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On boundary value problems in three-ion electrodiffusion

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Abstract

The existence of solutions to a class of two-point boundary value problems in three-ion electrodiffusion is investigated via an integro-differential formulation. Boundedness by upper and lower solutions corresponding to associated boundary value problems is considered and illustrated by Painlevé II solutions of a constrained version of the original boundary value problems.

Keywords: Three-ion electrodiffusion; Nonlinear boundary value problems; Third order ODE's; Topological methods

1. Introduction

Leuchtag in [1] presented an m-ion electrodiffusion model consisting of the nonlinear coupled system

$$\frac{\mathrm{d}n_i}{\mathrm{d}x} = \nu_i n_i p - c_i, \quad i = 1, \ldots, m,$$
$$\frac{\mathrm{d}p}{\mathrm{d}x} = \sum_{i=1}^{m} \nu_i n_i, \tag{1.1}$$

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where n_i is the number of ions with the same charge, p is the electric field, the ν_j are nonzero integral signed valencies while the c_i are real constants. In the three-ion case to be considered here, the model reduces to the following nonlinear third order equation for the electric field p:

$$\begin{aligned} &pp''' - p'p'' - (\nu_1+\nu_2+\nu_3)p^2p'' + (\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)p^3p' \\ &\quad - (\nu_1c_1+\nu_2c_2+\nu_3c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3p^5 + \nu_1\nu_2\nu_3(c_1+c_2+c_3)xp^3 \\ &\quad - \big[(\nu_2+\nu_3)\nu_1c_1 + (\nu_3+\nu_1)\nu_2c_2 + (\nu_1+\nu_2)\nu_3c_3\big]p^2 = 0. \end{aligned} \tag{1.2}$$

In previous work on the model (1.1), in the two-ion case a Bäcklund transformation was applied to an underlying Painlevé II reduction [2]. Here, the three-ion model is analyzed in connection with the existence of solutions to a class of two-point boundary value problems and the construction of upper and lower solutions [3]. The work complements that initiated in [4] on the three-ion model.

2. A two-point boundary value problem

Let us consider the following two-point boundary value problem for (1.2):

$$\begin{aligned} &pp''' - p'p'' - (\nu_1+\nu_2+\nu_3)p^2p'' + (\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)p^3p' \\ &\quad - (\nu_1c_1+\nu_2c_2+\nu_3c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3p^5 + \nu_1\nu_2\nu_3(c_1+c_2+c_3)xp^3 \\ &\quad - \big[(\nu_2+\nu_3)\nu_1c_1 + (\nu_3+\nu_1)\nu_2c_2 + (\nu_1+\nu_2)\nu_3c_3\big]p^2 = 0, \\ &p(0) = p_0, \qquad p(T) = p_T, \qquad p''(0) = r_0. \end{aligned} \tag{2.1}$$

If we set $u = p''/p$, $u_0 = r_0/p_0$, then for $p \neq 0$ this boundary value problem is equivalent to:

$$\begin{aligned} &p^2u' - (\nu_1+\nu_2+\nu_3)p^2p'' + (\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)p^3p' \\ &\qquad - (\nu_1c_1+\nu_2c_2+\nu_3c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3p^5 + \nu_1\nu_2\nu_3(c_1+c_2+c_3)xp^3 \\ &\quad = \big[(\nu_2+\nu_3)\nu_1c_1 + (\nu_3+\nu_1)\nu_2c_2 + (\nu_1+\nu_2)\nu_3c_3\big]p^2, \\ &p'' = pu, \qquad p(0) = p_0, \qquad p(T) = p_T, \qquad u(0) = u_0. \end{aligned} \tag{2.2}$$

