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A nonlocal nonlinear diffusion equation with blowing up boundary conditions ☆

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Abstract

We deal with boundary value problems (prescribing Dirichlet or Neumann boundary conditions) for a nonlocal nonlinear diffusion operator which is analogous to the porous medium equation. First, we prove existence, uniqueness and the validity of a comparison principle for these problems. Next, we impose boundary data that blow up in finite time and study the behavior of the solutions.

Keywords: Nonlocal diffusion; Neumann boundary conditions

1. Introduction

The aim of this paper is to study the asymptotic behavior of solutions of a nonlocal nonlinear diffusion operator under blowing up boundary conditions of Dirichlet or Neumann type.

First, let us introduce nonlocal diffusion problems. To this end, let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative, smooth function with $\int_{\mathbb{R}} J(r)\,dr = 1$, supported in $[-1, 1]$, symmetric, $J(r) = J(-r)$ and strictly decreasing in $[0, 1]$.

Nonlocal equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t), \tag{1}$$

and variations of it, have been recently used to model diffusion processes, see [2,4,8,14]. As stated in [8] if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping

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from location y to location x, then $(J * u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies Eq. (1). Equation (1), so-called nonlocal diffusion equation, shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed.

Another classical equation that has been used to model diffusion is the well-known porous medium equation $u_t = \Delta u^m$ with $m > 1$. This equation also shares several properties with the heat equation but there is a fundamental difference, in this case we have finite speed of propagation. Properties of solutions of the porous medium equation have been largely studied over the past years. See, for example, [1,13] and the corresponding bibliography.

In [6] a simple nonlocal model for diffusion that is analogous to the porous medium equation is studied. In this model the probability distribution of jumping from location y to location x is given by $J(\frac{x-y}{u(y,t)})\frac{1}{u(y,t)}$ when $u(y,t) > 0$ and 0 otherwise. In this case the rate at which individuals are arriving to position x from all other places is $\int_{\mathbb{R}} J(\frac{x-y}{u(y,t)})\,dy$ and the rate at which they are leaving location x to travel to all other sites is $-u(x,t) = -\int_{\mathbb{R}} J(\frac{y-x}{u(x,t)})\,dy$. As before this consideration, in the absence of external sources, leads immediately to the fact that the density u has to satisfy

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy - u(x,t). \tag{2}$$

In [3] we study this equation with homogeneous Neumann boundary conditions and prove that solutions exist globally and stabilize to the mean value of the initial data as $t \to \infty$.

The purpose of this paper is to continue the study of this nonlocal nonlinear evolution operator by prescribing nonhomogeneous Dirichlet or Neumann boundary conditions. In particular, we will look at the peaking phenomena, that is we impose that the boundary data blow up in finite time and study the asymptotic behavior of solutions. These type of boundary conditions appear in combustion processes, [11]. For the study of peaking for the porous medium equation we refer to [5,7,9,11,12]. For general references on blow-up problems see [10] and [12].

First, we deal with Dirichlet boundary conditions. We impose the value of $u(x,t)$ for $x < 0$, and obtain the following problem:

$$\begin{cases} u_t(x,t) = \int_{\mathbb{R}} J(\frac{x-y}{u(y,t)})\,dy - u(x,t), & x \geqslant 0,\ t \geqslant 0, \\ u(x,t) = f(x,t), & x < 0,\ t \geqslant 0, \\ u(x,0) = d + w_0(x), & x \geqslant 0. \end{cases}$$

We assume that $d \geqslant 0$ and w_0 is a nonnegative $L^1(0,+\infty)$ function. We will use the notation $\mathbb{R}^+ = (0,+\infty)$ and $\mathbb{R}^- = (-\infty,0)$. This problem can be written as, for $x \in \mathbb{R}^+$,

$$\begin{aligned} &u_t(x,t) = \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right) dy + \int_{\mathbb{R}^-} J\left(\frac{x-y}{f(y,t)}\right) dy - u(x,t), \\ &u(x,0) = d + w_0(x). \end{aligned} \tag{3}$$

In this model we are prescribing the values of $u(x,t)$ in $\mathbb{R}^-$ and impose that the equation is satisfied in $\mathbb{R}^+$. In this sense we are facing Dirichlet boundary conditions. Remark that since the problem is nonlocal it is not enough to prescribe only $u(0,t)$ and we have to impose values in the whole $\mathbb{R}^-$. Our first result states the existence and uniqueness of solutions and a comparison principle.

