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Oscillating solutions of a nonlinear fourth order ordinary differential equation

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Abstract

We study the existence of periodic solutions for a nonlinear fourth order ordinary differential equation. Under suitable conditions we prove the existence of at least one solution of the problem applying coincidence degree theory and the method of upper and lower solutions.

Keywords: Nonlinear ordinary differential equation; Multi-ion electrodiffusion theory; Semiconductors modelling

1. Introduction

In the last years there has been an increasing interest in higher order problems which arise in different models in Physics and Biology, such as the Swift–Hohenberg equation and the Extended Kolmogorov–Fisher equation, with $a > 0$ and $a < 0$ respectively (see e.g. [3,5,12,13]):

$$u^{(4)} + au'' + u^3 - u = 0.$$

On the other hand, we may also mention the classical nonlinear beam equations [6,7], equations from multi-ion electrodiffusion theory [9], or the one-dimensional stationary case of a quantum hydrodynamic model for semiconductors [8], namely

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$$-\delta^2\big(n(\log n)''\big)' + \left(\frac{J^2}{n} + Tn\right)' - nV' = -\frac{J}{\tau},$$
$$\lambda^2 V'' = n - C.$$

For the limit procedure $\delta \to 0$, after the change of variables $n = e^u$ and differentiation, the following equation is obtained:

$$\delta^2\left(u'' + \frac{(u')^2}{2}\right)'' + J^2\big(e^{-2u}u'\big)' - Tu'' + \frac{e^u - C}{\lambda^2} = \frac{J}{\tau}\big(e^{-u}\big)'.$$

In this work, we consider the problem:

$$u^{(4)} + au'' + g(t, u, u', u'', u''') = p(t) \tag{1}$$

under periodic boundary conditions

$$u^{(j)}(0) = u^{(j)}(T), \quad j = 0, \dots, 3,$$

for $p \in L^2(0, T)$ and g continuous, where T is a fixed positive constant. We remark that problem (1) is resonant, since the homogeneous problem $u^4 + au'' = 0$ admits any constant $c \in \mathbb{R}$ as a nontrivial periodic solution. In particular, if p is not orthogonal to constants in $L^2(0, T)$, then problem (1) has no solutions when g is bounded, with $\|g\|_{L^\infty} < \frac{1}{T}|\int_0^T p(t)\,dt|$. Indeed, if u is a solution, integrating the equation it follows that

$$\left|\int_0^T p(t)\,dt\right| = \left|\int_0^T g(t, u, u', u'', u''')\,dt\right| \leqslant T\|g\|_{L^\infty},$$

a contradiction. However, existence of solutions can be proved in the following context:

Theorem 1.1. *Let $a \notin (\frac{2\pi}{T}\mathbb{N})^2$ and $p \in L^2(0, T)$. Assume that g satisfies the condition*

$$\big|g(t, u, v, w, z)\big| \leqslant A + \varepsilon\big(|v| + |w| + |z|\big) \tag{2}$$

for some constant A and $\varepsilon < \frac{1}{c}$, where c is the constant given by Lemma 2.2 below. Moreover, assume that the limits

$$\limsup_{s\to+\infty} g(t, s, V) := g^+_{\sup}(t), \qquad \liminf_{s\to-\infty} g(t, s, V) := g^-_{\inf}(t), \tag{3}$$

or

$$\limsup_{s\to-\infty} g(t, s, V) := g^-_{\sup}(t), \qquad \liminf_{s\to+\infty} g(t, s, V) := g^+_{\inf}(t) \tag{4}$$

exist uniformly on $V \in [-k, k]^3$, with $k = \frac{(AT+\|p\|_{L^2}T^{1/2})c}{1-c\varepsilon}$.

