J. Math. Anal. Appl. 370 (2010) 530–537

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Extra invariance of shift-invariant spaces on LCA groups

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ARTICLE INFO

Article history:
Received 5 November 2009
Available online 15 May 2010
Submitted by R.H. Torres

Keywords:
Shift-invariant space
Translation invariant space
LCA groups
Range functions
Fiber spaces

ABSTRACT

This article generalizes recent results in the extra invariance for shift-invariant spaces to the context of LCA groups. Let G be a locally compact abelian (LCA) group and K a closed subgroup of G. A closed subspace of $L^2(G)$ is called K-invariant if it is invariant under translations by elements of K. Assume now that H is a countable uniform lattice in G and M is any closed subgroup of G containing H. In this article we study necessary and sufficient conditions for an H-invariant space to be M-invariant. As a consequence of our results we prove that for each closed subgroup M of G containing the lattice H, there exists an H-invariant space S that is exactly M-invariant. That is, S is not invariant under any other subgroup M' containing H. We also obtain estimates on the support of the Fourier transform of the generators of the H-invariant space, related to its M-invariance.

1. Introduction

Let G be a locally compact abelian (LCA) group and K a closed subgroup of G. For $y \in G$ let us denote by t_y the translation operator acting on $L^2(G)$. That is, $t_y f(x) = f(x-y)$ for $x \in G$ and $f \in L^2(G)$.

A closed subspace S of $L^2(G)$ satisfying that $t_k f \in S$ for every $f \in S$ and every $k \in K$ is called K-invariant.

In the case that G is $\mathbb{R}^d$ and K is $\mathbb{Z}^d$ the subspace S is called shift-invariant. Shift-invariant spaces are central in several areas such as approximation theory, wavelets, frames and sampling.

The structure of these spaces for the group $\mathbb{R}^d$ and $\mathbb{Z}^d$-translations has been studied in [8,15,10,13,3–5] and in the context of general LCA groups, in [12,6].

Independently of their mathematical interest, they are very important in applications. They provide models for many problems in signal and image processing.

A relevant question in the study of shift-invariant spaces in the line is whether the functions belonging to the space remain in the space when translated by a non-integer real number τ. It is easy to see that the set of parameters τ that leave the space invariant under translations by τ (the invariance set), forms a subgroup M of G. Clearly this subgroup contains the group $\mathbb{Z}$.

Spaces that are invariant under a subgroup M containing $\mathbb{Z}$, are easy to obtain. For example, if g is any function in $L^2(\mathbb{R})$, the space

$$S = \overline{\operatorname{span}}\{g(x-m)\colon m \in M\}$$

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[1] The research of the authors is partially supported by Grants: ANPCyT, PICT 2006–177, CONICET, PIP 5650, UBACyT X058 and X108.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.05.040

is a shift-invariant space that is M-invariant. However, the interest here is the more subtle question of recognizing when a given shift-invariant space is M-invariant.

Shift-invariant spaces that are M-invariant were completely characterized by Aldroubi et al. in [1], for every subgroup M such that $\mathbb{Z} \subseteq M \subseteq \mathbb{R}$.

More recently, the results have been extended to several variables by Anastasio et al. in [2].

The aim of this paper is to investigate whether these characterizations are still valid for the general context of LCA groups. This is important in order to obtain general conditions that can be applied to different cases, as, for example, the case of the classic groups such as the d-dimensional torus $\mathbb{T}^d$, the discrete group $\mathbb{Z}^d$, and the finite group $\mathbb{Z}_d$.

More precisely let $H \subseteq G$ be a countable uniform lattice and M be any closed subgroup of G satisfying that $H \subseteq M \subseteq G$. We want to study necessary and sufficient conditions in order that an H-invariant space $S \subseteq L^2(G)$ is M-invariant.

This article is organized as follows. In Section 2 we set the notation and give some definitions. We study the properties of the invariance set in Section 3. In Section 4 we describe the structure of M-invariant spaces and range functions in the context of LCA groups. The characterizations of M-invariance for H-invariant spaces are given in Section 5. Finally in Section 6 we give some applications.

