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L^p-representable functions on Banach spaces

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Abstract

In this paper we discuss the problem of integral representation of analytic functions over a complex Banach space E. We define the class of L^p*-representable* and ρ*-representable* functions and prove that they verify some growth conditions. The aim of this work is to characterize these classes of functions. It is also shown that it is possible to give an alternative representation for integral polynomials over a Hilbert space H using a universal Wiener measure.

Keywords: Integral representation; Cauchy integral formula; Holomorphic functions; Gaussian measures

0. Introduction

This note continues recent work on generalizations of the Cauchy integral formula to infinite-dimensional Banach spaces. In [13], I. Zalduendo and the author prove that given a separable Banach space X, there is an integral representation formula for a wide class of holomorphic functions f over X':

$$f(z) = \int_X e^{z(\gamma)} \widetilde{f(\gamma)}\, dW(\gamma) \quad \text{for all } z \in X',$$

where $\tilde{f}$ is a transform of f and W is a Wiener measure on X. In [5], S. Dineen and L. Nilsson obtain the same formula for X a fully nuclear space with a basis.

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The aim of this work is to characterize the class of "representable functions" in the Banach space setting. In Section 1 we summarize without proofs the relevant material related to the integral formula: complex valued random variables, Gaussian measure, Gross' and Fernique's theorems. We also introduce the space of L^p*-representable* functions and prove some preliminary results on them regarding growth conditions and analyticity. Section 2 gives a brief exposition about Hilbertian polynomials on a Hilbert space H (see [12,14]), introduces the spaces $\mathcal{H}_{hb}(H) \subset \mathcal{H}^2(B) \subset \mathcal{H}_h(H)$, and provides a detailed exposition of L^2*-representable* functions. We prove that every L^2*-representable* function on X' has a holomorphic extension to H and that it is possible to characterize them using the Hardy-type space $\mathcal{H}^2(B)$ on the unit ball of H. From this, it is also shown that using the theory of abstract Wiener spaces, it is possible to give an alternative representation for integral polynomials over a separable Hilbert space. In Section 3 we discuss the general case for L^p $(p > 1)$ and define the space of ρ*-representable* functions. We also obtain necessary conditions for L^p*-representability*. In Section 4 we study the (proper) inclusions between the spaces of L^2, L^p and ρ*-representable* functions giving some relevant examples and counterexamples. Finally, in Section 5 we apply the integral formula to functions which are holomorphic on a Banach space E, even if E is not a dual space, by using the Aron–Berner extension.

1. The formula

We begin this section setting notation and terminology. We need to consider Gaussian measures on Banach spaces and will use the theory of abstract Wiener spaces (see [10,15]). For a deeper discussion of the material in this section we refer the reader to [13]. Let H be a separable complex Hilbert space with orthonormal basis (e_n), and denote its inner product by $\langle\, , \rangle$. As always, H' can be identified with H via $I : H' \to H$ such that for $x \in H$ and $\phi \in H'$, $\phi(x) = \langle x, I(\phi)\rangle$. The isomorphism I is a *conjugate linear isometry*. Our need for analyticity will lead us to correct the lack of linearity of I with involutions in H and H'. If $x = \sum x_n e_n$ is an element of H, we denote $x^* = \sum \overline{x_n} e_n$. Similarly, if $\phi \in H'$, define ϕ^* so that $I(\phi^*) = I(\phi)^*$, and note that $\phi(x^*) = \overline{\phi^*(x)}$. These involutions depend on the basis chosen in H.

A complex-valued Gaussian random variable (with mean 0 and variance σ^2) is one whose density function $f : \mathbb{C} \to \mathbb{R}$ is

$$f(w) = \frac{1}{\pi\sigma^2} e^{-\frac{|w|^2}{\sigma^2}}.$$

Its real and imaginary parts are independent real-valued Gaussian random variables with mean 0 and variance $\frac{\sigma^2}{2}$. We will denote by $\Gamma(\sigma^2)$ the Gaussian measure induced on $\mathbb{C}$ by the density function f.

If P is a finite-rank orthogonal projector in H, a cylinder set in H is a set of the form

$$A = \{x \in H\colon\ Px \in B\},$$

where B is a Borel subset of PH. The collection of such sets is a field, but not a σ-field. We will denote by Ω the Gaussian cylinder measure defined on cylinder sets:

$$\Omega(A) = \frac{1}{\pi^n} \int\limits_B e^{-|w|^2}\, dw,$$

where n is the complex dimension of PH, and the integral is with respect to Lebesgue measure. This cylinder measure is not σ-additive, however, integrals of cylinder functions ($F : H \to \mathbb{C}$ of the form $F = h \circ P$) may be defined by setting

$$\int_A F \, d\Omega = \int_B h \, dG,$$

where G is standard n-dimensional Gaussian measure. Note also that the involution $*$ is Ω-preserving.

Elements $\phi \in H'$ are complex-valued Gaussian random variables with mean 0 and variance $\|\phi\|^2$. Note that real linear forms α are real-valued random variables with mean 0 and variance $\frac{1}{2}\|\alpha\|^2$.

A norm $\|\cdot\|$ on H with the property that for any $\varepsilon > 0$ there is a finite-rank orthogonal projector P_ε such that for all $P \perp P_\varepsilon$,

$$\Omega\left\{x \in H\colon \ \|Px\| > \varepsilon\right\} < \varepsilon,$$

is called *measurable* [10]. Upon completing $(H, \|\cdot\|)$ one obtains a Banach space X and $\iota : H \hookrightarrow X$ is called an abstract Wiener space. The inclusion ι is continuous and dense. Given $x_1', \ldots, x_n' \in X'$ and a Borel set $B \subset \mathbb{C}^n$, one defines, if $C_X = \{x \in X\colon \ (x_1'(x), \ldots, x_n'(x)) \in B\}$, and $C_H = C_X \cap H$,

$$\widetilde{\Omega}(C_X) := \Omega(C_H).$$

$\widetilde{\Omega}$ is σ-additive, and extends to a measure W (called Wiener measure) on the Borel σ-algebra $\mathcal{B}$ of X. We shall use the following important theorems.

Theorem 1.1. *(See Gross [9].) If X is a separable Banach space, there is a Hilbert space H such that $\iota : H \hookrightarrow X$ is an abstract Wiener space. Furthermore: there is a smaller abstract Wiener space $H \hookrightarrow X_0 \hookrightarrow X$ and an increasing sequence of finite-rank orthogonal projectors (p_n) converging to the identity in H; these extend to P_n on X_0 where they converge to the identity as well. Also, $W(X_0) = 1$.*

Theorem 1.2. *(See Fernique [8].) There is an $\varepsilon > 0$ such that*

$$\int_X e^{\varepsilon \|x\|^2} \, dW(x) < \infty.$$

If $\iota' : X' \to H'$ is the transpose of the inclusion ι, we note $\iota^* = * \circ \iota'$. In the following commutative diagram:

$$\begin{tikzcd} H \arrow[r, "\iota"] & X \\ H' \arrow[u, "I"] & X' \arrow[l, "\iota^*"] \arrow[u, "A"'] \end{tikzcd}$$

the arrow on the right, $A = \iota \circ I \circ \iota^*$ is called the *covariance operator* of W. We can choose $(z_n) \subset X'$ such that $\iota^*(z_n) = e_n'$, the orthonormal basis of H' dual to (e_n). We will denote by T the (densely-defined and unbounded) inverse of the operator A.