Theorem 1.1. *Let $d \geqslant 0$ and w_0 is a nonnegative $L^1(\mathbb{R}^+)$ function. Then, for every function $f \in C([0,T]; L^1(\mathbb{R}^-))$, there exists a unique solution $u \in C([0,T]; L^1(\mathbb{R}^+))$ of problem* (3).

Moreover, a comparison principle holds for continuous solutions: If u and v are two continuous solutions of (3) *with $u(x,0) \leqslant v(x,0)$, $\forall x \in \mathbb{R}^+$, then $u(x,t) \leqslant v(x,t)$, $\forall (x,t) \in \mathbb{R}^+ \times [0,\infty)$.*

Next, we deal with the peaking phenomena for this model. In fact, for simplicity, we consider the particular case where the function f is given by

$$f(x,t) = (T-t)^{-\alpha}, \quad \alpha > 0. \tag{4}$$

Notice that this function blows up in finite time, $t = T$. Our blow-up result for this problem reads as follows.

Theorem 1.2. *Let f be given by* (4). *Then the solution of* (3) *blows up at finite time T if and only if $\alpha \geqslant 1$.*

Moreover, blow-up is always global, and the asymptotic behaviour is given by $(T-t)^{\alpha-1}u(x,t) \to 1/(2(\alpha-1))$ if $\alpha > 1$ and by $(-\ln(T-t))^{-1}u(x,t) \to 1/2$ if $\alpha = 1$.

The total mass $M(t) = \int_{\mathbb{R}^+} u(x,t)\,dx$ blows up if and only if $\alpha \geqslant 1/2$.

Remark 1. Our ideas can be applied to more general boundary data. If $f = f(t)$ is increasing with $\lim_{t \nearrow T} f(t) = +\infty$ we obtain that u blows up if and only if f is not integrable up to T.

Remark 2. Theorem 1.2 shows that peaking phenomena for this model is different from the one for the porous medium equation. In fact, solutions of the porous medium equation, $u_t = (u^m)_{xx}$ with $u(0,t) = (T-t)^{-\alpha}$ blows up if and only if $\alpha > 0$. In this case the blow-up rate is $(T-t)^{-\alpha}$ and there exists a localization of the blow-up set if and only if $\alpha \leqslant 1/(m-1)$, [11].

Next, we impose Neumann boundary conditions. We deal with the problem

$$u_t(x,t) = \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right)dy - \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(x,t)}\right)dy + \int_{\mathbb{R}^-} J(x-y)f(y,t)\,dy$$

for $(x,t) \in \mathbb{R}^+ \times [0,\infty)$. In this model we assume that no individuals can jump outside the domain, $\mathbb{R}^+$, but it is prescribed the flux of individuals entering (or leaving) the domain through the term involving f (the datum). We can rewrite our problem as follows:

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right)dy + \int_{\mathbb{R}^-} J\left(\frac{x-y}{u(x,t)}\right)dy - u(x,t) \\ &\quad + \int_{\mathbb{R}^-} J(x-y)f(y,t)\,dy \quad \text{for } (x,t) \in \mathbb{R}^+ \times [0,\infty), \\ u(x,0) &= d + w(x,0) \quad \text{for } x \in \mathbb{R}^+. \end{aligned} \tag{5}$$

As before, we are considering a general class of initial conditions, that is $u(x,0) = d + w_0(x)$, with $d \geqslant 0$, $w_0 \in L^1(\mathbb{R}^+)$ and $w_0 \geqslant 0$.

Theorem 1.3. *For every $w_0 \in L^1(\mathbb{R}^+)$ and f nonnegative and integrable there exists a unique solution $u(x,t)$ of* (5). *Moreover, we have a comparison principle valid for continuous solutions.*

We will use the notation $f \sim g$ to mean that there exist finite positive constants c_1 and c_2 such that $c_1 f \leqslant g \leqslant c_2 f$. Concerning the blow-up problem we have

Theorem 1.4. *Let $u(x,t)$ be a solution of* (5) *with boundary datum f given by* (4). *Then u blows up if and only if $\alpha \geqslant 1$. The blow-up rate is given by $\|u(\cdot,t)\|_\infty \sim (T-t)^{-\alpha+1}$ if $\alpha > 1$ and $\|u(\cdot,t)\|_\infty \sim -\ln(T-t)$ if $\alpha = 1$. Blow-up is regional, the blow-up is given by $B(u) = [0,1]$.*

The total mass $M(t) = \int_{\mathbb{R}^+} u(x,t)\,dx$ blows up if and only if $\alpha \geqslant 1$.

