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Hypercyclic convolution operators on Fréchet spaces of analytic functions

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Abstract

A result of Godefroy and Shapiro states that the convolution operators on the space of entire functions on $\mathbb{C}^n$, which are not multiples of identity, are hypercyclic. Analogues of this result have appeared for some spaces of holomorphic functions on a Banach space. In this work, we define the space holomorphic functions associated to a sequence of spaces of polynomials and determine conditions on this sequence that assure hypercyclicity of convolution operators. Some known results come out as particular cases of this setting. We also consider holomorphic functions associated to minimal ideals of polynomials and to polynomials of the Schatten–von Neumann class.

Keywords: Spaces of holomorphic functions; Hypercyclic operators; Convolution operators

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0. Introduction

This note deals with convolution operators on Fréchet spaces of holomorphic functions associated to certain classes of polynomials. In particular, we are interested in determining when such operators are hypercyclic. Recall that convolution operators are those that commute with translations. Also, an operator $T : X \to X$ is hypercyclic if there exists $x \in X$ such that $\{T^n x\colon n \geqslant 1\}$ is dense in X.

In a seminal work, Godefroy and Shapiro [18] show that every convolution operator in $\mathcal{H}(\mathbb{C}^n)$ which is not a scalar multiple of identity is hypercyclic. In this way, they generalize classical results of Birkhoff [5] and MacLane [23] on the hypercyclicity of the translation and differentiation operators on $\mathcal{H}(\mathbb{C})$. Analogues of Godefroy and Shapiro's result for some particular spaces of holomorphic functions on Banach spaces are proved in [1,26,27].

Let E be a Banach space. Given a coherent sequence $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ of Banach spaces of k-homogeneous polynomials (see the definitions below), we define a Fréchet space $H_{b\mathfrak{A}}(E)$ of holomorphic functions of bounded type associated to $\mathfrak{A}(E)$, much in the spirit of holomorphy types introduced by Nachbin [25] (see also [13]). Under fairly general conditions, we characterize the convolution operators on this spaces of holomorphic functions and show that they are hypercyclic whenever they are not scalar multiples of the identity. We obtain some results of [1,26,27] as particular cases. Also, we see that spaces of holomorphic functions generated by polynomial minimal ideals are covered by our settings, if the dual space E' has the approximation property. We finally consider polynomials of the Schatten–von Neumann class in the sense of [11] and the associated space of holomorphic functions.

In the first section, we give the notions of coherence that will be used throughout and show some general properties. We define the Fréchet space $H_{b\mathfrak{A}}(E)$ of $\mathfrak{A}$-holomorphic functions of bounded type and present examples related to some common classes of homogeneous polynomials. The second section deals with duality questions for these spaces. Based on duality properties for each space $\mathfrak{A}_k(E)$, we characterize the dual of $H_{b\mathfrak{A}}(E)$ as the space of holomorphic functions of exponential type $\mathrm{Exp}_{\mathfrak{B}}(E')$ associated to some sequence $\mathfrak{B}$ of Banach spaces of polynomials. Minimal ideals of polynomials are particularly considered. Under certain hypotheses, $\mathrm{Exp}_{\mathfrak{B}}(E')$ is an algebra and its product can be identified with the convolution product in $H_{b\mathfrak{A}}(E)'$ (Section 3). In the fourth section, we characterize the convolution operators on $H_{b\mathfrak{A}}(E)$ and show that they are hypercyclic unless they are scalar multiples of the identity. We apply these results to the spaces of holomorphic functions of compact bounded type and of nuclearly entire functions of bounded type to obtain some results of [1,27]. We also consider holomorphic functions generated by coherent minimal ideals of polynomials. The last section deals with Schatten–von Neumann functions of bounded type, that is, those associated to the Schatten–von Neumann homogeneous polynomials defined by Cobos, Kühn and Peetre [11]. We use interpolation theory to show that these spaces satisfy our hypotheses. In particular, we obtain the hypercyclicity of convolution operators on the space of holomorphic function of Hilbert–Schmidt type [26].

We refer to [14,24] for notation and results regarding polynomials and holomorphic functions in general, and to [15,16] for polynomial ideals. For generalities on hypercyclic operators, we refer to the survey [19] and the references therein.

1. Definitions and properties

Throughout, E will denote a Banach space, E' its dual and $\mathcal{P}^k(E)$ the space of continuous k-homogeneous polynomials on E, with the usual norm. Also, for each k, $\mathfrak{A}_k(E)$ and $\mathfrak{B}_k(E)$

will be Banach spaces of k-homogeneous polynomials on E containing the finite type polynomials and continuously contained in $\mathcal{P}^k(E)$. In the sequel, all polynomials will be assumed to be continuous.

For $P \in \mathcal{P}^k(E)$, $\check{P}$ will denote the symmetric k-linear form associated to P. Also, if $a \in E$ and $r \in \mathbb{N}$, P_{a^r} is the $(k-r)$-homogeneous polynomial on E defined by

$$P_{a^r}(x) = \check{P}(\underbrace{a, \dots, a}_{r}, x, \dots, x).$$

In [9] we introduced the notion of coherent sequence of polynomial ideals. Here, we are interested in polynomials and analytic functions defined on a fixed Banach space. Therefore, we adapt the definition of coherence to our setting.

Definition 1.1. We say that the sequence $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ is *coherent* if there exist positive constants C and D such that the following conditions hold for all k:

(i) For each $P \in \mathfrak{A}_{k+1}(E)$ and $a \in E$, P_a belongs to $\mathfrak{A}_k(E)$ and
$$\|P_a\|_{\mathfrak{A}_k(E)} \leqslant C\|P\|_{\mathfrak{A}_{k+1}(E)}\|a\|.$$
(ii) For each $P \in \mathfrak{A}_k(E)$ and $\gamma \in E'$, γP belongs to $\mathfrak{A}_{k+1}(E)$ and
$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} \leqslant D\|\gamma\|\|P\|_{\mathfrak{A}_k(E)}.$$

Clearly, if some $\mathfrak{A}_k(E)$ is nonempty, then, by condition (i), $\mathfrak{A}_0(E)$ is the 1-dimensional space of constant functions on E. Note that the fact that $\mathfrak{A}_k(E)$ contains the finite type polynomials can be deduced from condition (ii). Also, condition (i) assures that $\mathfrak{A}_k(E)$ is continuously embedded in $\mathcal{P}^k(E)$ for every k.

Condition (ii) states that the product of a polynomial in $\mathfrak{A}(E)$ by a linear functional remains in $\mathfrak{A}(E)$. This is not necessarily the case if we multiply two polynomials in $\mathfrak{A}(E)$. Thus, we introduce the following:

Definition 1.2. A coherent sequence $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ is *multiplicative* if there exists $M > 0$ such that for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have $PQ \in \mathfrak{A}_{k+l}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leqslant M^{k+l}\|P\|_{\mathfrak{A}_k(E)}\|Q\|_{\mathfrak{A}_l(E)}.$$

It is not difficult to see that not every coherent sequence is multiplicative (see [10]).

The space of holomorphic functions of bounded type $H_b(E)$ is, in some sense, associated to the sequence $\{\mathcal{P}^k(E)\}_k$ of all homogeneous polynomials. Analogously, we can define the space of holomorphic functions of bounded type associated to any coherent polynomial sequence:

Definition 1.3. Let $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ be a coherent sequence. We define the space of $\mathfrak{A}$-holomorphic functions of bounded type on E by

$$H_{b\mathfrak{A}}(E) = \left\{ f \in H(E)\colon\ \frac{d^k f(0)}{k!} \in \mathfrak{A}_k(E) \text{ for all } k \text{ and } \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k(E)}^{\frac{1}{k}} \xrightarrow[k\to\infty]{} 0 \right\}.$$

Although the definition of $H_{b\mathfrak{A}}(E)$ involves the derivatives at 0, the same condition holds for the derivatives at any point $a \in E$, as the following lemma shows.