Then problem (1) *admits at least one solution, provided respectively that*:

$$\int_0^T g^+_{\sup}(t)\,dt < \int_0^T p(t)\,dt < \int_0^T g^-_{\inf}(t)\,dt \tag{5}$$

or

$$\int_0^T g^-_{\sup}(t)\,dt < \int_0^T p(t)\,dt < \int_0^T g^+_{\inf}(t)\,dt. \tag{6}$$

Remark 1.2. Conditions in Theorem 1.1 can be considered as an extension of Landesman–Lazer conditions for fourth order equations (see e.g. [11]; for a third order equation, see e.g. [1]). For example, if g is strictly monotone with respect to u, then condition (5) or (6) is also necessary: indeed, if g is, say, strictly nonincreasing with respect to u, and $u(t)$ is a solution, then $g^+_{\sup}(t) < g(t,u,u',u'',u''') < g^-_{\inf}(t)$, and thus

$$\int_0^T g^+_{\sup}(t)\,dt < \int_0^T p(t)\,dt < \int_0^T g^-_{\inf}(t)\,dt.$$

On the other hand, using the method of upper and lower solutions we obtain an existence result for $g = g(t,u,u'')$. For related results on this line, see e.g. [2,4].

For simplicity, we shall assume that g is continuously differentiable with respect to u and u''.

Theorem 1.3. *Let $g = g(t,u,u'')$ be continuously differentiable with respect to u and u'', and assume there exist functions $\alpha, \beta \in H^4_{\mathrm{per}}(0,T)$ such that*

$$\alpha^{(4)} + a\alpha'' + g(\cdot,\alpha,\alpha'') \leqslant p,$$
$$\beta^{(4)} + a\beta'' + g(\cdot,\beta,\beta'') \geqslant p$$

and

$$\alpha'' - K\alpha \geqslant \beta'' - K\beta$$

for some constant $K > 0$. Furthermore, assume that

$$\frac{\partial g}{\partial u}(t,u,v) + K\left(\frac{\partial g}{\partial u''}(t,u,v) + a + K\right) \leqslant 0 \tag{7}$$

for every $(t,u,v) \in \mathcal{C}$, where $\mathcal{C}$ is the set of all the vectors $(t,u,v) \in [0,T] \times \mathbb{R}^2$ satisfying

$$\alpha(t) \leqslant u \leqslant \beta(t),$$
$$\alpha''(t) - K\alpha(t) \geqslant v - Ku \geqslant \beta''(t) - K\beta(t).$$

Then there exists a solution $u \in H^4_{\mathrm{per}}(0,T)$ of (1), with $\alpha \leqslant u \leqslant \beta$.

Remark 1.4. The technical condition (7) is due to the fact that, unlike the second order case, a general maximum principle cannot be deduced for fourth order equations. We obtain a restricted maximum principle in Lemma 4.1, which is the key for the proof of Theorem 1.3.

The paper is organized as follows. In the next section we establish the general setting of the problem in the context of Mawhin coincidence degree theory and establish some a priori estimates for the problem. In Section 3 we verify the conditions of the continuation theorem of Section 2, and complete the proof of Theorem 1.1. Finally, in Section 4 we give a proof of Theorem 1.3 by monotonicity methods.

2. Coincidence degree theory—General setting of the problem

For the sake of completeness, we summarize in this section the main aspects of coincidence degree theory.

Let X and Y be real normed spaces, $L : \mathrm{Dom}(L) \subset X \to Y$ a linear Fredholm mapping of index 0, and $N : X \to Y$ continuous.

Next, set two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{R}(P) = \mathrm{Ker}(L)$ and $\mathrm{Ker}(Q) = \mathrm{R}(L)$ and an isomorphism $J : \mathrm{R}(Q) \to \mathrm{Ker}(L)$. It is readily seen that

$$L_P := L|_{\mathrm{Dom}(L)\cap\mathrm{Ker}(P)} : \mathrm{Dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{R}(L)$$

is one-to-one; we denote its inverse by K_P. If Ω is a bounded open subset of X, N is called L-compact on Ω if $QN(\Omega)$ is bounded and $K_P(I - Q)N : \Omega \to X$ is compact.