Remark 3. Also in this case the blow-up phenomena for our model is different from the one for the porous medium equation, see [5].

2. The Dirichlet problem

2.1. Existence and uniqueness

As in [6], existence and uniqueness follow from a fixed point argument, we give some details here for the reader's convenience. For some $t_0 > 0$ fixed, we consider the space $C([0,t_0]; L^1(\mathbb{R}^+))$ with the norm

$$|||w||| = \max_{0\leqslant t\leqslant t_0} \|w(\cdot,t)\|_{L^1(\mathbb{R}^+)}.$$

Let $X_{t_0} = \{w \in C([0,t_0]; L^1(\mathbb{R}^+))/\ w \geqslant 0\}$ that is closed in $C([0,t_0]; L^1(\mathbb{R}^+))$. We will look for a solution of 3 of the form $u(x,t) = d + w(x,t)$, where w is a fixed point of the operator $T_{w_0} : X_{t_0} \to X_{t_0}$, given by

$$T_{w_0}(w)(x,t) = \int_0^t e^{(s-t)} \int_{\mathbb{R}^+} J\left(\frac{x-y}{w(y,s)+d}\right) dy\,ds + \int_0^t e^{(s-t)} \int_{\mathbb{R}^-} J\left(\frac{x-y}{f(y,s)}\right) dy\,ds$$
$$+ e^{-t} w_0(x) - d\left(1-e^{-t}\right)$$

with $f \in C([0,T); L^1(\mathbb{R}^-))$.

Lemma 2.1. *Let $w_0, z_0 \in L^1(\mathbb{R}^+)$ be nonnegative functions, $w, z \in X_{t_0}$ and $f, g \in C([0,T); L^1(\mathbb{R}^-))$, then*

$$\left|\left|\left|T_{w_0}(w) - T_{z_0}(z)\right|\right|\right| \leqslant \left(1-e^{-t_0}\right)|||w-z||| + \|w_0 - z_0\|_{L^1(\mathbb{R}^+)} + \left(1-e^{-t_0}\right) \max_{0\leqslant t\leqslant t_0} \left\|f(\cdot,t) - g(\cdot,t)\right\|_{L^1(\mathbb{R}^-)}. \tag{6}$$

Proof. To obtain a bound for $|||T_{w_0}(w) - T_{z_0}(z)|||$ let us proceed as follows,

$$\int_{\mathbb{R}^+} \left|T_{w_0}(w)(x,t) - T_{z_0}(z)(x,t)\right| dx$$
$$\leqslant \int_0^t e^{s-t} \int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^-} \left(J\left(\frac{x-y}{f(y,s)}\right) - J\left(\frac{x-y}{g(y,s)}\right)\right) dy \right| dx\,ds + e^{-t} \int_{\mathbb{R}^+} |w_0 - z_0|(y)\,dy.$$

To study the first term let us consider $A^+(s) = \{y \in \mathbb{R}^+/\ w(y,s) \geqslant z(y,s)\}$ and $A^-(s) = \{y \in \mathbb{R}^+/\ w(y,s) \leqslant z(y,s)\}$. We obtain

$$\int_{\mathbb{R}^+} \left| \int_{\mathbb{R}^+} \left(J\left(\frac{x-y}{w(y,s)+d}\right) - J\left(\frac{x-y}{z(y,s)+d}\right)\right) dy \right| dx$$
$$= \int_{\mathbb{R}^+} \int_{A^+(s)} \left(J\left(\frac{x-y}{w(y,s)+d}\right) - J\left(\frac{x-y}{z(y,s)+d}\right)\right) dy\,dx$$
$$+ \int_{\mathbb{R}^+} \int_{A^-(s)} \left(J\left(\frac{x-y}{z(y,s)+d}\right) - J\left(\frac{x-y}{w(y,s)+d}\right)\right) dy\,dx.$$

We can apply Fubini's Theorem to obtain

$$\int_{\mathbb{R}^+} \int_{A^+(s)} \left(J\left(\frac{x-y}{w(y,s)+d}\right) - J\left(\frac{x-y}{z(y,s)+d}\right)\right) dy\,dx$$
$$\leqslant \int_{A^+(s)} \int_{\mathbb{R}} \left(J\left(\frac{x-y}{w(y,s)+d}\right) - J\left(\frac{x-y}{z(y,s)+d}\right)\right) dx\,dy$$
$$= \int_{A^+(s)} \left(w(y,s) - z(y,s)\right) dy.$$

