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An optimization problem with volume constraint in Orlicz spaces ☆

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Abstract

We consider the optimization problem of minimizing $\int_\Omega G(|\nabla u|)\,dx$ in the class of functions $W^{1,G}(\Omega)$, with a constraint on the volume of $\{u > 0\}$. The conditions on the function G allow for a different behavior at 0 and at ∞. We consider a penalization problem, and we prove that for small values of the penalization parameter, the constrained volume is attained. In this way we prove that every solution u is locally Lipschitz continuous and that the free boundary, $\partial\{u > 0\} \cap \Omega$ is smooth.

Keywords: Optimal design problems; Free boundaries; Orlicz spaces

1. Introduction

We begin with a few historical remarks. In the paper [1], Aguilera, Alt and Caffarelli study an optimal design problem with a volume constraint. The authors prove the regularity of minimizers by introducing a penalization term in the energy functional (the Dirichlet integral) and minimizing the penalized functional without the volume constraint. The authors start by observing that, for fixed values of the penalization parameter, the penalized functional is very similar to the one considered in the paper [3] and they obtain the regularity results by using techniques very similar to the ones in [3]. Then, they prove that for small values of the penalization parameter, the constrained volume is attained. In this way, all the regularity results apply to the solution of the optimal design problem.

The method we have just described has been applied to other problems with similar success. See, for instance, [2,9,12,18] where the differential equation satisfied by the minimizers is nondegenerate, uniformly elliptic, and [8], where the equation involved may be degenerate or singular elliptic, but it still has the property of being homogeneous.

In this article we show that the same kind of results can be obtained for problems where the differential equation satisfied by the minimizers is nonlinear degenerate or singular elliptic, and possibly not homogeneous. More precisely, the operator we study here has the form $\mathcal{L}u = \operatorname{div}(g(|\nabla u|)\frac{\nabla u}{|\nabla u|})$ where g satisfies the natural conditions introduced by Lieberman in [14]. These conditions generalize the so-called natural conditions of Ladyzhenskaya and Ural'tseva.

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In [14] the author studies the regularity of weak solutions of the equation

$$\mathcal{L}u = 0, \tag{1.1}$$

and proves that, under his conditions, the solutions of (1.1) are $C^{1,\beta}$.

The conditions imposed to g are the following: $g \in C^1(\mathbb{R}_{\geqslant 0})$, $g(t) > 0$ for $t > 0$ and

$$0 < \delta \leqslant \frac{tg'(t)}{g(t)} \leqslant g_0, \quad \forall t > 0, \tag{1.2}$$

for certain constants δ and g_0. Observe that $\delta = g_0 = p - 1$ when $g(t) = t^{p-1}$, and conversely, if $\delta = g_0$ then g is a power. For more examples of functions satisfying (1.2) see [15].

Condition (1.2) ensures that Eq. (1.1) is equivalent to a uniformly elliptic equation in nondivergence form with ellipticity constants independent of the solution u on sets where $\nabla u \neq 0$. This condition does not imply any kind of homogeneity on the function G (the primitive of g) and, moreover, it allows for a different behavior of the function g when $|\nabla u|$ is close to zero or infinity.

We describe now, more precisely, the problem that we study.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $0 \leqslant \varphi_0 \in W^{1,G}(\Omega)$ a Dirichlet datum, with $\varphi_0 \geqslant c_0 > 0$ in $\bar{A}$, where A is a nonempty relatively open subset of $\partial\Omega$ such that $A \cap \partial\Omega$ is C^2. Here $W^{1,G}(\Omega)$ is a Sobolev–Orlicz space (see Appendix A). Let $0 < \alpha < |\Omega|$ and

$$\mathcal{K}_\alpha = \big\{u \in W^{1,G}(\Omega) / \big|\{u > 0\}\big| = \alpha,\ u = \varphi_0 \text{ on } \partial\Omega\big\}.$$

Our problem is to minimize $\mathcal{J}(u) = \int_\Omega G(|\nabla u|)\,dx$ in $\mathcal{K}_\alpha$, with $g = G'$ satisfying (1.2).

One difficulty for the proof of the regularity of the minimizers in these type of problems, is that it is hard to make enough volume preserving perturbations without an a priori knowledge of the regularity of $\partial\{u > 0\}$.

In order to solve our original problem using nonvolume preserving perturbations we follow the idea of [1] and consider the following penalized problem: We let

$$\mathcal{K} = \big\{u \in W^{1,G}(\Omega) / u = \varphi_0 \text{ on } \partial\Omega\big\}$$

and

$$\mathcal{J}_\varepsilon(u) = \int_\Omega G\big(|\nabla u|\big)\,dx + F_\varepsilon\big(\big|\{u > 0\}\big|\big), \tag{1.3}$$

where

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s - \alpha) & \text{if } s < \alpha, \\ \frac{1}{\varepsilon}(s - \alpha) & \text{if } s \geqslant \alpha. \end{cases}$$

Then, the penalized problem is:

$$\text{find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v \in \mathcal{K}} \mathcal{J}_\varepsilon(v). \tag{P_ε}$$

To prove the existence of minimizers we use compact immersion theorems in Sobolev–Orlicz spaces and direct minimization. The regularity of the minimizers and of their free boundaries $\partial\{u_\varepsilon > 0\}$ follows by showing that any minimizer u_ε is a solution of the free boundary problem

$$\begin{cases} \mathcal{L}u_\varepsilon = 0 & \text{in } \{u_\varepsilon > 0\} \cap \Omega, \\ u_\varepsilon = 0, \quad \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda_\varepsilon & \text{on } \partial\{u_\varepsilon > 0\} \cap \Omega, \end{cases} \tag{1.4}$$

in the sense defined in [15], where λ_ε is a positive constant. The properties of the definition of weak solution are not difficult to establish since the minimization problem studied in [15] is very similar to (P_ε). The only difference is that in (P_ε) the functional is linear in $|\{u > 0\}|$ while here the term F_ε is piecewise linear and zero at the value α. With these properties we have that the free boundary is locally a $C^{1,\beta}$ surface in a neighborhood of $\mathcal{H}^{N-1}$—almost every point (see Corollary 2.1).

For a subclass of functions satisfying (1.2) we improve the regularity result for the case $N = 2$. Indeed, in that case the whole free boundary is regular. Full regularity of the free boundary in dimension 2 was proved in [1] and [4] in the case of uniformly elliptic operators, in [6] for the p-laplacian with $2-\delta \leqslant p < \infty$ for a small $\delta > 0$, and also in [12] for a penalization problem. In dimension 3 for p close to 2 a similar result was proved by A. Petrosyan (see [17]).

As in [1], the reason why this penalization method is so useful is that there is no need to pass to the limit in the penalization parameter ε for which regularity estimates uniform in ε would be needed. In fact, we show that for small values of ε the right volume is already attained. That is, $|\{u_\varepsilon > 0\}| = \alpha$ for small ε. This step is where the proof is different from previous work on similar problems, since here the function g may not be homogeneous (see Lemma 3.3).

Finally, the fact that for small ε any minimizer of $\mathcal{J}_\varepsilon$ satisfies $|\{u_\varepsilon > 0\}| = \alpha$ implies that any minimizer of our original optimization problem is also a minimizer of $\mathcal{J}_\varepsilon$ and, therefore, that it is locally Lipschitz continuous with smooth free boundary.