Lemma 1.4. *Let $\mathfrak{A}(E)=\{\mathfrak{A}_k(E)\}_k$ be a coherent sequence and $f\in H_{b\mathfrak{A}}(E)$. Then, for all $a\in E$,*

$$\lim_{k\to\infty}\left\|\frac{d^k f(a)}{k!}\right\|_{\mathfrak{A}_k(E)}^{\frac{1}{k}}=0.$$

Therefore, the function $\tau_a(f)=f(a+\cdot)$ belongs to $H_{b\mathfrak{A}}(E)$.

Proof. Let $f=\sum_{k=0}^{\infty}P_k\in H_{b\mathfrak{A}}(E)$, with $P_k=\frac{d^k f(0)}{k!}\in\mathfrak{A}_k(E)$. Then

$$f(x+a)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}\check{P}_k\big(a^{k-j},x^j\big)=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}\check{P}_k\big(a^{k-j},x^j\big).$$

Thus $\frac{d^j f(a)}{j!}=\sum_{k=j}^{\infty}\binom{k}{j}(P_k)_{a^{k-j}}$. The partial sums of this series belong to $\mathfrak{A}_k(E)$ since $\mathfrak{A}(E)$ is a coherent sequence. For $0<\varepsilon<1$, let $N\in\mathbb{N}$ be such that $\|P_k\|_{\mathfrak{A}_k(E)}^{\frac{1}{k}}\leqslant\frac{\varepsilon}{C\|a\|}$ if $k\geqslant N$. We have

$$\begin{aligned}\sum_{k=\max(j,N)}^{\infty}\left\|\binom{k}{j}(P_k)_{a^{k-j}}\right\|_{\mathfrak{A}_j(E)}&\leqslant\sum_{k=\max(j,N)}^{\infty}\binom{k}{j}C^{k-j}\|P_k\|_{\mathfrak{A}_k(E)}\|a\|^{k-j}\\&\leqslant\left(\frac{\varepsilon}{C\|a\|}\right)^j\sum_{k=\max(j,N)}^{\infty}\binom{k}{j}\varepsilon^{k-j}<\infty.\end{aligned}$$

Therefore, the series converges in $\mathfrak{A}_j(E)$ and $\frac{d^j f(a)}{j!}\in\mathfrak{A}_j(E)$.

Also, for $j\geqslant N$,

$$\left\|\frac{d^j f(a)}{j!}\right\|_{\mathfrak{A}_j(E)}^{\frac{1}{j}}\leqslant\frac{\varepsilon}{C\|a\|}\left(\sum_{k=j}^{\infty}\binom{k}{j}\varepsilon^{k-j}\right)^{\frac{1}{j}}\xrightarrow[j\to\infty]{}\frac{\varepsilon}{C\|a\|(1-\varepsilon)}.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\|\frac{d^j f(a)}{j!}\|_{\mathfrak{A}_j(E)}^{\frac{1}{j}}\xrightarrow[j\to\infty]{}0$. □

Lemma 1.5. *Let $\mathfrak{A}(E)=\{\mathfrak{A}_k(E)\}_k$ be a coherent multiplicative sequence. If $f,g\in H_{b\mathfrak{A}}(E)$, then $fg\in H_{b\mathfrak{A}}(E)$. Therefore, $H_{b\mathfrak{A}}(E)$ is an algebra.*

Proof. Take $f,g\in H_{b\mathfrak{A}}(E)$ with Taylor expansions $f=\sum_{k=0}^{\infty}P_k$ and $g=\sum_{k=0}^{\infty}Q_k$, where $P_k,Q_k\in\mathfrak{A}_k(E)$. Then $\frac{d^k(fg)(0)}{k!}=\sum_{j=0}^{k}P_jQ_{k-j}$ belongs to $\mathfrak{A}_k(E)$, since $\mathfrak{A}(E)$ is multiplicative. For each $\varepsilon>0$, take $A>0$ such that

$$\max\big\{\|P_k\|_{\mathfrak{A}_k(E)},\|Q_k\|_{\mathfrak{A}_k(E)}\big\}\leqslant A\varepsilon^k,$$

for every $k\geqslant 0$. Then,

$$\begin{aligned}\left\|\frac{d^k(fg)(0)}{k!}\right\|_{\mathfrak{A}_k(E)}^{\frac{1}{k}}&\leqslant\left(\sum_{j=0}^{k}M^k\|P_j\|_{\mathfrak{A}_j(E)}\|Q_{k-j}\|_{\mathfrak{A}_{k-j}(E)}\right)^{\frac{1}{k}}\\&\leqslant\big((k+1)A^2\big)^{\frac{1}{k}}M\varepsilon\xrightarrow[k\to\infty]{}M\varepsilon.\end{aligned}$$

Therefore $\|\frac{d^k(fg)(0)}{k!}\|_{\mathfrak{A}_k(E)}^{\frac{1}{k}}\to 0$ and thus $fg\in H_{b\mathfrak{A}}(E)$. □

We define in $H_{b\mathfrak{A}}(E)$ a sequence of seminorms $\{p_n\}_n$,

$$p_n(f) = \sum_{k=0}^{\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k(E)} n^k,$$

for $f \in H_{b\mathfrak{A}}(E)$.

It is easy to see that $(H_{b\mathfrak{A}}(E), \{p_n\}_n)$ is a Fréchet space. Moreover, for each $f \in H_{b\mathfrak{A}}(E)$, the partial sums of the Taylor series expansion of f about the origin converges to f in $H_{b\mathfrak{A}}(E)$.

Next, we present examples of coherent multiplicative sequences. The coherence is shown in [9]. It is immediate that the sequences of Examples 1.6 and 1.8 are multiplicative. For the other examples see [14, Exercise 2.63] and [10]. Moreover, Boyd and Lasalle showed that the product of two integral polynomials with values in a Banach algebra is also integral [4].

Example 1.6. Let $\mathfrak{A}(E)$ be the sequence of homogeneous polynomials, $\mathfrak{A}_k(E) = \mathcal{P}^k(E)$, $k \geqslant 1$. Then, $\mathfrak{A}(E)$ is coherent and multiplicative and $H_{b\mathfrak{A}}(E) = H_b(E)$.

Example 1.7. Let $\mathfrak{A}(E)$ be the sequence of nuclear polynomials, $\mathfrak{A}_k(E) = \mathcal{P}_N^k(E)$, $k \geqslant 1$. Then, $\mathfrak{A}(E)$ is coherent and multiplicative and $H_{b\mathfrak{A}}(E)$ is the space of nuclearly entire functions of bounded type $H_{Nb}(E)$ defined by Gupta and Nachbin (see [14,20]).

Example 1.8. Suppose $\mathfrak{A}(E)$ is the sequence of extendible polynomials, that is, $\mathfrak{A}_k(E) = \mathcal{P}_e^k(E)$, $k \geqslant 1$. Then, $\mathfrak{A}(E)$ is coherent and multiplicative. Moreover, an application of [8, Proposition 14] gives that $H_{b\mathfrak{A}}(E)$ is the space of all $f \in H(E)$ such that, for any Banach space $G \supset E$, there is an extension $\tilde{f} \in H_b(G)$ of f.

Example 1.9. Let $\mathfrak{A}(E)$ be the sequence of integral polynomials, $\mathfrak{A}_k(E) = \mathcal{P}_I^k(E)$, $k \geqslant 1$. Then, $\mathfrak{A}(E)$ is coherent and multiplicative and $H_{b\mathfrak{A}}(E)$ is the space of integral holomorphic functions of bounded type $H_{bI}(E)$ defined in [12].

2. Coherent sequences and duality

Let $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ be a coherent sequence. The Borel transform $\beta : H_{b\mathfrak{A}}(E)' \to H(E')$ assigns to each element $\varphi \in H_{b\mathfrak{A}}(E)'$ the holomorphic function $\beta(\varphi) \in H(E')$, given by $\beta(\varphi)(\gamma) = \varphi(\exp \circ \gamma) = \varphi(e^\gamma)$.