Analogously

$$\int_{\mathbb{R}^+}\int_{A^-(s)} \left(J\left(\frac{x-y}{z(y,s)+d}\right) - J\left(\frac{x-y}{w(y,s)+d}\right)\right) dy\,dx \leqslant \int_{A^-(s)} \big(z(y,s)-w(y,s)\big)\,dy.$$

Therefore, the first integral satisfies the following bound:

$$\int_{\mathbb{R}^+}\left|\int_{\mathbb{R}^+} \left(J\left(\frac{x-y}{w(y,s)+d}\right) - J\left(\frac{x-y}{z(y,s)+d}\right)\right) dy\right| dx \leqslant \int_{\mathbb{R}^+} \big|w(y,s)-z(y,s)\big|\,dy.$$

To study the second term we argue in a similar way considering $B^+(s) = \{y \in \mathbb{R}^- /\ f(y,s) \geqslant g(y,s)\}$ and $B^-(s) = \{y \in \mathbb{R}^- /\ f(y,s) \leqslant g(y,s)\}$. In this case we obtain

$$\int_{\mathbb{R}^+}\left|\int_{\mathbb{R}^-} \left(J\left(\frac{x-y}{f(y,s)}\right) - J\left(\frac{x-y}{g(y,s)}\right)\right) dy\right| dx \leqslant \int_{\mathbb{R}^-} \big|f(y,s)-g(y,s)\big|\,dy.$$

Summing up, we get

$$\begin{aligned}\int_{\mathbb{R}^+} \big|T_{w_0}(w)(x,t) - T_{z_0}(z)(x,t)\big|\,dx &\leqslant \int_0^t e^{s-t}\int_{\mathbb{R}^+} \big|w(y,s)-z(y,s)\big|\,dy\,ds \\ &\quad + \int_0^t e^{s-t}\int_{\mathbb{R}^-} \big|f(y,s)-g(y,s)\big|\,dy\,ds + e^{-t}\int_{\mathbb{R}^+} |w_0-z_0|(y)\,dy.\end{aligned}$$

From where it follows that

$$\begin{aligned}\big\|T_{w_0}(w)(\cdot,t) - T_{z_0}(z)(\cdot,t)\big\|_{L^1(\mathbb{R}^+)} &\leqslant \big(1-e^{-t}\big)\big\|w(\cdot,t)-z(\cdot,t)\big\|_{L^1(\mathbb{R}^+)} \\ &\quad + \big(1-e^{-t}\big)\big\|f(\cdot,t)-g(\cdot,t)\big\|_{L^1(\mathbb{R}^-)} + e^{-t}\|w_0-z_0\|_{L^1(\mathbb{R}^+)}.\end{aligned}$$

Hence, we obtain (6) as we wanted to prove. □

Now we are ready to prove existence and uniqueness of solutions of (3).

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R}^+)$ and $f \in C([0,T]; L^1(\mathbb{R}^-))$. Then there exists a unique solution $u \in C([0,T]; L^1(\mathbb{R}^+))$ of* (3).

Proof. First, we observe that $T_{w_0} : X_{t_0} \to X_{t_0}$. Indeed, for $w \in X_{t_0}$ we get

$$\begin{aligned}T_{w_0}(w)(x,t) &\geqslant \int_0^t e^{(s-t)}\int_{\mathbb{R}^+} J\left(\frac{x-y}{d}\right) dy\,ds + e^{-t}w_0(x) - d\big(1-e^{-t}\big) \\ &= e^{-t}w_0(x) \geqslant 0.\end{aligned}$$

From Lemma 2.1 we obtain that T_{w_0} is a strict contraction for $t_0 > 0$. Therefore there exists a unique fixed point of T_{w_0} in X_{t_0}. This shows that there exists a unique solution in $[0,t_0]$. Arguing in the same way taking as initial datum $u(x,t_0)$ we get a unique solution defined in $[0,2t_0]$. We may continue and obtain a solution defined for $0 < t < T$. □

Remark 4. Solutions of (3) depend continuously on the initial data. In fact, if u and v are solutions of (3) with initial data u_0 and v_0, respectively, and the same boundary data, then

$$\max_{0\leqslant t\leqslant t_0} \big\|u(\cdot,t)-v(\cdot,t)\big\|_{L^1(\mathbb{R}^+)} \leqslant e^{t_0}\big\|u(\cdot,0)-v(\cdot,0)\big\|_{L^1(\mathbb{R}^+)}.$$