The paper is organized as follows: We begin our analysis of problem (P_ε) for fixed ε in Section 2 where we prove the existence of a minimizer, local Lipschitz regularity and nondegeneracy near the free boundary (Theorem 2.1) and we also prove that minimizers are weak solutions of a free boundary problem—as defined in [15]—(Remark 2.1). As a consequence, the free boundary is a $C^{1,\beta}$ surface in a neighborhood of $\mathcal{H}^{N-1}$—almost every point in the free boundary (Corollary 2.1). For the case $N = 2$ and for the subclass of functions satisfying (1.2) we prove that their whole free boundary is regular (Corollary 2.2). In Section 3 we show that for small values of ε we recover our original optimization problem.

At the end of the paper we include three appendices with auxiliary results on Orlicz spaces, $\mathcal{L}$-subharmonic functions and blow-up sequences.

2. The penalized problem

2.1. Regularity of minimizers and their free boundaries

We begin by discussing the existence of extremals and their regularity. Next, we give some properties of the minimizers. Since the functional $\mathcal{J}_\varepsilon$ is very similar to the one in [15], some of the proofs follow as in [15] so we skip them altogether. Then, we prove that any minimizer of $\mathcal{J}_\varepsilon$ is a weak solution of (1.4), as defined in [15]. From this result we establish that the free boundary is smooth.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Then there exists a solution to the problem (P_ε). Moreover, any solution u_ε has the following properties*:

(1) *u_ε is locally Lipschitz continuous in Ω and, for $D \Subset \Omega$, $\|\nabla u\|_{L^\infty(D)} \leqslant C$ with $C = C(N, g_0, \delta, \operatorname{dist}(\partial\Omega, D), \varepsilon)$.*
(2) *$\mathcal{L}u_\varepsilon = 0$ in $\{u_\varepsilon > 0\}$.*
(3) *There are constants $0 < c_{\min} \leqslant C_{\max}$ and $\gamma \geqslant 1$ such that, for balls $B_r(x) \subset D$ with $x \in \partial\{u_\varepsilon > 0\}$,*

$$c_{\min} \leqslant \frac{1}{r}\left(\fint_{B_r(x)} u_\varepsilon^\gamma \, dx \right)^{1/\gamma} \leqslant C_{\max}.$$

(4) *For every $D \Subset \Omega$ there exist constants $C, c > 0$ such that, for every $x \in D \cap \{u_\varepsilon > 0\}$,*

$$c \operatorname{dist}\bigl(x, \partial\{u_\varepsilon > 0\}\bigr) \leqslant u_\varepsilon(x) \leqslant C \operatorname{dist}\bigl(x, \partial\{u_\varepsilon > 0\}\bigr).$$

(5) *For every $D \Subset \Omega$ there exists a constant $c > 0$ such that, for $x \in \partial\{u_\varepsilon > 0\}$ and $B_r(x) \subset D$,*

$$c \leqslant \frac{|B_r(x) \cap \{u_\varepsilon > 0\}|}{|B_r(x)|} \leqslant 1 - c.$$

The constants may depend on ε.

Proof. Observe that if $A \leqslant B$ then, $\varepsilon(B - A) \leqslant F_\varepsilon(B) - F_\varepsilon(A) \leqslant \frac{1}{\varepsilon}(B - A)$. Then, the proof follows as in Sections 3–5 in [15]. □

From now on we drop the subscript ε and denote by u (instead of u_ε) a solution of (P_ε).

Theorem 2.2 *(Representation Theorem). Let $u \in \mathcal{K}$ be a solution of (P_ε). Then,*

(1) $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ *for every* $D \Subset \Omega$.
(2) *There exists a Borel function q_u such that*

$$\mathcal{L}u = q_u \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}.$$

(3) *For $D \Subset \Omega$ there are constants $0 < c \leqslant C < \infty$ depending on N, Ω, D and ε such that, for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*

$$c \leqslant q_u(x) \leqslant C, \qquad cr^{N-1} \leqslant \mathcal{H}^{N-1}\big(B_r(x) \cap \partial\{u > 0\}\big) \leqslant Cr^{N-1}.$$

(4) $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) = 0$.

Proof. For the proof, see Sections 6 and 7 in [15]. Observe that $D \cap \partial\{u > 0\}$ has finite perimeter, thus, the reduce boundary $\partial_{\mathrm{red}}\{u > 0\}$ is defined as well as the measure theoretic normal $\nu(x)$ for $x \in \partial_{\mathrm{red}}\{u > 0\}$ (see [7]). □

Lemma 2.1. *Let $x_0, x_1 \in \partial\{u > 0\}$ and $\rho_k \to 0^+$. For $i = 0, 1$, let $x_{i,k} \to x_i$ with $u(x_{i,k}) = 0$ such that $B_{\rho_k}(x_{i,k}) \subset \Omega$ and such that the blow-up sequence*

$$u_{i,k}(x) = \frac{1}{\rho_k} u(x_{i,k} + \rho_k x)$$

has a limit $u_i(x) = \lambda_i (x \cdot \nu_i)^-$, with $0 < \lambda_i < \infty$ and ν_i a unit vector. Then $\lambda_0 = \lambda_1$.

Proof. It follows as in [8] by using the results in Appendix C. □

Lemma 2.2. *Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let*

$$\lambda = \lambda(x_0) := \limsup_{\substack{x \to x_0 \\ u(x) > 0}} \big|\nabla u(x)\big|.$$

Then, there exist sequences $y_k \in \Omega \cap \partial\{u > 0\}$, $d_k \to 0$, and a unit vector ν such that the blow-up sequence with respect to $B_{d_k}(y_k)$ has a limit u_0 with

$$u_0(x) = \lambda (x \cdot \nu)^-.$$

Proof. It follows as the proof of Theorem 2.3 in [8] by using the results in Appendices B and C. □

Lemma 2.3. *For $\mathcal{H}^{N-1}$-a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, there exists a sequence $\gamma_n \to 0$ such that, if u_n is the blow-up sequence with respect to $B_{\gamma_n}(x_0)$, we have that*

$$u_n \to \lambda^*\big(x \cdot \nu(x_0)\big)^-$$

with $\nu(x_0)$ the outward unit normal to $\partial\{u > 0\}$ at x_0 in the measure theoretic sense and $\lambda^* = g^{-1}(q_u(x_0))$.

Proof. Suppose that $\nu(x_0) = e_N$. As in Theorem 3.5 in [4] and Theorem 5.5 in [5] we can prove, by using the boundary regularity results of solutions of $\mathcal{L}v = 0$ (see [14]) that, for $\mathcal{H}^{N-1}$-a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, any blow-up limit of u with respect to sequences of balls $B_{\rho_k}(x_0)$, $\rho_k \to 0$, satisfies

$$\begin{cases} \mathcal{L}u_0 = 0 & \text{in } \{x_N < 0\}, \\ u_0 = 0, \quad g\big(|\nabla u_0|\big) = q_u(x_0) & \text{on } \{x_N = 0\}. \end{cases} \tag{2.1}$$

In particular, $u_0(x) = \lambda^* x_N^- + o(|x|)$ with $\lambda^* = g^{-1}(q_u(x_0))$.

