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Landesman–Lazer type conditions for a system of p-Laplacian like operators

Pablo Amster [a,b,*], Pablo De Nápoli [a,b]

[a] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*
[b] *Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina*

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Abstract

We study the existence of periodic solutions for a nonlinear second order system of ordinary differential equations of p-Laplacian type. Assuming suitable Nagumo and Landesman–Lazer type conditions we prove the existence of at least one solution applying topological degree methods. We extend a celebrated result by Nirenberg for resonant systems.

Keywords: p-Laplacian systems; Landesman–Lazer conditions

1. Introduction

We study the nonlinear system of second order differential equations

$$\phi(u')' = f(t, u, u'), \quad t \in (0, T), \tag{1}$$

under periodic boundary conditions

$$u(0) = u(T), \qquad u'(0) = u'(T). \tag{2}$$

Following the pioneering work of Manásevich and Mawhin [6] we assume that $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

* Corresponding author.
E-mail addresses: pamster@dm.uba.ar (P. Amster), pdenapo@dm.uba.ar (P. De Nápoli).

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(1) For any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, we have that

$$\big\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \big\rangle > 0. \tag{3}$$

(2) There exists a function $\alpha : (0, +\infty) \to (0, \infty)$ such that $\alpha(s) \to +\infty$ as $s \to +\infty$ and

$$\big\langle \phi(x), x \big\rangle \geqslant \alpha\big(|x|\big)|x| \quad \text{for all } x \in \mathbb{R}^N. \tag{4}$$

We remark that (3) and (4) imply that ϕ is an homeomorphism onto $\mathbb{R}^N$ (for details see [6]). Equation (1) is usually referred in the literature as a p-Laplacian type equation; indeed, the most standard examples in which the previous conditions hold are the N-dimensional p-Laplacian given by $\phi(x) = |x|^{p-2}x$ (with $p > 1$), and a system of one-dimensional p-Laplacians, namely: $\phi(x) = (|x_1|^{p_1-2}x_1, \dots, |x_N|^{p_N-2}x_N)$ ($p_j > 1$).

Without loss of generality, we may assume that $\phi(0) = 0$. For simplicity, we shall also assume that $f : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}^N$ is a continuous function.

We obtain solutions of (1)–(2) under Landesman–Lazer type conditions applying topological degree methods [7].

There exists a vast literature on Landesman–Lazer type conditions for resonant problems, starting at the pioneering work [5] for a resonant elliptic second order scalar equation under Dirichlet conditions (for a survey on Landesman–Lazer conditions see, e.g., [8]). In [10], Nirenberg extended these results to systems of elliptic equations. Nirenberg's result can be adapted to our problem (1)–(2) in the following way:

Theorem 1.1. *Let $\phi(x) = x$ and $f(t, u, u') = p(t) - cu' - g(u)$, and assume that the radial limits $g_v := \lim_{r\to+\infty} g(rv)$ exist uniformly respect to $v \in S^{N-1}$, the unit sphere of $\mathbb{R}^N$. Then* (1)–(2) *has at least one T-periodic solution if the following conditions hold*:

- *$g_v \neq \overline{p} := \frac{1}{T}\int_0^T p(t)\,dt$ for any $v \in S^{N-1}$.*
- *The degree of the mapping $\theta : S^{N-1} \to S^{N-1}$ given by*

  $$\theta(v) = \frac{g_v - \overline{p}}{|g_v - \overline{p}|}$$

  is different from 0.

In [12] Ortega and Sánchez gave an interesting example which shows that, in some sense, the existence of radial limits of g is necessary. More precisely, they have shown a system with ϕ and f as in Theorem 1.1 for which no periodic solution exists, although the following conditions are fulfilled for some $R > 0$:

- $g(u) \neq \overline{p}$ for $|u| \geqslant R$.
- The degree of the mapping $\theta_R : S^{N-1} \to S^{N-1}$ given by

  $$\theta_R(v) = \frac{g(Rv) - \overline{p}}{|g(Rv) - \overline{p}|}$$

  is different from 0.

Despite this example, we shall show that the assumption on the existence of radial limits can be replaced by a weaker condition (see condition (F1)).

As usual, when topological methods are applied, it is essential to obtain a priori bounds for the solutions. When f is a bounded function, a priori bounds can be deduced directly from Landesman–Lazer type conditions. However, if f is unbounded as a function of u', an extra assumption is required. We introduce a Nagumo type condition, which allows to establish a priori bounds for the derivatives. Nagumo condition was first introduced in [11] for a (linear) scalar equation, and generalized in many ways for systems of linear equations (see, e.g., [1]). A Nagumo condition for a scalar equation and general ϕ was introduced for example in [2]. The case of an N-dimensional p-Laplacian was studied in [9], where a priori bounds are obtained using Nagumo and Hartman type conditions. In this paper we shall assume a slightly different condition (see condition (N)), which can be regarded as an extension of the Nagumo assumption introduced in [3].