The following continuation theorem is due to Mawhin [10]:

Theorem 2.1. *Let L be a Fredholm mapping of index zero and N be L-compact on a bounded domain $\Omega \subset X$. Suppose*

1. *$Lx \neq \lambda Nx$ for each $\lambda \in (0, 1]$ and each $x \in \partial\Omega$.*
2. *$QNx \neq 0$ for each $x \in \mathrm{Ker}(L) \cap \partial\Omega$.*
3. *$d(JQN, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, where d denotes the Brouwer degree.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}(L) \cap \Omega$.

We shall denote by $H^n_{\mathrm{per}}(0, T)$ the usual Sobolev spaces of periodic functions, namely

$$H^n_{\mathrm{per}}(0, T) = \big\{u \in H^n(0, T):\ u^{(j)}(0) = u^{(j)}(T),\ j = 0, \dots, n - 1\big\}.$$

Then we may consider $X = H^3_{\mathrm{per}}(0, T)$, $Y = L^2(0, T)$ and L, N the operators given by

$$Lu = u^{(4)} + au'',$$
$$Nu = p - g(.\,, u, u', u'', u''').$$

It is clear from (2) that N is well defined, and its continuity follows by dominated convergence. Setting $\mathrm{Dom}(L) = H^4_{\mathrm{per}}(0, T)$, it is immediate to prove that

$$\mathrm{Ker}(L) = \mathbb{R},$$
$$\mathrm{R}(L) = \left\{\varphi \in L^2(0, T):\ \int_0^T \varphi(t)\, dt = 0\right\}.$$

Thus L is a Fredholm mapping of index zero. Moreover, we may take the projectors $P : X \to X$ and $Q : Y \to Y$ as the usual average functions

$$Pu = \bar{u} := \frac{1}{T}\int_0^T u(t)\, dt, \qquad Q\varphi = \bar{\varphi} := \frac{1}{T}\int_0^T \varphi(t)\, dt.$$

Hence, for $\varphi \in \mathrm{R}(L)$ it follows that $K_P(\varphi)$ is the unique solution $u \in H^4_{\mathrm{per}}(0, T)$ of the problem

$$u^{(4)} + au'' = \varphi,$$
$$\bar{u} = 0.$$

We shall make use of the following estimate.

Lemma 2.2. *There exists a constant c such that*

$$\|u\|_{H^4} \leqslant c\|Lu\|_{L^2} \tag{8}$$

for every $u \in H^4_{\mathrm{per}}(0,T)$ *such that* $\bar{u} = 0$.

Proof. If $a = 0$ the result follows from the well-known Poincaré–Wirtinger inequality: indeed, if $u \in H^4_{\mathrm{per}}(0,T)$ satisfies $\bar{u} = 0$ we have that

$$\|u\|_{L^2} \leqslant c\|u'\|_{L^2},$$

and by periodicity

$$\|u^{(j)}\|_{L^2} \leqslant c\|u^{(j+1)}\|_{L^2}, \quad j = 1, 2, 3.$$

If $a < 0$, from the equality

$$\int_0^T u\big(u^{(4)} + au''\big)\,dt = \int_0^T (u'')^2 - a(u')^2\,dt,$$

and Cauchy–Schwarz inequality, we deduce that $\|u''\|_{L^2} \leqslant c\|Lu\|_{L^2}$. Moreover,

$$\|u\|_{H^4} \leqslant c\|u^{(4)}\|_{L^2} \leqslant c\big(\|Lu\|_{L^2} - a\|u''\|_{L^2}\big)$$

and the proof follows.

For $a > 0$, we deduce as before that

$$\|u''\|_{L^2} \leqslant \|Lu\|_{L^2} + a\|u'\|_{L^2}$$

and it suffices to prove that $\|u'\|_{H^1} \leqslant c\|Lu\|_{L^2}$. By contradiction, suppose that $u_n^{(4)} + au_n'' \to 0$ for the L^2-norm, and $\|u_n'\|_{L^2} = 1$. Then $\{u_n\}$ is bounded for the H^4-norm, and choosing a subsequence we may assume that u_n converges for the H^3-norm to some function u. Note that $u_n'' + au_n \to 0$, then $u'' + au = 0$, and as $a \notin (\frac{2\pi}{T}\mathbb{N})^2$ we deduce that $u = 0$, a contradiction. □

Lemma 2.3. *Let L and N be as before and assume that p and g satisfy the assumptions of Theorem* 1.1. *Then N is L-compact on Ω for any bounded domain* $\Omega \subset H^3_{\mathrm{per}}(0,T)$.