2. Notation

Let G be an arbitrary locally compact Hausdorff abelian group (LCA) written additively. We will denote by m_G its Haar measure. The dual group of G, that is, the set of continuous characters on G, is denoted by $\hat{G}$. The value of the character $\gamma \in \hat{G}$ at the point $x \in G$, is written by (x, γ).

The Fourier transform of a Haar integrable function f on G, is the function $\hat{f}$ on $\hat{G}$ defined by

$$\hat{f}(\gamma) = \int_G f(x)(x, -\gamma)\, dm_G(x), \quad \gamma \in \hat{G}.$$

When the Haar measures m_G and $m_{\hat{G}}$ are normalized such that the Inversion Formula holds (see [14]), the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\hat{G})$, the so-called Plancharel transformation. We also denote this transformation by "$\wedge$".

Note that the Fourier transform satisfies $\widehat{t_x f}(\cdot) = (-x, \cdot)\hat{f}(\cdot)$.

For a subgroup K of G, the set

$$K^* = \{\gamma \in \hat{G}\colon (k, \gamma) = 1,\ \forall k \in K\}$$

is called the *annihilator* of K. Since every character in $\hat{G}$ is continuous, K^* is a closed subgroup of $\hat{G}$.

We will say that a closed subspace $V \subseteq L^2(G)$ is K*-invariant* if

$$f \in V \quad \Rightarrow \quad t_k f \in V, \quad \forall k \in K.$$

For a subset $\mathcal{A} \subseteq L^2(G)$, define

$$E_K(\mathcal{A}) = \{t_k\varphi\colon \varphi \in \mathcal{A},\ k \in K\} \quad \text{and} \quad S_K(\mathcal{A}) = \overline{\operatorname{span}}\, E_K(\mathcal{A}).$$

We call $S_K(\mathcal{A})$ the K-invariant space generated by $\mathcal{A}$. If $\mathcal{A} = \{\varphi\}$, we simply write $S_K(\varphi)$, and we call $S_K(\varphi)$ a *principal* K*-invariant space*.

Let L be a subset of G. We will say that a function f defined on G is L-periodic if $t_\ell f = f$ for all $\ell \in L$. A subset $B \subseteq G$ is L-periodic if its indicator function (denoted by χ_B) is L-periodic.

When two LCA groups G_1 and G_2 are topologically isomorphic we will write $G_1 \approx G_2$.

3. The invariance set

Here and subsequently G will be an LCA group and H a countable uniform lattice in G, that is, a countable discrete subgroup of G with compact quotient group G/H.

For simplicity of notation throughout this paper we will write Γ instead of $\hat{G}$.

The aim of this work is to characterize the extra invariance of an H-invariant space. For this, given $S \subseteq L^2(G)$ an H-invariant space, we define the *invariance set* as

$$M = \{x \in G\colon t_x f \in S,\ \forall f \in S\}. \tag{1}$$

If $\mathcal{A}$ is a set of generators for S, it is easy to check that $m \in M$ if and only if $t_m\varphi \in S$ for all $\varphi \in \mathcal{A}$.

In case that $M = G$, Wiener's theorem (see [8,15,10]) states that there exists a measurable set $E \subseteq \Gamma$ satisfying

$$S = \{f \in L^2(G)\colon \operatorname{supp}(\hat{f}) \subseteq E\}.$$

We want to describe S when M is not all G. We will first study the structure of the set M.

Proposition 3.1. *Let S be an H-invariant space of $L^2(G)$ and let M be defined as in* (1)*. Then M is a closed subgroup of G containing H.*

For the proof of this proposition we will need the following lemma. Recall that a semigroup is a nonempty set with an associative additive operation.

Lemma 3.2. *Let K be a closed semigroup of G containing H, then K is a group.*

Proof. Let π be the quotient map from G onto G/H. Since K is a semigroup containing H, we have that $K+H=K$, thus

$$\pi^{-1}\big(\pi(K)\big)=\bigcup_{k\in K}k+H=K+H=K. \tag{2}$$

This shows that $\pi(K)$ is closed in G/H and therefore compact.

By [9, Theorem 9.16], we have that a compact semigroup of G/H is necessarily a group, thus $\pi(K)$ is a group and consequently K is a group. □

Proof of Proposition 3.1. Since S is an H-invariant space, $H\subseteq M$.