Remark 5. Solutions of (3) depend continuously on the boundary data. In fact, if u and v are solutions of (3) with boundary data f and g, respectively, and the same initial datum, then

$$\max_{0\leqslant t\leqslant t_0}\left\|u(\cdot,t)-v(\cdot,t)\right\|_{L^1(\mathbb{R}^+)}\leqslant\left(e^{t_0}-1\right)\max_{0\leqslant t\leqslant t_0}\left\|f(\cdot,t)-g(\cdot,t)\right\|_{L^1(\mathbb{R}^-)}.$$

Remark 6. The function u is a solution of (3) if and only if

$$u(x,t)=\int_0^t e^{(s-t)}\int_{\mathbb{R}^+}J\left(\frac{x-y}{u(y,s)}\right)dy\,ds+\int_0^t e^{(s-t)}\int_{\mathbb{R}^-}J\left(\frac{x-y}{f(y,s)}\right)dy\,ds+e^{-t}u_0(x).$$

Notice that the solution does not have to be continuous at $x=0$, even if the initial datum u_0 is continuous.

Next, we state a comparison principle. The proof is similar to the one given in [6], we omit the details.

Theorem 2.2. *Let u and v be two continuous solutions of* (3). *If $u(x,0)\leqslant v(x,0)$ for all $x\in\mathbb{R}^+$, then $u(x,t)\leqslant v(x,t)$ for all $(x,t)\in\mathbb{R}^+\times[0,\infty)$.*

2.2. Blow-up analysis

In this subsection we consider $f(x,t)=(T-t)^{-\alpha}$, which blows up at finite time T. In this case Eq. (3) reads

$$u_t(x,t)=\int_{\mathbb{R}^+}J\left(\frac{x-y}{u(y,t)}\right)dy+\int_{\mathbb{R}^-}J\left((x-y)(T-t)^{\alpha}\right)dy-u(x,t).$$

Proof of Theorem 1.2. Taking account that the first integral is positive and performing the change of variables $r=(x-y)(T-t)^{\alpha}$ in the second integral we obtain

$$u_t(x,t)+u(x,t)\geqslant\frac{1}{(T-t)^{\alpha}}\int_{x(T-t)^{\alpha}}^{\infty}J(r)\,dr.$$

Then

$$u(x,t)\geqslant e^{-t}u_0+e^{-t}\int_0^t\frac{e^s}{(T-s)^{\alpha}}\int_{x(T-t)^{\alpha}}^{\infty}J(r)\,dr\,ds.$$

On the other hand, there exists $\delta>0$ such that for $T-\delta\leqslant t<T$,

$$\int_{x(T-t)^{\alpha}}^{\infty}J(r)\,dr\geqslant\frac{1}{4}.$$

Hence, for $T-\delta\leqslant t<T$,

$$u(x,t)\geqslant e^{-T}u_0+\frac{e^{-T}}{4}\int_0^t\frac{1}{(T-s)^{\alpha}}\,ds.$$

Therefore u blows up a time T for all $x\in\mathbb{R}^+$ and $\alpha\geqslant 1$ and

$$u(x,t)\geqslant\begin{cases}C(T-t)^{-\alpha+1}, & \alpha>1,\\ -C\ln(T-t), & \alpha=1.\end{cases}$$

In order to obtain the upper bound we compare with $w(x,t)=A(T-t)^{-\alpha+1}$. To prove that w is a supersolution we need that

$$\frac{A(\alpha-1)}{(T-t)^{\alpha}} \geqslant \int_0^{\infty} J\left(\frac{(x-y)(T-t)^{\alpha-1}}{A}\right) dy + \frac{1}{(T-t)^{\alpha}} \int_{x(T-t)^{\alpha}}^{\infty} J(r)\, dr + \frac{A}{(T-t)^{\alpha-1}}. \tag{7}$$

Performing the change of variables $z = \frac{(x-y)(T-t)^{\alpha-1}}{A}$ in the first integral we obtain

$$\begin{aligned}\int_0^{\infty} J\left(\frac{(x-y)(T-t)^{\alpha-1}}{A}\right) dy &= \int_{-\infty}^{\frac{x(T-t)^{\alpha-1}}{A}} \frac{A}{(T-t)^{\alpha-1}} J(z)\, dz \\ &\leqslant \int_{\mathbb{R}} \frac{A}{(T-t)^{\alpha-1}} J(z)\, dz = \frac{A}{(T-t)^{\alpha-1}}.\end{aligned}$$