Proof. By (2) and the imbedding $H^3(0,T) \hookrightarrow C^2([0,T])$, for $u \in \Omega$ it follows that

$$\|QNu\|_{L^2} = |T|^{1/2}\left|\bar{p} - \frac{1}{T}\int_0^T g(t,u,u',u'',u''')\right| \leqslant M$$

for some constant M. Moreover, if $\varphi = (I - Q)Nu = Nu - \overline{Nu}$, let $v = K_P(\varphi)$ the unique element of $H^4_{\mathrm{per}}(0,T)$ satisfying

$$Lv = \varphi, \quad \bar{v} = 0.$$

From the previous lemma, $\|v\|_{H^4} \leqslant c\|\varphi\|_{L^2} \leqslant C$ for some constant C, and compactness of $K_P(I-Q)N$ follows from the imbedding $H^4_{\mathrm{per}}(0,T) \hookrightarrow H^3_{\mathrm{per}}(0,T)$. □

3. Proof of Theorem 1.1

In this section we choose an appropriate $\Omega \subset H^3_{\mathrm{per}}(0,T)$ for which the conditions of Theorem 2.1 are fulfilled. First, we establish an a priori bound for the solutions of the equation $Lu = \lambda Nu$:

Lemma 3.1. *Let us assume that the conditions of Theorem* 1.1 *hold. Then there exists a positive constant R_0 such that if $Lu = \lambda Nu$ for some $\lambda \in (0,1]$, then $\|u\|_{H^3} < R_0$.*

Proof. We shall proceed by contradiction. Suppose there exists a sequence $\{u_n\} \subset H^3_{\mathrm{per}}(0,T)$ such that $\|u_n\|_{H^3} \to \infty$, and

$$Lu_n = \lambda_n N u_n$$

for some $\lambda_n \in (0,1]$. We may write $u_n = v_n + \bar{u}_n$, and as $Lv_n = Lu_n = \lambda_n N u_n$ from Lemma 2.2 we have:

$$\|v_n\|_{H^4} \leqslant c\|Nu_n\|_{L^2} = c\big\|p - g(t, u_n, u_n', u_n'', u_n''')\big\|_{L^2}.$$

Using condition (2) we deduce that

$$\|v_n\|_{H^4} \leqslant C + c\varepsilon\|v_n\|_{H^3},$$

where $C = c(\|p\|_{L^2} + AT^{1/2})$. Thus, $\|v_n\|_{H^4} \leqslant \frac{C}{1-c\varepsilon}$ and $|\bar{u}_n| \to \infty$. Moreover, writing $v_n^{(j)}(t) = \int_{t_0}^t v_n^{(j+1)}$ for some t_0, it follows that

$$\big\|v_n^{(j)}\big\|_{C([0,T])} \leqslant T^{1/2}\big\|v_n^{(j+1)}\big\|_{L^2} \leqslant k \quad \text{for } j = 1,2,3.$$

On the other hand, integrating the equality

$$u^{(4)} + au_n'' = \lambda_n N u_n$$

it follows that

$$\int_0^T p - g(t, u_n, u_n', u_n'', u_n''')\,dt = 0. \tag{9}$$

Taking a subsequence if necessary, assume for example that $\bar{u}_n \to +\infty$ and that (5) holds. By Fatou Lemma, we obtain from (9) that

$$\bar{p} = \limsup_{n\to\infty} \int_0^T g(t, u_n, u_n', u_n'', u_n''')\,dt \leqslant \int_0^T g^+_{\sup}(t)\,dt,$$

a contradiction. The proof is analogous for the other cases. □

Proof of Theorem 1.1. Set $\Omega = B_{\tilde{R}}(0) \subset H^3_{\mathrm{per}}(0,T)$, with $\tilde{R} \geqslant R_0$, the constant given by the previous lemma. In order to verify condition 2 in Theorem 2.1, let us observe that $\mathrm{Ker}(L) \cap \partial\Omega = \{\pm R\}$, where $R = \tilde{R}/\sqrt{T}$, and

$$QN(\pm R) = \frac{1}{T}\int_0^T p - g(t, \pm R, 0, 0, 0)\,dt.$$

It follows as before that $QN(\pm R) \neq 0$ for R large.