The paper is organized as follows. In Section 2 we give some notations and preliminary results. In particular, we recall the continuation theorem that will be used in the proofs. In Section 3, we introduce appropriate Landesman–Lazer type conditions and prove the existence of solutions of (1)–(2) for f bounded. Finally, in Section 4 we study the general case assuming a Nagumo type condition for f.

2. Some notations and preliminary results

We denote by C_T^1 the space of T-periodic functions in $C^1([0,T])$. The results we recall in this section are proved in [6]:

Proposition 2.1. *For $l \in C([0,T])$, let us define*

$$G_l(a) = \frac{1}{T}\int_0^T \phi^{-1}\big(a + l(t)\big)\,dt.$$

If ϕ satisfies conditions (3) *and* (4)*, then the function G_l has the following properties*:

(1) *For any fixed $l \in C([0,T])$, the equation*

$$G_l(a) = 0$$

has a unique solution $a = a(l)$.

(2) *The function $a : C([0,T]) \to \mathbb{R}^N$ thus defined, is continuous and sends bounded sets into bounded sets.*

Moreover, the following continuation theorem provides an analogue of the Mawhin coincidence degree theory (see [7]) for p-Laplacian type operators. An abstract version of the theory for more general nonlinear operators can be found in [4].

Theorem 2.1. *Let $\Omega \subset C_T^1$ an open set. Assume that*:

(1) *For $\lambda \in (0,1]$ the problem*

$$\phi(u')' = \lambda f(t,u,u') \tag{5}$$

has no solutions on $\partial\Omega$.

(2) *The equation*

$$F(u) = \frac{1}{T}\int_0^T f(t,u,0)\,dt = 0$$

has no solutions in $\partial\Omega \cap \mathbb{R}^N$.

(3) $$d_B\big(F, \Omega \cap \mathbb{R}^N, 0\big) \neq 0, \tag{6}$$

where d_B *denotes the Brouwer degree.*

Then problem (1)–(2) *has at least one solution in* Ω.

3. Existence results for bounded f

Throughout this section, we shall assume that f is bounded. In this case, we first observe that for any $u \in C_T^1$, if $l(t) = \int_0^t f(s,u,u')\,dt$ and $0 \leqslant \lambda \leqslant 1$ then $|a(\lambda l)| \leqslant k$ for some constant k depending only on $\|f\|_C$. Thus, if

$$\phi(u')' = \lambda f(t,u,u')$$

then $\phi(u'(t)) = a(\lambda l) + \lambda l(t)$, and we deduce that $|\phi(u')| \leqslant k + T\|f\|_C$. Hence $\|u'\|_C \leqslant M$ for some constant M.

Our Landesman–Lazer type condition reads as follows:

Condition (F1): There exists a family $\{(U_j, w_j)\}_{j=1,\dots,K}$ where U_j is an open subset of S^{N-1} and $w_j \in S^{N-1}$, such that $\{U_j\}$ covers S^{N-1} and the limit

$$\limsup_{s\to+\infty}\big\langle f(t,su,v), w_j\big\rangle := \overline{f}_{u,j}(t) \tag{7}$$

exists uniformly for $u \in U_j$ and $v \in \mathbb{R}^N$ with $|v| \leqslant M$ (M as before).

Remark 3.1. In particular, condition (F1) holds trivially if $f = p(t) - g(u)$, and radial limits for g exist uniformly as in Theorem 1.1. As condition (F1) may be hard to verify, we shall give a more explicit one (see condition (F2)).

Remark 3.2. If condition (F1) holds, a straightforward computation shows that the mapping $u \mapsto \overline{f}_{u,j}(t)$ is continuous in U_j for each fixed t. Indeed, if $\varepsilon > 0$ set $s_0 > 0$ and a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\big\langle f(t,su,v), w_j\big\rangle - \overline{f}_{u,j}(t) < \frac{\varepsilon}{3} \quad \text{for } s \geqslant s_0,$$

and

$$\overline{f}_{u,j}(t) - \big\langle f(t,s_n u,v), w_j\big\rangle < \frac{\varepsilon}{3} \quad \text{for } n \in \mathbb{N},$$

for every $u \in U_j$ and $|v| \leqslant M$. Fixing n such that $s_n \geqslant s_0$ we obtain:

$$\begin{aligned}\big|\overline{f}_{u,j}(t) - \overline{f}_{u_0,j}(t)\big| \leqslant{}& \big|\overline{f}_{u,j}(t) - \big\langle f(t,s_n u,v), w_j\big\rangle\big| \\ &+ \big|\big\langle f(t,s_n u,v), w_j\big\rangle - \big\langle f(t,s_n u_0,v), w_j\big\rangle\big| \\ &+ \big|\big\langle f(t,s_n u_0,v), w_j\big\rangle - \overline{f}_{u_0,j}(t)\big|.\end{aligned}$$