Take now $u_{0,j}$, a blow-up sequence of u_0 with respect to balls $B_{\mu_j}(0)$. We may assume that $u_{0,j} \to u_{00}$. Then,

$$u_{00} = \lambda^* x_N^-.$$

Now, we want to construct a blow-up sequence of u with limit u_{00}. Observe that

$$\left|\frac{1}{\rho_k\mu_j}u(x_0+\rho_k\mu_j x)-u_{00}(x)\right|\leqslant\frac{1}{\mu_j}\big|u_k(\mu_j x)-u_0(\mu_j x)\big|+\big|u_{0,j}(x)-u_{00}(x)\big|.$$

Since $u_k\to u_0$ and $u_{0,j}\to u_{00}$ uniformly on compacts sets we have that for $j\geqslant j_n$, $|u_{0,j}(x)-u_{00}(x)|<1/n$ and, for $k\geqslant k_{j,n}$, $|u_k(\mu_j x)-u_0(\mu_j x)|<\mu_j/n$ if $|x|\leqslant n$. We may suppose that $j_n\geqslant n$ and $k_{j,n}\geqslant n$. Now, taking $j=j_n$, $k=k_{j_n,n}$, and $\gamma_n=\rho_{k_{j_n,n}}\mu_{j_n}$, we have that $\gamma_n\to 0$ and $|u_{\gamma_n}(x)-u_{00}(x)|<2/n$ in B_n. The result follows. □

Theorem 2.3. *Let $u\in\mathcal{K}$ be a solution to (P_ε) and q_u the function in Theorem 2.2. Then there exists a constant λ_u such that*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}}\big|\nabla u(x)\big|=\lambda_u\quad\text{for every } x_0\in\Omega\cap\partial\{u>0\},\tag{2.2}$$

$$q_u(x_0)=g(\lambda_u),\quad \mathcal{H}^{N-1}\text{-a.e. } x_0\in\Omega\cap\partial_{\mathrm{red}}\{u>0\}.\tag{2.3}$$

Proof. It follows as in [12] by using Lemmas 2.1, 2.2 and 2.3. □

Now we can prove the asymptotic development of minimizers, namely,

Theorem 2.4. *For every $x_0\in\partial_{\mathrm{red}}\{u>0\}$,*

$$u(x_0+x)=\lambda_u\big(x\cdot\nu(x_0)\big)^-+o\big(|x|\big)\quad\text{as } x\to 0.$$

Proof. The proof follows as that of Theorem 7.1 in [15]. We let u_0 a blow-up limit of u at the point $x_0\in\partial_{\mathrm{red}}\{u>0\}$. Assume $\nu(x_0)=e_N$. First, by the definition of normal direction in the measure theoretic sense and the uniform nondegeneracy of u (Theorem 2.2(3)) we deduce that $u_0=0$ in $\{x_N>0\}$ and $u_0>0$ in $\{x_N<0\}$. So that, $\mathcal{L}u_0=0$ in $\{x_N<0\}$. Then, by the regularity results in [14] and the nondegeneracy property (Theorem 2.2(3)) we have, for a positive constant λ^*,

$$u_0(x)=\lambda^* x_N^-+o\big(|x|\big).$$

By making a second blow up as in Lemma 2.3 and applying Lemmas 2.1 and 2.2 and Theorem 2.3 we deduce that $\lambda^*=\lambda_u$.

On the other hand, by (2.2), $|\nabla u_0|\leqslant\lambda_u$. Thus,

$$u_0(x)\leqslant\lambda_u x_N^-.$$

Now, by a careful application of the strong maximum principle (see the proof of Theorem 7.1 in [15]) we conclude that

$$u_0(x)=\lambda_u x_N^-.$$

The proof is complete. □

Remark 2.1. Now we have that, by Theorems 2.1(1)–(3), 2.2(2) and 2.3, any minimizer satisfies all the properties of the definition of weak solution I in [15]. Moreover, by Theorem 2.4, the free boundary is flat at every point in $\partial_{\mathrm{red}}\{u>0\}$. Therefore, by Theorem 9.3 and Remark 9.2 in [15], we obtain the following regularity result for the free boundary $\partial\{u>0\}$:

Corollary 2.1. *Let $u\in\mathcal{K}$ be a solution to (P_ε). Then, $\mathcal{A}=\partial_{\mathrm{red}}\{u>0\}$ is relatively open with respect to $\partial\{u>0\}$, $\mathcal{A}$ is a $C^{1,\beta}$ surface and the remainder of the free boundary has zero $\mathcal{H}^{N-1}$-measure.*

2.2. *Full regularity in the case $N = 2$*

We will prove that in dimension two, for the subclass of functions satisfying (1.2) and (2.4), the whole free boundary is a $C^{1,\beta}$ surface.

The class that we consider consists of those functions satisfying condition (1.2) and such that

$$\text{there exist constants } t_0 > 0 \text{ and } k > 0 \text{ so that} \quad g(t) \leqslant kt \quad \text{for } t \leqslant t_0. \tag{2.4}$$

Observe that this condition is satisfied, for example, if $\delta \geqslant 1$. Also (2.4) holds when $g_0 \geqslant 1$ and there exists a constant C such that $\limsup_{t \to 0} \frac{g(t)}{t^{g_0}} = C$.

To prove the full regularity, we will use the following two lemmas. These lemmas hold for any dimension and for any δ and g_0.

Lemma 2.4. *Let $u \in \mathcal{K}$ be a local minimizer. Given $D \Subset \Omega$, there exist constants $C = C(N, D, \lambda_u)$, $r_0 = r_0(N, D) > 0$ and $\gamma = \gamma(N, D) > 0$ such that, if $x_0 \in D \cap \partial\{u > 0\}$ and $r < r_0$, then*

$$\sup_{B_r(x_0)} |\nabla u| \leqslant \lambda_u + Cr^{\gamma}.$$

Proof. The proof is similar to the proof of Theorem 7.1 in [5]. Here we make a little modification by using a result in [13] to avoid adding any new hypothesis to the function g.

Let $U_\rho = (G(|\nabla u|) - G(\lambda_u) - \rho)^+$ and $U_0 = (G(|\nabla u|) - G(\lambda_u))^+$. By Theorem 2.3 we know that U_ρ vanishes in a neighborhood of the free boundary. Since $U_\rho > 0$ implies that $G(|\nabla u|) > G(\lambda_u) + \rho$, the closure of $\{U_\rho > 0\}$ is contained in $\{G(|\nabla u|) > G(\lambda_u) + \rho/2\}$.

Let $v = G(|\nabla u|)$. By Lemma 1 in [13] we have that v satisfies

$$Mv := D_i\big(b_{ij}(\nabla u)D_j v\big) \geqslant 0 \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda_u) + \rho/2\big\},$$

where b_{ij} is defined in (B.1).