Finally, consider $J = \mathrm{id} : \mathbb{R} \to \mathbb{R}$. From the previous computations the degree $d(JQN, \operatorname{Ker}(L) \cap \Omega, 0)$ is well defined. Furthermore, as

$$JQN(R) \sim \bar{p} - g_+$$

and

$$JQN(-R) \sim \bar{p} - g_-,$$

we conclude from the hypothesis that the function $JQN : (-R, R) \to \mathbb{R}$ changes sign. Thus $d(JQN, \operatorname{Ker}(L) \cap \Omega, 0) = \pm 1$, and assertion 3 in Theorem 2.1 is proved. □

Remark 3.2. From condition (2), if the limits in (3) or (4) exist they are bounded functions, although g might be unbounded. However, (2) can be replaced by the condition

$$\left|g(t, u, v, w, z)\right| \leqslant A + \varepsilon\big(|u| + |v| + |w| + |z|\big)$$

with $\varepsilon < \frac{1}{c(T+1)}$, if we assume that the limits in (3) or (4) are uniform for $V \in \mathbb{R}^3$. Indeed, for v_n as in the previous proof we have that

$$\|v_n\|_{H^4} \leqslant C + c\varepsilon \|v_n\|_{H^4} + c\varepsilon T^{1/2} |\bar{u}_n|.$$

If $|\bar{u}_n|$ is bounded, the proof follows as before; otherwise

$$\left|u_n(t) - \bar{u}_n\right| = \left|\int_{t_0}^{t} u_n'(t)\,dt\right| \leqslant T^{1/2} \|u_n'\|_{L^2} \leqslant K + \frac{c\varepsilon T |\bar{u}_n|}{1 - c\varepsilon}.$$

If for example $\bar{u}_n \to +\infty$ then

$$\bar{u}_n \left(1 - \frac{c\varepsilon T}{1 - c\varepsilon}\right) \leqslant K + u_n(t).$$

Hence $\inf_{t\in[0,T]} u_n(t) \to +\infty$, and the rest of the proof follows as before. The proof is analogous if $\bar{u}_n \to -\infty$. Note that in this case the limits in (3) or (4) belong to $L^1(0, T)$, although not necessarily to $L^\infty(0, T)$.

Example 3.3. It is easy to verify that conditions of Theorem 1.1 hold for the equation

$$u^{(4)} + \frac{\varphi(t, u', u'', u''')}{1 + u^2} = \arctan u + p(t)$$

with φ bounded, and $-\frac{\pi}{2} < \bar{p} < \frac{\pi}{2}$.

4. Upper and lower solutions—Proof of Theorem 1.3

In this section we prove Theorem 1.3, and present an example for which the conditions are satisfied. We begin with a maximum principle for fourth order problems.

Lemma 4.1. *Let $\lambda, \mu > 0$ be such that $\lambda^2 \geqslant 4\mu$ and let $K^\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2}$. If $u \in H^4_{\mathrm{per}}(0, T)$ verifies that*

$$u^{(4)} - \lambda u'' + \mu u \geqslant 0,$$

then

$$u'' - K^{\pm}u \leqslant 0.$$

In particular, $u \geqslant 0$.

Proof. Let $v = u'' - K^+u$, then

$$v'' - K^-v = u^{(4)} - \lambda u'' + \mu u \geqslant 0,$$

and by the classical maximum principle, $v \leqslant 0$ and $u \geqslant 0$. In the same way, we conclude that $u'' - K^+u \leqslant 0$. □

Moreover, we shall need the following estimate.