Given a complex Banach space E, for $k \in \mathbb{N}$ we shall denote by $\mathcal{L}(^k E)$ the Banach space of all continuous k-linear mappings from $E^k = E \times \cdots \times E$ (k times) into $\mathbb{C}$, with respect to the pointwise vector operations and the norm defined by

$$\|A\| = \sup_{x_1 \neq 0, \ldots, x_k \neq 0} \frac{|A(x_1, \ldots, x_k)|}{\|x_1\| \cdots \|x_k\|}.$$

For $k = 0$ we define $\mathcal{L}(^0 E)$ to be the set of constant mappings from E into $\mathbb{C}$. A continuous k-homogeneous polynomial P from E into $\mathbb{C}$, is a mapping $P : E \to \mathbb{C}$ for which there exists $A \in \mathcal{L}(^k E)$ such that $P(x) = A(x, \ldots, x)$ for all $x \in E$. We let $\mathcal{P}(^k E)$ denote the Banach space of all continuous k-homogeneous polynomials from E into $\mathbb{C}$, with respect to the pointwise operations and the norm defined by

$$\|P\| = \sup_{x \neq 0} \frac{|P(x)|}{\|x\|^k}.$$

A power series from E to $\mathbb{C}$ about $x_0 \in E$, is a series in $x \in E$ of the form $S(x) = \sum_{k=0}^{\infty} P_k(x - x_0)$, where $P_k \in \mathcal{P}(^k E)$ for all $k \geqslant 0$. The radius of convergence of a power series about x_0, is the largest r, $0 \leqslant r \leqslant \infty$, such that the power series is uniformly convergent on every $B_\rho(x_0)$ for $0 \leqslant \rho < r$.

The radius of convergence of the power series is given by the Cauchy–Hadamard formula: $r = \frac{1}{\limsup \|P_k\|^{1/k}}$.

A mapping $f : E \to \mathbb{C}$ is holomorphic on E if, for every $x_0 \in E$, there exists a power series $\sum_{k=0}^{\infty} f_k(x - x_0)$ from E to $\mathbb{C}$ and some $\delta > 0$ such that $f(x) = \sum_{k=0}^{\infty} f_k(x - x_0)$ uniformly for $x \in B_\delta(x_0)$. This convergent power series is called the Taylor series of f at x_0, and the space of holomorphic functions from E to $\mathbb{C}$ is denoted by $\mathcal{H}(E)$.

Let $\mathcal{H}_b(E) = \{f \in \mathcal{H}(E)\colon \|f\|_\rho := \sup_{x \in B_\rho(0)} |f(x)| < \infty \text{ for all } 0 < \rho < \infty\}$. The space $\mathcal{H}_b(E)$, endowed with the topology generated by the semi-norms, $\|\cdot\|_\rho$, is a Fréchet space and functions in $\mathcal{H}_b(E)$ are called holomorphic functions of bounded type.

The following theorem provides an integral formula for a wide class of holomorphic functions and can be found in [13].

Theorem 1.3. *Let X be a separable Banach space and $\iota : H \hookrightarrow X$ the abstract Wiener space given by Gross' theorem. Denote by $\|\cdot\|_0$ the norm of X_0.*

Given $F : H \to \mathbb{C}$ holomorphic and $F^\sharp(x) = \overline{F(x^*)}$ such that

- (i) *$F^\sharp = \widetilde{F}^\sharp \circ \iota$, with $\widetilde{F}^\sharp : X \to \mathbb{C}$ some $L^p(W)$-integrable function $(p > 1)$ which is $\|.\|_0$-continuous on X_0.*
- (ii) *$\widetilde{F}^\sharp \circ P_n$ is almost surely bounded by some $g \in L^p(W)$.*

Set $f = F \circ I \circ \iota^* : X' \to C$, and $f^\sharp = F^\sharp \circ I \circ \iota^*$. Note that on the dense subspace Im A

$$f^\sharp \circ T = \widetilde{F}^\sharp \circ A \circ T = \widetilde{F}^\sharp.$$

If $f^\sharp \circ T$ denotes the class of $\widetilde{F}^\sharp$ in $L^p(W)$, then we have

$$f(z) = \int_X e^{z(\gamma)} \overline{\left(f^\sharp \circ T\right)}(\gamma)\, dW(\gamma) \quad \text{for all } z \in X'.$$

Definition 1. Let $p > 1$. A function $f \in \mathcal{H}(X')$ is called L^p*-representable* if

$$f(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma) \quad \text{with } g \in L^p(W).$$

The remainder of this section will be devoted to the proof of some properties of L^p-*representable* functions.

Lemma 1.4. *All continuous polynomials on X are in $L^p(W)$ for all $1 \leqslant p < \infty$.*

Proof. The inclusion follows from Fernique's theorem, because for any continuous k-homogeneous polynomial P,

$$\left|P(\gamma)\right|^p \leqslant \|P\|^p \|\gamma\|^{pk} \leqslant C\|P\|^p e^{\|\gamma\|} \in L^1(W). \qquad \square$$

The next lemma gives an estimation of the growth of an L^p-*representable* function.

Lemma 1.5. *Let f an L^p*-representable *function, q the conjugate of p, then*

$$\left|f(z)\right| \leqslant \|g\|_p e^{\frac{q}{4}\|\iota^*(z)\|^2}.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}
\left|f(z)\right| &\leqslant \|g\|_p \left\|e^{z(\gamma)}\right\|_q \leqslant \|g\|_p \left(\int_{\mathbb{C}} \left|e^w\right|^q \, d\Gamma\left(\left\|\iota^*(z)\right\|^2\right)(w)\right)^{\frac{1}{q}} \\
&= \|g\|_p \left(\int_{\mathbb{C}} e^{q \operatorname{Re}(w)} \, d\Gamma\left(\left\|\iota^*(z)\right\|^2\right)(w)\right)^{\frac{1}{q}} \\
&= \|g\|_p \left[\int_{\mathbb{R}} e^{qt} e^{-\frac{t^2}{\|\iota^*(z)\|^2}} \frac{dt}{\sqrt{\pi}\|\iota^*(z)\|}\right]^{\frac{1}{q}} = \|g\|_p e^{\frac{q}{4}\|\iota^*(z)\|^2}. \qquad \square
\end{aligned}$$

These results will be needed in Section 2.

Proposition 1.6. *Given $f(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma)$ for $g \in L^p(W)$, $p > 1$, then $f(z) = \sum_{k=0}^{\infty} f_k(z)$, where $f_k(z) = \int_X \frac{z(\gamma)^k}{k!} \overline{g(\gamma)}\, dW(\gamma)$, and $f \in \mathcal{H}_b(X')$.*

Proof. Consider $S_N(z,\gamma) = \sum_{k=0}^{N} \frac{1}{k!} z(\gamma)^k$, since $\lim_{N\to\infty} |e^{z(\gamma)} - S_N(z,\gamma)|^q = 0$ for all $z \in X'$, $\gamma \in X$ and we have that

$$\left|e^{z(\gamma)} - S_N(z,\gamma)\right|^q \leqslant \left|\sum_{k=N+1}^{\infty} \frac{1}{k!} z(\gamma)^k\right|^q \leqslant \left(\sum_{k=N+1}^{\infty} \frac{1}{k!} \|z\|^k \|\gamma\|^k\right)^q \leqslant e^{q\|z\|\|\gamma\|}$$

for all $q > 1$, according to Fernique's theorem, this last expression is integrable, consequently $S_N(z,\gamma) \to e^{z(\gamma)}$ in L^q and we conclude that

$$f(z)=\sum_{k=0}^{\infty}\int_X \frac{z(\gamma)^k}{k!}\overline{g(\gamma)}\,dW(\gamma).$$