If $\varphi \in \mathfrak{A}_k(E)'$, we have two natural ways to identify φ with an element in $H_{b\mathfrak{A}}(E)'$:

$$\begin{aligned} H_{b\mathfrak{A}}(E) &\xrightarrow{\tilde{\varphi}} \mathbb{C} \\ f &\mapsto \varphi\left(\frac{d^k f(0)}{k!}\right) \end{aligned} \qquad \text{or} \qquad \begin{aligned} H_{b\mathfrak{A}}(E) &\xrightarrow{\bar{\varphi}} \mathbb{C} \\ f &\mapsto \varphi\big(d^k f(0)\big). \end{aligned}$$

Thus, the Borel transform induces two different "polynomial" Borel transforms: $\beta_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ where $\beta_k(\varphi) = \beta(\tilde{\varphi})$ and $B_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ given by $B_k(\varphi) = \beta(\bar{\varphi})$. Note that for $\gamma \in E'$, $\beta_k(\varphi)(\gamma) = \varphi(\frac{\gamma^k}{k!})$ and $B_k(\varphi)(\gamma) = \varphi(\gamma^k)$.

In the polynomial setting it is more common to use the mapping B_k than the mapping β_k. Moreover, it is not necessary to deal with holomorphic functions in order to define the polynomial Borel transform B_k. Indeed, for a Banach space of k-homogeneous polynomials $\mathfrak{A}_k(E)$, we can define $B_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ by $B_k(\varphi)(\gamma) = \varphi(\gamma^k)$, for every $\gamma \in E'$. Also, we can express the holomorphic Borel transform β in terms of the B_k's:

$$\beta(\varphi) = \sum_{k=0}^{\infty} \frac{B_k(\varphi|_{\mathfrak{A}_k(E)})}{k!}.$$

Notation. *In the sequel, the expression*

$$\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$$

will always mean that the polynomial Borel transform $B_k : \mathfrak{A}_k(E)' \to \mathfrak{B}_k(E')$ is an isometric isomorphism.

The following lemma states that in order to have this duality, $\mathfrak{A}_k(E)$ must be "small":

Lemma 2.1. *If $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$, then finite type polynomials are dense in $\mathfrak{A}_k(E)$:*

$$\overline{\mathcal{P}_f^k(E)}^{\mathfrak{A}_k} = \mathfrak{A}_k(E).$$

Proof. Suppose there exists $P \in \mathfrak{A}_k(E) - \overline{\mathcal{P}_f^k(E)}^{\mathfrak{A}_k}$. Then there is $\varphi \in \mathfrak{A}_k(E)'$ such that $\varphi(P) = 1$ and $\varphi|_{\mathcal{P}_f^k(E)} \equiv 0$. For every $\gamma \in E'$, $\varphi(\gamma^k) = 0$ and then $B_k(\varphi)(\gamma) = 0$. Thus $B_k(\varphi) = 0$ and therefore $\varphi = 0$ in $\mathfrak{A}_k(E)$, which is a contradiction. □

Since the Taylor expansion about the origin of a function $f \in H_{b\mathfrak{A}}(E)$ converges in $H_{b\mathfrak{A}}(E)$, we have

Corollary 2.2. *Let $\{\mathfrak{A}_k(E)\}_k$ and $\{\mathfrak{B}_k(E')\}_k$ be coherent sequences such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. Then finite type polynomials are dense in $H_{b\mathfrak{A}}(E)$.*

Examples 2.3. We exhibit two simple situations of coherent sequences where the duality $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ holds.

First, if $\mathcal{P}_A^n(E)$ is the space of approximable n-homogeneous polynomials, then $\mathcal{P}_A^n(E)'$ is (isometrically) the space of integral polynomials $\mathcal{P}_I^n(E')$ [13].

Second, if E' has the approximation property, the dual of $\mathcal{P}_N^n(E)$ coincides with $\mathcal{P}^n(E')$ [20].

Note that in both cases, the sequence of dual spaces (i.e. $\{\mathcal{P}^n(E')\}_n$ and $\{\mathcal{P}_I(E')\}_n$, respectively) is coherent and multiplicative.

Remark 2.4. Now we use results from [15] on minimal, maximal and dual (or adjoint) polynomial ideals to show how to obtain other examples in which the duality relation $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ holds.

Suppose $\mathfrak{A}_k$ is a minimal ideal and let α_k be its associated k-symmetric tensor norm. If E' has the bounded approximation property, then $\mathfrak{A}_k(E)$ identifies isometrically with $\bigotimes_{s,\alpha_k}^k E'$ and then $\mathfrak{A}_k(E)'$ is isometrically isomorphic to $\mathfrak{B}_k(E') = \mathfrak{A}_k^{\mathrm{dual}}(E')$ via the Borel transform B_k, see [15, Corollary 4.3].

On the other hand, if we start with a maximal ideal $\mathfrak{B}_k$, let $\mathfrak{A}_k = (\mathfrak{B}_k^{\mathrm{dual}})^{\min}$. Again, if E' has the bounded approximation property, the Borel transform B_k is an isometric isomorphism between $\mathfrak{A}_k(E)'$ and $\mathfrak{B}_k(E')$.

The following proposition states that if the duals of $\mathfrak{A}_k(E)$ form a coherent sequence of spaces of polynomials, then $\{\mathfrak{A}_k(E)\}_k$ inherits the coherence.

Proposition 2.5. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence with constants C and D, and suppose $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. Then, $\{\mathfrak{A}_k(E)\}_k$ is a coherent sequence with constants D and C.*

Proof. First, observe that if $\xi \in E'$ and $a \in E$, then $(\xi^{k+1})_a = \xi(a)\xi^k$. Thus, for every $\psi \in \mathfrak{B}_k(E')$,

$$B_k^{-1}(\psi)\big((\xi^{k+1})_a\big) = \xi(a)B_k^{-1}(\psi)\big(\xi^k\big) = \xi(a)\psi(\xi) = (a\psi)(\xi) = B_{k+1}^{-1}(a\psi)\big(\xi^{k+1}\big).$$

This implies that, for every $P \in \mathcal{P}_f^{k+1}(E)$, $B_k^{-1}(\psi)(P_a) = B_{k+1}^{-1}(a\psi)(P)$ and

$$\|P_a\|_{\mathfrak{A}_k(E)} = \sup_{\|\psi\|_{\mathfrak{B}_k(E')}=1} \big|B_{k+1}^{-1}(a\psi)(P)\big| \leqslant D\|a\|\|P\|_{\mathfrak{A}_{k+1}(E)}.$$

By the density result in Lemma 2.1, we obtain that for every $P \in \mathfrak{A}_{k+1}(E)$ and every $a \in E$, P_a belongs to $\mathfrak{A}_k(E)$ and $\|P_a\|_{\mathfrak{A}_k(E)} \leqslant D\|a\|\|P\|_{\mathfrak{A}_{k+1}(E)}$.

To prove the second condition of coherence, note that if γ and ξ are in E' and $\psi \in \mathfrak{B}_{k+1}(E')$ we have, by the polarization formula,

$$\begin{aligned}
&B_{k+1}^{-1}(\psi)\big(\gamma\xi^k\big) \\
&\quad = \frac{1}{2^{k+1}(k+1)!} \sum_{\varepsilon_1,\ldots,\varepsilon_{k+1}=\pm 1} \varepsilon_1\cdots\varepsilon_{k+1} B_{k+1}^{-1}(\psi)\big(\big(\varepsilon_1\gamma + (\varepsilon_2+\cdots+\varepsilon_{k+1})\xi\big)^{k+1}\big) \\
&\quad = \frac{1}{2^{k+1}(k+1)!} \sum_{\varepsilon_1,\ldots,\varepsilon_{k+1}=\pm 1} \varepsilon_1\cdots\varepsilon_{k+1} \psi\big(\varepsilon_1\gamma + (\varepsilon_2+\cdots+\varepsilon_{k+1})\xi\big) \\
&\quad = \psi_\gamma(\xi) = B_k^{-1}(\psi_\gamma)\big(\xi^k\big).
\end{aligned}$$