Lemma 4.2. *Let* $\lambda, \mu > 0$ *and* $Lu = u^{(4)} - \lambda u'' + \mu u$. *Then there exists a constant* c *such that*

$$\|u\|_{H^4} \leqslant c\|Lu\|_{L^2}$$

for every $u \in H^4_{\mathrm{per}}(0, T)$.

Proof. Multiplying by u and integrating by parts, it is immediate that

$$\|u\|_{H^2} \leqslant c\|Lu\|_{L^2}.$$

On the other hand, if $Lu \to 0$ for the H^2-norm, then $u \to 0$ for the H^2-norm and hence $u^{(4)} \to 0$. As $\|u'''\|_{L^2} \leqslant c\|u^{(4)}\|_{L^2}$, the result follows. □

Proof of Theorem 1.3. Set $\lambda > K$ such that

$$\lambda \geqslant -\left(a + \inf_{(t,u,v)\in\mathcal{C}} \frac{\partial g}{\partial u''}(t, u, v)\right),$$

and define

$$\mu = K(\lambda - K).$$

Then $K = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2}$. Moreover, if we consider the closed and convex set

$$B = \left\{u \in C^2_{\mathrm{per}}\big([0, T]\big)\colon\ \alpha''(t) - K\alpha(t) \geqslant u''(t) - Ku(t) \geqslant \beta''(t) - K\beta(t)\right\},$$

then the classical maximum principle implies that $\alpha \leqslant u \leqslant \beta$ for any $u \in B$. Writing

$$u'(t) = \int_{t_0}^{t} u''(s)\, ds,$$

it follows that B is bounded in $C^2_{\mathrm{per}}([0, T])$.

Let us define a fixed point operator $T : C^2_{\mathrm{per}}([0, T]) \to C^2_{\mathrm{per}}([0, T])$ in the following way: for $v \in C^2_{\mathrm{per}}([0, T])$, define Tv as the unique solution $u \in H^4_{\mathrm{per}}(0, T)$ of the linear problem

$$u^{(4)} - \lambda u'' + \mu u = p - (a + \lambda)v'' + \mu v - g(\cdot, v, v'').$$

From Lemma 4.2 and the compact imbedding $H^4(0, T) \hookrightarrow C^2([0, T])$, it is easy to prove that T is well defined and compact. Moreover, for $v \in B$ and $u = Tv$, then

$$(u-\alpha)^{(4)} - \lambda(u-\alpha)'' + \mu(u-\alpha)$$
$$\geqslant \mu(v-\alpha) - (a+\lambda)(v-\alpha)'' - \left[g(\cdot, v, v'') - g(\cdot, \alpha, \alpha'')\right].$$

As $(v-\alpha)'' \leqslant K(v-\alpha)$, by the mean value theorem we obtain:

$$(u-\alpha)^{(4)} - \lambda(u-\alpha)'' + \mu(u-\alpha) \geqslant \left[\mu - \frac{\partial g}{\partial u}(\xi) - K\left(a + \lambda + \frac{\partial g}{\partial u''}(\xi)\right)\right](v-\alpha) \geqslant 0$$

for some mean value $\xi = \xi(t) \in \mathcal{C}$. Hence $(u-\alpha)'' \leqslant K(u-\alpha)$, and in the same way, we see that $(u-\beta)'' \geqslant K(u-\beta)$. Thus, $T(B) \subset B$, and the proof follows by Schauder theorem. □

Next we present a simple example for which the conditions of Theorem 1.3 are satisfied.

Example 4.3. Consider the nonhomogeneous Extended Kolmogorov–Fisher equation

$$u^{(4)} + au'' + u^3 - u = p(t), \tag{10}$$

where $a < -2\sqrt{2}$. Setting $R = \sqrt{(4+a^2)/12} > 1$ and $K = -a/2$, it suffices to consider $\alpha = -R$, $\beta = R$, and the existence of a T-periodic solution of (10) follows for any $p \in L^\infty(0, T)$ satisfying $R - R^3 < p(t) < R^3 - R$.

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