We first proceed to show that M is closed. Let $x_0\in G$ and let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in M converging to x_0. Then

$$\lim_{\lambda}\|t_{x_\lambda}f-t_{x_0}f\|_2=0.$$

Since S is closed, it follows that $t_{x_0}f\in S$, thus $x_0\in M$.

It is easy to check that M is a semigroup of G, hence we conclude from Lemma 3.2 that M is a group. □

4. The structure of principal *M*-invariant spaces

4.1. Preliminaries

Shift-invariant spaces in $L^2(\mathbb{R}^d)$ are completely characterized using fiberization techniques and range functions (see [5]). This theory has been extended to general LCA groups in [6]. In what follows we state some definitions and properties given in that work.

We will assume that G is a second countable LCA group and H a countable uniform lattice in G.

The fact that G is second countable, G/H is compact and $\widehat{G/H}\approx H^*$, implies that H^* is countable and discrete. Moreover, since $\Gamma/H^*\approx\hat{H}$, H^* is a countable uniform lattice in Γ. Therefore, there exists a measurable section Ω of Γ/H^* with finite m_Γ-measure (see [11] and [7]).

Let $L^2(\Omega,\ell^2(H^*))$ be the space of all measurable functions $\Phi:\Omega\to\ell^2(H^*)$ such that

$$\|\Phi\|_2^2:=\int_\Omega\big\|\Phi(\omega)\big\|^2_{\ell^2(H^*)}\,dm_\Gamma(\omega)<\infty.$$

The following proposition shows that the space $L^2(\Omega,\ell^2(H^*))$ is isometric (up to a constant) to $L^2(G)$.

Proposition 4.1. *The mapping $\mathcal{T}_H:L^2(G)\to L^2(\Omega,\ell^2(H^*))$ defined as*

$$\mathcal{T}_Hf(\omega)=\big\{\hat{f}\big(\omega+h^*\big)\big\}_{h^*\in H^*}$$

is an isomorphism that satisfies $\|\mathcal{T}_Hf\|_2=\|f\|_{L^2(G)}$.

For $f\in L^2(G)$, the sequence $\mathcal{T}_Hf(\omega)=\{\hat{f}(\omega+h^*)\}_{h^*\in H^*}$ is *the H-fiber of f at ω*. Given a subspace V of $L^2(G)$ and $\omega\in\Omega$, the *H-fiber space* of V at ω is

$$J_H(V)(\omega)=\overline{\big\{\mathcal{T}_Hf(\omega):\ f\in V\big\}},$$

where the closure is taken in the norm of $\ell^2(H^*)$.

The map that assigns to each ω the fiber space $J_H(V)(\omega)$ is known in the literature as the *range function* of V.

The following proposition characterizes H-invariant spaces in terms of range functions and fibers.

Proposition 4.2. *If S is an H-invariant space in $L^2(G)$, then*

$$S=\big\{f\in L^2(G):\ \mathcal{T}_Hf(\omega)\in J_H(S)(\omega)\ \text{for a.e.}\ \omega\in\Omega\big\}.$$

Moreover, if $S=S_H(\mathcal{A})$ for a countable set $\mathcal{A}\subseteq L^2(G)$, then, for almost every $\omega\in\Omega$,

$$J_H(S)(\omega)=\overline{\operatorname{span}}\big\{\mathcal{T}_H\varphi(\omega):\ \varphi\in\mathcal{A}\big\}.$$

In this article we will use fiberization techniques for a more general case, since the subspaces will be invariant under a closed subgroup which is not necessarily discrete.

The above results from [6] can be extended straightforwardly to the case that the spaces are invariant under a closed subgroup M of G containing a countable uniform lattice H as follows.

Since $H \subseteq M$, we have that $M^* \subseteq H^*$ and, in particular, M^* is discrete. Thus, there exists a countable section $\mathcal{N}$ of H^*/M^*. Then, the set given by

$$\mathcal{D} = \bigcup_{\sigma \in \mathcal{N}} \Omega + \sigma \tag{3}$$

is a σ-finite measurable section of the quotient Γ/M^*. Using this section of Γ/M^* it is possible to obtain, in a way analogous to the discrete case, the following:

Proposition 4.3.