This implies that if P is a finite type k-homogeneous polynomial on E, then $B_{k+1}^{-1}(\psi)(\gamma P) = B_k^{-1}(\psi_\gamma)(P)$. And thus, for every $P \in \mathcal{P}_f^k(E)$,

$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} = \sup_{\|\psi\|_{\mathfrak{B}_k(E')}=1} \big|B_k^{-1}(\psi_\gamma)(P)\big| \leqslant C\|\gamma\|\|P\|_{\mathfrak{A}_k(E)}.$$

Therefore, again by Lemma 2.1, for every $P \in \mathfrak{A}_k(E)$ the polynomial γP belongs to $\mathfrak{A}_{k+1}(E)$ and $\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} \leqslant C\|\gamma\|\|P\|_{\mathfrak{A}_k(E)}$. □

In order to study the dual of $H_{b\mathfrak{A}}(E)$, we need the following

Definition 2.6. Let $\mathfrak{B}(E) = \{\mathfrak{B}_k(E)\}_k$ be a coherent sequence. We define the holomorphic functions of $\mathfrak{B}$-exponential type on E,

$$\mathrm{Exp}_{\mathfrak{B}}(E) = \Big\{ f \in H(E)\colon\ d^k f(0) \in \mathfrak{B}_k(E) \text{ for all } k \text{ and } \limsup_{k\to\infty} \big\|d^k f(0)\big\|_{\mathfrak{B}_k}^{\frac{1}{k}} < \infty \Big\}.$$

A classical result of Gupta states that, for E' with the approximation property, the Borel transform defines a duality between the space of nuclearly entire functions of bounded type over E, $H_{Nb}(E)$, and the space of holomorphic mappings of exponential type on E', $\mathrm{Exp}(E')$ [14,20]. In an analogous way, we prove the following:

Proposition 2.7. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence and let $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. Then the Borel transform is a vector space isomorphism between $H_{b\mathfrak{A}}(E)'$ and $\mathrm{Exp}_{\mathfrak{B}}(E')$.*

Proof. Let $\varphi \in H_{b\mathfrak{A}}(E)'$. Since φ is continuous, there are constants $c, R > 0$ such that $|\varphi(g)| \leqslant c p_R(g)$, for every $g \in H_{b\mathfrak{A}}(E)$. In particular, for each k, if g belongs to $\mathfrak{A}_k(E)$, we get $|\varphi(g)| \leqslant cR^k \|g\|_{\mathfrak{A}_k(E)}$. Then $\|\varphi_{|\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \leqslant cR^k$, for every $k \geqslant 1$. Moreover, since $\frac{d^k\beta(\varphi)(0)}{k!}(\gamma) = \varphi_{|\mathfrak{A}_k(E)}(\frac{\gamma^k}{k!})$ we have that $d^k\beta(\varphi)(0) = B_k(\varphi_{|\mathfrak{A}_k(E)})$. Then $\|d^k\beta(\varphi)(0)\|^{\frac{1}{k}}_{\mathfrak{B}_k(E')} = \|\varphi_{|\mathfrak{A}_k(E)}\|^{\frac{1}{k}}_{\mathfrak{A}_k(E)'} \leqslant c^{\frac{1}{k}} R$. Therefore, $\beta(\varphi) \in \mathrm{Exp}_{\mathfrak{B}}(E')$.

The Borel transform β is injective as a consequence of Corollary 2.2. To see that it is also surjective, let $\psi \in \mathrm{Exp}_{\mathfrak{B}}(E')$ and $A = \sup_k \|d^k\psi(0)\|^{\frac{1}{k}}_{\mathfrak{B}_k(E')}$. For each $g \in H_{b\mathfrak{A}}(E)$, we define

$$\varphi(g) = \sum_{k=0}^{\infty} B_k^{-1}\big(d^k\psi(0)\big)\left(\frac{d^k g(0)}{k!}\right).$$

Since

$$\big|\varphi(g)\big| \leqslant \sum_{k=0}^{\infty} \big\|d^k\psi(0)\big\|_{\mathfrak{B}_k(E')} \left\|\frac{d^k g(0)}{k!}\right\|_{\mathfrak{A}_k(E)} \leqslant \sum_{k=0}^{\infty} A^k \left\|\frac{d^k g(0)}{k!}\right\|_{\mathfrak{A}_k(E)} = p_A(g),$$

we have $\varphi \in H_{b\mathfrak{A}}(E)'$. Finally, simple computations show that $\beta(\varphi) = \psi$. $\square$

3. Multiplication in $H_{b\mathfrak{A}}(E)'$

Suppose that $\mathfrak{B}(E) = \{\mathfrak{B}_k(E)\}_k$ is a coherent multiplicative sequence. Then, we can see that $\mathrm{Exp}_{\mathfrak{B}}(E)$ is an algebra. Indeed, if $f, g \in \mathrm{Exp}_{\mathfrak{B}}(E)$, with $A_1 = \sup_k \|d^k f(0)\|^{\frac{1}{k}}_{\mathfrak{B}_k(E)}$ and $A_2 = \sup_k \|d^k g(0)\|^{\frac{1}{k}}_{\mathfrak{B}_k(E)}$, we have $d^k(fg)(0) \in \mathfrak{B}_k(E)$ and $\|d^k(fg)(0)\|^{\frac{1}{k}}_{\mathfrak{B}_k(E)} \leqslant M(A_1 + A_2)$, where M is the multiplicative constant of the sequence $\mathfrak{B}(E)$ (see the proof of Lemma 1.5 for some details).

This fact allows us to introduce a multiplication on $H_{b\mathfrak{A}}(E)'$, just copying the product in $\mathrm{Exp}_{\mathfrak{B}}(E)$ via the Borel transform:

Definition 3.1. Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and let $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. For $\varphi, \psi \in H_{b\mathfrak{A}}(E)'$ we define the product $\odot$ in $H_{b\mathfrak{A}}(E)'$, by

$$\varphi \odot \psi = \beta^{-1}\big(\beta(\varphi)\beta(\psi)\big).$$

We would also like to define the "natural" convolution product in $H_{b\mathfrak{A}}(E)'$: if $\varphi, \psi \in H_{b\mathfrak{A}}(E)'$, then $\varphi * \psi \in H_{b\mathfrak{A}}(E)'$ is the linear functional given by

$$\varphi * \psi(f) = \psi\big(x \mapsto \varphi(\tau_x f)\big).$$

We prove that this convolution product is well defined in a few steps. First we define:

Definition 3.2. Let $\varphi \in H_{b\mathfrak{A}}(E)'$ and let $f \in H_{b\mathfrak{A}}(E)$. We define $\varphi * f : E \to \mathbb{C}$ as follows,

$$\varphi * f(x) = \varphi(\tau_x f) = \varphi\big(f(x + \cdot)\big).$$

Therefore, the desired convolution product can be rewritten as $\varphi * \psi(f) = \psi(\varphi * f)$. For this to be well defined, we must show that $\varphi * f \in H_{b\mathfrak{A}}(E)$ and that $f \to \varphi * f$ is continuous. This will be done in the following lemma and theorem.