Taking $\delta > 0$ small enough, the second term in the right-hand side is less than $\frac{\varepsilon}{3}$ for $|u - u_0| < \delta$, and it follows that $|\overline{f}_{u,j}(t) - \overline{f}_{u_0,j}(t)| < \varepsilon$.

Theorem 3.3. *Assume that f is bounded, and that condition* (F1) *holds. Then the periodic boundary value problem* (1)–(2) *admits at least one solution, provided that*:

(1) *For each $u \in S^{N-1}$ there exists j such that $u \in U_j$ and*

$$\int_0^T \overline{f}_{u,j}(t)\,dt < 0.$$

(2) *There exists a constant R_0 such that $d_B(F, B_R, 0) \neq 0$ for any $R \geqslant R_0$, where $B_R \subset \mathbb{R}^N$ denotes the open ball of radius R centered at* 0, *and F is defined as in Theorem* 2.1.

Remark 3.4. It follows from the proof below that $F(u) \neq 0$ for $u \in \mathbb{R}^N$ with $|u|$ large. Thus, the Brouwer degree in condition (2) is well defined.

Proof of Theorem 3.3. We claim that the periodic solutions of $\phi(u')' = \lambda f(t, u, u')$ with $0 < \lambda \leqslant 1$ are a priori bounded for the C^1-norm. Indeed, otherwise there exist sequences $\lambda_n \in (0, 1]$ and $\{u_n\} \in C_T^1$ such that $\phi(u_n')' = \lambda_n f(t, u_n, u_n')$ and $\|u_n\|_{C^1} \to \infty$. From the previous considerations $\|u_n'\|_C \leqslant M$, and thus $u_n - u_n(0)$ is bounded, $|u_n(0)| \to \infty$. In particular, $|u_n(t)| \to \infty$ uniformly in t. We may assume that $u_n(t) \neq 0$, and define $z_n(t) = \frac{u_n(t)}{|u_n(t)|}$. Taking a subsequence if necessary, we may assume that $z_n \to u$ uniformly in t for some $u \in S^{N-1}$. From condition (1), $\int_0^T \overline{f}_{u,j}\,dt < -\varepsilon < 0$ for some j, then for each fixed t we obtain:

$$\begin{aligned}
&\big\langle f\big(t, u_n(t), u_n'(t)\big), w_j\big\rangle - \overline{f}_{u,j}(t) \\
&\quad = \big\langle f\big(t, |u_n(t)| z_n(t), u_n'(t)\big), w_j\big\rangle - \overline{f}_{z_n(t),j}(t) + \overline{f}_{z_n(t),j}(t) - \overline{f}_{u,j}(t) \\
&\quad < \frac{\varepsilon}{T}
\end{aligned}$$

when n is large enough. Then by Fatou Lemma,

$$\limsup_{n\to\infty} \int_0^T \big\langle f\big(t, u_n(t), u_n'(t)\big), w_j\big\rangle\,dt \leqslant \int_0^T \overline{f}_{u,j}(t) + \frac{\varepsilon}{T}\,dt < 0,$$

a contradiction since $\int_0^T f(t, u_n(t), u_n'(t))\,dt = \int_0^T \phi(u_n')'(t)\,dt = 0$. In the same way, it is easy to see that $F(u) \neq 0$ for $u \in \mathbb{R}^N$ with $|u|$ large.

Thus, fixing $\Omega = B_R(0) \subset C_T^1$ with R large enough, the proof follows from Theorem 2.1. □

In the next result we shall consider a particular case of the previous theorem. Let us first note that condition (F1) implies, for any fixed $x \in \mathbb{R}^N$ and $u_0 \in S^{N-1}$, that

$$\limsup_{s\to+\infty} \big\langle f(t, x + su, v), w_j\big\rangle = \overline{f}_{u,j}(t)$$

for some j, uniformly for u in a neighborhood of u_0 and $|v| \leqslant M$. The following condition is stronger than (F1), since we impose a uniformity condition with respect to x. However, it has the advantage that it allows to compute the Brouwer degree explicitly.