Hence U_ρ satisfies

$$MU_\rho \geqslant 0 \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda_u) + \rho/2\big\}.$$

Now, extend the operator M to a uniformly elliptic operator in divergence-form,

$$\widetilde{M}w := D_i\big(\tilde{b}_{ij}(x)D_j w\big) \quad \text{in } \Omega,$$

with measurable coefficients such that

$$\tilde{b}_{ij}(x) = b_{ij}(\nabla u) \quad \text{in } \big\{G\big(|\nabla u|\big) > G(\lambda_u) + \rho/2\big\}.$$

Then, we have

$$\widetilde{M}U_\rho \geqslant 0 \quad \text{in } \Omega.$$

Let $D \Subset \Omega$ and let $r_0 = \operatorname{dist}(D, \partial\Omega)$, $x_0 \in D \cap \partial\{u > 0\}$. For $0 < r < r_0$, let

$$h_\rho(r) = \sup_{B_r(x_0)} U_\rho, \qquad h_0(r) = \sup_{B_r(x_0)} U_0.$$

Then, $h_\rho(r) - U_\rho$ is a $\widetilde{M}$-supersolution in the ball $B_r(x_0)$ and

$$\begin{aligned} h_\rho(r) - U_\rho &\geqslant 0 && \text{in } B_r(x_0), \\ &= h_\rho(r) && \text{in } B_r(x_0) \cap \{u = 0\}. \end{aligned}$$

By Theorem 2.1, $|B_r(x_0) \cap \{u = 0\}| \geqslant cr^N$. Then, applying the weak Harnack inequality (see [10, Theorem 8.18]) with $1 \leqslant p < N/(N-2)$, we get

$$\inf_{B_{r/2}(x_0)} \big(h_\rho(r) - U_\rho\big) \geqslant cr^{-N/p} \big\| h_\rho(r) - U_\rho \big\|_{L^p(B_r(x_0))} \geqslant ch_\rho(r).$$

Letting now $\rho \to 0$ we obtain

$$\inf_{B_{r/2}(x_0)} \big(h_0(r) - U_0\big) \geqslant ch_0(r),$$

for some $0 < c < 1$. Or, equivalently,

$$\sup_{B_{r/2}(x_0)} U_0 \leqslant (1-c)h_0(r).$$

Therefore,

$$h_0\left(\frac{r}{2}\right) \leqslant (1-c)h_0(r),$$

from which it follows that $h_0(r) \leqslant Cr^\gamma$ for some $C > 0$, $0 < \gamma < 1$. That is

$$G\big(|\nabla u|\big) \leqslant G(\lambda_u) + Cr^\gamma \quad \text{in } B_r(x_0)$$

and, therefore,

$$|\nabla u| \leqslant \lambda_u + Cr^\gamma \quad \text{in } B_r(x_0).$$

The conclusion of the lemma follows. □

Lemma 2.5. *Let x_1 be a regular free boundary point.*
Take

$$\tau_\rho(x) = \begin{cases} x + \rho^2 \phi(\frac{|x-x_1|}{\rho})\nu_u(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere}, \end{cases}$$

where $\phi \in C_0^\infty(-1,1)$ with $\phi'(0) = 0$.
Let

$$\delta = \rho^2 \int_{B_\rho(x_1)\cap\partial\{u>0\}} \phi\left(\frac{|x-x_1|}{\rho}\right) d\mathcal{H}^{N-1}, \tag{2.5}$$

and let $v_\rho(x) = u(\tau_\rho^{-1}(x))$. Then,

$$\int_{B_\rho(x_1)} \big(G\big(|\nabla v_\rho|\big) - G\big(|\nabla u|\big)\big)\,dx = -l\rho^{N+1}\Phi(\lambda_u) + o\big(\rho^{N+1}\big), \tag{2.6}$$

where $l = \lim_{\rho\to 0}\frac{\delta}{\rho^{N+1}}$ and $\Phi(t) = g(t)t - G(t)$.

Proof. The proof follows the lines of Theorem 3.1 in [8]. □

It is in the following lemma where we need to impose condition (2.4).

Lemma 2.6. *Let $\Phi(t) = g(t)t - G(t)$, and g satisfying condition* (2.4). *Let $D \Subset \Omega$, $x_0 \in \partial\{u > 0\}$ such that $B_\mu(x_0) \subset D$. Take $v = \max(u - t\eta, 0)$, where $t > 0$, $\eta \in C_0^\infty(\Omega)$, $\eta = 0$ in $\Omega \setminus B_{\mu(x_0)}$ and $|\nabla\eta| \leqslant C/t$. Then,*

$$\int_{B_\mu(x_0)\cap\{u>0\}} \big(G\big(|\nabla v|\big) - G\big(|\nabla u|\big)\big)\,dx \leqslant \int_{B_\mu(x_0)\cap\{0<u\leqslant t\eta\}} \Phi\big(|\nabla u|\big)\,dx + C_0 t^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\,dx$$

for $C_0 = C_0(N, \delta, g_0, \operatorname{dist}(\partial\Omega, D), \varepsilon, C)$.

Proof. The proof follows as in Theorem 4.3 in [4]. We only have to make the following observations. First, for $0 \leqslant t \leqslant 1$, we have that $|\nabla u - t\nabla\eta| \leqslant |\nabla u| + C \leqslant C_1 + C$, where C_1 is the constant in Theorem 2.1(1). On the other hand, if g satisfies (2.4) and if $F(s) = \frac{g(s)}{s}$, then for $0 \leqslant s \leqslant C_1 + C$, there exists a constant C_0 such that $F(s) \leqslant C_0$. Therefore, we have that $F(|\nabla u - t\nabla\eta|)$ is bounded by C_0. The rest of the proof follows as in [4]. □

Now, following ideas from [12], using Lemmas 2.4–2.6, we prove, for $N = 2$ and g satisfying (2.4), the following:

Theorem 2.5. *Let $N = 2$, g satisfying* (2.4) *and u a minimizer. Then, for any ball B_r centered at the free boundary we have*

$$\fint_{B_r\cap\{u>0\}} \big(\Phi(\lambda_u) - \Phi(|\nabla u|)\big)^+ \to 0 \quad \text{as } r \to 0,$$

where $\Phi(t) = g(t)t - G(t)$.

Proof. Let $0 < r < \mu \leqslant 1$, $t > 0$ and v_0 be the function defined in Lemma 2.6. By Theorem 2.1, $u \leqslant Cr$ in $B_r(x_0)$. Take $t = Cr$ and let $\delta_t = |\{0 < u \leqslant t\eta\} \cap B_\mu(x_0)|$.