Lemma 3.3. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and suppose $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Let $P \in \mathfrak{A}_k(E)$ and $\varphi \in \mathfrak{A}_{k-l}(E)'$, $k \geqslant l$. Then the l-homogeneous polynomial $x \mapsto \varphi(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and*

$$\left\| x \mapsto \varphi(P_{x^l}) \right\|_{\mathfrak{A}_l(E)} \leqslant M^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Proof. If P is a finite type polynomial, then $x \mapsto \varphi(P_{x^l})$ is also a finite type polynomial and thus belongs to $\mathfrak{A}_l(E)$. We can therefore define a linear operator

$$T : \left(\mathcal{P}_f^k(E), \|\cdot\|_{\mathfrak{A}_k(E)}\right) \to \mathfrak{A}_l(E)$$

$$P = \sum_{j=1}^N \gamma_j^k \mapsto \left[x \mapsto \varphi(P_{x^l})\right].$$

If $P = \sum_{j=1}^N \gamma_j^k$ and $\psi \in \mathfrak{A}_l(E)'$, then

$$\psi\left(T(P)\right) = \sum_{j=1}^N \left(B_{k-l}(\varphi) B_l(\psi)\right)(\gamma_j) = B_k^{-1}\left(B_{k-l}(\varphi) B_l(\psi)\right)(P).$$

Then, for every $\psi \in \mathfrak{A}_l(E)'$,

$$\left|\psi\left(T(P)\right)\right| \leqslant M^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|\psi\|_{\mathfrak{A}_l(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Therefore, T is continuous and, by Lemma 2.1, can be extended to $\mathfrak{A}_k(E)$. By density, it easily follows that $T(P)(x) = \varphi(P_{x^l})$ for every $x \in E$ and every $P \in \mathfrak{A}_k(E)$. □

Theorem 3.4. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Let $f \in H_{b\mathfrak{A}}(E)$ and $\varphi \in H_{b\mathfrak{A}}(E)'$. Then $\varphi * f$ belongs to $H_{b\mathfrak{A}}(E)$ and the application*

$$T_\varphi : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$$

$$f \mapsto \varphi * f$$

is a continuous linear operator.

Proof. If $f = \sum_{k=0}^\infty P_k$, $P_k \in \mathfrak{A}_k(E)$, then

$$\varphi * f(x) = \varphi(\tau_x f) = \varphi\left(\sum_{k=0}^\infty \sum_{l=0}^k \binom{k}{l} (P_k)_{x^l}\right) = \sum_{l=0}^\infty \sum_{k=l}^\infty \binom{k}{l} \varphi\left((P_k)_{x^l}\right).$$

From Lemma 3.3, the polynomial $x \mapsto \varphi((P_k)_{x^l})$ belongs to $\mathfrak{A}_l(E)$. Also, proceeding as in the beginning of the proof of Proposition 2.7, we have that there exist constants $c, R > 0$ such that $\|\varphi|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \leqslant cR^k$. Let $\varepsilon > 0$. Since $f \in H_{b\mathfrak{A}}(E)$, there exists $c_\varepsilon > 0$ such that $\|P_k\|_{\mathfrak{A}_k(E)} \leqslant c_\varepsilon \varepsilon^k$ for all k. Then $\frac{d^l(\varphi * f)(0)}{l!}$ belongs to $\mathfrak{A}_l(E)$ because $\frac{d^l(\varphi * f)(0)}{l!}(x) = \sum_{k=l}^\infty \binom{k}{l} \varphi((P_k)_{x^l})$ and, by Lemma 3.3,

$$\begin{aligned}\sum_{k=l}^{\infty}\binom{k}{l}\|x\mapsto\varphi(P_{x^l})\|_{\mathfrak{A}_l(E)} &\leqslant \sum_{k=l}^{\infty}\binom{k}{l}M^k\|\varphi|_{\mathfrak{A}_{k-l}(E)}\|_{\mathfrak{A}_{k-l}(E)'}\|P_k\|_{\mathfrak{A}_k(E)}\\ &\leqslant c\sum_{k=l}^{\infty}\binom{k}{l}M^kR^{k-l}\|P_k\|_{\mathfrak{A}_k(E)}\\ &\leqslant cc_\varepsilon(M\varepsilon)^l\sum_{k=l}^{\infty}\binom{k}{l}(MR\varepsilon)^{k-l}\\ &=\frac{cc_\varepsilon(M\varepsilon)^l}{(1-MR\varepsilon)^{l+1}}<\infty,\end{aligned}$$

for arbitrary small $\varepsilon>0$. Moreover, since

$$\left\|\frac{d^l(\varphi*f)(0)}{l!}\right\|_{\mathfrak{A}_l(E)}^{\frac{1}{l}}\leqslant\frac{(cc_\varepsilon)^{\frac{1}{l}}M\varepsilon}{(1-MR\varepsilon)^{\frac{l+1}{l}}}\xrightarrow[l\to\infty]{}\frac{M\varepsilon}{1-MR\varepsilon},$$

for every ε sufficiently small, it follows that $\varphi*f\in H_{b\mathfrak{A}}(E)$. Finally, the application T_φ is continuous because

$$\begin{aligned}p_r(\varphi*f)&=\sum_{l=0}^{\infty}\left\|\frac{d^l(\varphi*f)(0)}{l!}\right\|_{\mathfrak{A}_l(E)}r^l\leqslant c\sum_{l=0}^{\infty}r^l\sum_{k=l}^{\infty}\binom{k}{l}M^kR^{k-l}\|P_k\|_{\mathfrak{A}_k(E)}\\ &=c\sum_{k=0}^{\infty}\|P_k\|_{\mathfrak{A}_k(E)}M^k\sum_{l=0}^{k}\binom{k}{l}r^lR^{k-l}=c\sum_{k=0}^{\infty}\|P_k\|_{\mathfrak{A}_k(E)}M^k(R+r)^k\\ &=cp_{M(R+r)}(f).\qquad\square\end{aligned}$$

We can now define the following:

Definition 3.5. For $\varphi,\psi\in H_{b\mathfrak{A}}(E)'$, the *convolution product* $\varphi*\psi\in H_{b\mathfrak{A}}(E)'$ is defined by

$$\varphi*\psi(f)=\psi\big(x\mapsto\varphi(\tau_xf)\big)=\psi(\varphi*f),$$

for $f\in H_{b\mathfrak{A}}(E)$.

As a consequence of Theorem 3.4 we have

Corollary 3.6. *If $\varphi\in H_{b\mathfrak{A}}(E)'$, then the application*

$$\begin{aligned}M_\varphi: H_{b\mathfrak{A}}(E)'&\to H_{b\mathfrak{A}}(E)'\\ \psi&\mapsto\psi*\varphi\end{aligned}$$

is a continuous linear operator when we consider the strong dual topology on $H_{b\mathfrak{A}}(E)'$.

Proof. Just note that M_φ is the transpose of T_φ. $\square$

Next we show that the two products defined on $H_{b\mathfrak{A}}(E)'$ are actually the same.

Proposition 3.7. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and $\{\mathfrak{A}_k(E)\}_k$ such that $\mathfrak{A}_k(E)'=\mathfrak{B}_k(E')$ for every k. Let $\varphi,\psi\in H_{b\mathfrak{A}}(E)'$. Then $\varphi\odot\psi=\varphi*\psi$.*

Proof. Since finite type polynomials are dense in $H_{b\mathfrak{A}}(E)$, it is sufficient to verify that, for each $\gamma \in E'$ and $k \geqslant 0$, $\varphi \odot \psi(\gamma^k) = \varphi * \psi(\gamma^k)$.

For $g \in \mathrm{Exp}_{\mathfrak{B}}(E')$, $\beta^{-1}(g)(\gamma^k) = B_k^{-1}(d^k g(0))(\gamma^k) = d^k g(0)(\gamma)$. Then,

$$\begin{aligned}
\varphi \odot \psi(\gamma^k) &= \beta^{-1}\big(\beta(\varphi)\beta(\psi)\big)(\gamma^k) = d^k\big(\beta(\varphi)\beta(\psi)\big)(0)(\gamma) \\
&= \sum_{j=0}^{k} \binom{k}{j} d^j\big(\beta(\varphi)\big)(0)(\gamma) d^{k-j}\big(\beta(\psi)\big)(0)(\gamma) \\
&= \sum_{j=0}^{k} \binom{k}{j} \varphi(\gamma^j)\psi(\gamma^{k-j}).
\end{aligned}$$

On the other hand, since $(\varphi * \gamma^k)(x) = \varphi(\tau_x \gamma^k) = \sum_{j=0}^{k} \binom{k}{j}\varphi(\gamma^j)\gamma(x)^{k-j}$, we obtain

$$\varphi * \psi(\gamma^k) = \psi(\varphi * \gamma^k) = \sum_{j=0}^{k} \binom{k}{j} \varphi(\gamma^j)\psi(\gamma^{k-j}) = \varphi \odot \psi(\gamma^k). \qquad \square$$

As an immediate consequence, we have:

Corollary 3.8. *The convolution product in $H_{b\mathfrak{A}}(E)'$ is commutative.*

4. Convolution operators and hypercyclicity

As usual, by a convolution operator T, we mean a continuous operator that commutes with translations τ_x, i.e., $T \circ \tau_x = \tau_x \circ T$. Next we characterize the convolution operators on $H_{b\mathfrak{A}}(E)$.

