, lower density and packing measure are related as follows.

Proposition 4. *Let E be a central Cantor set such that $0 < \mathcal{P}^t(E) < \infty$. Then, its lower density $\Theta_*^t(\mu_E, \cdot)$ is μ_E almost everywhere the reciprocal of $\mathcal{P}^t(E)$; in particular, it is μ_E almost everywhere constant.*

Proof. For each $\sigma \in D_k$ and $k \geqslant 1$, the set $I_\sigma \cap E$ is a translation of $I_{0^k} \cap E$. Hence, the translation invariance of packing measures implies that $\mathcal{P}^t(E) = 2^k \mathcal{P}^t(E \cap I_\sigma)$. If we define $\nu = (\mathcal{P}^t(E))^{-1} \mathcal{P}^t|_E$, then ν and μ_E coincide on each I_σ, and by regularity, these measures are identical.

It is known (see [15] or [10], Theorem 6.10) that $\Theta_*^t(\mathcal{P}^t|_E, x) = 1$ for $\mathcal{P}^t$ a.e. $x \in E$. Then, $\Theta_*^t(\mu_E, x) = (\mathcal{P}^t(E))^{-1}$ for μ_E a.e. $x \in E$. □

As a consequence of the previous Proposition, the proof of Theorem 1 is the computation of the lower density of μ_E which is our next result. Define:

$$\overline{B}_t := \limsup_{n\to\infty} 2^n (s_n + y_n)^t.$$

The following theorem is valid.

Theorem 5. *Let E be a central Cantor set such that $\mathcal{P}^t(E) < \infty$. Then,*

(1) $\Theta_*^t(\mu_E, x) \geqslant (2^t \overline{B}_t)^{-1}$ *for all* $x \in E$;
(2) *if condition* (5) *holds, then* $\Theta_*^t(\mu_E, x) \leqslant (2^t \overline{B}_t)^{-1}$ *for* μ_E *a.e.* $x \in E$.

In particular, $\Theta_*^t(\mu_E, x) = (2^t \overline{B}_t)^{-1}$ *for* μ_E *a.e.* $x \in E$.

In [14] it is computed $\Theta^{*s}(\mu_E, x)$, where the conditions (a) $r_k \leqslant 1/3$ $\forall k$ and (b) $0 < \lim_{n\to\infty} 2^n s_n^s < \infty$ are assumed. By (3), condition (a) implies $\dim_P E \leqslant \log 2/\log 3$. Furthermore, (b) implies that the Hausdorff and packing dimensions of E must agree. In the same article, an example is given showing that some bound on the dimension is needed. Recently, in the particular case in which $r_k = a$ for all k and a is at most slightly greater than 1/3, Wang et al. [16] computed $\Theta^{*s}(\mu_E, x)$ and $\Theta^s_*(\mu_E, x)$ *for all x*.

In Section 4, we compute the upper density without imposing condition (b). Precisely, if

$$\underline{B}_s := \liminf_{n\to\infty} 2^n (s_n + y_n)^s,$$

we have:

Theorem 6. *Let E a central Cantor set with $r_n \leqslant 1/3$ and $0 < \mathcal{H}^s(E) < \infty$. Then,*

(1) $\Theta^{*s}(\mu_E, x) \leqslant 2^{1-s}\underline{B}_s^{-1}$ *for all* $x \in E$;
(2) $\Theta^{*s}(\mu_E, x) \geqslant 2^{1-s}\underline{B}_s^{-1}$ *for* μ_E *a.e.* $x \in E$.

In particular, $\Theta^{*s}(\mu_E, x) = 2^{1-s}\underline{B}_s^{-1}$ *for* μ_E *a.e.* $x \in E$.

If the limit $B = \lim_{n\to\infty} 2^n s_n^s$ exists and is finite and positive (which implies that Hausdorff and packing dimensions agree), then $\underline{B}_s = \overline{B}_s = (2^{1/s} - 1)^s B$ (see [14], Lemma 2.4). This implies, by (4) and Theorem 6, that the Hausdorff measure is (up to a constant) the inverse of the upper density, which gives an idea of duality between Hausdorff and packing measures. This is not true in the general case. In [15] was proved that the upper density is not related with Hausdorff measure but with centered Hausdorff measure, which is defined as

$$\mathcal{C}^s(E) = \sup\{C^s(F):\ F \subset E\}$$

where

$$C^s(E) := \sup_{\delta>0}\left\{\inf\left\{\sum_i |B_i|^s:\ E \subset \bigcup_i B_i,\ B_i \text{ is a ball centered in } E,\ |B_i| \leqslant \delta\right\}\right\}.$$

As a consequence of Theorem 6 we have:

Theorem 7. *If E is a central Cantor set with $r_k \leqslant 1/3$ for all k large enough, then* $\mathcal{C}^s(E) = 2^{s-1}\liminf_{n\to\infty} 2^n (s_n + y_n)^s$.

Finally, in Section 5 we discuss condition (5). We give an example where this hypothesis is not satisfied but still the proof of Theorem 1 can be modified to conclude that formula (6) holds. However, this formula is not true for central Cantor sets in general, since we have the following result.