This in turn, is equivalent to the second order integro-differential boundary value problem:

$$\begin{aligned} &p'' = -C_3 + \varphi(x)p + C_1\big(p' - p'(0)\big)p + \frac{C_2}{2}p^3p\int_0^x \big(C_4p^2(t) + C_5t\big)p(t)\,\mathrm{d}t, \\ &p(0) = p_0, \qquad p(T) = p_T, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &C_1 = \nu_1+\nu_2+\nu_3, \qquad C_2 = -(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3), \qquad C_3 = \nu_1c_1+\nu_2c_2+\nu_3c_3, \\ &C_4 = \frac{1}{2}\nu_1\nu_2\nu_3, \qquad C_5 = -\nu_1\nu_2\nu_3(c_1+c_2+c_3), \\ &C_6 = (\nu_2+\nu_3)\nu_1c_1 + (\nu_3+\nu_1)\nu_2c_2 + (\nu_1+\nu_2)\nu_3c_3, \end{aligned}$$

and

$$\varphi(x) = u_0 - \frac{1}{2}C_2p_0^2 + \frac{C_3}{p_0} + C_6x.$$

Here, we shall assume that the associated linear operator $\mathcal{L}p := p'' - \varphi p$ is below resonance for the Dirichlet conditions, that is

$$\varphi_i := \inf_{0\leqslant x\leqslant T} \varphi(x) = \frac{r_0 + C_3}{p_0} - C_2\frac{p_0^2}{2} + \min\{0, C_6T\} > -\left(\frac{\pi}{T}\right)^2. \tag{2.4}$$

By standard results, condition (2.4) implies that $\mathcal{L} : H^2 \cap H_0^1(0,T) \to L^2(0,T)$ is invertible. In particular, a straightforward computation shows that if $p \in H^2 \cap H_0^1(0,T)$ then

$$\|p\|_{L^2} \leqslant \frac{1}{(\frac{\pi}{T})^2 + \varphi_i}\|\mathcal{L}p\|_{L^2}$$

and

$$\|p'\|_{L^2} \leqslant \frac{\frac{\pi}{T}}{(\frac{\pi}{T})^2 + \varphi_i}\|\mathcal{L}p\|_{L^2}.$$

Furthermore, writing $p'' = \mathcal{L}p + \varphi p$, we also deduce that

$$\|p''\|_{L^2} \leqslant (1+k)\|\mathcal{L}p\|_{L^2},$$

where

$$k = \frac{\max\{|\frac{r_0+C_3}{p_0} - C_2\frac{p_0^2}{2}|, |\frac{r_0+C_3}{p_0} - C_2\frac{p_0^2}{2} + C_6T|\}}{(\frac{\pi}{T})^2 + \varphi_i}.$$

Hence,

$$\|p'\|_C \leqslant T^{1/2}\|p''\|_{L^2} \leqslant T^{1/2}(1+k)\|\mathcal{L}p\|_{L^2},$$

and setting

$$N = T^{1/2}\max\left\{\frac{\frac{\pi}{T}}{(\frac{\pi}{T})^2 + \varphi_i}, 1+k\right\}$$

we conclude that

$$\|p\|_{C^1} := \max\bigl\{\|p\|_C, \|p'\|_C\bigr\} \leqslant N\|\mathcal{L}p\|_{L^2}.$$

In order to establish a sufficient condition for the existence of solutions, let us define a polynomial $\gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\gamma(M) = 2|C_1|M^2 + \frac{|C_2|}{2}M^3 + TM^2\bigl(|C_4|M^2 + |C_5|T\bigr). \tag{2.5}$$

From the limits

$$\lim_{M\to 0^+}\frac{\gamma(M)}{M} = 0, \qquad \lim_{M\to +\infty}\frac{\gamma(M)}{M} = +\infty,$$

it is clear that the function $M - NT^{1/2}\gamma(M)$ achieves a positive maximum $A_{\max}$ at some value $M_{\max} > 0$.