(i) *The mapping $\mathcal{T}_M : L^2(G) \to L^2(\mathcal{D}, \ell^2(M^*))$ defined as*

$$\mathcal{T}_M f(\delta) = \{\hat{f}(\delta + m^*)\}_{m^* \in M^*}$$

is an isomorphism that satisfies $\|\mathcal{T}_M f\|_2 = \|f\|_{L^2(G)}$.

(ii) *Let S be an M-invariant space generated by a countable set $\mathcal{A}$. For each $\delta \in \mathcal{D}$, define the M-*fiber space *of S at δ as*

$$J_M(S)(\delta) = \overline{\operatorname{span}}\{\mathcal{T}_M \varphi(\delta):\ \varphi \in \mathcal{A}\}.$$

If P and P_δ are the orthogonal projections onto S and $J_M(S)(\delta)$ respectively, then, for every $g \in L^2(G)$,

$$\mathcal{T}_M(Pg)(\delta) = P_\delta(\mathcal{T}_M g(\delta)) \quad \text{a.e. } \delta \in \mathcal{D}.$$

(iii) *If S is an M-invariant space in $L^2(G)$, then*

$$S = \{f \in L^2(G):\ \mathcal{T}_M f(\delta) \in J_M(S)(\delta) \text{ for a.e. } \delta \in \mathcal{D}\}.$$

4.2. Principal M-invariant spaces

We prove now the following characterization of principal M-invariant spaces. This result extends the $\mathbb{R}^d$ case.

Theorem 4.4. *Let $f \in L^2(G)$ and let M be a closed subgroup of G containing H. If $g \in S_M(f)$, then there exists an M^*-periodic function η such that $\hat{g} = \eta \hat{f}$.*

Conversely, if η is an M^*-periodic function such that $\eta \hat{f} \in L^2(\Gamma)$, then the function g defined by $\hat{g} = \eta \hat{f}$ belongs to $S_M(f)$.

Proof. Let us call $S = S_M(f)$ and let P and P_δ be the orthogonal projections onto S and $J_M(S)(\delta)$ respectively. Given $g \in S$, we first define η_g in $\mathcal{D}$ as

$$\eta_g(\delta) = \begin{cases} \frac{\langle \mathcal{T}_M g(\delta), \mathcal{T}_M f(\delta)\rangle}{\|\mathcal{T}_M f(\delta)\|_2^2} & \text{if } \delta \in E_f, \\ 0 & \text{otherwise}, \end{cases}$$

where E_f is the set $\{\delta \in \mathcal{D}:\ \|\mathcal{T}_M f(\delta)\|_2^2 \neq 0\}$. Then, since $\{\mathcal{D} + m^*\}_{m^* \in M^*}$ forms a partition of Γ, we can extend η_g to all Γ in an M^*-periodic way.

Now, by Proposition 4.3 we have that

$$\mathcal{T}_M g(\delta) = \mathcal{T}_M(Pg)(\delta) = P_\delta(\mathcal{T}_M g(\delta)) = \eta_g(\delta)\mathcal{T}_M f(\delta).$$

Since η_g is an M^*-periodic function, $\hat{g} = \eta_g \hat{f}$ as we wanted to prove.

Conversely, if $\hat{g} = \eta \hat{f}$, with η an M^*-periodic function, then $\mathcal{T}_M g(\delta) = \eta(\delta)\mathcal{T}_M f(\delta)$. By Proposition 4.3, $g \in S$. □

5. Characterization of *M*-invariance

If $H \subseteq M \subseteq G$, where H is a countable uniform lattice in G and M is a closed subgroup of G, we are interested in describing when an H-invariant space S is also M-invariant.

Let Ω be a measurable section of Γ/H^* and $\mathcal{N}$ a countable section of H^*/M^*. For $\sigma \in \mathcal{N}$ we define the set B_σ as

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{m^* \in M^*} (\Omega + \sigma) + m^*. \tag{4}$$

Therefore, each B_σ is an M^*-periodic set.

Since Ω tiles Γ by H^* translations and $\mathcal{N}$ tiles H^* by M^* translations, it follows that $\{B_\sigma\}_{\sigma\in\mathcal{N}}$ is a partition of Γ.
Now, given an H-invariant space S, for each $\sigma \in \mathcal{N}$, we define the subspaces

$$U_\sigma = \{ f \in L^2(G):\ \hat{f} = \chi_{B_\sigma} \hat{g},\ \text{with } g \in S \}. \tag{5}$$

5.1. Characterization of M-invariance in terms of subspaces

The main theorem of this section characterizes the M-invariance of S in terms of the subspaces U_σ.