Theorem 8. *Given $0 < t < 1$, there exists a central Cantor set E such that*

$$\mathcal{P}^t(E) < 2^t \overline{B}_t.$$

Therefore, it is necessary to ask some separation condition for the conclusion in Theorem 1 remains valid.

In the next section we prove Theorem 1.

3. Lower density and packing measure

We will note by $a(\sigma)$ and $b(\sigma)$ to the endpoints of the interval I_σ. For the first part of Theorem 5 we need the following lemma.

Lemma 9. *If a_j, b_j are positive numbers and $0 < t < 1$ then*:

$$\min\left\{\frac{a_j}{b_j^t}:\ 1 \leqslant j \leqslant k\right\} \leqslant \frac{a_1 + a_2 + \cdots + a_k}{(b_1 + b_2 + \cdots + b_k)^t}.$$

Proof. Let m be the term on the left. It follows that $mb_j^t \leqslant a_j$ for $1 \leqslant j \leqslant k$ and, in consequence (remember $t < 1$),

$$m(b_1 + b_2 + \cdots + b_k)^t \leqslant m\left(b_1^t + b_2^t + \cdots + b_k^t\right) \leqslant a_1 + a_2 + \cdots + a_k$$

and the lemma follows. □

Proof of Theorem 5(1). Given $\varepsilon > 0$, let k_0 be such that $2^k (s_k + y_k)^t < \overline{B}_t + \varepsilon$ for all $k \geqslant k_0$. Fix $x \in E$ and $r > 0$. There exists $\sigma \in D$ such that

$$I_\sigma \subseteq B(x, r) \quad \text{but} \quad I_{\tilde\sigma} \nsubseteq B(x, r) \quad \text{whenever } |\tilde\sigma| < |\sigma|. \tag{8}$$

Put $n=|\sigma|$. So, $\mu_E(B(x,r))\geqslant 2^{-n}$. We assume r is small enough so that $n\geqslant k_0$. If $r\leqslant s_n+y_n$, then

$$\frac{\mu_E(B(x,r))}{(2r)^t}\geqslant\frac{2^{-n}}{2^t(s_n+y_n)^t}\geqslant\frac{1}{2^t(\overline{B}_t+\varepsilon)}.$$

Then, it remains to consider the case in which $r>s_n+y_n$. Notice that there are at most two words which verify (8). We will only analyze the case in which the last letter in σ is zero and $x\geqslant a(\sigma)$, since the other cases are analogous. We have

$$r+x>a(\sigma)+s_n+y_n=a(\sigma|_{(n-1)}1).$$

Then $B(x,r)$ contains a portion of the interval $I_{\sigma|_{(n-1)}1}$. We divide the proof in two cases, according the right endpoint of the ball $B(x,r)$ belongs or not to the set E.

Assume first that $x+r\notin E$. In this case, there exists $\tau\in D$ such that $x+r\in[b(\sigma|_{n-1}1\tau0),a(\sigma|_{n-1}1\tau1)]$. Define:

$$n_1=\min\{i\geqslant 1:\ \tau_i=1\},$$
$$n_{j+1}=\min\{i>n_j:\ \tau_i=1\}\quad\text{if the set is not empty.}$$

If L is the maximum of the indices for which n_j is defined, we have that

$$\mu_E\big(B(x,r)\big)\geqslant 2^{-n}+\sum_{j=1}^{L}2^{-(n_j+n)}+2^{-(n+|\tau|+1)}.$$

On the other hand, we have

$$x+r\leqslant a(\sigma|_{n-1}1\tau1)=a(\sigma)+s_n+y_n+\sum_{j=1}^{L}(s_{n_j+n}+y_{n_j+n})+s_{n+|\tau|+1}+y_{n+|\tau|+1}.$$

Since $x\geqslant a(\sigma)$, using the last two inequalities, Lemma 9 and $n\geqslant k_0$, we have:

$$\begin{aligned}\frac{\mu_E(B(x,r))}{(2r)^t}&\geqslant\frac{\sum_{j=0}^{L}2^{-(n_j+n)}+2^{-(n+|\tau|+1)}}{2^t(\sum_{j=0}^{L}(s_{n_j+n}+y_{n_j+n})+s_{n+|\tau|+1}+y_{n+|\tau|+1})^t}\\&\geqslant 2^{-t}\min\left(\left\{\frac{2^{-(n_j+n)}}{s_{n_j+n}+y_{n_j+n}}:\ 0\leqslant j\leqslant L\right\}\cup\left\{\frac{2^{-(n+|\tau|+1)}}{s_{n+|\tau|+1}+y_{n+|\tau|+1}}\right\}\right)\\&\geqslant 2^{-t}(\overline{B}_t+\varepsilon)^{-1},\end{aligned}\tag{9}$$

where $n_0=0$.