On the other hand, as $x > 0$ the second integral in (7) satisfies

$$\frac{1}{(T-t)^{\alpha}} \int_{x(T-t)^{\alpha}}^{\infty} J(r)\, dr \leqslant \frac{1}{(T-t)^{\alpha}} \int_0^{\infty} J(r)\, dr = \frac{1}{2(T-t)^{\alpha}}.$$

Summing up, we have that for t near T, the function w is a supersolution if

$$A > \frac{1}{2(\alpha-1)} \quad \text{for } \alpha > 1.$$

Moreover, if $u(x,t)$ is a solution of (3), then taking A larger if necessary, $u(x,t) \leqslant A(T-t)^{1-\alpha}$ for all $x \in \mathbb{R}^+$. For the case $\alpha = 1$ we consider as supersolution the function $w(x,t) = -A\ln(T-t)$.

Up to now, we have proved that $u(x,t) \sim (T-t)^{1-\alpha}$. Next, we use this fact to obtain the asymptotic behaviour. Assume that $\alpha > 1$ (the case $\alpha = 1$ is analogous). We have

$$\begin{aligned} u_t(x,t) &\leqslant \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right) dy + \int_{\mathbb{R}^-} J\big((x-y)(T-t)^{\alpha}\big)\, dy \\ &\leqslant \int_{\mathbb{R}^+} J\left(\frac{x-y}{C(T-t)^{1-\alpha}}\right) dy + \frac{1}{2(T-t)^{\alpha}} \leqslant \frac{C}{(T-t)^{\alpha-1}} + \frac{1}{2(T-t)^{\alpha}}.\end{aligned}$$

Integrating in $[0,t]$ and taking limits we obtain

$$\limsup_{t\nearrow T} (T-t)^{\alpha-1} u(x,t) \leqslant \frac{1}{2(\alpha-1)}.$$

To get the lower bound we observe that

$$\begin{aligned} u_t(x,t) &\geqslant -u(x,t) + \int_{\mathbb{R}^-} J\big((x-y)(T-t)^{\alpha}\big)\, dy \\ &\geqslant -\frac{C}{(T-t)^{\alpha-1}} + \frac{1}{(T-t)^{\alpha}} \int_{x(T-t)^{\alpha}}^{\infty} J(z)\, dz.\end{aligned}$$

Integrating and taking limits we get

$$\liminf_{t\nearrow T} (T-t)^{\alpha-1} u(x,t) \geqslant \frac{1}{2(\alpha-1)}.$$

We have proved that

$$\lim_{t\nearrow T} (T-t)^{\alpha-1} u(x,t) = \frac{1}{2(\alpha-1)}.$$

If $\alpha < 1$, taking $v(x,t) = C_1 - C_2(T-t)^{1-\alpha}$ as a supersolution, we obtain that there exists a constant $K = K(T, \alpha, u_0)$ such that $u(x,t) \leqslant K$ for all $(x,t) \in \mathbb{R}^+ \times (0,T)$.

Next, we study the behavior of the mass, $M(t) = \int_0^\infty u(x,t)\,dx$, which satisfies the equation

$$M'(t) = \int_0^\infty \int_0^\infty J\left(\frac{x-y}{u(y,t)}\right) dy\,dx + \int_0^\infty \frac{1}{(T-t)^\alpha} \int_{x(T-t)^\alpha}^\infty J(r)\,dr\,dx - M(t). \tag{8}$$

Applying Fubini's Theorem in the first integral it is easy to check that

$$\int_0^\infty \int_0^\infty J\left(\frac{x-y}{u(y,t)}\right) dx\,dy \leqslant M(t).$$

For the second integral we observe that for $r \geqslant x(T-t)^\alpha \geqslant 1$, then $J(r) = 0$. Therefore,

$$\begin{aligned}\int_0^\infty \frac{1}{(T-t)^\alpha} \int_{x(T-t)^\alpha}^\infty J(r)\,dr\,dx &= \int_0^{\frac{1}{(T-t)^\alpha}} \frac{1}{(T-t)^\alpha} \int_{x(T-t)^\alpha}^\infty J(r)\,dr\,dx \\ &= \int_0^1 \int_\theta^\infty \frac{1}{(T-t)^{2\alpha}} J(r)\,dr\,d\theta = \frac{B}{(T-t)^{2\alpha}}.\end{aligned}$$