Now, let us take x_1 far from x_0 and such that $\partial\{u > 0\} \cap B_{r_1}(x_1)$ is regular for small r_1. Let ρ be such that (2.5) is satisfied for $\delta = \delta_t$, and consider $v_1 = v_\rho$ defined in $B_{r_1}(x_1)$ as in Lemma 2.5. Then, the function

$$v = \begin{cases} v_0 & \text{in } B_\mu(x_0),\\ v_1 & \text{in } B_{r_1}(x_1),\\ u & \text{elsewhere} \end{cases}$$

is admissible for our minimization problem and $|\{v > 0\}| = |\{u > 0\}|$. Therefore, by Lemmas 2.5 and 2.6, we have

$$0 \leqslant \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u) = \int_{B_\rho(x_0)} \big(G(|\nabla v|) - G(|\nabla u|)\big)\,dx + \int_{B_{r_1}(x_1)} \big(G(|\nabla v|) - G(|\nabla u|)\big)\,dx$$

$$\leqslant \int_{B_\mu(x_0)\cap\{0<u\leqslant t\eta\}} \Phi(|\nabla u|) + Ct^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\,dx - l\rho^3\Phi(\lambda_u) + o(\rho^3).$$

By the definition of δ_t we have

$$\int_{B_\mu(x_0)\cap\{0<u\leqslant t\eta\}} \big(\Phi(\lambda_u) - \Phi(|\nabla u|)\big)\,dx \leqslant Ct^2 \int_{B_\mu(x_0)\cap\{u>t\eta\}} |\nabla\eta|^2\,dx + o(\rho^3) + (\delta_t - l\rho^3)\Phi(\lambda_u).$$

Now choose

$$\eta(x) = \begin{cases} \frac{\log(\mu/|x-x_0|)}{\log(\mu/r)} & \text{in } B_\mu(x_0)\setminus B_r(x_0),\\ 1 & \text{in } B_r(x_0),\\ 0 & \text{in } \Omega\setminus B_\mu(x_0). \end{cases}$$

Observe that the condition $|\nabla\eta| \leqslant C/t$ is satisfied if we choose μ such that $\mu \geqslant 2r$.

By our election of t and η, we have

$$\int_{B_r(x_0)\cap\{u>0\}} \big(\Phi(\lambda_u) - \Phi(|\nabla u|)\big)^+\,dx \leqslant \int_{B_\mu(x_0)} \big(\Phi(|\nabla u|) - \Phi(\lambda_u)\big)^+\,dx + \frac{Cr^2}{\log(\mu/r)} + o(\rho^3) + (\delta_t - l\rho^3)\Phi(\lambda_u).$$

By Lemma 2.4, we have that $\Phi(|\nabla u|) - \Phi(\lambda_u) \leqslant \Phi(\lambda_u + Cr^\gamma) - \Phi(\lambda_u) = \Phi'(\xi)Cr^\gamma$ for some $\lambda_u \leqslant \xi \leqslant \lambda_u + Cr^\gamma$. As $\Phi'(t) = g'(t)t \leqslant g_0 g(t)$, and g is nondecreasing, we have $\Phi'(\xi) \leqslant g_0 g(\xi) \leqslant g_0 g(\lambda_u + Cr^\gamma)$.

Therefore, by the definition of l, we have

$$\fint_{B_r(x_0)\cap\{u>0\}} \big(\Phi(\lambda_u) - \Phi(|\nabla u|)\big)^+\,dx \leqslant C\left(\frac{(\mu^{\gamma+2} + o(\rho^3))}{r^2} + \frac{1}{\log(\mu/r)}\right),$$

where $C = C(\lambda_u)$. As, by Theorem 2.1(5), $\delta_t \leqslant c\mu^2$ we have that $o(\rho^3) = o(\mu^2)$. Taking $r = \mu h(\mu)^\beta$, where $h(\mu) = \max(\mu, \frac{o(\mu^2)}{\mu^2})$ with $\beta < \min\{\gamma/2, 1/2\}$, we obtain the desired result. □

Corollary 2.2. *Let $N = 2$, g satisfying* (2.4) *and $u \in \mathcal{K}$ be a solution to (P_ε). Then $\partial\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω.*

Proof. The proof follows now as in [3], we give the proof here for the readers' convenience. Let u_k be a blow-up sequence converging to u_0. Since, $\nabla u_k \to \nabla u_0$ a.e. in $\mathbb{R}^N$, we conclude from Theorems 2.3 and 2.5 that $|\nabla u_0| = \lambda_u$ in $B_1 \cap \{u_0 > 0\}$. And then

$$0 = \mathcal{L}u_0 = \operatorname{div}\left(\frac{g(|\nabla u_0|)}{|\nabla u_0|}\nabla u_0\right) = \frac{g(\lambda_\varepsilon)}{\lambda_u}\Delta u_0 \quad \text{in } \{u_0 > 0\}.$$

Therefore, u_0 is harmonic in $\{u_0 > 0\}$. On the other hand, if we take $v = |\nabla u_0|^2$, we have that $v = \lambda_u^2$ in $\{u_0 > 0\}$ and, in particular, $\Delta v = 0$ in $\{u_0 > 0\}$. Since $\Delta v = |D^2 u_0|^2$, we conclude that ∇u_0 is constant in each connected component of $\{u_0 > 0\}$. Therefore, by Lemma C.1(6) and (8), we have

$$u_0 = \lambda_u (x \cdot \nu_0)^- + q\big((x \cdot \nu_0) - s\big)^+$$

for some ν_0 and $q, s \geqslant 0$. Since $\{u_0 = 0\}$ has positive density at the origin, we have that $s > 0$ or $q = 0$. Therefore, we have proved that any blow-up sequence has a subsequence that converges to the half-linear function $u_0 = \lambda_u (x \cdot \nu_0)^-$ in some neighborhood of the origin. Then, applying Theorem 9.3 and Remark 9.2 in [15] we have the desired result. □

Remark 2.2. Since the functional in [15] is linear in $|\{u > 0\}|$, we can also prove, for the minimizers of the problem treated in [15], the full regularity of the free boundary when $N = 2$. We only have to use Theorem 2.4, Lemma 2.6 (to treat the first term of the functional) and the result follows as in [3].

3. Behavior of the minimizer for small ε

Since we want to analyze the dependence of the problem with respect to ε, we will again denote by u_ε a solution to problem (P_ε).

To complete the analysis of the problem, we will now show that if ε is small enough, then

$$\big|\{u_\varepsilon > 0\}\big| = \alpha.$$

To this end, we need to prove that the constant $\lambda_\varepsilon := \lambda_{u_\varepsilon}$ is bounded from above and below by positive constants independent of ε. We perform this task in a series of lemmas.

Lemma 3.1. *Let $u_\varepsilon \in \mathcal{K}$ be a solution of (P_ε). Then, there exists a constant $C > 0$ independent of ε such that*

$$\lambda_\varepsilon \leqslant C.$$

Proof. The proof is similar to the one of Theorem 3 in [1].

First, we will prove that there exist $C, c > 0$, independent of ε, such that

$$c \leqslant \big|\{u_\varepsilon > 0\}\big| \leqslant C\varepsilon + \alpha.$$

In fact, by taking $\bar{u} \in W^{1,G}(\Omega)$ such that $|\{\bar{u} > 0\}| \leqslant \alpha$ we have that $\mathcal{J}_\varepsilon(u_\varepsilon) \leqslant \mathcal{J}_\varepsilon(\bar{u}) \leqslant C$. Hence, $F_\varepsilon(|\{u_\varepsilon > 0\}|) \leqslant C$. Thus we obtain the bound from above. We also have that $\int_\Omega G(|\nabla u_\varepsilon|)$ is bounded.