Now consider the case $x+r\in E$. If $x+r$ is the endpoint of a basic interval, then the existence of the τ and the prove below is still valid. If not, then there is an infinite word $\omega\in\overline{D}$ such that $x+r\in I_{\omega'|_k}$ for any $k\geqslant 1$, where $\omega'=\sigma|_{n-1}1\omega$. Similarly to (9) we define:

$$n_1=\min\{i\geqslant 1:\ \omega_i=1\},\qquad n_{j+1}=\min\{i>n_j:\ \omega_i=1\}.$$

In this case, (n_j) is not bounded and for any L, then

$$\mu_E\big(B(x,r)\big)\geqslant 2^{-n}+\sum_{j=1}^{L}2^{-(n+n_j)}.$$

We also have that

$$x+r\leqslant b\big(\omega'|_{n_L}\big)=a(\sigma)+\sum_{j=0}^{L}(s_{n+n_j}+y_{n+n_j})+s_{n_L}.$$

Since $s_{n_L}\to 0$ when $L\to\infty$, taking L large enough, we have:

$$\frac{\mu_E(B(x,r))}{(2r)^t}\geqslant\frac{\sum_{j=0}^{L}2^{-(n_j+n)}}{2^t(\sum_{j=0}^{L}(s_{n_j+n}+y_{n_j+n}))^t}-\varepsilon.$$

Similarly to (9), this is bounded by $2^{-t}(\overline{B}_t+\varepsilon)^{-1}-\varepsilon$. □

For the second part of Theorem 5 we need the following lemma.

Lemma 10. *There exists $L > 0$ such that*

$$s_n + y_n \leqslant y_{n-\ell} \quad \text{for } L \leqslant \ell < n \text{ and all } n \text{ large enough}. \tag{10}$$

Proof. Firstly note that the inequality

$$s_n + y_n \leqslant y_{n-\ell} \tag{11}$$

is equivalent to

$$r_{n-\ell} \cdots r_{n-1} \leqslant \frac{1 - 2r_{n-\ell}}{1 - r_n}$$

since $s_n + y_n = (1 - r_n)(r_1 \cdots r_{n-1})$ and $y_{n-\ell} = (1 - 2r_{n-\ell})(r_1 \cdots r_{n-\ell-1})$. We have

$$r_{n-\ell} \cdots r_{n-1} \leqslant \frac{1}{2^\ell} \quad \text{and} \quad 1 - 2\beta \leqslant \frac{1 - 2r_{n-\ell}}{1 - r_n},$$

hence (11) holds if $2^{-\ell} \leqslant 1 - 2\beta$, or equivalently $\ell \geqslant \log_{1/2}(1 - 2\beta)$. If we choose $L = \lceil \log_{1/2}(1 - 2\beta) \rceil$, then the lemma follows. □

Proof of Theorem 5(2). We begin constructing a set $A \subset E$ of full measure, that is, $\mu_E(A) = 1$. Then we show that each point in this set verifies the stated inequality.

Let (n_k) be an increasing sequence such that

$$\lim_{k\to\infty} 2^{n_k}(s_{n_k} + y_{n_k})^t = \limsup_{n\to\infty} 2^n (s_n + y_n)^t. \tag{12}$$

We assume that $n_{k+1} - n_k > k$ for all k.

For each $k \geqslant 1$, let j be such that $2^j \leqslant k < 2^{j+1}$. Then, with L as in Lemma 10, we define the set

$$A_k = \{x \in E \colon \sigma_{n_k - L}(x) = 1,\ \sigma_{n_k - L+1}(x) = \cdots = \sigma_{n_k - L+j}(x) = 0\}.$$

Note that $\mu(A_k) = 2^{-j-1}$ and therefore $\sum_i \mu_E(A_i) = \infty$. Moreover, our assumption on the sequence implies that the events A_k are independent. Hence, Borel–Cantelli Lemma implies that the upper limit

$$A = \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k$$

has full measure.

Now fix $x \in A$. Then $x \in A_k$ for infinite values of k, and for each of these values we define $r_k = s_{n_k} + y_{n_k} - s_{n_k - L+j}$, where $2^j \leqslant k < 2^{j+1}$.

Set $\sigma = \sigma(x)$ and $m = n_k - L$. Then $x - a(\sigma|_m) \leqslant s_{m+j}$ since $a(\sigma|_m) = a(\sigma|_{m+j})$. Moreover, if $j \geqslant L$, we have $a(\sigma|_m) = a(\sigma|_{n_k})$. Then

$$x + r_k = x - s_{m+j} + s_{n_k} + y_{n_k} \leqslant a(\sigma|_{n_k}) + s_{n_k} + y_{n_k} = a\big(\sigma|_m 0^{L-1} 1\big),$$

where $0^{L-1} \in D_{L-1}$ is the word with $L - 1$ zeroes. Furthermore, the gap to the left of $I_{\sigma|_{n_k}}$ has length y_i for some $1 \leqslant i \leqslant m$, and by Lemma 10 we have

$$x - r_k = x + s_{m+j} - s_{n_k} - y_{n_k} > a(\sigma|_m) - y_i.$$

Then $B(x, r_k) \cap E \subset I_{\sigma|_{n_k}} \cap E$ (and possibly the point $a(\sigma|_m 0^{L-1} 1)$). For the opposite inclusion, if j is sufficiently large ($j \geqslant 2L$ works), then by Lemma 10 we have

$$s_{m+j} < s_{m+j} + y_{m+j} \leqslant y_{n_k},$$

hence

$$x + r_k > x + s_{n_k} \geqslant b(\sigma|_{n_k});$$

on the other hand,

$$x - r_k \leqslant a(\sigma|_{n_k}) + 2s_{m+j} - s_{n_k} - y_{n_k} < a(\sigma|_{n_k}).$$