Summing up, we obtain

$$M'(t) \leqslant \frac{B}{(T-t)^{2\alpha}},$$

and by integration we conclude the upper bound

$$M(t) \leqslant \begin{cases} C, & \alpha < 1/2, \\ -C\ln(T-t), & \alpha = 1/2, \\ C(T-t)^{-2\alpha+1}, & \alpha > 1/2. \end{cases}$$

From the positivity of the first integral in (8), we also have

$$M'(t) \geqslant \frac{B}{(T-t)^{2\alpha}} - M(t),$$

which, by integration, gives us the inverse inequality. □

3. The Neumann problem

3.1. Existence and uniqueness

As in the previous section, the existence and uniqueness of solutions of (5) is a consequence of a fixed point argument. Let us consider the operator $L_{w_0}: X_{t_0} \to X_{t_0}$ given by

$$\begin{aligned}L_{w_0}(w)(x,t) = {}& \int_0^t e^{(s-t)} \int_{\mathbb{R}^+} J\left(\frac{x-y}{w(y,s)+d}\right) dy\,ds + \int_0^t e^{(s-t)} \int_{\mathbb{R}^-} J\left(\frac{x-y}{w(x,s)+d}\right) dy\,ds \\ &+ \int_0^t e^{(s-t)} \int_{\mathbb{R}^-} J(x-y) f(y,s)\,dy\,ds + e^{-t} w_0(x) - d\big(1-e^{-t}\big).\end{aligned}$$

Lemma 3.1. *Let w_0, z_0 be nonnegative functions such that $w_0, z_0 \in L^1(\mathbb{R}^+)$, $w, z \in X_{t_0}$ and $f, g \in C([0,t_0], L^1(\mathbb{R}^-))$, then*

$$\left|\!\left|\!\left| L_{w_0}(w) - L_{z_0}(z) \right|\!\right|\!\right| \leqslant \left(1 - e^{-t_0}\right) |\!|\!| w - z |\!|\!| + \|w_0 - z_0\|_{L^1(\mathbb{R}^+)} + \left(1 - e^{-t_0}\right) \max_{0 \leqslant t \leqslant t_0} \left\| f(\cdot,t) - g(\cdot,t) \right\|_{L^1(\mathbb{R}^-)}.$$

Proof. It is analogous to the proof of Lemma 2.1 in [6]. □

The next theorem shows existence and uniqueness of solutions of (5).

Theorem 3.1. *For every $u_0 \in L^1(\mathbb{R}^+)$ and $f \in C([0,T]; L^1(\mathbb{R}^-))$ nonnegative there exists a unique solution $u(x,t)$ of* (5).

Proof. It is similar to the proof of Theorem 2.1 (see also [6]). □

We have some consequences of the previous arguments that we collect as remarks.

Remark 7. The solution of (5) depends continuously on the initial datum. If u and v are solutions of (5) with initial data u_0 and v_0, respectively, then

$$\max_{0 \leqslant t \leqslant t_0} \left\| u(\cdot,t) - v(\cdot,t) \right\|_{L^1(\mathbb{R}^+)} \leqslant e^{t_0} \left\| u(\cdot,0) - v(\cdot,0) \right\|_{L^1(\mathbb{R}^+)}.$$

Remark 8. The solution of (5) depends continuously on the boundary datum. If u and v are solutions of (5) with boundary data f and g, respectively, and with the same initial datum, than

$$\max_{0 \leqslant t \leqslant t_0} \left\| u(\cdot,t) - v(\cdot,t) \right\|_{L^1(\mathbb{R}^+)} \leqslant \left(e^{t_0} - 1\right) \max_{0 \leqslant t \leqslant t_0} \left\| f(\cdot,t) - g(\cdot,t) \right\|_{L^1(\mathbb{R}^-)}.$$

Remark 9. The function u is a solution of (5) if and only if

$$u(x,t) = \int_0^t e^{(s-t)} \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,s)}\right) dy\, ds + \int_0^t e^{(s-t)} \int_{\mathbb{R}^-} J\left(\frac{x-y}{u(x,s)}\right) dy\, ds$$
$$+ \int_0^t e^{(s-t)} \int_{\mathbb{R}^-} J(x-y) f(y,s)\, dy\, ds + e^{-t} u_0(x).$$

Notice that the solution does not have to be continuous at $x = 0$, even if the initial datum u_0 is continuous.

As before, we have a comparison principle valid for continuous solutions. Again we omit the details of the proof.