As $u_\varepsilon = \varphi_0$, on $\partial\Omega$ by Lemma A.3, we have $\|\nabla u_\varepsilon - \nabla\varphi_0\|_G \leqslant C$ and, by Lemma A.4, we also have $\|u_\varepsilon - \varphi_0\|_G \leqslant C$. Then, $\|u_\varepsilon\|_{W^{1,G}(\Omega)} \leqslant C$. Using the Sobolev trace theorem, Hölder inequality and the embedding Theorem A.1, we have, for $q < \delta + 1$,

$$\int_{\partial\Omega} \varphi_0^q \, d\mathcal{H}^{N-1} \leqslant C\big|\{u_\varepsilon > 0\}\big|^{\frac{\delta+1-q}{\delta+1}} \|u_\varepsilon\|^q_{W^{1,\delta+1}(\Omega)} \leqslant C\big|\{u_\varepsilon > 0\}\big|^{\frac{\delta+1-q}{\delta+1}} \|u_\varepsilon\|^q_{W^{1,G}(\Omega)} \leqslant C\big|\{u_\varepsilon > 0\}\big|^{\frac{\delta+1-q}{\delta+1}},$$

and thus we obtain the bound from below.

The rest of the proof follows as in Lemma 3.1 in [8]. □

Lemma 3.2. *Let $u_\varepsilon \in \mathcal{K}$ be a solution of (P_ε), $B_r \Subset \Omega$ and v a solution of*

$$\mathcal{L}v = 0 \quad \text{in } B_r, \qquad v = u_\varepsilon \quad \text{on } \partial B_r.$$

Then, there exists a positive constant $\gamma = \gamma(\delta, g_0, N)$ such that

$$\int_{B_r} \left|\nabla(u_\varepsilon - v)\right|^q dx \geqslant C\left|B_r \cap \{u_\varepsilon = 0\}\right| \left(\frac{1}{r}\left(\fint_{B_r} u_\varepsilon^\gamma \, dx\right)^{1/\gamma}\right)^q$$

for all $q \geqslant 1$, where C is a constant independent of ε.

Proof. The proof follows the lines of the proof of Lemma 3.2 in [8]. The only difference is that in the present situation we have to use the weak Harnack inequality for solutions of $\mathcal{L}v = 0$ (see [14, Theorem 1.3]). □

Without loss of generality, from now on we will suppose that $g_0 \geqslant 1$.

Lemma 3.3. *Let u_ε and v be as in Lemma 3.2. Then, if r is small enough (depending on ε), we have*

$$\int_{B_r} \left(G\left(|\nabla u_\varepsilon|\right) - G\left(|\nabla v|\right)\right) dx \geqslant C \int_{B_r} |\nabla u_\varepsilon - \nabla v|^{g_0+1} \, dx \tag{3.1}$$

for some constant C independent of ε.

Proof. First, we will use an inequality proved in [15] (see Theorem 2.3). Let

$$A_1 = \left\{x \in B_r\colon |\nabla u_\varepsilon - \nabla v| \leqslant 2|\nabla u_\varepsilon|\right\}, \qquad A_2 = \left\{x \in B_r\colon |\nabla u_\varepsilon - \nabla v| > 2|\nabla u_\varepsilon|\right\},$$

then $B_r = A_1 \cup A_2$ and we have that

$$\int_{B_r} \left(G\left(|\nabla u_\varepsilon|\right) - G\left(|\nabla v|\right)\right) dx \geqslant C\left(\int_{A_2} G\left(|\nabla u_\varepsilon - \nabla v|\right) dx + \int_{A_1} F\left(|\nabla u_\varepsilon|\right)|\nabla u_\varepsilon - \nabla v|^2 \, dx\right). \tag{3.2}$$

Therefore, by using that $g_0 \geqslant 1$ and property (g1) in Lemma A.1, we have

$$\begin{aligned} &G\left(|\nabla u_\varepsilon - \nabla v|\right) \geqslant C|\nabla u_\varepsilon - \nabla v|^{g_0+1}, \\ &F\left(|\nabla u_\varepsilon|\right) \geqslant C|\nabla u_\varepsilon|^{g_0-1} \geqslant C|\nabla u_\varepsilon - \nabla v|^{g_0-1} \quad \text{in } A_1, \end{aligned} \tag{3.3}$$

if $|\nabla u_\varepsilon| \leqslant 1$ and $|\nabla v - \nabla u_\varepsilon| \leqslant 1$.

On the other hand, by Lemma 3.1 and (2.2), we have that for small r (depending on ε), $|\nabla u_\varepsilon|$ is bounded by a constant independent of ε. By Lemma 5.1 in [14] there exist $C_0, C_1 = C_0, C_1(N, g_0, \delta)$ such that

$$\sup_{B_r} G\left(|\nabla v|\right) \leqslant \frac{C_0}{r^N} \int_{B_{2r}} G\left(|\nabla v|\right) dx \leqslant \frac{C_1}{r^N} \int_{B_{2r}} \left(1 + G\left(|\nabla u_\varepsilon|\right)\right) dx \leqslant \bar{C}$$

with $\bar{C}$ is independent of ε if r is small (depending on ε). Therefore, (3.3) holds for every $x \in B_r$ with a constant C independent of ε. Combining (3.2) and (3.3) we obtain the desired result. □

Lemma 3.4. *For every $\varepsilon > 0$ there exists a neighborhood of A in Ω such that $u_\varepsilon > 0$ in this neighborhood.*

Proof. The proof follows the lines of that of Lemma 3.4 in [8]. However, one observation is in order. When applying Schwartz symmetrization, we use the fact that this symmetrization preserves the distribution function and strictly decreases the functional $\int_B G(|\nabla u|)\, dx$, unless the function is already radially symmetric and radially decreasing. These facts hold by Corollary 2.35, in Section II.8 of [11]. The rest of the proof follows without changes. □

Lemma 3.5. *Let $u_\varepsilon \in \mathcal{K}$ be a solution of (P_ε). Then,*

$$\lambda_\varepsilon \geqslant c > 0$$

where c is independent of ε.

Proof. The proof follows as in [8] by using Lemmas 3.2–3.4 and Lemma C.1. □

With these uniform bounds on λ_ε, we can prove the main result in this section:

Theorem 3.1. *Under the hypotheses of Lemma* 3.5*, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$, $|\{u_\varepsilon > 0\}| = \alpha$. Therefore, u_ε is a minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$.*

Proof. It follows as in Theorem 3.1 in [8] by using Lemmas 3.1 and 3.5. □

As a corollary we have

Corollary 3.1. *Any minimizer u of $\mathcal{J}$ in $\mathcal{K}_\alpha$ is a locally Lipschitz continuous function, $\partial_{\mathrm{red}}\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has vanishing $\mathcal{H}^{N-1}$-measure. Moreover, if $N = 2$ and g satisfies* (2.4)*, $\partial\{u > 0\}$ is a $C^{1,\beta}$ surface locally in Ω.*

Proof. Let u be a minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$. Let $\varepsilon > 0$ small. Then, there exists a solution u_ε to (P_ε) and $|\{u_\varepsilon > 0\}| = \alpha$. Hence, $\mathcal{J}_\varepsilon(u) = \mathcal{J}(u) \leqslant \mathcal{J}(u_\varepsilon) = \mathcal{J}_\varepsilon(u_\varepsilon)$. Therefore, u is a solution of (P_ε), and the regularity result follows from Corollary 2.1. □

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Appendix A. Properties of *G* and Orlicz spaces

The following results are all included in [15].