We conclude that $\mu_E(B(x,r_k)) = 2^{-n_k}$. Hence

$$\frac{\mu_E(B(x,r_k))}{(2r_k)^t} = \frac{2^{-n_k}}{2^t(s_{n_k}+y_{n_k}-s_{m+j})^t} = \frac{1}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}\frac{1}{(1-\frac{s_{m+j}}{s_{n_k}+y_{n_k}})^t}. \tag{13}$$

Note that

$$\frac{s_{m+j}}{s_{n_k}+y_{n_k}} \leqslant \frac{1}{2^{j-L}}\frac{s_{n_k}}{s_{n_k}+y_{n_k}}.$$

Then, taking limit in k in (13), we conclude the proof. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. If $\mathcal{P}^t(E) = 0$ or $\mathcal{P}^t(E) = \infty$, then the theorem follows from (2) and inequalities (7). If $0 < \mathcal{P}^t(E) < \infty$, then it follows from Proposition 4 and Theorem 5. □

4. Upper density and centered Hausdorff measure

In this section we prove Theorems 6 and 7.

Proof of Theorem 6(1). Fix $\varepsilon > 0$ and $x \in E$. There is k_0 such that

$$\underline{B}_s - \varepsilon < 2^k(s_k+y_k)^s \tag{14}$$

whenever $k \geqslant k_0$.

Fix $r > 0$. There is a $\sigma \in D$ with the following property:

$$B(x,r) \supseteq I_\sigma \quad \text{but} \quad B(x,r) \nsupseteq I_{\tilde\sigma} \quad \text{whenever } |\tilde\sigma| < |\sigma|.$$

Put $n = |\sigma|$. By choosing r small enough we can assume $n \geqslant k_0$. Our hypothesis implies (y_k) is decreasing and $x \in I_\sigma$. In consequence $r > \max\{a(\sigma)-x+s_n, x-a(\sigma)\}$. We can assume the last letter in σ is a zero, since the other case is analogous. With this assumption, $\mu_E(B(x,r)) = \mu_E[a(\sigma), x+r]$. We will divide into two cases.

Case 1: $r \leqslant a(\sigma)-x+s_n+y_n$. In this case $\mu_E(B(x,r)) = 2^{-n}$. Then,

$$\frac{\mu_E(B(x,r))}{(2r)^s} \leqslant 2^{-n}2^{-s}\big(\max\{a(\sigma)-x+s_n, x-a(\sigma)\}\big)^{-s}.$$

It is enough to prove that $\max\{a(\sigma)-x+s_n, x-a(\sigma)\} \geqslant s_{n+1}+y_{n+1}$. By *reductio ad absurdum*, suppose that $a(\sigma)-x+s_n < s_{n+1}+y_{n+1}$ and $x-a(\sigma) < s_{n+1}+y_{n+1}$. This implies that $a(\sigma)+s_{n+1} < x < a(\sigma)+s_{n+1}+y_{n+1}$ what is a contradiction since $x \in E \cap I_\sigma$.

Case 2: $r > a(\sigma)-x+s_n+y_n$. In this case, $x+r \in I_{\sigma|_{n-1}1}$ and

$$\mu_E\big(B(x,r)\big) = 2^{-n} + \mu_E\big(\big[a(\sigma)+s_n+y_n, x+r\big]\big).$$

Assume first that $x+r \notin E$. So, there is a finite word τ such that $x+r \in [b(\sigma|_{n-1}1\tau 0), a(\sigma|_{n-1}1\tau 1)]$. Associated to τ we define:

$$n_1 = \min\{i \geqslant 1:\ \tau_i = 1\},$$
$$n_{j+1} = \min\{i > n_j:\ \tau_i = 1\} \quad \text{if the set is not empty.}$$

Let L be the maximum of the indices for which n_j is defined. We have:

$$\frac{\mu_E(B(x,r))}{(2r)^s} \leqslant \frac{2^{-n}+\sum_{j=1}^L 2^{-(n_j+n)}+2^{-(n+|\tau|+1)}}{2^s(a(\sigma)-x+s_n+y_n+\sum_{j=1}^L(s_{n_j+n}+y_{n_j+n})+s_{n+|\tau|+1})^s}.$$

Put $n_0 = 0$. Since $a(\sigma)-x+s_n \geqslant 0$, we want to prove that for any $\tau \in D$,

$$\frac{\sum_{j=0}^L 2^{-(n_j+n)}+2^{-(n+|\tau|+1)}}{(y_n+\sum_{j=1}^L(s_{n_j+n}+y_{n_j+n})+s_{n+|\tau|+1})^s} \leqslant 2(\underline{B}_s-\varepsilon)^{-1}. \tag{15}$$

In order to prove (15), we consider two subcases. Put $N = n+|\tau|+1$.