Since $z(\cdot)$ is a complex-valued Gaussian random variable (mean 0 and variance $\|i^*z\|^2$) for all $z\in X'$, choosing $r\in\mathbb{N}$ such that $q\leqslant 2r$ we can compute

$$\begin{aligned}\big\|z(\cdot)^k\big\|_q &\leqslant \big\|z(\cdot)^k\big\|_{2r}=\bigg(\int_X \big|z(\gamma)^k\big|^{2r}\,dW(\gamma)\bigg)^{1/2r}\\ &=\bigg(\int_{\mathbb{C}}|w|^{2rk}\,d\Gamma\big(\big\|i^*z\big\|^2\big)\bigg)^{1/2r}=\bigg(\int_{\mathbb{R}^2}\big(x^2+y^2\big)^{rk}e^{-\frac{x^2+y^2}{\|i^*z\|^2}}\frac{dx\,dy}{\pi\|i^*z\|^2}\bigg)^{1/2r}\\ &=\bigg(\int_0^{+\infty}\rho^{2rk}e^{-\frac{\rho^2}{\|i^*z\|^2}}\frac{2\rho}{\|i^*z\|^2}\,d\rho\bigg)^{1/2r}=\sqrt[2r]{(kr)!}\,\big\|i^*z\big\|^k,\end{aligned}$$

then

$$\big|f_k(z)\big|\leqslant\frac{\|z(\gamma)^k\|_q}{k!}\|g\|_p\leqslant\frac{\sqrt[2r]{(kr)!}}{k!}\big\|i^*z\big\|^k\|g\|_p\leqslant\frac{\sqrt[2r]{(kr)!}}{k!}\|z\|^k\|g\|_p,$$

hence

$$\|f_k\|\leqslant\frac{\sqrt[2r]{(kr)!}}{k!}\|g\|_p.$$

From this we conclude that $\limsup\sqrt[k]{\|f_k\|}=0$ and according to the Cauchy–Hadamard formula f is bounded on bounded sets so $f\in\mathcal{H}_b(X')$. $\square$

This result follows immediately from Proposition 1.6. However, the following alternative proof will be useful.

Theorem 1.7. *If f is an L^p-*representable *function, then there exists $F\in\mathcal{H}_b(H)$ satisfying $F\circ I\circ\iota^*(z)=f(z)$. Moreover, this extension is unique.*

Proof. There are unique k-homogeneous polynomials f_k such that $f(z)=\sum_{k\geqslant 0}f_k(z)$, and the f_k may be expressed (see [4])

$$f_k(z)=\frac{1}{2\pi i}\int_{|\lambda|=r}\frac{f(\lambda z)}{\lambda^{k+1}}\,d\lambda.$$

Thus for all $r>0$, using Lemma 1.5 we have

$$\begin{aligned}\big|f_k(z)\big|&\leqslant\frac{1}{2\pi}\int_{|\lambda|=r}\frac{|f(\lambda z)|}{r^{k+1}}|d\lambda|\leqslant\frac{1}{2\pi}\int_{|\lambda|=r}\frac{Ce^{\frac{q}{4}\|\iota^*(\lambda z)\|^2}}{r^{k+1}}|d\lambda|\\ &\leqslant\frac{C}{2\pi}\int_0^{2\pi}\frac{e^{\frac{q}{4}r^2\|\iota^*z\|^2}}{r^{k+1}}r\,dt=\frac{Ce^{\frac{q}{4}r^2\|\iota^*z\|^2}}{r^k}.\end{aligned}$$

Since the function $r \mapsto \frac{e^{\frac{q}{4}r^2\|\iota^* z\|^2}}{r^k}$ attains its minimum at $r_k = \sqrt{\frac{2k}{q\|\iota^* z\|^2}}$ we conclude that

$$\left|f_k(z)\right| \leqslant C\frac{q^{\frac{k}{2}}e^{\frac{k}{2}}}{(2k)^{\frac{k}{2}}}\left\|\iota^* z\right\|^k = C\sqrt{\frac{(qe)^k}{(2k)^k}}\left\|I \circ \iota^*(z)\right\|^k.$$

As $I \circ \iota^*(X')$ is dense in H, we conclude that there is a unique continuous extension of f_k to H. If F_k denote this extension, then $F(x) = \sum_{k\geqslant 0} F_k(x)$ extends f. Moreover $F \in \mathcal{H}_b(H)$ because

$$\limsup \|F_k\|^{\frac{1}{k}} \leqslant \limsup \sqrt[2k]{C^2\frac{(qe)^k}{(2k)^k}} = 0. \qquad \square$$

In the sequel, we will denote by $F = \sum_{k\geqslant 0} F_k$ the unique extension of f in $\mathcal{H}_b(H)$.

The following theorem provides information about how the integral formula acts over the polynomials of finite type $\mathcal{P}_f({}^k X)$.

Theorem 1.8. *If $\varphi \in X'$, then $\int_X e^{z(\gamma)}\overline{\varphi(\gamma)^k}\,dW(\gamma) = [(\iota^*\varphi)(I \circ \iota^*(z))]^k$.*

Proof. According to Theorem 1.3, we have $\widetilde{F^\sharp}(\gamma) = \varphi(\gamma)^k$ and from Lemma 1.4, $\widetilde{F^\sharp}(\gamma) \in L^p$ for all $p > 1$. Since

$$F^\sharp(x) = \left[\varphi \circ \iota(x)\right]^k \quad \text{and} \quad F(x) = \overline{F^\sharp\left(x^*\right)},$$

we see that

$$F(x) = \overline{\left[\varphi \circ \iota\left(x^*\right)\right]}^k = \overline{\left[\iota'\varphi\left(x^*\right)\right]}^k = \left[\left(\iota^*\varphi\right)(x)\right]^k,$$

hence

$$f(z) = F \circ I \circ \iota^*(z) = \left[\left(\iota^*\varphi\right)\left(I \circ \iota^*(z)\right)\right]^k. \qquad \square$$

2. *L^2-representable* functions

Dwyer [7] defined Hilbert–Schmidt k-functionals and H.-H. Kuo studied k-linear functionals of Hilbert–Schmidt type in [10] and [11]. O. Lopushansky and A. Zagorodnyuk study, in [12], the space of Hilbertian k-homogeneous polynomials $\mathcal{P}_h({}^k H)$ over H and define their ℓ^2-sum, the Hardy-type space $\mathcal{H}^2(B)$, consisting of all functions F that can be expanded in Taylor series $F(x) = \sum_{k\geqslant 0} F_k(x)$ with F_k a k-homogeneous Hilbertian polynomial and $\|F\|_2^2 = \sum_{k\geqslant 0}\|F_k\|_h^2 < \infty$. These are Hilbert spaces, and $\mathcal{H}^2(B)$ is dual to the symmetric Fock space, which plays an important role in quantum mechanics. Given a separable Hilbert space $(H, \langle\,,\rangle)$ with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ it is possible to define a norm h on the algebraic tensor product $\bigotimes^k H$ of H such that the completion $\bigotimes_h^k H$ of $\bigotimes^k H$ under this norm is a Hilbert space (see [2,6,14]). Moreover, if $v \in \bigotimes_h^k H$, then v can be uniquely represented as

$$v = \sum_{i=1}^{\infty} \alpha_i e_{1i} \otimes \cdots \otimes e_{ki},$$

the inner product $(v \mid w)_k$ on $\bigotimes_h^k H$ can be defined by

$$(v \mid w)_k = \sum_{i,j} \alpha_i \overline{\beta_j} \langle e_{1i}, e_{1j} \rangle \cdots \langle e_{ki}, e_{kj} \rangle, \quad \alpha_i, \beta_j \in \mathbb{C},$$

where

$$v = \sum_{i=1}^{\infty} \alpha_i e_{1i} \otimes \cdots \otimes e_{ki}, \qquad w = \sum_{j=1}^{\infty} \beta_j e_{1j} \otimes \cdots \otimes e_{kj} \in \bigotimes\nolimits_h^k H$$

and its norm can be computed as $(v, v)^{\frac{1}{2}} = (\sum_{i=1}^{\infty} |\alpha_i|^2)^{\frac{1}{2}}$.