Lemma A.1. *Let $g \geqslant 0$ satisfy* (1.2)*. Then, if $G(t) = \int_0^t g(s)\,ds$,*

(g1) $\min\{s^\delta, s^{g_0}\}g(t) \leqslant g(st) \leqslant \max\{s^\delta, s^{g_0}\}g(t)$,
(g2) *G is convex and C^2,*
(g3) $\frac{tg(t)}{1+g_0} \leqslant G(t) \leqslant tg(t)$, $\forall t \geqslant 0$.

Lemma A.2. *If $\widetilde{G}$ is such that $\widetilde{G}'(t) = g^{-1}(t)$, then*

$$\frac{(1+\delta)}{\delta}\min\bigl\{s^{1+1/\delta}, s^{1+1/g_0}\bigr\}\widetilde{G}(t) \leqslant \widetilde{G}(st) \leqslant \frac{\delta}{1+\delta}\max\bigl\{s^{1+1/\delta}, s^{1+1/g_0}\bigr\}\widetilde{G}(t). \qquad (\widetilde{G}1)$$

We recall that the functional

$$\|u\|_G = \inf\left\{k > 0\colon \int_\Omega G\left(\frac{|u(x)|}{k}\right)dx \leqslant 1\right\}$$

is a norm in the Orlicz space $L^G(\Omega)$, which is the linear hull of the Orlicz class

$$K_G(\Omega) = \left\{u \text{ measurable: } \int_\Omega G\bigl(|u|\bigr)\,dx < \infty\right\}.$$

Observe that this set is convex since G is a convex function (property (g2)). The Orlicz–Sobolev space $W^{1,G}(\Omega)$ consists of those functions in $L^G(\Omega)$ whose distributional derivatives ∇u also belong to $L^G(\Omega)$. And we have that $\|u\|_{W^{1,G}} = \max\{\|u\|_G, \|\nabla u\|_G\}$ is a norm in this space.

Theorem A.1. *$L^G(\Omega) \hookrightarrow L^{1+\delta}(\Omega)$ continuously.*

Lemma A.3. *There exists a constant $C = C(g_0, \delta)$ such that*

$$\|u\|_G \leqslant C \max\left\{\left(\int_\Omega G(|u|)\,dx\right)^{1/(\delta+1)}, \left(\int_\Omega G(|u|)\,dx\right)^{1/(g_0+1)}\right\}.$$

Lemma A.4. *If $u \in W^{1,1}(\Omega)$ with $u = 0$ on $\partial\Omega$ and $\int_\Omega G(|\nabla u|)\,dx$ is finite, then*

$$\int_\Omega G\left(\frac{|u|}{R}\right) dx \leqslant \int_\Omega G(|\nabla u|)\,dx \quad \text{for } R = \operatorname{diam}\Omega.$$

Appendix B. Some results on $\mathcal{L}$-solutions with linear growth

In this section we will state some properties of $\mathcal{L}$-subsolutions. From now on, we note $B_r^+ = B_r(0) \cap \{x_N > 0\}$.

Remark B.1. Let u be such that $\mathcal{L}u = 0$. Then, in the set $\{|\nabla u| > 0\}$, u satisfies a linear nondivergence uniformly elliptic equation, $Tu = 0$, where

$$Tv = b_{ij}(\nabla u) D_{ij} v = 0 \tag{B.1}$$

with

$$b_{ij} = \delta_{ij} + \left(\frac{g'(|\nabla u|)|\nabla u|}{g(|\nabla u|)} - 1\right)\frac{D_i u D_j u}{|\nabla u|^2},$$

and the matrix $b_{ij}(\nabla u)$ is β-elliptic in $\{|\nabla u| > 0\}$, where $\beta = \max\{\max\{g_0, 1\}, \max\{1, 1/\delta\}\}$.

Lemma B.1. *Let $0 < r \leqslant 1$. Let $u \in C(\overline{B_r^+})$ be such that $\mathcal{L}u = 0$ in B_r^+ and $0 \leqslant u \leqslant \alpha x_N$ in B_r^+, $u \leqslant \delta_0 \alpha x_N$ on $\partial B_r^+ \cap B_{r_0}(\bar{x})$ with $\bar{x} \in \partial B_r^+$, $\bar{x}_N > 0$ and $0 < \delta_0 < 1$.*

Then, there exist $0 < \gamma < 1$ and $0 < \varepsilon \leqslant 1$, depending only on r and N such that

$$u(x) \leqslant \gamma \alpha x_N \quad \text{in } B_{\varepsilon r}^+.$$

Proof. See Lemma B.1 in [16]. □

Theorem B.1. *Let u be a Lipschitz function in $\mathbb{R}^N$ with Lipschitz constant L such that*

(1) *$u \geqslant 0$ in $\mathbb{R}^N$, $\mathcal{L}u = 0$ in $\{u > 0\}$.*
(2) *$\{x_N < 0\} \subset \{u > 0\}$, $u = 0$ in $\{x_N = 0\}$.*
(3) *There exists $0 < \lambda_0 < 1$ such that $\frac{|\{u=0\} \cap B_R(0)|}{|B_R(0)|} > \lambda_0$, $\forall R > 0$.*

Then $u = 0$ in $\{x_N > 0\}$.

Proof. The proof will be divided into several steps.

Step 1. Let $u_0(x) = \frac{u(Tx)}{T}$, with $T > 0$, to be chosen later.

Then, the function u_0 satisfies the same properties as u with the same constants L and λ_0.

Let $\beta = \frac{\lambda_0}{2^{N-1}} < 1$. Then, by properties (2) and (3) with $R = 1$, we have that there exists $x_0 \in B_1(0)$, with $x_{0,N} > \beta$ such that $u_0(x_0) = 0$. Since u_0 is Lipschitz with constant L, we have $u_0(x) \leqslant L|x - x_0|$. Thus, if we take $r_0 = \frac{\beta}{4}$, we have $u_0(x) \leqslant \frac{L\beta}{4}$ for $|x - x_0| < r_0$. There holds that $x_N \geqslant \frac{3\beta}{4}$ in $B_{r_0}(x_0)$. Hence, we have

$$u_0(x) \leqslant \frac{L x_N}{3} \quad \text{on } \partial B_{R_1}^+ \cap B_{r_0}(x_0),$$

where $R_1 = |x_0| > \beta$.

By property (1) and Lemma 8.1 in [15], $\mathcal{L}u_0 \geqslant 0$. By property (2), $0 \leqslant u_0(x) \leqslant Lx_N$.

Taking $\delta_0 = 1/3$, $\bar{x} = x_0$, $\alpha = L$ and $r = R_1$ in Lemma B.1, we have that there exist $0 < \gamma_1 < 1$ and $0 < \varepsilon_1 \leqslant 1$, depending only on r_0 and $x_{0,N}$, such that

$$0 \leqslant u_0(x) \leqslant \gamma_1 L x_N \quad \text{in } B^+_{R_1\varepsilon_1}. \tag{B.2}$$

Observe that, since $x_{0,N} > \beta$, γ_1 and ε_1 depend only on λ_0.