Case 2.1: *τ is not constantly* 0 *nor constantly* 1 (*when* $|\tau|>1$). We follow the ideas in [13] and use induction in $|\tau|$. If $|\tau|=0$, then the left side of (15) becomes $\frac{2^{-n}+2^{-N}}{(y_n+s_N)^s}$. Define

$$\lambda := \frac{y_n - y_N}{s_n - s_N + y_n - y_N}.$$

Since $r_k \leqslant 1/3$ for all k, we have $\lambda \geqslant 1/2$. Using concavity of the function t^s, estimate (14) and $\lambda \geqslant 1/2$ we obtain:

$$(y_n+s_N)^s \geqslant \lambda(s_n+y_n)^s + (1-\lambda)(s_N+y_N)^s \geqslant (\underline{B}_s-\varepsilon)\big(\lambda 2^{-n}+(1-\lambda)2^{-N}\big) \geqslant (\underline{B}_s-\varepsilon)1/2\big(2^{-n}+2^{-N}\big).$$

So, the case $|\tau|=0$ is proved.

Now, assume $|\tau|>0$. Put $\Lambda := (y_n + \sum_{j=1}^{L}(s_{n_j+n}+y_{n_j+n}) + s_N)^s$. We have

$$\Lambda \geqslant \lambda\left(y_n + \sum_{j=1}^{L-1}(s_{n_j+n}+y_{n_j+n}) + s_{n_L+n}\right)^s + (1-\lambda)\left(y_n + \sum_{j=1}^{L}(s_{n_j+n}+y_{n_j+n}) + 2s_N + y_N\right)^s$$

with

$$\lambda = \frac{s_N+y_N}{y_{n_L+n}+2s_N+y_N}.$$

Note that $\lambda \leqslant 1/2$ since (y_k) is decreasing.

As τ is not constantly 0, applying the inductive hypothesis to $\tau|_{n_L-1}$ (or $\tau=\emptyset$ if $n_L=1$), we obtain

$$\left(y_n + \sum_{j=1}^{L-1}(s_{n_j+n}+y_{n_j+n}) + s_{n_L+n}\right)^s \geqslant \frac{(\underline{B}_s-\varepsilon)}{2}\left(\sum_{j=0}^{L-1}2^{-(n_j+n)} + 2^{-(n_L+n)}\right).$$

Moreover, as τ is not constantly 1, put $J=\max\{j\colon\ \tau_j=0\}$. Applying the inductive hypothesis to $\tau|_{J-1}$, we obtain

$$\left(y_n + \sum_{j=1}^{L}(s_{n_j+n}+y_{n_j+n}) + 2s_N + y_N\right)^s \geqslant \frac{(\underline{B}_s-\varepsilon)}{2}\left(\sum_{j=0}^{\tilde{L}}2^{-(n_j+n)} + 2^{-(J+n)}\right),$$

where $\tilde{L}=\max\{j\colon\ n_j<J\}$.

Using the last three inequalities we have:

$$\begin{aligned}\Lambda &\geqslant \frac{(\underline{B}_s-\varepsilon)}{2}\left\{\sum_{j=0}^{L}2^{-(n_j+n)} + (1-\lambda)\left(\sum_{j=0}^{\tilde{L}}2^{-(n_j+n)} + 2^{-(J+n)} - \sum_{j=0}^{L}2^{-(n_j+n)}\right)\right\}\\ &\geqslant \frac{(\underline{B}_s-\varepsilon)}{2}\left(\sum_{j=0}^{L}2^{-(n_j+n)} + (1-\lambda)2^{-(n+|\tau|)}\right).\end{aligned}$$

Since $\lambda \leqslant 1/2$ the proof is complete.

Case 2.2. If $\tau_i=0$ for all i then the proof is exactly the same as in the case $|\tau|=0$. If $\tau_i=1$ for all i, then (considering the same convex combination as above) the proof is direct.

Finally, we consider $x+r\in E$. If x is an endpoint of a basic interval, we still have existence of a word τ as before, and the proof is still valid. If not, then there is a word $\omega\in\overline{D}$ such that $x+r\in I_{\omega'|N}$ for any N, where $\omega' := \sigma|_{n-1}1\omega$. Define:

$$n_1=\min\{i\geqslant 1\colon\ \omega_i=1\},\qquad n_{j+1}=\min\{i>n_j\colon\ \omega_i=1\}.$$

Put $n_0=0$. If N is large enough, using (15), we obtain

$$\frac{\mu_E(B(x,r))}{(2r)^s} \leqslant \frac{\sum_{j=0}^{\infty}2^{-(n_j+n)}}{2^s(y_n+\sum_{j=1}^{\infty}(s_{n_j+n}+y_{n_j+n}))^s} \leqslant \frac{\sum_{j=0}^{L}2^{-(n_j+n)}+2^{-N}}{2^s(y_n+\sum_{j=1}^{L}(s_{n_j+n}+y_{n_j+n})+s_N)^s.} + \varepsilon \leqslant \frac{2}{2^s(\underline{B}_s-\varepsilon)} + \varepsilon,$$

where $L=\max\{i\colon\ n_i<N-n\}$. $\square$

Proof of Theorem 6(2). We proceed in a similar fashion to the proof of Theorem 5(2). Consider an increasing sequence with $n_{k+1}-n_k>k$ such that

$$\lim 2^{n_k}(s_{n_k}+y_{n_k})^s = \underline{B}_s.$$

For each $k \geqslant 1$, let j be such that $2^j \leqslant k < 2^{j+1}$. We define the set

$$A_k = \{x \in E\colon \sigma_{n_k}(x) = 1,\ \sigma_{n_k+1}(x) = \cdots = \sigma_{n_k+j}(x) = 0\}.$$

Note that $\mu_E(A_k) = 2^{-(j+1)}$, so the series $\sum \mu_E(A_k)$ diverges. Since the events A_k were chosen independent Borel–Cantelli Lemma applies and we can conclude that the set $A = \bigcap_{n\geqslant 1} \bigcup_{k\geqslant n} A_k$ has full measure.