Proposition 2.1. *(See Lopushansky and Zagorodnyuk [12].) Let $\mathcal{G}_k$ denotes the permutation group of the set $\{1, 2, \ldots, k\}$ and $x^k = x \otimes \cdots \otimes x$ (k times).*

(1) *There exists a unique continuous orthogonal projection S_k on $\bigotimes_h^k H$ such that*

$$S_k(e_{1j} \otimes \cdots \otimes e_{kj}) = \frac{1}{k!} \sum_{\theta \in \mathcal{G}_k} e_{\theta(1)j} \otimes \cdots \otimes e_{\theta(k)j}.$$

(2) *The space $(H_h^k)'$ of continuous linear functionals on $H_h^k = S_k(\bigotimes_h^k H)$ is isometrically isomorphic to some vector subspace $\mathcal{P}_h({}^k H) \subset \mathcal{P}({}^k H)$ of k-homogeneous continuous polynomials on H.*
(3) *H_h^k is the closure of the linear span of $\{x^k\}_{x \in H} \subset \bigotimes_h^k H$.*

Given $\phi \in (H_h^k)'$, from the Riesz Theorem, there exists a unique $w \in H_h^k$ such that $\phi(v) = (v \mid w)_k$ for all $v \in H_h^k$. Define $P_\phi(x) = (x^k \mid w)_k$, it follows that $|P_\phi(x)| = |\phi(x^k)| \leqslant \|\phi\| \|x^k\| = \|\phi\| \|x\|^k$, so P_ϕ is bounded and continuous. Denote by $\mathcal{P}_h({}^k H)$ the vector subspace $\{P_\phi \colon \phi \in (H_h^k)'\} \subset \mathcal{P}({}^k H)$ with norm $\|P_\phi\|_h = \|\phi\|$. Note that for $P \in \mathcal{P}_h({}^k H)$, the usual polynomial norm is bounded by the Hilbertian norm: $\|P\| \leqslant \|P\|_h$.

We recall from Section 1 that the sequence $(z_n) \subset X'$ was chosen satisfying $(\iota^* z_n) = (e_n')$, and that these form an orthonormal basis for H'. Given a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = k$ set

$$z^\alpha(\gamma) = \prod_{j=1}^{\infty} \big[z_j(\gamma)\big]^{\alpha_j} \quad \text{and} \quad \big(\iota^* z\big)^\alpha(x) = \prod_{j=1}^{\infty} \big[e_j'(x)\big]^{\alpha_j}.$$

For simplicity of notation the closure of the span of $\{z^\alpha(\cdot)\}_{|\alpha|=k}$ in $L^p(W)$ will be denoted by $L_k^p(W)$. Since $\int_X z^\alpha(\gamma) \overline{z^\beta(\gamma)} \, dW = \delta_{\alpha\beta} \alpha!$ (see [13]), we have $L_n^2(W) \perp L_m^2(W)$ whenever $n \neq m$.

S. Dineen defines polynomials of integral type over a Banach space in [3]. $P \in \mathcal{P}({}^k E)$ is an integral polynomial if there exist a regular Borel measure μ of finite variation on (B, w^*) such that

$$P(z) = \int_B \varphi(z)^k \, d\mu,$$

here B denotes the unit ball in E'. The measure μ is said to represent P and the space $\mathcal{P}_I({}^k E)$ of all such polynomials is a Banach space when is normed by $\|P\|_I = \inf\{|\mu| \colon \mu \text{ represents } P\}$. For a deeper discussion we refer the reader to [4].

For the proofs of the following results we refer the reader to O. Lopushansky and A. Zagorodnyuk [12].

Theorem 2.2.

(a) *The polynomials* $\sqrt{\frac{|\alpha|!}{\alpha!}}(\iota^* z)^\alpha$ *form an orthonormal basis for* $\mathcal{P}_h({}^k H)$.
(b) $\mathcal{P}_f({}^n H) \subset \mathcal{P}_I({}^n H) \subset \mathcal{P}_h({}^n H)$.
(c) *Given* $Q_n \in \mathcal{P}_h({}^n H)$ *and* $Q_m \in \mathcal{P}_h({}^m H)$, *then* $Q_n Q_m \in \mathcal{P}_h({}^{n+m} H)$ *and* $\|Q_n Q_m\|_h \leqslant \|Q_n\|_h \|Q_m\|_h$.

We now examine the set of L^2-*representable* functions. As we will see, they are intimately related to Hilbertian polynomials and the Hardy space $\mathcal{H}^2(B)$. For the L^p-*representable* functions, we have to introduce the following vector spaces of holomorphic functions:

Hilbertian germs

$$\mathcal{H}_h(H) = \left\{ F = \sum_{k \geqslant 0} F_k \in \mathcal{H}(H)\colon\ F_k \in \mathcal{P}_h\big({}^k H\big),\ \limsup \sqrt[k]{\|F_k\|_h} < \infty \right\}$$

and

Hilbertian entire mappings of bounded type

$$\mathcal{H}_{hb}(H) = \big\{ F \in \mathcal{H}_h(H)\colon\ \limsup \sqrt[k]{\|F_k\|_h} = 0 \big\}.$$

We endow the space $\mathcal{H}_{hb}(H)$ with the topology τ generated by the family of semi-norms:

$$\|F\|_{h,\rho} = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \big\|\hat{d}^k F(0)\big\|_h.$$

Proposition 2.3. $(\mathcal{H}_{hb}(H), \tau)$ *is a Fréchet space.*

Proof. The proof is standard. □

Consider the operators

$$\mathcal{T}_2 : L^2(W) \to \mathcal{H}_b(X') \quad \text{and} \quad T_2 : L^2(W) \to \mathcal{H}_b(H),$$

where

$$\mathcal{T}_2(g)(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma)$$

and $T_2(g)$ is defined by Theorem 1.7 as the unique extension of $\mathcal{T}_2(g)$ to $\mathcal{H}_b(H)$.

Remark 1. From Theorem 1.8, it follows that $T_2(\frac{z^\alpha}{\sqrt{\alpha!}}) = \frac{(i^* z)^\alpha}{\sqrt{\alpha!}}$. Since $\{\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}(i^* z)^\alpha\}_{|\alpha|=n}$ form an orthonormal basis for $\mathcal{P}_h({}^n H)$ and $\{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}\}_{|\alpha|=n}$ is an orthonormal set in $L^2(W)$, it follows that $T_2|_{L^2_k(W)} : L^2_k(W) \to \mathcal{P}_h({}^k H)$ is an isomorphism. We have proved more, namely $\|g_k\|_2^2 = k! \|T_2(g_k)\|_h^2$ holds for any $g_k \in L^2_k(W)$.

We first prove a lemma in order to characterize the set

$$\left\{ g \in L^2(W)\colon\ \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma) = 0,\ \forall z \in X' \right\} = \operatorname{span}\big\{e^{z(\cdot)}\big\}^{\perp}_{z \in X'}.$$

Lemma 2.4. *The following equalities hold*:

$$\overline{\operatorname{span}\{e^{z(\gamma)}\}_{z\in X'}}^{\|\cdot\|_2} \overset{(1)}{=} \overline{\operatorname{span}\{z(\gamma)^k\}_{k\geqslant 0,\, z\in X'}}^{\|\cdot\|_2} \overset{(2)}{=} \overline{\operatorname{span}\{z^\alpha(\gamma)\}_{k\geqslant 0,\, |\alpha|=k}}^{\|\cdot\|_2}.$$

Proof. (1)($\subseteq$) Was proved in Proposition 1.6.