Now, take $u_1(x) = \frac{u_0(R_1\varepsilon_1 x)}{R_1\varepsilon_1}$. Then, u_1 satisfies the properties of u_0 with the same constants L and λ_0.

Therefore, there exists $x_1 \in B_1(0)$, with $x_{1,N} > \beta$ such that $u_1(x_1) = 0$. By (1), $u_1(x) \leqslant L|x - x_1|$. Thus, if we take $r_1 = \frac{\gamma_1\beta}{4}$, we have $u_1(x) \leqslant \frac{\gamma_1 L\beta}{4}$ for $|x - x_1| < r_1$. As $\gamma_1 \leqslant 1$, there holds that $x_N \geqslant \frac{3\beta}{4}$ in $B_{r_1}(x_1)$. Thus, we have that

$$u_1(x) \leqslant \frac{\gamma_1 L x_N}{3} \quad \text{on } \partial B^+_{R_2} \cap B_{r_1}(x_1),$$

where $R_2 = |x_1| > \beta$.

By property (1), $\mathcal{L}u_1 \geqslant 0$. And, by (B.2), $0 \leqslant u_1(x) \leqslant \gamma_1 L x_N$ in B^+_1.

Taking $\delta_0 = 1/3$, $\bar{x} = x_1$, $\alpha = \gamma_1 L$ and $r = R_2$ in Lemma B.1, we have that there exist $0 < \gamma_2 < 1$ and $0 < \varepsilon_2 \leqslant 1$, depending only on λ_0 such that $u_1(x) \leqslant \gamma_2\gamma_1 L x_N$ in $B^+_{R_2\varepsilon_2}$.

Inductively, we construct a sequence u_k, such that u_k satisfies the same hypotheses as u_0 with the same constants L and λ_0 and such that

$$0 \leqslant u_{k-1} \leqslant \alpha_k x_N \quad \text{in } B^+_{R_k\varepsilon_k}, \tag{B.3}$$

where $\alpha_k = L\prod_{i=1}^k \gamma_i$, and $0 < \gamma_i, \varepsilon_i < 1$ depend only on λ_0. From the construction we have $u_k(x) = \frac{u_{k-1}(R_k\varepsilon_k x)}{R_k\varepsilon_k}$.

Therefore, for any $k \geqslant 1$,

$$u_0 \leqslant \alpha_k x_N \quad \text{in } B^+_{\delta_k}, \tag{B.4}$$

where $\delta_k = \prod_{i=1}^k R_i\varepsilon_i$.

Step 2. Let us see that $\alpha_k \to 0$ as $k \to \infty$. Suppose, by contradiction, that this does not hold. Then, since α_k is decreasing, there exists $\alpha_0 > 0$ such that $\alpha_k \geqslant \alpha_0$ for $k \geqslant 1$. We have $\alpha_{k+1} = \gamma_{k+1}\alpha_k$, and $r_k = \frac{\beta}{4}\alpha_k \geqslant \frac{\beta}{4}\alpha_0$. Thus, we can take in Lemma (B.1) $u = u_k$, $r_0 = \frac{\beta}{4}\alpha_0$ $\gamma = \gamma_k$. We can think that γ_{k+1} was taken as the minimum over the γ's such that the conclusion of the lemma is satisfied. Therefore, $\gamma_{k+1} \leqslant \gamma_1 < 1$ for every k. Then, $\alpha_k \leqslant L\gamma_1^k$ for all $k \geqslant 1$. Therefore, $\alpha_k \to 0$; a contradiction.

Step 3. Now we can prove that $u(x) = 0$ in $\{x_N > 0\}$. Suppose that there exists ξ with $\xi_N > 0$ such that $u(\xi) > 0$. Then, since $\alpha_k \to 0$, there exists $k \geqslant 1$ such that $u(\xi) > \alpha_k\xi_N$. Now, for this fixed k, take $T > |\xi|\beta^{-k}(\prod_{i=1}^k \varepsilon_i)^{-1}$. Then, since $R_i > \beta$, we have that $|\xi| < T\delta_k$. Thus, if we take $\bar{\xi} = \frac{\xi}{T}$ we have that $u_0(\bar{\xi}) > \alpha_k\bar{\xi}_N$. But, on the other hand, since $|\bar{\xi}| < \delta_k$ by (B.4), we have $u_0(\bar{\xi}) \leqslant \alpha_k\bar{\xi}_N$, which is a contradiction. □

Appendix C. Blow-up limits

Now we give the definition of blow-up sequence, and we collect some properties of the limits of these blow-up sequences for certain classes of functions that are used throughout the paper.

Let u be a function with the following properties:

(C1) u is Lipschitz in Ω with constant $L > 0$, $u \geqslant 0$ in Ω and $\mathcal{L}u = 0$ in $\Omega \cap \{u > 0\}$.

(C2) Given $0 < \kappa < 1$, there exist two positive constants C_κ and r_κ such that, for every ball $B_r(x_0) \subset \Omega$ and $0 < r < r_\kappa$,

$$\frac{1}{r}\left(\fint_{B_r(x_0)} u^\gamma\, dx\right)^{1/\gamma} \leqslant C_\kappa \quad \text{implies that } u \equiv 0 \text{ in } B_{\kappa r}(x_0).$$

(C3) There exist constants $r_0 > 0$ and $0 < \lambda_1 \leqslant \lambda_2 < 1$ such that, for every ball $B_r(x_0) \subset \Omega$ x_0 on $\partial\{u > 0\}$ and $0 < r < r_0$,

$$\lambda_1 \leqslant \frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leqslant \lambda_2.$$

Definition C.1. Let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call u_k a blow-up sequence with respect to $B_{\rho_k}(x_k)$.

Since u is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that for a subsequence,

$$u_k \to u_0 \quad \text{in } C^\alpha_{\mathrm{loc}}\big(\mathbb{R}^N\big) \text{ for every } 0 < \alpha < 1,$$
$$\nabla u_k \to \nabla u_0 \quad *\text{-weakly in } L^\infty_{\mathrm{loc}}\big(\mathbb{R}^N\big),$$

and u_0 is Lipschitz in $\mathbb{R}^N$.

Lemma C.1. *If u satisfies properties* (C1)–(C3)*, then*

(1) $u_0 \geqslant 0$ *in* Ω *and* $\mathcal{L}u_0 = 0$ *in* $\{u_0 > 0\}$.
(2) $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance.*
(3) $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.
(4) *If* $K \Subset \{u_0 = 0\}$, *then* $u_k = 0$ *in* K *for* k *big enough.*
(5) *If* $K \Subset \{u_0 > 0\} \cup \{u_0 = 0\}^\circ$, *then* $\nabla u_k \to \nabla u_0$ *uniformly in* K.
(6) *There exists a constant* $0 < \lambda < 1$ *such that*

$$\frac{|B_R(y_0) \cap \{u_0 = 0\}|}{|B_R(y_0)|} \geqslant \lambda, \quad \forall R > 0,\ \forall y_0 \in \partial\{u_0 > 0\}.$$

(7) $\nabla u_k \to \nabla u_0$ *a.e. in* $\mathbb{R}^N$.
(8) *If* $x_k \in \partial\{u > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$.

Proof. The proof follows as in [8] and [12]. □

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