We will prove that our thesis is valid for $x \in A$. So, pick $x \in A$ and for those k for which $x \in A_k$, define $r_k = s_{n_k} + y_{n_k} + s_{n_k+j}$. Then $B(x, r_k)$ contains the interval $[a(\sigma|_{n_k}) - s_{n_k} - y_{n_k}, a(\sigma|_{n_k}) + s_{n_k} + y_{n_k}]$ whose measure is 2^{-n_k+1}. So,

$$\frac{\mu_E(B(x,r_k))}{(2r_k)^s} = \frac{2\cdot 2^{-n_k}}{2^s(s_{n_k} + y_{n_k})^s(1 + \frac{s_{n_k+j}}{s_{n_k}+y_{n_k}})^s}.$$

Taking limit in k, we obtain the desired result. □

Proof of Theorem 7. If $\liminf_{n\to\infty} 2^n(s_n + y_n)^s$ is zero or infinity, so $\liminf_{n\to\infty} 2^n s_n^s$ and $\mathcal{H}^s(E)$ are, in view of (7) and Theorem (see [13]). Since $\mathcal{H}^s(E) \leqslant \mathcal{C}^s(E) \leqslant 2^s\mathcal{H}^s(E)$ (see [15], Lemma 3.3), $\mathcal{C}^s(E)$ is zero or infinity.

If $\liminf_{n\to\infty} 2^n(s_n + y_n)^s$ is neither zero nor infinity, then $0 < \mathcal{H}^s(E) < \infty$ (moreover, we are in the hypothesis of Theorem 6).

As in Proposition 4, since $\mathcal{C}^s$ is also invariant by translations, the measure $\nu := (\mathcal{C}^s(E))^{-1}\mathcal{C}^s|_E$ coincides with μ_E. Using that $\Theta^{*s}(\mathcal{C}^s|_E, x) = 1$, for $\mathcal{C}^s$ a.e. $x \in E$ (see [15], Corollary 7.1) we conclude $\Theta^{*s}(\mu_E, x) = (\mathcal{C}^s(E))^{-1}$ for μ_E a.e. $x \in E$. The thesis follows comparing this and the value of the density obtained in Theorem 6. □

In the next section we discuss the hypothesis of Theorem 1.

5. On the separation condition

In view of Lemma 10 and the proof of Theorem 1, the hypothesis of this theorem can be replaced by: *there exists $L > 0$ such that*

$$s_n + y_n \leqslant y_{n-\ell}, \quad \text{for } L \leqslant \ell < n \text{ and all } n \text{ large enough.}$$

It may happen that there is no such L. For example, when $r_k \geqslant c > 0$ for all k and there is a subsequence (k_i) such that $r_{k_i} \to 1/2$. However, if there is some control on the subsequence, the proof of Theorem 1 can still be adapted.

Example 1 *(Example of a Cantor set such that $0 < \mathcal{P}^t(E) = \overline{B}_t < \infty$ and there is a subsequence of the ratios that tends to $1/2$).* Let $0 < a < 1/2$ and $\beta_{2k} = (1 - \epsilon_k)/2$, where $\epsilon_k \to 0$; below we impose conditions on a and ϵ_k. For $k \geqslant 1$ we define

$$r_k = \begin{cases} a, & k \text{ odd},\\ \beta_k, & k \text{ even},\end{cases}$$

and let E be the corresponding Cantor set. Notice that

$$s_{2n} = (a/2)^n \prod_{j=1}^{n}(1 - \epsilon_j) \quad \text{and} \quad s_{2n+1} = a(a/2)^n \prod_{j=1}^{n}(1 - \epsilon_j).$$

If $t = \log 4/\log(2/a)$ we have $2a^t = 2^{1-t}$. Then, if (ϵ_j) is a summable sequence, it is easily verified from (2) that $0 < \mathcal{P}^t(E) < \infty$.

Also, by the identity $s_k + y_k = s_{k-1} - s_k$, we have that

$$2^{2n}(s_{2n} + y_{2n})^t = (2a^t)^n(2^{1-t})^{n-1}\prod_{j=1}^{n-1}(1 - \epsilon_j)^t 2(1 - \beta_{2n})^t$$

and

$$2^{2n+1}(s_{2n+1} + y_{2n+1})^t = 2(1 - a)^t(2a^t)^n(2^{1-t})^n\prod_{j=1}^{n}(1 - \epsilon_j)^t.$$

Therefore, $\overline{B}_t$ is obtained by taking limit to any subsequence with odd subindices.