Condition (F2): Let $\{e_1, \dots, e_N\}, \{w_1, \dots, w_N\} \subset S^{N-1}$ be two arbitrary bases of $\mathbb{R}^N$, and assume that the limits

$$\limsup_{s\to+\infty} \langle f(t, x + se_i, v), w_i \rangle := \overline{f}_i(t) \tag{8}$$

and

$$\liminf_{s\to-\infty} \langle f(t, x + se_i, v), w_i \rangle := \underline{f}_i(t) \tag{9}$$

exist uniformly respect to $x \in \operatorname{span}\{e_j\colon j \neq i\}$ and $v \in \mathbb{R}^N$ with $|v| \leqslant M$ (M as before).

Remark 3.5. It is easy to see that (F2) implies (F1). Indeed, if $u \in S^{N-1}$ then $u = x + \alpha e_i$ with $x \in \operatorname{span}\{e_j\colon j \neq i\}$ for some i and $\alpha \neq 0$. Fix $\delta < |\alpha|$, and consider $\tilde{u} = \tilde{x} + \tilde{\alpha} e_i \in U := B_\delta(u) \cap S^{N-1}$. Then

$$\limsup_{s\to+\infty} \langle f(t, s\tilde{u}, v), w_i \rangle = \limsup_{s\to+\infty} \langle f(t, s\tilde{x} + s\tilde{\alpha} e_i, v), w_i \rangle = \overline{f}_i(t)$$

if $\alpha > 0$, and

$$\limsup_{s\to+\infty} \langle f(t, s\tilde{u}, v), -w_i \rangle = -\liminf_{s\to-\infty} \langle f(t, -s\tilde{x} - s\tilde{\alpha} e_i, v), w_i \rangle = -\underline{f}_i(t)$$

if $\alpha < 0$, uniformly for $\tilde{u} \in U$ and $|v| \leqslant M$. Thus, the result follows from the compactness of S^{N-1}.

Theorem 3.6. *Assume that f is bounded and that* (F2) *holds. Then problem* (1)–(2) *admits at least one solution, provided that*

$$\int_0^T \overline{f}_i(t)\,dt < 0 < \int_0^T \underline{f}_i(t)\,dt$$

for each $i = 1, \dots, N$.

Proof. From Remark 3.5 and the hypothesis, it is clear that condition (1) in Theorem 3.3 holds. In order to compute the Brouwer degree $d_B(F, B_R, 0)$ for large R, consider the homotopy $H : [0, 1] \times \mathbb{R}^N \to \mathbb{R}^N$ given by

$$H(\lambda, u) = \lambda F(u) - (1 - \lambda) C u,$$

where $C : \mathbb{R}^N \to \mathbb{R}^N$ is the isomorphism uniquely defined by the identities $\langle C e_i, w_j \rangle = \delta_{ij}$. Suppose that $H(\lambda, u) = 0$ for some $\lambda \in [0, 1]$ and $|u| = R$. Writing $u = \sum_{j=1}^N a_j e_j$ we deduce that $|a_i|$ is large for some i. Suppose for example that $a_i \gg 0$, then

$$0 = \langle \lambda F(u) - (1 - \lambda) C u, w_i \rangle = \lambda \langle F(u), w_i \rangle - (1 - \lambda) a_i.$$

On the other hand,

$$\limsup_{s\to+\infty} \int_0^T \langle f(t, x + se_i, 0), w_i \rangle\,dt \leqslant \int_0^T \overline{f}_i(t)\,dt < 0.$$

Thus $\langle F(u), w_i \rangle < 0$, which yields a contradiction. The proof is analogous if $a_i \ll 0$. We conclude that, for R large,

$$d_B(F, B_R, 0) = (-1)^N d_B(C, B_R, 0) = \pm 1,$$

and the proof is complete. □

Example 3.7. Let

$$f_i(t,u,v) = \mu_i(t,u_i) + \frac{\theta_i(t,u,v)}{u_i^2+1}$$

with μ_i, θ_i bounded, and

$$\limsup_{s\to+\infty} \mu_i(t,s) := \overline{\mu}_i(t),$$
$$\liminf_{s\to-\infty} \mu_i(t,s) := \underline{\mu}_i(t).$$

Furthermore, assume that

$$\int_0^T \overline{\mu}_i(t)\,dt < 0 < \int_0^T \underline{\mu}_i(t)\,dt.$$

Thus, if we set $\{e_1,\dots,e_N\}$ and $\{w_1,\dots,w_N\}$ as the canonical basis of $\mathbb{R}^N$, the assumptions of the previous theorem hold, with $\overline{f}_i = \overline{\mu}_i$, $\underline{f}_i = \underline{\mu}_i$. Note, however, that radial limits of f do not necessarily exist.