(1)($\supseteq$) The proof is by induction in k. For $k=0$, taking $z=0$ we see that the constants are included. Assume the inclusion holds for $n\leqslant k$; we will prove it for $k+1$,

$$\left|\frac{e^{\lambda z(\gamma)}-\sum_{j=0}^{k}\frac{\lambda^j z(\gamma)^j}{j!}}{\lambda^{k+1}}-\frac{z(\gamma)^{k+1}}{(k+1)!}\right|\leqslant\left|\sum_{j=k+2}^{\infty}\lambda^{(j-k-1)}\frac{z(\gamma)^j}{j!}\right|$$

which is for $|\lambda|\leqslant 1$ less than or equal to

$$|\lambda|\left(\sum_{j=n+2}^{\infty}\frac{\|z\|^j\|\gamma\|^j}{j!}\right)\leqslant|\lambda|e^{\|z\|\|\gamma\|}\to 0\quad\text{in } L^2(W)\text{ as }\lambda\to 0.$$

(2)($\supseteq$) Polarization formula (see [4]).

(2)($\subseteq$) Fix $z_0\in X'$, we know that $(\iota^* z_0)^n\in\mathcal{P}_h({}^nH)$. Therefore we can write

$$\left(\iota^* z_0\right)^n=\sum_{|\alpha|=n}a_\alpha\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}\left(\iota^* z\right)^\alpha,\quad\text{where }\{a_\alpha\}\subset\mathbb{C}\text{ and }\sum_{|\alpha|=n}|a_\alpha|^2=\left\|\left(\iota^* z_0\right)^n\right\|_h.$$

We first define $h(\gamma)=\sum_{|\alpha|=n}\overline{a_\alpha}\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}z^\alpha(\gamma)$. It follows that $h\in L^2(W)$, because $\{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}\}_{|\alpha|=n}$ is an orthonormal set in $L^2(W)$ and we have that $\sum_{|\alpha|=n}|\alpha|!|\overline{a_\alpha}|^2=n!\|(\iota^* z_0)^n\|_h^2<\infty$.

Theorem 1.8 now shows that

$$\begin{aligned}\int_X e^{z(\gamma)}\overline{\left(z_0(\gamma)^n-h(\gamma)\right)}\,dW&=\int_X e^{z(\gamma)}\left(\overline{z_0(\gamma)}^n-\sum_{|\alpha|=n}a_\alpha\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}\overline{z^\alpha(\gamma)}\right)dW\\&=\left[\left(\iota^* z_0\right)\left(I\circ\iota^*(z)\right)\right]^n-\sum_{|\alpha|=n}a_\alpha\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}\left(\iota^* z\right)^\alpha\left(I\circ\iota^*(z)\right)\\&=0\quad\text{for all } z\in X',\end{aligned}$$

since we have proved ($\supseteq$) for the second equality and (1), it follows that

$$\left(z_0(\gamma)^n-h(\gamma)\right)\in\overline{\operatorname{span}\{e^{z(\gamma)}\}_{z\in X'}}^{\|\cdot\|_2}\cap\operatorname{span}\{e^{z(\gamma)}\}_{z\in X'}^{\perp}=\{0\}$$

and then $z_0(\gamma)^n\in\overline{\operatorname{span}\{z^\alpha(\gamma)\}_{k\geqslant 0,\,|\alpha|=k}}^{\|\cdot\|_2}$. $\square$

Let us mention two important consequences.

Proposition 2.5. *If $f(z)=\sum_{k\geqslant 0}f_k(z)$ is an L^2-*representable *function, then each f_k is an L^2-*representable *function and $F_k\in\mathcal{P}_h({}^kH)$, $\forall k\geqslant 0$.*

Proof. By definition, there exist $g\in L^2(W)$ such that

$$f(z)=\int_X e^{z(\gamma)}\overline{g(\gamma)}\,dW.$$

Let π_k denote the orthogonal projection onto $L^2_k(W)$. By Proposition 1.6

$$f_k(z) = \frac{1}{k!}\int_X z(\gamma)^k \overline{g(\gamma)}\,dW = \langle \pi_k e^{z(\cdot)}, g(\cdot)\rangle_{L^2(W)} = \langle e^{z(\cdot)}, \pi_k g\rangle_{L^2(W)}$$

hence $f_k(z) = \int_X e^{z(\gamma)}\overline{\pi_k g(\gamma)}\,dW$ is L^2-*representable*.

According to Remark 1, we obtain $F_k \in T_2(L^2_k(W)) = \mathcal{P}_h({}^k H)$. □

The following result gives a precise characterization of the set of L^2-*representable* functions.

Theorem 2.6. *The following are equivalent*:

(i) f *is an* L^2-representable *function.*
(ii) $F = \sum_{k\geqslant 0} F_k$, *where* $\sum_{k\geqslant 0}\sqrt{k!}F_k \in \mathcal{H}^2(B)$.

Proof. (i) ⇒ (ii). By Proposition 2.5, $F_k \in \mathcal{P}_h({}^k H)$ for all $k \geqslant 0$. Moreover, for $f(z) = \int_X e^{z(\gamma)}\overline{g(\gamma)}\,dW$ we have $\sum_{k\geqslant 0} k!\|F_k\|_h^2 = \sum_{k\geqslant 0}\|p_k(g)\|_2^2 \leqslant \|g\|_2^2 < \infty$.

(ii) ⇒ (i). Since $\sum_{k\geqslant 0}\sqrt{k!}F_k \in \mathcal{H}^2(B)$, we have $F_k \in \mathcal{P}_h({}^k H)$ for all $k \geqslant 0$ and by Remark 1, there exist functions $g_k \in L^2_k(W)$ such that

$$F_k\big(I\circ \iota^* z\big) = \int_X e^{z(\gamma)}\overline{g_k(\gamma)}\,dW \quad\text{and}\quad \|g_k\|_2^2 = k!\|F_k\|_h^2.$$

Therefore, $g = \sum_{k\geqslant 0} g_k \in L^2(W)$ and $f(z) = \int_X e^{z(\gamma)}\overline{g(\gamma)}\,dW$. □

Let us mention an important consequence of the theorem. If we restrict our attention to integral polynomials over a separable Hilbert space, it is possible to give an alternative representation for them. The advantage of this representation is the use of a universal measure (Wiener measure).

Theorem 2.7. *Let H a separable Hilbert space, then there exist an abstract Wiener space $\iota: H \hookrightarrow X$ and a Wiener measure W, such that every $P \in \mathcal{P}_I({}^k H)$ has an alternative representation*

$$P(x) = \int_X x(\gamma)^k \widetilde{P(\gamma)}\,dW. \tag{1}$$

Proof. We first recall that $\mathcal{P}_I({}^k H) \subset \mathcal{P}_h({}^k H)$ (see [12]). For any measurable norm defined on H, call X the completion of H with this norm. Then $\iota: H \hookrightarrow X$ is an abstract Wiener space (see [10, 15]).

We can consider $x \in H$ a "measurable linear functional" over X. Moreover, $x: X \to \mathbb{C}$ is a complex-valued Gaussian random variable with mean 0 and variance $\|x\|_H^2$ (see [10,15]).