Let us define $\epsilon_j = j^{-2}$. We will mimic the proof of Theorem 5(2). In this case we cannot find L as in Lemma 10, but recalling that $s_n + y_n \leqslant y_{n-\ell}$ iff $r_{n-\ell}\cdots r_{n-1} \leqslant (1 - 2r_{n-\ell})/(1 - r_n)$, and noting that

$$r_{n-\ell}\cdots r_{n-1} \leqslant (a/2)^{\lfloor \ell/2 \rfloor} \quad \text{and} \quad (1-2r_{n-\ell})/(1-r_n) > 1-2\beta_{n-\ell} = (n-\ell)^{-2},$$

then, we need $(a/2)^{\lfloor \ell/2 \rfloor} \leqslant (n-\ell)^{-2}$. Hence, if $L_n = \lceil 4\log n/\log(2/a) \rceil$, we have

$$s_n + y_n \leqslant y_{n-\ell}, \quad \text{for all } L_n \leqslant \ell < n.$$

Set $n_k = k(k+1)+1$ and $L_k := L_{n_k}$; as before, for $k \geqslant 1$, let

$$A_k = \{x \in E\colon \sigma_{n_k-L_k}(x) = 1,\ \sigma_{n_k-L_k+1}(x) = \cdots = \sigma_{n_k-L_k+j}(x) = 0\},$$

where j is such that $2^j \leqslant k < 2^{j+1}$. For k large enough, the independence of these events holds since $n_k - L_k + j < n_{k+1} - L_{k+1}$ for all k large enough. Then, Borel–Cantelli Lemma applies and $A = \bigcap_{n\geqslant 1}\bigcup_{k\geqslant n} A_k$ has full measure. The rest of the proof is the same as before, but we must note that $j - L_k \to \infty$ as $k \to \infty$. In fact, since $\log k^2 + 1 > \log(k(k+1)+1)$ for k large enough, we have

$$j - L_k > \frac{\log k}{\log 2} - 4\frac{\log(k(k+1)+1)}{\log(\frac{2}{a})} - 2 > \log k\left(\frac{1}{\log 2} - \frac{12}{\log(\frac{2}{a})}\right) - 2,$$

which tends to ∞ if $a < 2^{-11}$.

We conclude the paper with the proof of Theorem 8, which shows that the formula from Theorem 1 is not true for central Cantor sets in general.

Proof of Theorem 8. Let E be the Cantor set given by the sequence (r_k) defined as follows. Let $0 < t < 1$ and

$$r_k = \begin{cases} \beta_n, & 2^n < k < 2^{n+1}, \\ \alpha_n, & k = 2^n, \end{cases}$$

where $\beta_n = 1/2 - \epsilon_n$ with $\epsilon_n \searrow 0$ (ϵ_n will be specified later), and let α_n be such that

$$\alpha_n \beta_n^{2^n-1} = \left(\frac{1}{2^{2^n}}\right)^{1/t};$$

it is easily verified that $\alpha_n \to 0$.

Firstly we claim that for all n large enough, if $2^n < l < 2^{n+1}$, then

$$2^l(s_l + y_l)^t < 2^{l+1}(s_{l+1} + y_{l+1})^t. \tag{16}$$

In fact, $s_{l+1} + y_{l+1} = (s_l + y_l)r_l(1 - r_{l+1})/(1 - r_l)$. We have two cases.

Case 1. If $r_{l+1} = \beta_n$, then

$$2^{l+1}(s_{l+1} + y_{l+1})^t = 2^l(s_l + y_l)^t 2\beta_n^t \tag{17}$$

and (16) holds since $\beta_n^t > 1/2$ if n is large enough.

Case 2. If $r_{l+1} = \alpha_{n+1}$ (i.e. $l + 1 = 2^{n+1}$), then

$$2^{l+1}(s_{l+1} + y_{l+1})^t = 2^l(s_l + y_l)^t 2\beta_n^t\left(\frac{1-\alpha_{n+1}}{1-\beta_n}\right)^t, \tag{18}$$

and the claim holds since the last quotient tends to 2.

Furthermore, if $n_k = 2^k$, then

$$\begin{aligned} 2^{n_k}(s_{n_k} + y_{n_k})^t &= 2^{n_k}(s_{n_k-1} - s_{n_k})^t \\ &= 2^{n_k}\left(\left(\prod_{j=1}^{k-1}\beta_j^{(2^j-1)}\right)\prod_{j=0}^{k-1}\alpha_j - \left(\prod_{j=1}^{k-1}\beta_j^{(2^j-1)}\right)\prod_{j=0}^{k}\alpha_j\right)^t \\ &= 4\alpha_0^t(1-\alpha_k)^t. \end{aligned}$$

Then, from (16), the sequence (n_k) reaches the upper limit, that is

$$\overline{B} = \limsup_{n\to\infty} 2^n(s_n + y_n)^t = \lim_{k\to\infty} 2^{n_k}(s_{n_k} + y_{n_k})^t,$$

and also, $0 < \mathcal{P}^t(E) < \infty$.

Now we show that $\Theta_*^t(\mu_E, x) \geqslant C(2^t\overline{B})^{-1}$ for μ_E a.e. $x \in E$, with $C > 1$, which implies $\mathcal{P}^t(E) < 2^t\overline{B}$. *Here we do not care about the optimality of* C.