Proposition 4.1. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence and $\{\mathfrak{A}_k(E)\}_k$ such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then for every convolution operator $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ there exists a unique functional $\varphi \in H_{b\mathfrak{A}}(E)'$ such that*

$$T(f) = \varphi * f,$$

for every $f \in H_{b\mathfrak{A}}(E)$. If, in addition, $\{\mathfrak{B}_k(E')\}_k$ is multiplicative, then for every $\varphi \in H_{b\mathfrak{A}}(E)'$, T_φ is a convolution operator on $H_{b\mathfrak{A}}(E)$.

Proof. Let $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ be a convolution operator and let $\varphi = \delta_0 \circ T$, i.e. $\varphi(f) = T(f)(0)$ for $f \in H_{b\mathfrak{A}}(E)$. Then $\varphi \in H_{b\mathfrak{A}}(E)'$ and

$$T(f)(x) = \big[\tau_x T(f)\big](0) = T(\tau_x f)(0) = \varphi(\tau_x f) = \varphi * f(x),$$

for every $f \in H_{b\mathfrak{A}}(E)$ and $x \in E$. The uniqueness of φ follows from the identity

$$T(e^\gamma) = \varphi * e^\gamma = \big[x \mapsto \varphi(\tau_x e^\gamma)\big] = \varphi(e^\gamma)e^\gamma = \beta(\varphi)(\gamma)e^\gamma \tag{1}$$

and the fact that β is one to one.

The second assertion is immediate by Theorem 3.4. $\square$

Proposition 4.1 and the results in the previous sections allow us to establish the following:

Corollary 4.2. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and let $\{\mathfrak{A}_k(E')\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then $\psi \mapsto T_{\beta^{-1}(\psi)}$ is an algebra isomorphism from $\mathrm{Exp}_{\mathfrak{B}}(E')$ onto the algebra of convolution operators on $H_{b\mathfrak{A}}(E)$.*

Now we are ready to prove the announced result about hypercyclicity of convolution operators. We follow the steps of the proof of [26, Theorem 3.1].

Theorem 4.3. *Suppose that E' is separable. Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence and $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then, every convolution operator $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ which is not a scalar multiple of the identity is hypercyclic.*

Proof. Let $\varphi \in H_{b\mathfrak{A}}(E)'$ be the linear functional given in Proposition 4.1, which satisfies $T(f) = \varphi * f$. Since T is not a scalar multiple of the identity, it follows that φ is not a scalar multiple of δ_0.

Since E' is separable, by Corollary 2.2, $H_{b\mathfrak{A}}(E)$ is separable. Therefore, we can use the Hypercyclicity Criterion [17,22].

First, note that $\mathrm{span}\{e^\gamma\colon \gamma \in U\}$ is dense in $H_{b\mathfrak{A}}(E)$ for any nonempty open set $U \subset E'$. Indeed, if $\psi \in H_{b\mathfrak{A}}(E)'$ and $\psi(e^\gamma) = 0$ for every $\gamma \in U$, then $\beta(\psi) \equiv 0$ in U and we have $\beta(\psi) = 0$. This means that ψ is 0.

Also, the fact that φ is not a scalar multiple of δ_0 implies that $\beta(\varphi)$ is not a constant function. Indeed, if $\beta(\varphi)$ was constant, then $\lambda = \varphi(1) = \beta(\varphi)(0) = \beta(\varphi)(\gamma) = \varphi(e^\gamma)$ for all $\gamma \in E'$. But, on the other hand, $\lambda = \lambda\delta_0(e^\gamma)$ for all $\gamma \in E'$ and we would have that $\varphi = \lambda\delta_0$.

We will now prove that T satisfies the Hypercyclicity Criterion. Let

$$V = \bigl\{\gamma \in E'\colon \bigl|\beta(\varphi)(\gamma)\bigr| < 1\bigr\} \quad \text{and} \quad W = \bigl\{\gamma \in E'\colon \bigl|\beta(\varphi)(\gamma)\bigr| > 1\bigr\}.$$

Then $V, W \subset E'$ are open sets, and they are nonempty. Indeed, if $W = \emptyset$ ($V = \emptyset$), then $\beta(\varphi)$ ($\frac{1}{\beta(\varphi)}$) would be a nonconstant bounded entire function. Let

$$H_V(E) = \mathrm{span}\bigl\{e^\gamma,\ \gamma \in V\bigr\} \quad \text{and} \quad H_W(E) = \mathrm{span}\bigl\{e^\gamma,\ \gamma \in W\bigr\}.$$

As we have observed, $H_V(E)$ and $H_W(E)$ are dense in $H_{b\mathfrak{A}}(E)$.

For $\gamma \in V$, $T(e^\gamma) = \beta(\varphi)(\gamma)e^\gamma$ (see (1)). Then $T(H_V(E)) \subset H_V(E)$. Also, $T^n(e^\gamma) = \beta(\varphi)(\gamma)^n e^\gamma$, and since $|\beta(\varphi)(\gamma)| < 1$ for $\gamma \in V$, we obtain that $T^n(f) \xrightarrow[n\to\infty]{} 0$, for every $f \in H_V(E)$.

For $\gamma \in W$, let $S(e^\gamma) = \frac{e^\gamma}{\beta(\varphi)(\gamma)}$. Since $\{e^\gamma,\ \gamma \in W\}$ is linearly independent (see the proof of [1, Lemma 2.3]), we can linearly extend S to $H_W(E)$. Then $S(H_W(E)) \subset H_W(E)$ and $S^n(e^\gamma) = \frac{e^\gamma}{\beta(\varphi)(\gamma)^n}$. Thus $S^n(f) \xrightarrow[n\to\infty]{} 0$, for every $f \in H_W(E)$, since $|\beta(\varphi)(\gamma)| > 1$ for $\gamma \in W$.

Finally, $TS(e^\gamma) = T(\frac{e^\gamma}{\beta(\varphi)(\gamma)}) = e^\gamma$ and therefore $TSf = f$ for all $f \in H_W(E)$.

By the Hypercyclicity Criterion, T is hypercyclic. □

By Theorem 3.4, if $\mathfrak{B}(E')$ is a coherent multiplicative sequence, each $\varphi \in H_{b\mathfrak{A}}(E)'$ defines a convolution operator $f \mapsto \varphi * f$. As mentioned in the previous proof, this operator is not a scalar multiple of the identity unless φ is a scalar multiple of δ_0. Therefore, we have:

Corollary 4.4. *Suppose that E' is separable. Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and $\{\mathfrak{A}_k(E)\}_k$ such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. For every $\varphi \in H_{b\mathfrak{A}}(E)'$ which is not a scalar multiple of δ_0, the operator*

$$\begin{aligned} T_\varphi : H_{b\mathfrak{A}}(E) &\to H_{b\mathfrak{A}}(E) \\ f &\mapsto T_\varphi(f) = \varphi * f \end{aligned}$$

is hypercyclic.

Now we apply the previous results to different spaces of holomorphic functions.