It is clear that the polynomial $P \in \mathcal{P}_h({}^k H)$ is an extension of $p \in \mathcal{P}({}^k X')$ defined by $p(z) = P\circ I\circ \iota^*(z)$. By Theorem 2.6, p is L^2-*representable*:

$$p(z) = \int_X e^{z(\gamma)}\overline{\big[(T_2|_{L^2_k(W)})^{-1}(P)\big](\gamma)}\,dW = \int_X \frac{z(\gamma)^k}{k!}\overline{\big[(T_2|_{L^2_k(W)})^{-1}(P)\big](\gamma)}\,dW$$

and then calling $\widetilde{P} = \frac{\overline{[(T_2|_{L^2_k(W)})^{-1}(P)]}}{k!}$, we have proved (1) for the dense subset $I \circ \iota^*(X') \subset H$ (Gross [9]). We only need to show that the right side is a continuous polynomial of the variable x on H. But

$$\left| \int_X x(\gamma)^k \widetilde{\overline{P(\gamma)}}\, dW \right| \leqslant \big\| \widetilde{\overline{P(\cdot)}} \big\|_2 \big\| x(\cdot)^k \big\|_2 = \big\| \widetilde{\overline{P(\cdot)}} \big\|_2 \sqrt{k!} \|x\|_H^k. \qquad \square$$

3. L^p-*representable* functions ($1 < p < 2$)

Our next objective is to provide a detailed exposition of L^p-*representable* functions for $1 < p < 2$. Many results are similar to the L^2 case but the proofs are a bit more involved.

Clearly, every L^{p_0}-*representable* function for $1 < p_0 < 2$ is also L^p-*representable* for all $p \in (1; p_0)$, hence we need only considerer the cases in which $q = \frac{p}{p-1}$ is an even integer.

Theorem 3.1. *If $f = \sum f_k$ is an L^p-*representable *function for $1 < p < 2$, then each f_k is L^2-*representable *and there exist $F_k \in \mathcal{P}_h({}^k H)$ satisfying $F_k \circ I \circ \iota^*(z) = f_k(z)$ for all $k \geqslant 0$.*

Proof. Let $g \in L^p(W)$ such that

$$f(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma) \quad \text{with } g \in L^p(W),\ 1 < p < 2.$$

We define $M_k : L^q_k(W) \to \mathbb{C}$ by

$$M_k(\psi) = \int_X \psi(\gamma) \overline{g(\gamma)}\, dW.$$

Given $\phi \in \mathcal{P}_f({}^k X)$ we first compute (assuming $q = 2r$, $r \in \mathbb{N}$):

$$\begin{aligned} \|\phi\|_q &= \left[\int_X |\phi(\gamma)|^q\, dW \right]^{\frac{1}{q}} = \|\phi^r\|_2^{\frac{2}{q}} = \left[(kr)! \big\| T_2(\phi^r) \big\|_h^2 \right]^{\frac{1}{q}} \\ &= \sqrt[q]{(kr)!} \big\| T_2(\phi)^r \big\|_h^{\frac{1}{r}} \leqslant \sqrt[q]{(kr)!} \big\| T_2(\phi) \big\|_h = \frac{\sqrt[q]{(kr)!}}{\sqrt{k!}} \|\phi\|_2. \end{aligned}$$

Since $L^q(W) \hookrightarrow L^2(W)$ we conclude that $\|\cdot\|_q$ and $\|\cdot\|_2$ are equivalent norms over $\mathcal{P}_f({}^k X)$. Therefore we have $\widetilde{M_k} : L^2_k(W) \to \mathbb{C}$ continuous and $\|\widetilde{M_k}\| \leqslant \frac{\sqrt[q]{(kr)!}}{\sqrt{k!}} \|g\|_p$. By Riesz' Theorem there exist unique $g_k \in L^2_k(W)$ such that

$$M_k(\psi) = \int_X \psi(\gamma) \overline{g(\gamma)}\, dW = \int_X \psi(\gamma) \overline{g_k(\gamma)}\, dW \quad \text{and} \quad \|g_k\|_2 \leqslant \frac{\sqrt[q]{(kr)!}}{\sqrt{k!}} \|g\|_p.$$

Therefore $f(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW = \sum_{k \geqslant 0} \frac{1}{k!} \int_X z(\gamma)^k \overline{g(\gamma)}\, dW$ and

$$f_k(z) = \frac{1}{k!} M_k\big(z(\cdot)^k\big) = \frac{1}{k!} \int_X z(\gamma)^k \overline{g_k(\gamma)}\, dW = \int_X e^{z(\gamma)} \overline{g_k(\gamma)}\, dW$$

hence f_k is an L^2-*representable* function and the existence of $F_k \in \mathcal{P}_h({}^k H)$ is proved. $\square$

In light of Theorem 3.1 we should now ask if $g(\gamma) = \sum_{k \geqslant 0} g_k(\gamma) \in L^2(W)$, and so, if any L^p-*representable* function is also L^2-*representable*. This is not the case, as will see in Section 4. We will need the following definition.

Definition 2. A function $f \in \mathcal{H}(X')$ is called ρ-*representable* if

$$f(z) = \int_X e^{\rho z(\gamma)} \overline{g(\gamma)}\, dW(\gamma) \quad \text{for } g \in L^2(W).$$

Theorem 3.2. *Given $p > 1$, there exist $\rho_0(p)$ such that for any $\rho > \rho_0(p)$, any f which is* L^p-representable *is also* ρ-representable.

Proof. Without loss of generality we can assume $q = 2r$, $r \in \mathbb{N}$. Let $\rho_0(p) = \sqrt{\frac{q}{2}}$ and $g \in L^p(W)$ such that $f(z) = \int_X e^{z(\gamma)} \overline{g(\gamma)}\, dW(\gamma)$.

Taking $\{g_k\}_{k \geqslant 0} \subset L^2(W)$ as in Theorem 3.1, for $\rho^2 > r = \frac{q}{2}$ we assert that $\sum_{k=0}^N \frac{1}{\rho^k} g_k$ defines a Cauchy sequence in $L^2(W)$. Let $a_k = \frac{\sqrt[r]{(kr)!}}{k!} \|g\|_p^2$. Since $\frac{kr}{k+1} \leqslant \frac{a_{k+1}}{a_k} = \frac{\sqrt[r]{(kr+1)\cdots(kr+r)}}{k+1} \leqslant r$, we conclude that the series $\sum_{k \geqslant 0} \frac{a_k}{\rho^{2k}}$ is convergent for any $\rho^2 > \rho_0(p)^2 = r$.

Given $\varepsilon > 0$, if $N > M$ we have

$$\left\| \sum_{k=M+1}^N \frac{1}{\rho^k} g_k \right\|_2^2 = \sum_{k=M+1}^N \frac{1}{\rho^{2k}} \|g_k\|_2^2 \leqslant \sum_{k>M} \frac{a_k}{\rho^{2k}} < \varepsilon \quad \text{for } M \text{ large enough.}$$

If we write $\tilde{g} = \sum_{k=0}^\infty \frac{1}{\rho^k} g_k$, then $\tilde{g} \in L^2(W)$, and

$$\begin{aligned}
\int_X e^{\rho z(\gamma)} \overline{\tilde{g}(\gamma)}\, dW &= \sum_{k \geqslant 0} \int_X \rho^k \frac{z(\gamma)^k}{k!} \overline{\tilde{g}(\gamma)}\, dW \\
&= \sum_{k \geqslant 0} \left(\sum_{n \geqslant 0} \int_X \rho^k \frac{z(\gamma)^k}{k!} \frac{\overline{g_n(\gamma)}}{\rho^n}\, dW \right) \\
&= \sum_{k \geqslant 0} \int_X \frac{z(\gamma)^k}{k!} \overline{g_k(\gamma)}\, dW = \sum_{k \geqslant 0} f_k(z) = f(z)
\end{aligned}$$

which is our claim. $\square$

Remark 2. The proof above gives more, namely

(a) $\|\tilde{g}\|_2 = \sqrt{\sum_{k \geqslant 0} \frac{\|g_k\|_2^2}{\rho^{2k}}} \leqslant \sqrt{\sum_{k \geqslant 0} \frac{1}{\rho^{2k}} \frac{\sqrt[r]{(kr)!}}{k!} \|g\|_p^2} \leqslant C(\rho) \|g\|_p$ hence $g \mapsto \tilde{g}$ is continuous, and

(b) if f is L^p-*representable*, then for $\rho > \rho_0(p)$, we have $f(z) = v(\rho z)$ for some $v(z)$ which is L^2-*representable*.