Let $x \in E$ and let r be small enough. Then $I_\sigma \subset B(x,r)$ for some $\sigma \in D$ but $I_{\tilde{\sigma}} \not\subseteq B(x,r)$ if $|\tilde{\sigma}| < |\sigma|$. Set $n = |\sigma|$. Note that $r < s_{n-1}$. We need to separate the proof in two cases.

Case 1. Suppose $n \neq n_j$ $\forall j$. Set $n_k = \min\{n_j\colon n_j > n\}$. Then, using (17) and (18), we obtain

$$\begin{aligned}\frac{\mu_E(B(x,r))}{(2r)^t} &\geqslant \frac{1}{2^t 2^n s_{n-1}^t}\\ &= \frac{2^{n_k-n}}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}\left(\frac{s_n+y_n}{s_{n-1}}\right)^t \prod_{j=1}^{n_k-n}\frac{(s_{n+j}+y_{n+j})^t}{(s_{n+j-1}+y_{n+j-1})^t}\\ &= \frac{2^{n_k-n}}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}(1-\beta_{k-1})^t\left(\beta_{k-1}^t\right)^{n_k-n}\left(\frac{1-\alpha_k}{1-\beta_{k-1}}\right)^t\\ &= \frac{1}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}\left(2\beta_{k-1}^t\right)^{n_k-n}(1-\alpha_k)^t.\end{aligned}$$

Note that $n_k - n \geqslant 1$. Moreover, given $1 < C < 2^{1-t}$, then $2\beta_{k-1}^t(1-\alpha_k)^t \geqslant C$ for all k large enough, hence

$$\frac{\mu_E(B(x,r))}{(2r)^t} \geqslant \frac{C}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}$$

if r is small enough.

Case 2. We construct a set A of full measure such that on *each* level n_k (that is, whenever $n = n_k$) we have

$$\frac{\mu_E(B(x,r))}{(2r)^t} \geqslant 2^t\left(\frac{1}{2^t 2^{n_k}(s_{n_k}+y_{n_k})}\right), \quad \text{for } x \in A. \tag{19}$$

Then, this inequality together the previous case implies the theorem.

We assume that $r \geqslant (s_{n_k}+y_{n_k})/2$, otherwise (19) is immediate. First note that for k large enough,

$$2s_{n_k} + y_{n_{k-1}+l} \leqslant \frac{s_{n_k}+y_{n_k}}{2}, \quad \text{for } 1 \leqslant l < n_{k-1}. \tag{20}$$

In fact,

$$\begin{aligned}2s_{n_k} + y_{n_{k-1}+l} &= 2\alpha_0\alpha_k\left(\frac{1}{2^{2^k-2}}\right)^{1/t} + \alpha_0\alpha_{k-1}\beta_{k-1}^{l-1}\left(\frac{1}{2^{2^{k-1}-2}}\right)^{1/t}(2\epsilon_{k-1})\\ &\leqslant 2\alpha_0\left(\frac{1}{2^{2^k-2}}\right)^{1/t}\left(\alpha_k + \alpha_{k-1}\left(2^{2^{k-1}}\right)^{1/t}\epsilon_{k-1}\right)\\ &= \frac{s_{n_k}+y_{n_k}}{2}4\left(\alpha_k + \alpha_{k-1}\left(2^{2^{k-1}}\right)^{1/t}\epsilon_{k-1}\right),\end{aligned}$$

and (20) holds if we choose $\epsilon_{k-1} \leqslant (2^{2^{k-1}})^{-1/t}$.

Now, let

$$\widetilde{D}_{n_k} = \left\{\sigma \in D_{n_k}\colon \sigma = \tau 01^l \text{ or } \sigma = \tau 10^l,\ 1 \leqslant l < n_{k-1},\ \tau \in D_{n_k-(l+1)}\right\}$$

and define

$$A_k = \bigcup_{\sigma \in \widetilde{D}_{n_k}} I_\sigma \cap E \quad \text{and} \quad A = \bigcup_{n \geqslant 1}\bigcap_{k \geqslant n} A_k.$$

Note that if $x \in A$, then, for all k large enough, x belongs to a basic interval of level n_k which is next to a gap of length $y_{n_{k-1}+l}$. Hence, inequality (20) implies that $B(x,r)$ contains two basic intervals of level n_k. Then, (19) holds because

$$\frac{\mu_E(B(x,r))}{(2r)^t} \geqslant \frac{2}{2^t 2^{n_k} s_{n_k-1}^t} = \frac{2(1-\alpha_k)^t}{2^t 2^{n_k}(s_{n_k}+y_{n_k})^t}.$$

Finally, the events A_k are independent and

$$\mu_E(A_k) = \frac{\#\widetilde{D}_{n_k}}{2^{n_k}} = \frac{2\sum_{j=1}^{n_{k-1}-1}\#D_{n_k-(l+1)}}{2^{n_k}} = 1 - \frac{2}{2^{n_{k-1}}}.$$

Hence,

$$\mu_E(A) = \lim_{n\to\infty} \prod_{k\geqslant n} \left(1 - \frac{2}{2^{n_{k-1}}}\right) = 1,$$

which concludes the proof. □

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