Theorem 5.1. *If $S \subseteq L^2(G)$ is an H-invariant space and M is a closed subgroup of G containing H, then the following are equivalent.*

(i) *S is M-invariant.*
(ii) *$U_\sigma \subseteq S$ for all $\sigma \in \mathcal{N}$.*

Moreover, in case any of these hold we have that S is the orthogonal direct sum

$$S = \dot{\bigoplus_{\sigma\in\mathcal{N}}} U_\sigma .$$

Now we state a lemma that we need to prove Theorem 5.1.

Lemma 5.2. *Let S be an H-invariant space and $\sigma \in \mathcal{N}$. Assume that the subspace U_σ defined in (5) satisfies $U_\sigma \subseteq S$. Then, U_σ is an M-invariant space and in particular is H-invariant.*

Proof. Let us prove first that U_σ is closed. Suppose that $f_j \in U_\sigma$ and $f_j \to f$ in $L^2(G)$. Since $U_\sigma \subseteq S$ and S is closed, f must be in S. Further,

$$\|\widehat{f_j} - \hat{f}\|_2^2 = \|(\widehat{f_j} - \hat{f})\chi_{B_\sigma}\|_2^2 + \|(\widehat{f_j} - \hat{f})\chi_{B_\sigma^c}\|_2^2 = \|\widehat{f_j} - \hat{f}\chi_{B_\sigma}\|_2^2 + \|\hat{f}\chi_{B_\sigma^c}\|_2^2.$$

Since the left-hand side converges to zero, we must have that $\hat{f}\chi_{B_\sigma^c} = 0$ a.e. $\gamma \in \Gamma$. Then, $\hat{f} = \hat{f}\chi_{B_\sigma}$. Consequently $f \in U_\sigma$, so U_σ is closed.

Now we show that U_σ is M-invariant. Given $m \in M$ and $f \in U_\sigma$, we will prove that $(m,\cdot)\hat{f}(\cdot) \in \hat{U}_\sigma$.

Since $f \in U_\sigma$, there exists $g \in S$ such that $\hat{f} = \chi_{B_\sigma}\hat{g}$. Hence,

$$(m,\cdot)\hat{f}(\cdot) = (m,\cdot)(\chi_{B_\sigma}\hat{g})(\cdot) = \chi_{B_\sigma}(\cdot)\big((m,\cdot)\hat{g}(\cdot)\big). \tag{6}$$

If we were able to find an H^*-periodic function ℓ_m verifying

$$(m,\gamma) = \ell_m(\gamma) \quad \text{a.e. } \gamma \in B_\sigma, \tag{7}$$

then, we can rewrite (6) as

$$(m,\cdot)\hat{f}(\cdot) = \chi_{B_\sigma}(\cdot)(\ell_m\hat{g})(\cdot).$$

Theorem 4.4 can then be applied for the uniform lattice H. Thus, since ℓ_m is H^*-periodic, we obtain that $\ell_m\hat{g} \in \widehat{S_H(g)} \subseteq \hat{S}$ and so, $(m,\cdot)\hat{f}(\cdot) \in \hat{U}_\sigma$.

Now we define the function ℓ_m as follows. For each $h^* \in H^*$, set

$$\ell_m(\omega + h^*) = (m, \omega + \sigma) \quad \text{a.e. } \omega \in \Omega. \tag{8}$$

It is clear that ℓ_m is H^*-periodic.

Since $(m,\cdot)$ is M^*-periodic,

$$(m, \omega + \sigma) = (m, \omega + \sigma + m^*) \quad \text{a.e. } \omega \in \Omega,\ \forall m^* \in M^*.$$

Thus, (7) holds.

Note that, since $H \subseteq M$, the H-invariance of U_σ is a consequence of the M-invariance. □

Proof of Theorem 5.1. (i) ⇒ (ii): Fix $\sigma \in \mathcal{N}$ and $f \in U_\sigma$. Then $\hat{f} = \chi_{B_\sigma}\hat{g}$ for some $g \in S$. Since χ_{B_σ} is an M^*-periodic function, by Theorem 4.4, we have that $f \in S_M(g) \subseteq S$, as we wanted to prove.