Example 4.5. In [1] the authors study differentiation operators in $H_{bc}(E)$, the space of holomorphic functions of compact bounded type on E (that is: $f = \sum P_n \in H_{bc}(E)$ whenever each P_n is an approximable n-homogeneous polynomial and $\|P_n\|^{\frac{1}{n}} \to 0$, where $\|\cdot\|$ denotes the usual norm). They show that if the differentiation operator is constructed from an entire function of exponential type on $\mathbb{C}$, then it is hypercyclic. By Examples 2.3, this result is a particular case of Corollary 4.4. Indeed, every such differentiation operator in $H_{b\mathfrak{A}}(E)$ is a convolution operator: if $\Phi(z) = \sum c_n z^n$ is an exponential type function and $a \in E$, we define $h(\gamma) = \sum c_n \gamma(a)^n$, then $h \in \mathrm{Exp}_{\mathfrak{B}}(E')$ and $\beta^{-1}(h)(f) = \sum c_n d^n f(0)(a)$ for all $f \in H_{b\mathfrak{A}}(E)$. Therefore,

$$\beta^{-1}(h) * f(x) = h(\tau_x f) = \sum c_n \big(d^n \tau_x f\big)(0)(a) = \sum c_n d^n f(x)(a).$$

Example 4.6. Consider the space $H_{Nb}(E)$ of nuclearly entire functions of bounded type. If E' is separable with the approximation property, Examples 2.3 and Proposition 4.1 assert that the convolution operators on $H_{Nb}(E)$ are precisely the operators T_φ, $\varphi \in H_{Nb}(E)'$. Such an operator is hypercyclic whenever it is not a scalar multiple of the identity by Theorem 4.3. This last statement answers a question of Aron and Markose in [2]. For E a dual Banach space and a slightly different definition of nuclear polynomials, Petersson obtained a stronger version of this result [27].

Example 4.7. Let $\{\mathfrak{A}_k\}_k$ be a sequence of minimal ideals. If E' has the bounded approximation property, $\mathfrak{A}_k(E)'$ can, by Remark 2.4, be identified with $\mathfrak{B}_k(E') = \mathfrak{A}_k^{\mathrm{dual}}(E')$. Therefore, if E' is also separable and $\{\mathfrak{B}_k(E')\}_k$ is coherent and multiplicative, the convolution operators on $H_{b\mathfrak{A}}(E)$ are those of the form T_φ, $\varphi \in H_{b\mathfrak{A}}(E)'$. They are hypercyclic if they are not scalar multiples of the identity.

For example, we can take $\mathfrak{A}_k$ to be the minimal ideal associated to the tensor norm η_k [7,21]. In this case, $\{\mathfrak{B}_k\}_k$ is the coherent multiplicative sequence of extendible polynomials.

In the next section, we present other examples in which the hypotheses of Theorem 4.3 and Corollary 4.4 hold. Namely, we consider the holomorphic functions of bounded type associated to the Schatten–von Neumann polynomials in the sense of Cobos, Kühn and Peetre [11].

5. Holomorphic Schatten–von Neumann functions of bounded type

Suppose $\mathcal{H}$ is a separable Hilbert space. Let us first recall the definition of Hilbert–Schmidt n-homogeneous polynomials on $\mathcal{H}$, which will be denoted $\mathcal{S}_2^n(\mathcal{H})$. For finite type polynomials it is possible to define an inner product in the following way: if $y, z \in \mathcal{H}$, $\langle\langle\cdot, y\rangle^n, \langle\cdot, z\rangle^n\rangle = \langle z, y\rangle^n$, then $\mathcal{S}_2^n(\mathcal{H})$ is the completion of the space of finite type polynomials $\mathcal{P}_f^n(\mathcal{H})$ with this inner product.

Note that if $\{e_i\}_i$ is an orthonormal basis of $\mathcal{H}$ and $P, Q \in \mathcal{S}_2^n(\mathcal{H})$, then

$$\langle P, Q\rangle = \sum_{i_1,\dots,i_n=1}^{\infty} \check{P}(e_{i_1},\dots,e_{i_n})\overline{\check{Q}(e_{i_1},\dots,e_{i_n})}.$$

Also note that the Borel transform is an isometric isomorphism between $(\mathcal{S}_2^n(\mathcal{H}))'$ and $\mathcal{S}_2^n(\mathcal{H}')$.

Cobos, Kühn and Peetre in [11] define Schatten–von Neumann classes of multilinear functionals on $\mathcal{H}$. We adapt their definition to homogeneous polynomials on $\mathcal{H}$. To this end, throughout this section we will denote by $\mathcal{S}_1^n(\mathcal{H})$ and $\mathcal{S}_\infty^n(\mathcal{H})$ the spaces of n-homogeneous nuclear and approximable polynomials on $\mathcal{H}$, respectively. Since $\mathcal{H}'$ has the approximation property and the Radon–Nikodym property, the Borel transform is an isometric isomorphism between $(\mathcal{S}_1^n(\mathcal{H}))'$ and $\mathcal{P}^n(\mathcal{H}')$, and also between $(\mathcal{S}_\infty^n(\mathcal{H}))'$ and $\mathcal{S}_1^n(\mathcal{H}')$.

We use the complex interpolation method [3,6] to define Schatten polynomials. Following [11], we define:

Definition 5.1. The Schatten–von Neumann p-class of n-homogeneous polynomials on $\mathcal{H}$ is defined as

$$\mathcal{S}_p^n(\mathcal{H}) = \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})\big]_\theta,$$

with $\frac{1}{p} = 1-\theta$ and $0<\theta<1$. Here, $[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\theta$ denotes the space obtained by complex interpolation from the pair $(\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H}))$, with parameter θ.

The following result, which is the polynomial version of [11, Theorem 3.1] and can be proved analogously, shows that this definition is consistent with the definition of $\mathcal{S}_2^n(\mathcal{H})$.

Proposition 5.2. *We have the following isometric isomorphisms*:

$$\big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{P}^n(\mathcal{H})\big]_{\frac{1}{2}} \overset{1}{=} \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})\big]_{\frac{1}{2}} \overset{1}{=} \mathcal{S}_2^n(\mathcal{H}).$$

For the real method of interpolation the previous proposition holds with equivalent norms (see [11]).

In the proof of [11, Theorem 4.5], the reflexivity of $\mathcal{S}_p^n(\mathcal{H})$ is proven. In fact, this can be seen as a consequence of the following result:

Proposition 5.3. *If $1<p,q<\infty$ are such that $\frac{1}{p}+\frac{1}{q}=1$, then the Borel transform is an isometric isomorphism between $(\mathcal{S}_p^n(\mathcal{H}))'$ and $\mathcal{S}_q^n(\mathcal{H}')$.*

Proof. We know that the statement holds when $p=2$. Next, assume $1<p<2$.

By the Reiteration Theorem [3, 4.6.1] for the complex method,

$$\mathcal{S}_p^n(\mathcal{H}) = \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})\big]_\theta = \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})\big]_\eta,$$

where $\theta = \frac{\eta}{2}$. Then $\mathcal{S}_p^n(\mathcal{H}) = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})]_{2\theta}$.

In the following two cases, the Borel transform is an isomorphism

$$B_n : \big(\mathcal{S}_1^n(\mathcal{H})\big)' \to \mathcal{P}^n(\mathcal{H}') \quad \text{and} \quad B_n : \big(\mathcal{S}_2^n(\mathcal{H})\big)' \to \mathcal{S}_2^n(\mathcal{H}').$$

Then $B_n : [(\mathcal{S}_1^n(\mathcal{H}))', (\mathcal{S}_2^n(\mathcal{H}))']_{2\theta} \to [\mathcal{P}^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')]_{2\theta}$ is an isomorphism.

Since $\mathcal{S}_2^n(\mathcal{H})$ is reflexive, and by a duality theorem [3, Corollary 4.5.2], we have

$$\big[\big(\mathcal{S}_1^n(\mathcal{H})\big)', \big(\mathcal{S}_2^n(\mathcal{H})\big)'\big]_{2\theta} = \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})\big]'_{2\theta} = \big(\mathcal{S}_p^n(\mathcal{H})\big)'.$$

On the other hand,

$$\big[\mathcal{P}^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')\big]_{2\theta} = \big[\mathcal{S}_\infty^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')\big]_{2\theta} = \big[\mathcal{S}_1^n(\mathcal{H}'), \mathcal{S}_\infty^n(\mathcal{H}')\big]_\nu,$$

with $\nu = \frac{1}{2}2\theta + (1-2\theta) = 1-\theta$ (the first equality follows from [3, Theorem 4.2.2] and the last one from the Reiteration Theorem). Therefore, $B_n : (\mathcal{S}_p^n(\mathcal{H}))' \to [\mathcal{S}_1^n(\mathcal{H}'), \mathcal{S}_\infty^n(\mathcal{H}')]_\nu = \mathcal{S}_q^n(\mathcal{H}')$ is an isomorphism, with $\frac{1}{q} = 1-\nu = \theta$, that is, $\frac{1}{q} = 1 - \frac{1}{p}$.

