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J. Math. Anal. Appl. 310 (2005) 397–411
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Nonhomogeneous Neumann problem for the Poisson equation in domains with an external cusp ☆

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Received 10 May 2004

Available online 30 March 2005

Submitted by G. Komatsu

Abstract

In this work we analyze the existence and regularity of the solution of a nonhomogeneous Neumann problem for the Poisson equation in a plane domain Ω with an external cusp. In order to prove that there exists a unique solution in $H^1(\Omega)$ using the Lax–Milgram theorem we need to apply a trace theorem. Since Ω is not a Lipschitz domain, the standard trace theorem for $H^1(\Omega)$ does not apply, in fact the restriction of $H^1(\Omega)$ functions is not necessarily in $L^2(\partial\Omega)$. So, we introduce a trace theorem by using weighted Sobolev norms in Ω. Under appropriate assumptions we prove that the solution of our problem is in $H^2(\Omega)$ and we obtain an a priori estimate for the second derivatives of the solution.

☆ Supported by ANPCyT under grant PICT 03-05009, by CONICET under grant PIP 0660/98 and Fundación Antorchas.

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[1] The first author is partially supported by PICT 03-10724.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.01.065

Keywords: Cuspidal domains; Traces; Neumann problem; Regularity

1. Introduction

This paper deals with an elliptic equation in a domain with an external cusp. Since this kind of domains are not Lipschitz, the standard arguments to prove existence cannot be applied when nonhomogeneous Neumann boundary conditions are imposed on some part of the boundary. Indeed, to apply the Lax–Milgram theorem in this case one needs to use some trace theorem for Sobolev spaces. However, simple examples show that, for some cusps, there are functions in $H^1(\Omega)$ such that their restriction to the boundary are not in $L^2(\partial\Omega)$. Therefore the classic trace theorems for Lipschitz domains are not valid in this case.

We consider the following model problem: let Ω be the plane domain defined by

$$\Omega = \big\{(x, y)\colon\ 0 < x < 1,\ 0 < y < \varphi(x)\big\},$$

with $\varphi \in C^2(0,1)$, $\varphi, \varphi', \varphi'' > 0$ on $(0,1)$, $\varphi(0) = \varphi'(0) = 0$ (a typical example is $\varphi(x) = x^\alpha$, $\alpha > 1$), and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ the boundary of Ω, where

$$\Gamma_1 = \{0 \leqslant x \leqslant 1,\ y = 0\}, \qquad \Gamma_2 = \{x = 1,\ 0 \leqslant y \leqslant 1\}$$

and

$$\Gamma_3 = \big\{0 \leqslant x \leqslant 1,\ y = \varphi(x)\big\}$$

(see Fig. 1).

We seek u such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \Gamma_3, \end{cases} \tag{1.1}$$

where ν denotes the outside normal to Ω.

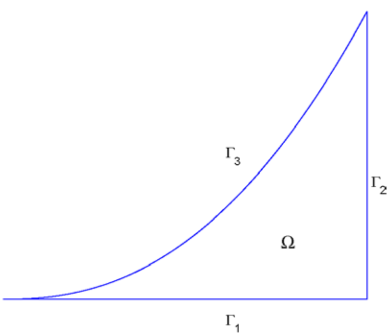

Fig. 1. Cuspidal domain.

In [6] the authors characterize the traces of the Sobolev spaces $W^{1,p}(\Omega)$, $1 \leqslant p < \infty$, for domains of the class considered here by using some weighted norm on the boundary. Existence of solutions of (1.1) can be derived from their results under certain hypothesis on the data. In order to obtain existence results for more general data we present a different kind of trace results by introducing a weighted Sobolev space in Ω such that the restriction to the boundary of functions in that space are in $L^p(\Gamma)$.

Once the existence of a solution is known, the question about its regularity arises naturally. For the Poisson problem with homogeneous boundary conditions on cuspidal domains it is known that, if the right-hand side of the equation is in $L^2(\Omega)$, then the solution belongs to $H^2(\Omega)$ (see [2,5]). We show that the technique introduced by Khelif in [5] can be extended to treat nonhomogeneous Neumann type boundary conditions. In this way we prove that the solution of our model problem belongs to the space $H^2(\Omega)$.

2. Existence and uniqueness of solution

In this section we prove some trace results and apply them to obtain existence and uniqueness of solution of our model problem using the Lax–Milgram theorem.