Remark 3.8. As in [12], from the mean value theorem for vector-valued integrals it follows that if (1)–(2) admits a solution, then 0 belongs to the closed convex hull of $f([0,T]\times\mathbb{R}^{2N})$. Indeed, if we consider the closed curve $\gamma(t) = f(t,u(t),u'(t))$, then $\frac{1}{T}\int_0^T \gamma(t)\,dt = 0$, and hence 0 belongs to the convex hull of the set $\{\gamma(t)\colon\ 0 \leqslant t \leqslant T\}$.

4. Nagumo-type conditions

In this section we study the existence of solutions for f not necessarily bounded, assuming a Nagumo type condition. Let $\{z_1,\dots,z_N\}$ and $\{w_1,\dots,w_N\}$ be two arbitrary bases of $\mathbb{R}^N$ and assume that ϕ satisfies the following:

Condition (Φ): For each $i = 1,\dots,N$ there exists a constant $R_i \geqslant 0$ such that if $\langle x, z_i\rangle = 0$ then $|\langle\phi(x), w_i\rangle| \leqslant R_i$.

Remark 4.1. Condition (Φ) is trivially satisfied if $\phi(x) = (\phi_1(x_1),\dots,\phi_N(x_N))$ (uncoupled case), taking $\{z_1,\dots,z_N\}$ and $\{w_1,\dots,w_N\}$ as the canonical basis of $\mathbb{R}^N$, and $R_i = 0$. More generally, one may consider any ϕ such that if $x_i = 0$ then $\phi_i(x) = 0$: for example, this is the case of the vector valued p-Laplacian given by $\phi(x) = |x|^{p-2}x$.

Assuming (Φ), we state our Nagumo type condition in the following way:

Condition (N): For each $i = 1,\dots,N$ there exists $M_i > R_i$ and a function $\psi_i : [0,+\infty) \to (0,+\infty)$ such that

$$\bigl|\bigl\langle f(t,u,v), w_i\bigr\rangle\bigr| \leqslant \psi_i\bigl(\bigl|\bigl\langle\phi(v), w_i\bigr\rangle\bigr|\bigr)$$

for arbitrary $(t,u,v) \in [0,T]\times\mathbb{R}^{2N}$ and

$$\int_{R_i}^{M_i} \frac{1}{\psi_i(s)}\,ds > T.$$

Thus, we obtain a priori bounds for the derivatives of the solutions. More precisely:

Lemma 4.2. *Assume that* (Φ) *and* (N) *hold. Then there exists a constant* M *such that if* $u \in C^1_T$ *satisfies* $\phi(u')' = \lambda f(t, u, u')$ *for some* $\lambda \in [0, 1]$ *then* $\|u'\|_C \leqslant M$.

Proof. We shall prove that in fact $|\langle \phi(u'), w_i\rangle| < M_i$ for $i = 1, \ldots, N$. Indeed, suppose for example that $\langle \phi(u'(\tilde{t})), w_i\rangle \geqslant M_i$ for some $\tilde{t}$ and some i. As $\langle u(0), z_i\rangle = \langle u(T), z_i\rangle$, by Rolle Theorem we deduce that $\langle u'(t), z_i\rangle = 0$ for some t. Using (Φ) and the continuity of $\phi(u')$ we conclude that $\langle \phi(u'(t_0)), w_i\rangle = R_i$ for some t_0, and $\langle \phi(u'(t_1)), w_i\rangle = M_i$ for some t_1. Furthermore, we may suppose that $\langle \phi(u'(t)), w_i\rangle \in (R_i, M_i)$ for any t between t_0 and t_1. Thus

$$T < \int_{R_i}^{M_i} \frac{1}{\psi_i(s)}\, ds = \int_{t_0}^{t_1} \frac{\lambda\langle f(t, u, u'), w_i\rangle}{\psi_i(\langle \phi(u'(t)), w_i\rangle)}\, dt.$$

By (N), this last term is less or equal that $|t_1 - t_0|$, a contradiction. The proof is analogous if $\langle \phi(u'(\tilde{t})), w_i\rangle \leqslant -M_i$ for some $\tilde{t}$. □

Thus we have:

Corollary 4.1. *Theorems* 3.3 *and* 3.6 *are still true if the condition 'f bounded' is replaced by* (Φ) *and* (N)*, and the constant* M *in respective conditions* (F1) *and* (F2) *is given by the previous lemma.*

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