(ii) ⇒ (i): Suppose that $U_\sigma \subseteq S$ for all $\sigma \in \mathcal{N}$. Note that Lemma 5.2 implies that U_σ is M-invariant, and we also have that the U_σ are mutually orthogonal since the sets B_σ are disjoint.

Suppose that $f \in S$. Then, since $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ is a partition of Γ, it follows that $\hat{f} = \sum_{\sigma \in \mathcal{N}} \hat{f} \chi_{B_\sigma}$. This implies that $f \in \dot{\bigoplus}_{\sigma \in \mathcal{N}} U_\sigma$ and consequently, S is the orthogonal direct sum

$$S = \dot{\bigoplus_{\sigma \in \mathcal{N}}} U_\sigma.$$

As each U_σ is M-invariant, so is S. □

5.2. Characterization of M-invariance in terms of H-fibers

In this section we will first express the conditions of Theorem 5.1 in terms of H-fibers. Then, we will give a useful characterization of the M-invariance for a finitely generated H-invariant space in terms of the Gramian.

If $f \in L^2(G)$ and $\sigma \in \mathcal{N}$, we define the function f^σ by

$$\widehat{f^\sigma} = \hat{f} \chi_{B_\sigma}.$$

Let P_σ be the orthogonal projection onto S_σ, where

$$S_\sigma = \{f \in L^2(G)\colon \operatorname{supp}(\hat{f}) \subseteq B_\sigma\}.$$

Therefore

$$f^\sigma = P_\sigma f \quad \text{and} \quad U_\sigma = P_\sigma(S) = \{f^\sigma\colon f \in S\}.$$

Moreover, if $S = S_H(\mathcal{A})$ with $\mathcal{A}$ a countable subset of $L^2(G)$, then

$$J_H(U_\sigma)(\omega) = \overline{\operatorname{span}}\{\mathcal{T}_H(\varphi^\sigma)(\omega)\colon \varphi \in \mathcal{A}\}. \tag{9}$$

Remark 5.3. Note that the fibers

$$\mathcal{T}_H(\varphi^\sigma)(\omega) = \{\chi_{B_\sigma}(\omega + h^*)\hat{\varphi}(\omega + h^*)\}_{h^* \in H^*}$$

can be described in a simple way as

$$\chi_{B_\sigma}(\omega + h^*)\hat{\varphi}(\omega + h^*) = \begin{cases} \hat{\varphi}(\omega + h^*) & \text{if } h^* \in \sigma + M^*, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\sigma \neq \sigma'$, $J_H(U_\sigma)(\omega)$ and $J_H(U_{\sigma'})(\omega)$ are orthogonal subspaces for a.e. $\omega \in \Omega$.

Combining Theorem 5.1 with Proposition 4.2 and (9) we obtain the following result.

Proposition 5.4. *Let S be an H-invariant space generated by a countable set $\mathcal{A} \subseteq L^2(G)$. The following statements are equivalent.*

(i) *S is M-invariant.*
(ii) *$\mathcal{T}_H(\varphi^\sigma)(\omega) \in J_H(S)(\omega)$ a.e. $\omega \in \Omega$ for all $\varphi \in \mathcal{A}$ and $\sigma \in \mathcal{N}$.*

Let $\Phi = \{\varphi_1, \ldots, \varphi_\ell\}$ be a finite collection of functions in $L^2(G)$. Then, the *Gramian* G_Φ of Φ is the $\ell \times \ell$ matrix of H^*-periodic functions

$$\begin{aligned}[G_\Phi(\omega)]_{ij} &= \langle \mathcal{T}_H(\varphi_i)(\omega), \mathcal{T}_H(\varphi_j)(\omega)\rangle \\ &= \sum_{h^* \in H^*} \widehat{\varphi_i}(\omega + h^*)\overline{\widehat{\varphi_j}(\omega + h^*)}\end{aligned} \tag{10}$$

for $\omega \in \Omega$.