Let $V = \{v \in H^1(\Omega)\colon\ v|_{\Gamma_2} = 0\}$. The variational problem associated with (1.1) is given by: Find $u \in V$ such that

$$a(u,v) = L_1(v) + L_2(v) \quad \forall v \in V,$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \qquad L_1(v) = \int_\Omega f v \quad \text{and} \quad L_2(v) = \int_{\Gamma_3} g v.$$

Using the Poincaré inequality, it is easy to see that the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on V. Therefore, in order to prove that there exists a unique solution in V using the Lax–Milgram theorem, we need to impose conditions on the data f and g which guarantee that the linear operators L_1 and L_2 are continuous on V. For the continuity of L_1 it is enough to assume that $f \in L^2(\Omega)$. On the other hand, the continuity of L_2 when $g \in L^2(\Gamma_3)$, in the case of a Lipschitz domain, is proved by using well-known results on restrictions of $H^1(\Omega)$ to the boundary. However, since our domain is not Lipschitz, the standard trace theorem for $H^1(\Omega)$ does not apply, in fact, the following example shows that for some cusps the restriction of $H^1(\Omega)$ functions is not necessarily in $L^2(\Gamma)$.

Example 2.1. Consider $\varphi(x) = x^\alpha$, $\alpha > 1$, and the function $u(x,y) = x^{-\gamma}$. Then, an easy computation shows that $u \in H^1(\Omega)$ iff $\gamma < \frac{\alpha-1}{2}$. However, $u \in L^2(\Gamma)$ iff $\gamma < \frac{1}{2}$. So, for $\alpha > 2$, taking $\frac{1}{2} \leqslant \gamma < \frac{\alpha-1}{2}$, we have examples of functions which are in $H^1(\Omega)$ and such that their restrictions to the boundary are not in $L^2(\Gamma)$.

In [6], Mazya et al. characterize the space of traces of $W^{1,p}(\Omega)$, for non-Lipschitz domains Ω of the type considered here, by using some weighted norms on the boundary. In particular, it follows from their results that there exists a constant C such that

$$\bigl\| u\varphi^{\frac{1}{2}} \bigr\|_{L^2(\Gamma)} \leqslant C \|u\|_{H^1(\Omega)}. \tag{2.1}$$

Indeed, the left-hand side agrees with the first term in the norm $\|\cdot\|_{TW_p^1(\Omega)}$, with $p=2$, introduced in [6, p. 108] which, as proved in that paper, is bounded by the H^1 norm.

The inequality (2.1) can be used to prove the continuity of L_2 under the assumption that $g\varphi^{-\frac{1}{2}} \in L^2(\Gamma_3)$, in fact we have

$$\begin{aligned}\left|L_2(u)\right| &= \left|\int_{\Gamma_3} g\varphi^{-\frac{1}{2}} u\varphi^{\frac{1}{2}}\right| \leqslant \left\|g\varphi^{-\frac{1}{2}}\right\|_{L^2(\Gamma_3)}\left\|u\varphi^{\frac{1}{2}}\right\|_{L^2(\Gamma_3)} \\ &\leqslant C\left\|g\varphi^{-\frac{1}{2}}\right\|_{L^2(\Gamma_3)}\|u\|_{H^1(\Omega)}.\end{aligned}$$

Let us observe that assuming continuity of g the condition $g\varphi^{-\frac{1}{2}} \in L^2(\Gamma_3)$ implies that g has to vanish at the origin, which does not seem to be a natural condition for the existence of a solution. Therefore, our goal is to relax the assumption on g by introducing a trace result of a different nature of those in [6]. More precisely, we want to give sufficient conditions to have traces in L^p of the boundary. In order to do that we introduce the weighted Sobolev space $W_\varphi^{1,p}(\Omega)$ as the closure of $C^\infty(\bar{\Omega})$ in the norm

$$\|u\|^p_{W_\varphi^{1,p}(\Omega)} := \left\|u\varphi^{-\frac{1}{p}}\right\|^p_{L^p(\Omega)} + \left\|\nabla u\varphi^{(\frac{p-1}{p})}\right\|^p_{L^p(\Omega)}.$$

In what follows we use the letter C to denote a generic constant which depends only on p.