The following result may be established:

Theorem 2.1. *Assume that* (2.4) *holds, and that* $\Theta \leqslant A_{\max}$, *where*

$$\Theta := \max\left\{|p_0|, |p_T|, \frac{|p_T - p_0|}{T}\right\} + N\bigl(|C_3|T^{1/2} + \|\varphi B\|_{L^2}\bigr) \tag{2.6}$$

and

$$B(t)=\left(\frac{p_T-p_0}{T}\right)t+p_0.$$

Then the boundary value problem (2.1) *admits at least one classical solution.*

Proof. Let us define an operator $\mathcal{T}: C^1[0,T]\to C^1[0,T]$ given by $\mathcal{T}q=p$, where p is the unique solution of the linear problem

$$\begin{cases} p''(x)-\varphi(x)p(x)=V(q)-C_3, \\ p(0)=p_0, \qquad p(T)=p_T, \end{cases}$$

with

$$V(q)(x):=C_1\big(q'(x)-q'(0)\big)q(x)+\frac{C_2}{2}q^3(x)+q(x)\int_0^x\big(C_4q^2(t)+C_5t\big)q(t)\,\mathrm{d}t.$$

By standard results, $\mathcal{T}$ is well defined and compact. Moreover, if θ is the unique function satisfying

$$\mathcal{L}\theta=-C_3, \qquad \theta(0)=p_0, \qquad \theta(T)=p_T,$$

then

$$\|\theta-B\|_{C^1}\leqslant N\|C_3-\varphi B\|_{L^2}\leqslant N\big(|C_3|T^{1/2}+\|\varphi B\|_{L^2}\big).$$

Hence, $\|\theta\|_{C^1}\leqslant\Theta$, where Θ is given by (2.6). Moreover, the following bound is obtained for $p=\mathcal{T}q$:

$$\|p-\theta\|_{C^1}\leqslant N\big\|\mathcal{L}(p-\theta)\big\|_{L^2}=N\big\|V(q)\big\|_{L^2}.$$

It is readily shown that if $\|q\|_{C^1}\leqslant M$ then $|V(q)(x)|\leqslant\gamma(M)$, with $\gamma(M)$ as in (2.5). Thus, $\|p\|_{C^1}\leqslant\Theta+NT^{1/2}\gamma(M)$, and since $\Theta\leqslant A_{\max}$ we conclude that if $\|q\|_{C^1}\leqslant M_{\max}$ then $\|p\|_{C^1}\leqslant M_{\max}$. The result now follows from Schauder's Fixed Point Theorem. □

As a particular consequence we obtain:

Corollary 2.1. *Assume that* $\min\{0,C_6T\}>L-(\frac{\pi}{T})^2$ *for some* $L>0$. *Then there exist constants* $\delta_0,\delta_T>0$ *such that the boundary value problem* (2.1) *admits at least one classical solution for any choice of the parameters* p_0, p_T, C_3 *and* r_0 *satisfying*:

$$|p_T|<\delta_T, \qquad \frac{|r_0|+|C_3|}{L}<|p_0|<\delta_0.$$

Proof. From the hypotheses, it is clear that if C_2 and C_6 are fixed then (2.4) holds for δ_0 small enough. Furthermore, it is observed that $\|\varphi\|_C$ and the constant N remain bounded as $p_0\to 0$, provided $|\frac{r_0+C_3}{p_0}|<L$. On the other hand, if $p_0,p_T\to 0$ then $\|B\|_{C^1}\to 0$ and $C_3\to 0$. Hence $\Theta\to 0$, and the result follows. □

3. The case $\nu_1 + \nu_2 + \nu_3 = 0$. Upper and lower solutions

Here we focus on the boundary value problem

$$p'' = -C_3 + \varphi(x)p + \frac{C_2}{2}p^3 + p\int_0^x \bigl(C_4 p^2(t) + C_5 t\bigr)p(t)\,\mathrm{d}t,$$
$$p(0) = p_0, \qquad p(T) = p_T, \tag{3.1}$$

corresponding to the case $C_1 = \sum \nu_i = 0$. Note that this condition implies that $C_2 \geqslant 0$.