Corollary 3.3. *Given $p > 1$ and $\theta \in \mathbb{C}$ such that $|\theta| < \frac{1}{\rho_0(p)}$, for any L^p*-representable *function f, we have that $\sum_{k\geqslant 0}\sqrt{k!}\theta^k F_k \in \mathcal{H}^2(B)$, and so*

(a) $\sum_{k\geqslant 0}\sqrt{k!}F_k \in \mathcal{H}_h(H)$, *and*
(b) $F = \sum_{k\geqslant 0} F_k \in \mathcal{H}_{hb}(H)$.

Proof. Given f, an L^p-*representable* function, and $|\theta| < \frac{1}{\rho_0(p)}$, from Remark 2 it follows that for any $\rho > \rho_0(p)$, one has $f(z) = v(\rho z)$ for some $v(z)$ which is L^2-*representable*. From this we can choose ρ such that $|\theta| < \frac{1}{\rho} < \frac{1}{\rho_0(p)}$ and $\|F_k\|_h = \rho^k \|V_k\|_h$, then we have

$$\sum_{k\geqslant 0} k!|\theta|^{2k}\|F_k\|_h^2 = \sum_{k\geqslant 0} k!|\theta|^{2k}\rho^{2k}\|V_k\|_h^2 \leqslant \sum_{k\geqslant 0} k!\|V_k\|_h^2 < \infty.$$

From this is clear that

(a) $\limsup \sqrt[k]{\sqrt{k!}\|F_k\|_h} < \infty$, and
(b) $\limsup \sqrt[k]{\|F_k\|_h} = 0$. $\square$

4. Examples

We have seen that any L^2-*representable* function is also L^p-*representable* ($1 < p < 2$) and any L^p-*representable* function is a ρ-*representable* function for $\rho > [\rho_0(p)]^2$. We give here some examples to show that both are proper inclusions.

Theorem 4.1. *If $1 < p < 2$, then there exist L^p*-representable *functions which are not L^2*-representable.

Proof. Without loss of generality we can assume $p = \frac{m}{n}$ for $m, n \in \mathbb{N}$.

We first show that the inclusion

$$\overline{\operatorname{span}\{z(\cdot)^k\}_{k\geqslant 0,\, z\in X'}}^{\|\cdot\|_2} \hookrightarrow \overline{\operatorname{span}\{z(\cdot)^k\}_{k\geqslant 0,\, z\in X'}}^{\|\cdot\|_p}$$

is not surjective, for which it is enough to see that it is not an open map. Choose $z \in X'$ such that $\|I \circ \iota^* z\|_H = 1$, and $k = 2jn$, $j \in \mathbb{N}$. Let us compute

$$\begin{aligned}\|z(\cdot)^k\|_p^p &= \int_X |z(\gamma)^k|^p \, dW = \int_{\mathbb{C}} |w|^{kp} e^{-|w|^2} \frac{dw}{\pi} = \int_0^{+\infty} r^{pk} e^{-r^2} 2r \, dr \\ &= \int_0^{+\infty} u^{\frac{pk}{2}} e^{-u} \, du = \int_0^{+\infty} u^{mj} e^{-u} \, du = (mj)!.\end{aligned}$$

Writing $b_j = \frac{\|z(\cdot)^{2jn}\|_p}{\|z(\cdot)^{2jn}\|_2} = \frac{[(mj)!]^{\frac{n}{m}}}{[(2jn)!]^{\frac{1}{2}}}$ we see that

$$\frac{b_{j+1}}{b_j} = \frac{[(jm+m)!]^{\frac{n}{m}}}{\sqrt{[2(j+1)n]!}} \frac{\sqrt{(2jn)!}}{[(jm)!]^{\frac{n}{m}}} \leqslant \frac{[m(j+1)]^n}{(2jn+1)^n},$$

and so

$$\limsup \frac{b_{j+1}}{b_j} \leqslant \left(\frac{m}{2n}\right)^n < 1.$$

Since $\lim_{j\to+\infty} b_j = 0$, we conclude that the inclusion is not bounded from below.

We proceed to show that for $X = H = \mathbb{C}$ we have an L^p-*representable* function which is not L^2-*representable*. We can choose $f \in L^p(\mathbb{C})$, with $f \notin L^2(\mathbb{C})$, such that there exists a sequence $\{f^{(n)}\}_{n\in\mathbb{N}} \subset \operatorname{span}\{z(\cdot)^k\}_{k\geqslant 0, z\in X'}$ for which $f = \lim_{n\to\infty} f^{(n)}$ in $L^p(W)$.

For $f^{(n)}$ as above the integral formula can be written (see [13])

$$f^{(n)}(w) = \int_{\mathbb{C}} e^{\langle w,\gamma\rangle} f^{(n)}(\gamma)\, dW. \tag{2}$$

From Lemma 1.5, we deduce that

$$\left|f^{(n)}(w) - f^{(m)}(w)\right| \leqslant e^{\frac{q|w|^2}{4}} \left\|f^{(n)} - f^{(m)}\right\|_p.$$

It follows that $\{f^{(n)}\}_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence in any bounded set. Let $g \in \mathcal{H}(\mathbb{C})$ such that $g(w) = \lim_{n\to\infty} f^{(n)}(w)$. Now, taking a subsequence if necessary, we can assume that $f^{(n)} \to f$ a.e. From this we obtain $g = f$ a.e. (hence $g \in L^p(W)$) and letting $n \to \infty$ in (2), we conclude that g is L^p-*representable* because

$$g(w) = \int_{\mathbb{C}} e^{\langle w,\gamma\rangle} g(\gamma)\, dW.$$

It remains to prove that g is not L^2-*representable*.

We know from Theorem 2.6, that

$$g(w) = \sum_{k\geqslant 0} \alpha_k w^k \quad \text{is } L^2\text{-}\textit{representable} \quad \Leftrightarrow \quad \sum_{k\geqslant 0} k!|\alpha_k|^2 < \infty.$$

Moreover, if $\sum_{k\geqslant 0} k!|\alpha_k|^2 < \infty$, then $\sum_{k=0}^{\infty} \alpha_k w^k$ is convergent in L^2, because

$$\left\|\sum_{k=0}^{\infty} \alpha_k w^k\right\|_2^2 = \sum_{k=0}^{\infty} \left\|\alpha_k w^k\right\|_2^2 = \sum_{k=0}^{\infty} k!|\alpha_k|^2.$$

From this we deduce that if g were an L^2-*representable* function, we would have $g \in L^2$, but this is impossible because $g = f$ a.e. □

Theorem 4.2. *Given* $1 < p \leqslant 2$ *and* $\rho > \rho_0(p)$, *then there exist an* ρ-representable *function which is not* L^p-representable.

Proof. If f is L^p-*representable*, then f satisfies the growth condition (Lemma 1.5) $|f(z)| \leqslant C_1 e^{\frac{q\|\iota^* z\|^2}{4}}$. Also, if $f(z) = v(\rho z)$ where v is an L^2-*representable* function, then $|f(z)| \leqslant C_2 e^{\frac{\rho^2\|\iota^* z\|^2}{2}}$.