Theorem 3.2. *If u and v are solutions of* (5) *with $u(x,0) \leqslant v(x,0)$ for all $x \in \mathbb{R}^+$, then $u(x,t) \leqslant v(x,t)$ for all $(x,t) \in \mathbb{R}^+ \times [0,\infty)$.*

3.2. Blow-up analysis

In this subsection we deal with solutions of (5) with $f(x,t) = (T-t)^{-\alpha}$, where $\alpha > 0$.

Proof of Theorem 1.4. We have

$$u_t(x,t) = \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right) dy + \int_{\mathbb{R}^-} J\left(\frac{x-y}{u(x,t)}\right) dy - u(x,t) + \int_{\mathbb{R}^-} J(x-y) \frac{1}{(T-t)^\alpha}\, dy.$$

Changing variables, $\zeta = x - y$, we obtain

$$u_t(x,t) + u(x,t) = \int_{\mathbb{R}^+} J\left(\frac{x-y}{u(y,t)}\right) dy + \int_{\mathbb{R}^-} J\left(\frac{x-y}{u(x,t)}\right) dy + \frac{1}{(T-t)^\alpha} \int_x^\infty J(\zeta)\, d\zeta.$$

Hence

$$u_t(x,t) + u(x,t) \geqslant \frac{1}{(T-t)^\alpha} \int_x^\infty J(\zeta)\, d\zeta.$$

Then

$$u(x,t) \geqslant e^{-t} u_0(x) + e^{-t} \int_0^t \frac{1}{(T-t)^\alpha} \int_x^\infty J(\zeta)\, d\zeta\, ds.$$

Let $A = \int_x^\infty J(\zeta)\, d\zeta$. If $x \geqslant 1$, then $A = 0$. Therefore, for $0 < x < 1$ we have

$$u(x,t) \geqslant e^{-t} u_0(x) + A e^{-t} \int_0^t \frac{1}{(T-t)^\alpha}\, ds$$

and hence

$$u(x,t) \geqslant C \int_0^t \frac{1}{(T-t)^\alpha}\, ds$$

for $x \in (0,1)$ and $0 < t < T$. We have proved that u blows up if $\alpha \geqslant 1$ and the estimates valid for $x \in (0,1)$:

(a) If $\alpha > 1$, then $u(x,t) \geqslant C(T-t)^{-\alpha+1}$.
(b) If $\alpha = 1$, then $u(x,t) \geqslant -C \ln(T-t)$.

In an analogous way, we obtain that if $D > \frac{1}{2(\alpha-1)}$, with $\alpha > 1$, then $z(x,t) = D(T-t)^{1-\alpha}$ is a supersolution of (5) for t close to T. Moreover, if $u(x,t)$ is a solution of (5), then taking D larger if needed, $u(x,t) \leqslant D(T-t)^{1-\alpha}$ for all $0 < x < 1$. If $\alpha = 1$ we obtain $u(x,t) \leqslant -D\ln(T-t)$, considering as supersolution $w(x,t) = -D\ln(T-t)$. Finally, if $\alpha < 1$, using as supersolution $v(x,t) = C_1 - C_2(T-t)^{1-\alpha}$, we get that there exists K such that $u(x,t) \leqslant K$.

To find the blow-up set we argue as follows: if $x > 1$, using the blow-up rate, we have

$$\begin{aligned} u_t(x,t) &\leqslant \int_0^1 J\left(\frac{x-y}{u(y,t)}\right) dy + \int_1^\infty J\left(\frac{x-y}{u(y,t)}\right) dy \\ &\leqslant (T-t)^{1-\alpha} \int_{\frac{x-1}{(T-t)^{1-\alpha}}}^{\frac{x}{(T-t)^{1-\alpha}}} J(z)\, dz + \int_1^\infty J\left(\frac{x-y}{u(y,t)}\right) dy. \end{aligned}$$

Notice that

$$(T-t)^{1-\alpha} \int_{\frac{x-1}{(T-t)^{1-\alpha}}}^{\frac{x}{(T-t)^{1-\alpha}}} J(z)\, dz \leqslant C.$$

Hence we can use a comparison argument with $\bar{u}(x,t) = Ke^t$ (with K large) in $[1,\infty)$ to end the proof.

Next, we study the behavior of the mass, $M(t)=\int_0^\infty u(x,t)\,dx$, which satisfies the equation

$$M'(t)=\frac{1}{(T-t)^\alpha}\int_0^\infty\int_{-\infty}^0 J(x-y)\,dy\,dx=\frac{C}{(T-t)^\alpha}.$$

Direct integration gives that M blows up if and only if $\alpha\geqslant 1$. □

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