The quantities α and β are termed lower and upper solutions respectively for the boundary value problem (3.1) if

$$\alpha'' \geqslant -C_3 + \varphi(x)\alpha + \frac{C_2}{2}\alpha^3 + \alpha\int_0^x \bigl(C_4\alpha^2(t) + C_5 t\bigr)\alpha(t)\,\mathrm{d}t, \tag{3.2}$$

$$\beta'' \leqslant -C_3 + \varphi(x)\beta + \frac{C_2}{2}\beta^3 + \beta\int_0^x \bigl(C_4\beta^2(t) + C_5 t\bigr)\beta(t)\,\mathrm{d}t, \tag{3.3}$$

and

$$\alpha(0) \leqslant p_0 \leqslant \beta(0), \qquad \alpha(T) \leqslant p_T \leqslant \beta(T). \tag{3.4}$$

Then we have:

Theorem 3.1. *Let α and β be respectively lower and upper solutions of the boundary value problem* (3.1), *with $0 \leqslant \alpha \leqslant \beta$, and assume that $\nu_1\nu_2(\nu_1 + \nu_2) > 0$, $c_1 + c_2 + c_3 \leqslant 0$. Then* (3.1) *admits a solution u with $\alpha \leqslant u \leqslant \beta$.*

Proof. From the assumptions, it readily follows that if $0 \leqslant p \leqslant q$ then

$$p(x)\int_0^x \bigl(C_4 p^2(t) + C_5 t\bigr)p(t)\,\mathrm{d}t \geqslant q(x)\int_0^x \bigl(C_4 q^2(t) + C_5 t\bigr)q(t)\,\mathrm{d}t.$$

Let us fix a nonnegative constant λ such that

$$\lambda \geqslant \frac{3}{2}C_2\|\beta\|_C^2 + \varphi_s,$$

with

$$\varphi_s := \sup_{0\leqslant x\leqslant T}\varphi(x) = \frac{r_0 + C_3}{p_0} - C_2\frac{p_0^2}{2} + \max\{0, C_6 T\}.$$

Then the function $\varphi p + \frac{C_2}{2}p^3 - \lambda p$ is nonincreasing in p for $\alpha(x) \leqslant p \leqslant \beta(x)$. Next, define a fixed point operator $\mathcal{T}$ such that for fixed $q \in C[0,T]$, $p = \mathcal{T}q$ is the unique solution of the linear problem

$$p'' - \lambda p = -C_3 + \varphi q + \frac{C_2}{2}q^3 - \lambda q + q\int_0^x \bigl(C_4 q^2(t) + C_5 t\bigr)q(t)\,\mathrm{d}t,$$
$$p(0) = p_0, \qquad p(T) = p_T.$$

By standard results, $\mathcal{T} : C[0,T] \to C[0,T]$ is well defined and compact. Moreover, if $\alpha \leqslant q \leqslant \beta$, for $p = \mathcal{T}q$ it follows that

$$p'' - \lambda p \leqslant -C_3 + \varphi\alpha + \frac{C_2}{2}\alpha^3 - \lambda\alpha + \alpha \int_0^x \big(C_4\alpha^2(t) + C_5 t\big)\alpha(t)\,\mathrm{d}t \leqslant \alpha'' - \lambda\alpha$$

and

$$p'' - \lambda p \geqslant -C_3 + \varphi\beta + \frac{C_2}{2}\beta^3 - \lambda\beta + \beta \int_0^x \big(C_4\beta^2(t) + C_5 t\big)\beta(t)\,\mathrm{d}t \geqslant \beta'' - \lambda\beta.$$

From (3.4) and the maximum principle we conclude that $\alpha \leqslant p \leqslant \beta$.

The result follows by applying Schauder's Theorem to the bounded, convex and closed set $\{u \in C[0,T]\colon \alpha \leqslant u \leqslant \beta\}$. □

An analogous result when $\alpha \leqslant \beta \leqslant 0$ may be established, namely:

Theorem 3.2. *Let α and β be respectively lower and an upper solutions of* (3.1)*, with $\alpha \leqslant \beta \leqslant 0$, and assume that $\nu_1\nu_2(\nu_1 + \nu_2) < 0$, $c_1 + c_2 + c_3 \leqslant 0$. Then the boundary value problem* (3.1) *admits a solution u with $\alpha \leqslant u \leqslant \beta$.*

4. An iterative quasilinearization method

Here, the existence of an ordered couple of lower and upper solutions is assumed and an iterative scheme that converges to a solution of problem (3.1) is constructed. Under appropriate conditions, the convergence is proved to be quadratic.