Choose $x_0 \in H$, $\|x_0\|_H = 1$ and $a \in (0,1)$ such that $a\rho^2 > [\rho_0(p)]^2$. Let $V \in \mathcal{H}_b(H)$ defined by $V(x) = \sum_{k\geqslant 0} \frac{a^k}{k!2^k} \langle x, x_0\rangle^{2k}$,

$$\left|V(x)\right| \leqslant \sum_{k\geqslant 0} \frac{a^k}{k!2^k}\|x\|_H^{2k} = e^{\frac{a\|x\|_H^2}{2}}.$$

It is easy to check that $v(z) = V \circ I \circ \iota^*(z)$ is L^2-*representable*:

$$\sum_{k\geqslant 0} k!\|V_k\|_h^2 = \sum_{k\geqslant 0} \frac{(2k)!}{k!^2}\left(\frac{a^2}{4}\right)^k < \infty$$

because this series has radius of convergence $R = 1/4$ and $0 < a < 1$.

Consider $f(z) = v(\rho z)$ which is clearly ρ-*representable*, we claim that f is not L^p-*representable*. For $\lambda > 0$,

$$V(\lambda\rho x_0) = \sum_{k\geqslant 0} \frac{a^k}{k!2^k}\langle\lambda\rho x_0, x_0\rangle^{2k} = \sum_{k\geqslant 0} \frac{a^k}{k!2^k}(\lambda\rho)^{2k} = e^{\frac{a\lambda^2\rho^2}{2}}$$

from this, recalling that $[\rho_0(p)]^2 > q/2$, since $a\rho^2 > [\rho_0(p)]^2$, we have

$$\left|V(\lambda\rho x_0)\right|e^{-\frac{q\lambda^2}{4}} = e^{\frac{(2a\rho^2-q)\lambda^2}{4}} \to_{\lambda\to+\infty} +\infty$$

therefore f is not an L^p-*representable* function. □

Summarizing, we have the following necessary conditions for L^p-*representability*:

- Growth condition: $|f(z)| \leqslant C_1 e^{\frac{q\|\iota^* z\|^2}{4}}$ (Lemma 1.5).
- Hilbertian–germ condition (HG): for each f_k we have $F_k \in \mathcal{P}_h({}^k H)$ and $R^{-1} = \limsup \sqrt[k]{k!\|F_k\|_h^2} \leqslant [\rho_0(p)]^2$ (Corollary 3.3).

The (HG) condition gives more information:

$$\begin{aligned}\left|F(x)\right| &\leqslant \sum_{k\geqslant 0}\|F_k\|\|x\|^k \leqslant \sum_{k\geqslant 0}\|F_k\|_h\|x\|^k \\ &\leqslant \left(\sum_{k\geqslant 0} k!\|F_k\|_h^2\theta^k\right)^{\frac{1}{2}}\left(\sum_{k\geqslant 0}\frac{\|x\|^{2k}}{\theta^k k!}\right)^{\frac{1}{2}} = C(\theta)e^{\frac{\|x\|^2}{2\theta}}.\end{aligned}$$

From this we deduce that (HG) is stronger than the growth condition when $R > 2/q$. Theorem 2.6 shows that if $R > 1$, then $f = F \circ I \circ \iota^*$ is L^2-*representable*. We leave as an open question if it is true that given a function $F \in \mathcal{H}_{hb}(H)$ satisfying the (HG) condition with $R \leqslant 1$, then $f = F \circ I \circ \iota^*$ is an L^r-*representable* function for $r < \frac{2}{2-R}$.

5. Aron–Berner extension

We have been working under the assumption that f is an entire mapping of bounded type on a dual space. Now suppose that $f \in \mathcal{H}_b(E)$ where E is a Banach space which has a separable dual. Our purpose is to give an integral representation

$$f(z) = \int_{E'} e^{\gamma(z)}\overline{g(\gamma)}\,dW(\gamma) \quad \text{for } g \in L^p(W),\ 1 < p. \tag{3}$$

This can be done through the Aron–Berner extension of f to the bidual E''. For the Aron–Berner construction we refer the reader to [1,4] and [16]. We need to recall only the following. If $f = \sum_k f_k$ is the Taylor series expansion of f (about 0, say). Then each k-homogeneous polynomial $f_k : E \to \mathbb{C}$ may be canonically extended to the bidual: $AB_k(f_k) : E'' \to \mathbb{C}$, and the Aron–Berner extension of f is defined to be

$$AB(f) = \sum_k AB_k(f_k).$$

Theorem 5.1. *$f \in \mathcal{H}_b(E)$ has an integral representation as* (3) *if and only if $AB(f)$ is L^p-*representable.

Proof. ($\Leftarrow$) Let $J : E \to E''$ the canonical inclusion. By definition $AB(f)(Jz) = f(z)$ and $[Jz](\gamma) = \gamma(z)$.

($\Rightarrow$) It is sufficient to show that $f_k(z) = \frac{1}{k!}\int_{E'} \gamma(z)^k \overline{g(\gamma)}\, dW(\gamma)$ has as Aron–Berner extension the polynomial $AB_k(f_k)(\zeta) = \frac{1}{k!}\int_{E'} \zeta(\gamma)^k \overline{g(\gamma)}\, dW(\gamma)$. We will use the following result due to I. Zalduendo [16].

Theorem 5.2. *If $Q \in \mathcal{P}(^k E'')$ and $Q|_E = P$, then $Q = AB_k(P)$ if and only if*

(a) *for each $x \in E$, $DQ(x)$ is w^*-*continuous*, and*
(b) *for each $z \in E''$ and $(x_\alpha) \subset E$ converging w^* to z, $DQ(z)(x_\alpha) \to DQ(z)(z)$.*

We will prove that $Q(\zeta) = \frac{1}{k!}\int_{E'} \zeta(\gamma)^k \overline{g(\gamma)}\, dW(\gamma)$ has w^*-continuous first-order differentials, therefore (a) and (b) are satisfied.

Let $r, s \in \mathbb{R}$ such that $1/r + 1/s + 1/p = 1$.

Since $DQ(\zeta)(\xi) = \frac{1}{(k-1)!}\int_{E'} \xi(\gamma)\zeta(\gamma)^{k-1}\overline{g(\gamma)}\, dW(\gamma)$ for all $\zeta, \xi \in E''$, we have

$$\left|DQ(\zeta)(\xi)\right| \leqslant \frac{1}{(k-1)!}\left\|\xi(\cdot)\right\|_r \left\|\zeta(\cdot)^{k-1}\right\|_s \left\|g(\cdot)\right\|_p \leqslant \frac{C(r,s,g,\zeta)}{(k-1)!}\left\|I \circ \iota^*\xi\right\|_H.$$

For each $\zeta \in E''$, $DQ(\zeta)(\cdot)$ has a unique extension to H. Calling this extension Φ_ζ, we then have

$$\begin{aligned} DQ(\zeta)(\xi) &= \Phi_\zeta\big(I \circ \iota^*\xi\big) = \big\langle I \circ \iota^*\xi, I(\Phi_\zeta)\big\rangle = \overline{\big\langle I(\Phi_\zeta), I \circ \iota^*\xi\big\rangle} \\ &= \overline{\iota^*\xi\big(I(\Phi_\zeta)\big)} = (\iota'\xi)\big(I(\Phi_\zeta)^*\big) = \xi\big(\iota \circ I\big(\Phi_\zeta^*\big)\big). \end{aligned}$$

Since $\iota \circ I(\Phi_\zeta^*) \in E'$, it follows that $DQ(\zeta)(\cdot)$ is w^*-continuous. □

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