Given a subspace V of $L^2(G)$, the *dimension function* is defined by

$$\dim_V : \Omega \to \mathbb{N}_0 \cup \{\infty\}, \quad \dim_V(\omega) := \dim(J_H(V)(\omega)).$$

We will also need the next result which is a straightforward consequence of Propositions 4.1 and 4.2.

Proposition 5.5. *Let S_1 and S_2 be H-invariant spaces. If $S = S_1 \dot{\oplus} S_2$, then*

$$J_H(S)(\omega) = J_H(S_1)(\omega) \dot{\oplus} J_H(S_2)(\omega), \quad \text{a.e. } \omega \in \Omega.$$

The converse of this proposition is also true, but will not be needed.

Now we give a slightly simpler characterization of M-invariance for the finitely generated case.

Theorem 5.6. *If S is an H-invariant space, finitely generated by Φ, then the following statements are equivalent.*

(i) *S is M-invariant.*
(ii) *For almost every $\omega \in \Omega$, $\dim_S(\omega) = \sum_{\sigma \in \mathcal{N}} \dim_{U_\sigma}(\omega)$.*
(iii) *For almost every $\omega \in \Omega$, $\operatorname{rank}[G_\Phi(\omega)] = \sum_{\sigma \in \mathcal{N}} \operatorname{rank}[G_{\Phi^\sigma}(\omega)]$, where $\Phi^\sigma = \{\varphi^\sigma\colon \varphi \in \Phi\}$.*

Proof. (i) $\Rightarrow$ (ii): By Theorem 5.1, $S = \dot{\bigoplus}_{\sigma \in \mathcal{N}} U_\sigma$. Then, (ii) follows from Proposition 5.5.

(ii) $\Rightarrow$ (i): Since $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ is a partition of Γ, $S \subseteq \dot{\bigoplus}_{\sigma \in \mathcal{N}} U_\sigma$. Then, by Remark 5.3 we have that

$$J_H(S)(\omega) \subseteq \dot{\bigoplus_{\sigma \in \mathcal{N}}} J_H(U_\sigma)(\omega).$$

Using (ii), we obtain that $J_H(S)(\omega) = \dot{\bigoplus}_{\sigma \in \mathcal{N}} J_H(U_\sigma)(\omega)$. The proof follows as a consequence of Proposition 5.4.

The equivalence between (ii) and (iii) follows from (9). $\square$

6. Applications of *M*-invariance

In this section we estimate the size of the supports of the Fourier transforms of the generators of a finitely generated H-invariant space which is also M-invariant.

We will not include the proof of the result stated bellow, since it follows readily from the $\mathbb{R}^d$ case (see [2, Section 6]).

Theorem 6.1. *Let S be an H-invariant space, finitely generated by the set $\{\varphi_1, \ldots, \varphi_\ell\}$, and define*

$$E_j = \{\omega \in \Omega\colon \dim_S(\omega) = j\}, \quad j = 0, \ldots, \ell.$$

If S is M-invariant and $\mathcal{D}'$ is any measurable section of Γ/M^*, then

$$m_\Gamma\big(\{y \in \mathcal{D}'\colon \widehat{\varphi_i}(y) \neq 0\}\big) \leqslant \sum_{j=0}^{\ell} m_\Gamma(E_j) j \leqslant \ell,$$

for each $i = 1, \ldots, \ell$.

Corollary 6.2. *Let $\varphi \in L^2(G)$ be given. If $S_H(\varphi)$ is M-invariant for some closed subgroup M of G such that $H \subsetneqq M$, then $\hat{\varphi}$ must vanish on a set of positive m_Γ-measure.*

Furthermore, if $m_\Gamma(\Gamma) = +\infty$, $\hat{\varphi}$ must vanish on a set of infinite m_Γ-measure.

Proof. Let

$$\mathcal{D} = \bigcup_{\sigma \in \mathcal{N}} \Omega + \sigma.$$

Then, $\mathcal{D}$ is a section of Γ/M^*.

By Theorem 6.1, we have that

$$m_\Gamma\big(\{y \in \mathcal{D}\colon \hat{\varphi}(y) \neq 0\}\big) \leqslant 1,$$

thus

$$\begin{aligned} m_\Gamma\big(\{y \in \Gamma\colon \hat{\varphi}(y) = 0\}\big) &= \sum_{m^* \in M^*} m_\Gamma\big(\{y \in \mathcal{D}\colon \hat{\varphi}(y) = 0\}\big) \\ &\geqslant \#\big(M^*\big)\#(\mathcal{N} - 1). \end{aligned} \tag{11}$$

Since $H \subsetneqq M$, it follows that $\#\mathcal{N} > 1$, so $m_\Gamma(\{y \in \Gamma\colon \hat{\varphi}(y) = 0\}) > 0$.