In this connection, it proves convenient to write $(3.1)_1$ as

$$p'' - \frac{C_2}{2}p^3 = F(p),$$

where the mapping $F : C[0,1] \to C[0,1]$ is given by

$$F(p) = -C_3 + \varphi p + p \int_0^x \big(C_4 p^2(t) + C_5 t\big)p(t)\,\mathrm{d}t. \tag{4.1}$$

It is readily seen that F is infinitely Fréchet differentiable, with

$$DF(p)[q] = \varphi q + p \int_0^x \big(3C_4 p^2(t) + C_5 t\big)q(t)\,\mathrm{d}t + q \int_0^x \big(C_4 p^2(t) + C_5 t\big)p(t)\,\mathrm{d}t \tag{4.2}$$

and

$$\begin{aligned} D^2F(p)[q,r] = {} & 6C_4 p \int_0^x pqr\,\mathrm{d}t + q \int_0^x \big(3C_4 p^2(t) + C_5 t\big)r(t)\,\mathrm{d}t \\ & + r \int_0^x \big(3C_4 p^2(t) + C_5 t\big)q(t)\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

As a preliminary we note appropriate comparison and existence-uniqueness results for the semi-linear operator $S_\lambda p := p'' - \frac{C_2}{2} p^3 - \lambda p$ with $\lambda, C_2 \geqslant 0$ (proofs are straightforward):

Lemma 4.1. *If $p, q \in H^2(0,T)$ satisfy*

$$S_\lambda p \geqslant S_\lambda q \quad a.e. \quad p(0) \leqslant q(0), \qquad p(T) \leqslant q(T),$$

then $p \leqslant q$.

Lemma 4.2. *Let $\xi \in L^2(0,T)$ and $p_0, p_T \in \mathbb{R}$. Then the boundary value problem*

$$S_\lambda p = \xi, \qquad p(0) = p_0, \qquad p(T) = p_T$$

admits a unique solution $p \in H^2(0,T)$.

In what follows, we shall consider only the situation when Theorem 3.1 applies. Analogous conclusions hold when Theorem 3.2 obtains.

In order to construct an iterative Newton-type scheme for the boundary value problem (3.1), the following result is required.

Lemma 4.3. *Let the assumptions of Theorem* 3.1 *hold. Then there exists $p \in C^2[0,T]$ such that $\alpha \leqslant p \leqslant \beta$, and*

$$\begin{cases} S_0 p = F(\alpha) + DF(\alpha)(p - \alpha), \\ p(0) = p_0, \quad p(T) = p_T. \end{cases} \tag{4.4}$$

Proof. Fix a nonnegative constant $\lambda \geqslant \varphi_s$, with φ_s as in Theorem 3.1, and define a function $\Pi : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$\Pi(x,p) = \begin{cases} p & \text{if } \alpha(x) \leqslant p \leqslant \beta(x), \\ \alpha(x) & \text{if } p < \alpha(x), \\ \beta(x) & \text{if } p > \beta(x). \end{cases}$$

From Lemma 4.2 and Schauder's Theorem, it is seen that the following quasilinear problem admits at least one solution:

$$\begin{cases} S_\lambda p = F(\alpha) + DF(\alpha)\big(\Pi(p) - \alpha\big) - \lambda \Pi(p), \\ p(0) = p_0, \quad p(T) = p_T. \end{cases}$$

We now write $S_\lambda p = F(\alpha) - \lambda\alpha + (DF(\alpha) - \lambda I)(\Pi(p) - \alpha)$, and from the choice of λ it follows immediately that $S_\lambda p \leqslant F(\alpha) - \lambda\alpha \leqslant S_\lambda \alpha$.