For the case $2 < p < \infty$, we have

$$\mathcal{S}_p^n(\mathcal{H}) = \big[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})\big]_\theta = \big[\mathcal{S}_2^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})\big]_\eta,$$

where $\eta = 2\theta - 1$. We proceed analogously to obtain the desired result. □

Corollary 5.4. *For $1 < p < \infty$, the Schatten–von Neumann classes $\mathcal{S}_p^n(\mathcal{H})$ are reflexive.*

In [10] it is shown that interpolation of coherent sequences is coherent. We will now show that interpolation of multiplicative coherent sequences is multiplicative.

Proposition 5.5. *Let E be Banach space and let $\{\mathfrak{A}_k^0(E)\}_k$, $\{\mathfrak{A}_k^1(E)\}_k$ be coherent multiplicative sequences (with constants M_0 and M_1, respectively). Then, the sequence $\{\mathfrak{A}_k^\theta(E)\}_k$ is multiplicative, where $\mathfrak{A}_k^\theta(E) = [\mathfrak{A}_k^0(E), \mathfrak{A}_k^1(E)]_\theta$, for every $0 < \theta < 1$ (with constant $M_0^{1-\theta} M_1^\theta$).*

Proof. Since $\{\mathfrak{A}_k^j(E)\}_k$ are multiplicative, for $j = 0, 1$, we can define a continuous bilinear mapping

$$\begin{aligned} \Phi_{k,l}^j : \mathfrak{A}_k^j(E) \times \mathfrak{A}_l^j(E) &\to \mathfrak{A}_{k+l}^j(E) \\ (P, Q) &\mapsto PQ. \end{aligned}$$

It follows that $\|\Phi_{k,l}^j\| \leqslant M_j^{k+l}$. Then, by the Multilinear Interpolation Theorem [3, Theorem 4.4.1], $(P, Q) \mapsto PQ$ defines a mapping

$$\Phi_{k,l}^\theta : \mathfrak{A}_k^\theta(E) \times \mathfrak{A}_l^\theta(E) \to \mathfrak{A}_{k+l}^\theta(E),$$

which is continuous and has norm less than or equal to $(M_0^{1-\theta} M_1^\theta)^{k+l}$. That is, if $P \in \mathfrak{A}_k^\theta(E)$, $Q \in \mathfrak{A}_l^\theta(E)$, then $PQ \in \mathfrak{A}_{k+l}^\theta(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}^\theta(E)} \leqslant \big(M_0^{1-\theta} M_1^\theta\big)^{k+l} \|P\|_{\mathfrak{A}_k^\theta(E)} \|Q\|_{\mathfrak{A}_l^\theta(E)}. \qquad \square$$

From Propositions 5.3 and 5.5 we have

Corollary 5.6. *For every $1 \leqslant p \leqslant \infty$, the sequence of Schatten–von Neumann p-classes of homogeneous polynomials $\{\mathcal{S}_p^k(\mathcal{H})\}_k$ is multiplicative.*

Let $H_{b\mathcal{S}_p}(\mathcal{H})$ be the holomorphic functions of bounded type of the Schatten–von Neumann p-class on $\mathcal{H}$, then by Theorems 3.4 and 4.3, we have

Corollary 5.7. *Let $\mathcal{H}$ be separable Hilbert space. For $1 \leqslant p \leqslant \infty$, every convolution operator on $H_{b\mathcal{S}_p}(\mathcal{H})$ which is not a scalar multiple of the identity is hypercyclic. Moreover, each $\varphi \in H_{b\mathcal{S}_p}(\mathcal{H})'$ defines a convolution operator T_φ.*

The case $p = 2$ of this result (Hilbert–Schmidt holomorphic functions of bounded type) was proved by Petersson in [26].

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References

[1] Richard Aron, Juan Bès, Hypercyclic differentiation operators, in: Function Spaces, Edwardsville, IL, 1998, in: Contemp. Math., vol. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 39–46.
[2] Richard Aron, Dinesh Markose, On universal functions, in: Satellite Conference on Infinite Dimensional Function Theory, J. Korean Math. Soc. 41 (1) (2004) 65–76.
[3] Jöran Bergh, Jörgen Löfström, Interpolation Spaces. An Introduction, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin, 1976.
[4] Christopher Boyd, Silvia Lasalle, personal communication.
[5] George D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris 189 (1929) 473–475.
[6] Alberto P. Calderón, Intermediate spaces and interpolation, the complex method, Stud. Math. 24 (1964) 113–190.
[7] Daniel Carando, Extendible polynomials on Banach spaces, J. Math. Anal. Appl. 233 (1) (1999) 359–372.
[8] Daniel Carando, Extendibility of polynomials and analytic functions on ℓ_p, Studia Math. 145 (1) (2001) 63–73.
[9] Daniel Carando, Verónica Dimant, Santiago Muro, Coherent sequences of polynomial ideals on Banach spaces, preprint.
[10] Daniel Carando, Verónica Dimant, Santiago Muro, Holomorphic functions and coherent polynomial ideals on Banach spaces, in preparation.
[11] Fernando Cobos, Thomas Kühn, Jaak Peetre, Schatten–von Neumann classes of multilinear forms, Duke Math. J. 65 (1) (1992) 121–156.
[12] Verónica Dimant, Pablo Galindo, Manuel Maestre, Ignacio Zalduendo, Integral holomorphic functions, Studia Math. 160 (1) (2004) 83–99.
[13] Seán Dineen, Holomorphy types on a Banach space, Studia Math. 39 (1971) 241–288.
[14] Seán Dineen, Complex analysis on infinite-dimensional spaces, Springer Monogr. Math., Springer-Verlag London, Ltd., London, 1999.
[15] Klaus Floret, Minimal ideals of n-homogeneous polynomials on Banach spaces, Results Math. 39 (3–4) (2001) 201–217.
[16] Klaus Floret, On ideals of n-homogeneous polynomials on Banach spaces, in: Proceedings of the Fest-Colloquium in honour of Professor A. Mallios, University of Athens, 2002, pp. 19–38.
[17] Robert M. Gethner, Joel H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. 100 (2) (1987) 281–288.
[18] Gilles Godefroy, Joel H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (2) (1991) 229–269.
[19] Karl-Goswin Grosse-Erdmann, Universal families and hypercyclic operators, Bull. Amer. Math. Soc. (N.S.) 36 (3) (1999) 345–381.
[20] Chaitan P. Gupta, On the Malgrange theorem for nuclearly entire functions of bounded type on a Banach space, Indag. Math. 32 (1970) 356–358.
[21] Pádraig Kirwan, Raymond A. Ryan, Extendibility of homogeneous polynomials on Banach spaces, Proc. Amer. Math. Soc. 126 (4) (1998) 1023–1029.
[22] Carol Kitai, Invariant closed sets for linear operators, Dissertation, Univ. Toronto, 1982.
[23] Gerald R. MacLane, Sequences of derivatives and normal families, J. Anal. Math. 2 (1952) 72–87.

[24] Jorge Mujica, Complex Analysis in Banach Spaces, Math. Stud., vol. 120, North-Holland, Amsterdam, 1986.
[25] Leopoldo Nachbin, Topology on Spaces of Holomorphic Mappings, Springer-Verlag, New York, 1969.
[26] Henrik Petersson, Hypercyclic convolution operators on entire functions of Hilbert–Schmidt holomorphy type, Ann. Math. Blaise Pascal 8 (2) (2001) 107–114.
[27] Henrik Petersson, Hypercyclic subspaces for Fréchet space operators, J. Math. Anal. Appl. 319 (2) (2006) 764–782.