If $m_\Gamma(\Gamma) = +\infty$, then either $m_\Gamma(\mathcal{D}) = +\infty$ or $\#M^* = +\infty$. In case that $\#M^* = +\infty$, by (11), $\hat{\varphi}$ must vanish on a set of infinite m_Γ-measure. If $m_\Gamma(\mathcal{D}) = +\infty$, since $m_\Gamma(\Omega) = 1$, it follows that $\#\mathcal{N} = +\infty$. Then, using again (11), we can conclude the same as before. $\square$

As a consequence of Theorem 6.1, in case that $M = G$, we obtain the following corollary.

Corollary 6.3. *If $\varphi \in L^2(G)$ and $S_H(\varphi)$ is G-invariant, then*

$$m_\Gamma\big(\operatorname{supp}(\hat{\varphi})\big) \leqslant 1.$$

6.1. Exactly M-invariance

Let M be a closed subgroup of G containing a countable uniform lattice H. The next theorem states that there exists an M-invariant space S that is *not* invariant under any vector outside M. We will say in this case that S is *exactly* M-invariant.

Note that because of Proposition 3.1, an M-invariant space is exactly M-invariant if and only if it is not invariant under any closed subgroup M' containing M.

Theorem 6.4. *For each closed subgroup M of G containing a countable uniform lattice H, there exists a shift-invariant space of $L^2(G)$ which is exactly M-invariant.*

Proof. Suppose that $0 \in \mathcal{N}$ and take $\varphi \in L^2(G)$ satisfying $\operatorname{supp}(\hat{\varphi}) = B_0$, where B_0 is defined as in (4). Let $S = S(\varphi)$.

Then, $U_0 = S$ and $U_\sigma = \{0\}$ for $\sigma \in \mathcal{N}$, $\sigma \neq 0$. So, as a consequence of Theorem 5.1, it follows that S is M-invariant.

Now, if M' is a closed subgroup such that $M \subsetneqq M'$, we will show that S cannot be M'-invariant.

Since $M \subseteq M'$, $(M')^* \subseteq M^*$. Consider a section $\mathcal{C}$ of the quotient $M^*/(M')^*$ containing the origin. Then, the set given by

$$\mathcal{N}' := \{\sigma + c\colon\ \sigma \in \mathcal{N},\ c \in \mathcal{C}\}$$

is a section of $H^*/(M')^*$ and $0 \in \mathcal{N}'$.

If $\{B'_{\sigma'}\}_{\sigma' \in \mathcal{N}'}$ is the partition defined in (4) associated to M', for each $\sigma \in \mathcal{N}$ it holds that $\{B'_{\sigma+c}\}_{c \in \mathcal{C}}$ is a partition of B_σ, since

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{c \in \mathcal{C}} \Omega + \sigma + c + \big(M'\big)^* = \bigcup_{c \in \mathcal{C}} B'_{\sigma+c}. \tag{12}$$

We will show now that $U'_0 \nsubseteq S$, where U'_0 is the subspace defined in (5) for M'. Let $g \in L^2(G)$ such that $\hat{g} = \hat{\varphi}\chi_{B'_0}$. Then $g \in U'_0$. Moreover, since $\operatorname{supp}(\hat{\varphi}) = B_0$, by (12), $\hat{g} \neq 0$.

Suppose that $g \in S$, then $\hat{g} = \eta\hat{\varphi}$ where η is an H^*-periodic function. Since $M \subsetneqq M'$, there exists $c \in \mathcal{C}$ such that $c \neq 0$. By (12), $\hat{g}$ vanishes in B'_c. Then, the H^*-periodicity of η implies that $\eta(\gamma) = 0$ a.e. $\gamma \in \Gamma$. So $\hat{g} = 0$, which is a contradiction.

This shows that $U'_0 \nsubseteq S$. Therefore, S is not M'-invariant. □

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