On the other hand, consider the Taylor expansion

$$F(z) = F(\alpha) + DF(\alpha)(z - \alpha) + R(z).$$

From the fact that $D^2F(p)[q,r] \leqslant 0$ for $p, q, r \geqslant 0$, it follows that $R(z) \leqslant 0$ for $z \geqslant \alpha$. Then

$$S_\lambda p = F\big(\Pi(p)\big) - R\big(\Pi(p)\big) - \lambda \Pi(p) \geqslant F(\beta) - \lambda\beta \geqslant S_\lambda \beta.$$

Hence, $\alpha \leqslant p \leqslant \beta$, and $S_0 p = F(\alpha) + DF(\alpha)(p - \alpha)$. Again, we may write $S_0 p = F(p) - R(p)$, where the Taylor remainder $R(p)$ is nonpositive, and conclude that $S_0 p \geqslant F(p)$. Thus p is a lower solution of the boundary value problem (3.1). □

Next, we define a sequence as follows. Start with $p_1 = \alpha$, then from Lemma 4.2 we may choose a lower solution p_2 with $p_1 \leqslant p_2 \leqslant \beta$ satisfying (4.4). Iteration of this process produces a nondecreasing sequence

$$p_1 \leqslant p_2 \leqslant p_3 \leqslant \cdots \leqslant \beta$$

such that

$$S_0 p_{n+1} = F(p_n) + DF(p_n)(p_{n+1} - p_n), \qquad p_{n+1}(0) = p_0, \qquad p_{n+1}(T) = p_T, \quad (4.5)$$

where $\{p_n\}$ converges pointwise to some function p. From standard results (Dini's Theorem), $p_n \to p$ uniformly, and use of (4.5) shows that p is a solution of (3.1). In order to prove the quadratic convergence of $\{p_n\}$ we impose an extra requirement:

Theorem 4.4. *Let the assumptions of Theorem* 3.1 *hold. Further, assume that*

$$\mu + \varphi + \frac{C_2}{2}\left(p_{n+1}^2 + p_{n+1}p_n + p_n^2\right) + \int_0^x \left(C_4\beta^2 + C_5 t\right)\beta \, dt \geqslant 0 \quad (4.6)$$

for some $n \geqslant 2$ and some constant $\mu < (\frac{\pi}{T})^2$, and that $k_0 k_1 < \frac{1}{T}$, where

$$k_0 = \frac{\frac{\pi}{T}}{(\frac{\pi}{T})^2 - \mu}, \qquad k_1 = \left\| \beta \int_0^x \left(3C_4\beta^2 + C_5 t\right) dt \right\|_C .$$

Then

$$\|p_{n+1} - p_n\|_C \leqslant k \|p_n - p_{n-1}\|_C^2$$

for some constant k independent of n. In particular, the sequence defined by (4.6) *converges quadratically to a solution of the boundary value problem* (3.1).

Proof. In the context the previous proof, define $E_n = p_n - p_{n-1}$. Then $\{E_n\}$ is pointwise non-increasing and tends to 0 as $n \to \infty$. Moreover, for $n \geqslant 2$

$$S_0 p_{n+1} - S_0 p_n = R_n + DF(p_n)(p_{n+1} - p_n),$$

where $R_n := F(p_n) - F(p_{n-1}) - DF(p_{n-1})(p_n - p_{n-1})$ is the Taylor remainder. It follows that

$$\begin{aligned} E_{n+1}'' + \mu E_{n+1} &= R_n + p_n \int_0^x \left(3C_4 p_n^2 + C_5 t\right) E_{n+1} \, dt \\ &\quad + \left(\mu + \varphi + \frac{C_2}{2}\left(p_{n+1}^2 + p_{n+1}p_n + p_n^2\right) + \int_0^x \left(C_4 p_n^2 + C_5 t\right) p_n \, dt\right) E_{n+1} \\ &\geqslant R_n + p_n \int_0^x \left(3C_4 p_n^2 + C_5 t\right) E_{n+1} \, dt. \end{aligned}$$

Thus, if $\Phi \in H^2 \cap H_0^1(0, T)$ denotes the unique solution of

$$\Phi'' + \mu\Phi = R_n + p_n \int_0^x \big(3C_4 p_n^2 + C_5 t\big) E_{n+1}\,\mathrm{d}t, \tag{4.7}$$

then the standard comparison principle implies that $E_{n+1} \leqslant \Phi$. Moreover,

$$\|\Phi\|_C \leqslant Tk_0 \|\Phi'' + \mu\Phi\|_C \leqslant Tk_0\big(\|R_n\|_C + k_1\|E_{n+1}\|_C\big),$$

and hence

$$\|E_{n+1}\|_C \leqslant \frac{Tk_0}{1 - Tk_0k_1}\|R_n\|_C.$$

Furthermore, from the Taylor expansion of F we deduce that

$$\|R_n\|_C \leqslant T\left(6|C_4|\|\beta\|_C^2 + |C_5|\frac{T}{2}\right)\|E_n\|_C^2.$$

Finally, note that if (4.6) holds, then it also holds for any $m \geqslant n$, and the result is established. □

5. Painlevé II

Here, we present Painlevé II solutions of a suitably constrained version of boundary value problem (2.1). Thus, we obtain particular nonconstant solutions of (2.1) such that

$$p'' + \lambda p^3 + \mu x p + \nu = 0 \tag{5.1}$$

for appropriate constants λ, μ and ν. Note that if λ is negative, the transformation $p(x) \mapsto Y(x) := \sigma p(\omega x)$ for suitable choice of σ and ω gives a solution of the standard Painlevé II equation $Y'' = 2Y^3 \pm xY + C$. One use of (5.1) to eliminate p''' and p'' in (2.1), it is seen that

$$\begin{aligned}
&-p\big[3\lambda p^2 p' + \mu p + \mu x p'\big] + p'\big[\lambda p^3 + \mu x p + \nu\big] + (\nu_1+\nu_2+\nu_3)p^2\big[\lambda p^3 + \mu x p + \nu\big] \\
&\quad + (\nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3)p^3 p' - (\nu_1 c_1 + \nu_2 c_2 + \nu_3 c_3)p' - \frac{1}{2}\nu_1\nu_2\nu_3 p^5 \\
&\quad + \nu_1\nu_2\nu_3(c_1+c_2+c_3)xp^3 - \big[(\nu_2+\nu_3)\nu_1 c_1 + (\nu_1+\nu_3)\nu_2 c_2 + (\nu_1+\nu_2)\nu_3 c_3\big]p^2 \\
&= 0,
\end{aligned}$$

whence we obtain:

$$\begin{aligned}
&\lambda = \frac{1}{2}(\nu_2\nu_3 + \nu_1\nu_3 + \nu_1\nu_2), \\
&\mu = (\nu_1 c_1 + \nu_2 c_2 + \nu_3 c_3)(\nu_1+\nu_2+\nu_3) \\
&\qquad - \big[(\nu_2+\nu_3)\nu_1 c_1 + (\nu_1+\nu_3)\nu_2 c_2 + (\nu_1+\nu_2)\nu_3 c_3\big], \\
&\nu = \nu_1 c_1 + \nu_2 c_2 + \nu_3 c_3, \\
&\mu(\nu_1+\nu_2+\nu_3) + \nu_1\nu_2\nu_3(c_1+c_2+c_3) = 0, \\
&(\nu_1+\nu_2)(\nu_1+\nu_3)(\nu_2+\nu_3) = 0.
\end{aligned} \tag{5.2}$$

In view of the latter condition, we proceed with the constraint $\nu_1 + \nu_2 = 0$, $\nu_3 \neq 0$ whence $c_3 = 0$, and

$$\lambda = -\nu_1^2/2, \qquad \mu = \nu_1^2(c_1+c_2), \qquad \nu = c_1\nu_1 + c_2\nu_2 = \nu_1(c_1 - c_2). \tag{5.3}$$

Finally, from the boundary conditions in (2.1) we obtain the constraint

$$r_0 = \frac{\nu_1^2}{2} p_0^3 - \nu_1(c_1 - c_2). \tag{5.4}$$

Analogous results hold by cyclic interchange for $\{\nu_2 + \nu_3 = 0,\ \nu_1 \neq 0,\ c_1 = 0\}$ and $\{\nu_3 + \nu_1 = 0,\ \nu_2 \neq 0,\ c_2 = 0\}$.

Conversely, if $\tilde{p}$ is a solution of (5.1) for some $\lambda < 0$, μ and ν, then we obtain a solution of (2.1) by setting $\tilde{p}_0 = \tilde{p}(0)$, $\tilde{p}_T = \tilde{p}(T)$, $\tilde{r}_0 = -\alpha \tilde{p}_0^3 - \gamma$, and

$$\tilde{\nu}_1 = -\tilde{\nu}_2 := \pm\sqrt{2|\lambda|},$$
$$\tilde{c}_1 = \frac{1}{2}\left(\frac{\nu}{\tilde{\nu}_1} - \frac{\mu}{2\lambda}\right), \qquad \tilde{c}_2 = -\frac{1}{2}\left(\frac{\nu}{\tilde{\nu}_1} + \frac{\mu}{2\nu}\right), \qquad \tilde{c}_3 = 0.$$

The constant $\tilde{\nu}_3$ may be chosen arbitrarily.

A Painlevé II solution of the boundary value problem (2.1) may be used as a lower or an upper solution for a related boundary problem, for which Theorem 3.1 or Theorem 3.2 applies. In particular, the following result holds:

Corollary 5.1. *Let $\tilde{p}$ be a nonnegative concave solution of* (5.1) *for some $\lambda < 0$, $\mu \leqslant 0 \leqslant \nu$. Fix a constant $\tilde{\nu}_3 > 0$ and assume that $\tilde{p}(0) > 0$. Then the boundary value problem* (3.1) *admits at least one solution p such that $0 \leqslant p \leqslant \tilde{p}$ for any choice of the parameters for which*:

(i) $0 < p_0 \leqslant \tilde{p}_0$, $0 \leqslant p_T \leqslant \tilde{p}_T$.

(ii) $\nu_1 + \nu_2 + \nu_3 = 0$, $c_1 + c_2 + c_3 \leqslant 0$.

(iii) $0 \leqslant \nu_1 c_1 + \nu_2 c_2 - c_3(\nu_1 + \nu_2) \leqslant \nu$.

(iv) $-2\lambda \leqslant \nu_1^2 + \nu_1\nu_2 + \nu_2^2$.

(v) $0 < \nu_1\nu_2(\nu_1 + \nu_2) \leqslant -2\lambda\tilde{\nu}_3$.

(vi) $\tilde{\nu}_3\nu \leqslant \nu_1\nu_2(\nu_1 + \nu_2)(c_1 + c_2 + c_3)$.

(vii) $$\frac{\tilde{r}_0 + \nu}{\tilde{p}_0} + \lambda\tilde{p}_0^2 + (\tilde{\nu}_3\nu - \mu)j \leqslant \frac{r_0 + \nu_1 c_1 + \nu_2 c_2 - c_3(\nu_1 + \nu_2)}{p_0} - \left(\nu_1^2 + \nu_1\nu_2 + \nu_2^2\right)\frac{p_0^2}{2} + \left[\nu_1^2 c_1 + \nu_2^2 c_2 - (\nu_1 + \nu_2)^2 c_3\right]j, \quad j = 0, 1.$$

From conditions (ii) *and* (iii) *it follows that $\tilde{C}_3 \geqslant C_3 \geqslant 0$. By* (iv), *$\alpha \equiv 0$ is a lower solution of* (3.1). *Moreover, it follows from* (vii) *that $\tilde{\varphi} \leqslant \varphi$, and as $\tilde{C}_1 = \tilde{\nu}_3 \geqslant 0$ and $\tilde{p}$ is concave and nonnegative, then $\tilde{C}_1(\tilde{p}' - \tilde{p}'(0))\tilde{p} \leqslant 0$. From* (iv), *we also deduce that $\tilde{C}_2 \leqslant C_2$. Finally, from* (ii), (v) *and* (vi) *we conclude that $\tilde{C}_4 \leqslant C_4 \leqslant 0$ and $\tilde{C}_5 \leqslant C_5 \leqslant 0$. This implies that $\tilde{p}$ is an upper solution of* (3.1).

It is noted that if $\tilde{\nu}_3 \gg 0$, then conditions (ii) *to* (vi) *are fulfilled for appropriate choices of ν_i and c_i. Moreover, condition* (vii) *holds if r_0/p_0 is large enough.*

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