Lemma 2.1. *There exists a constant C such that for any $u \in W_\varphi^{1,p}(\Omega)$ with $1 \leqslant p < \infty$,*

$$\|u\|_{L^p(\Gamma)} \leqslant C\left(\left\|u\varphi^{-\frac{1}{p}}\right\|_{L^p(\Omega)} + \left\|\nabla u\varphi^{(\frac{p-1}{p})}\right\|_{L^p(\Omega)}\right).$$

Proof. We will use the following change of variables which is a generalization of that introduced by Grisvard [3] for power type cusps. Let $\xi = \frac{1}{\varphi'(x)}$ and $\eta = \frac{y}{\varphi(x)}$ then, Ω is transformed in $\tilde{\Omega}$ given by

$$\tilde{\Omega} = \left\{(\xi,\eta)\colon\ \xi > \frac{1}{\varphi'(1)},\ 0 < \eta < 1\right\},$$

see Fig. 2.

We denote by $\tilde{\Gamma}_1 = \{(\xi,\eta)\colon\ \xi \geqslant \frac{1}{\varphi'(1)},\ \eta = 0\}$, $\tilde{\Gamma}_2 = \{(\xi,\eta)\colon\ \xi = \frac{1}{\varphi'(1)},\ 0 \leqslant \eta \leqslant 1\}$ and $\tilde{\Gamma}_3 = \{(\xi,\eta)\colon\ \xi \geqslant \frac{1}{\varphi'(1)},\ \eta = 1\}$.

First we give the proof for the case $p=1$. Writing $v(\xi,\eta) = u(x,y)$ we have

$$\begin{aligned}\int_{\Gamma_3} |u| &= \int_0^1 \left|u\big(x,\varphi(x)\big)\right|\sqrt{1+\varphi'(x)^2}\,dx \leqslant C\int_0^1 \left|u\big(x,\varphi(x)\big)\right|dx \\ &= C\int_{\frac{1}{\varphi'(1)}}^{\infty} \left|v(\xi,1)\right|J(\xi)\,d\xi,\end{aligned} \tag{2.2}$$

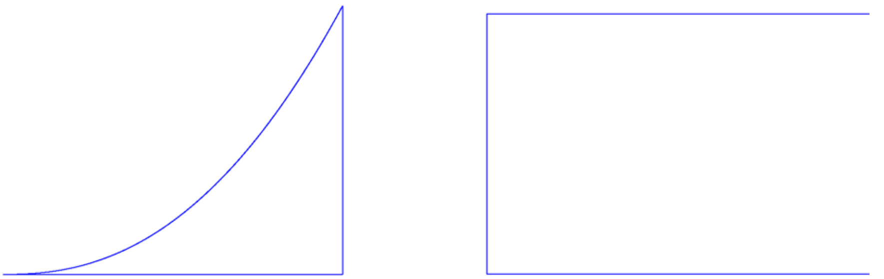

Fig. 2.

where

$$J(\xi)=\frac{\varphi'(x)^2}{\varphi''(x)}. \tag{2.3}$$

Applying the following standard trace inequality in $\tilde{\Omega}$:

$$\|w\|_{L^1(\tilde{\Gamma}_3)} \leqslant C\left(\|w\|_{L^1(\tilde{\Omega})}+\left\|\frac{\partial w}{\partial \eta}\right\|_{L^1(\tilde{\Omega})}\right),$$

to the function $w(\xi,\eta)=v(\xi,\eta)J(\xi)$, we get

$$\int_{\frac{1}{\varphi'(1)}}^{\infty}\left|v(\xi,1)\right|J(\xi)\,d\xi \leqslant C\left(\int_{\tilde{\Omega}}\left|v(\xi,\eta)\right|J(\xi)\,d\xi\,d\eta+\int_{\tilde{\Omega}}\left|\frac{\partial v(\xi,\eta)}{\partial \eta}\right|J(\xi)\,d\xi\,d\eta\right)$$

and therefore, changing variables and using (2.2) and (2.3), we have

$$\int_{\Gamma_3}|u| \leqslant C\left(\int_{\Omega}|u|\varphi(x)^{-1}\,dx\,dy+\int_{\Omega}\left|\frac{\partial u}{\partial y}\right|dx\,dy\right).$$

Applying the same argument on Γ_1 and a standard trace theorem on Γ_2, we obtain

$$\|u\|_{L^1(\Gamma)} \leqslant C\big(\|u\varphi^{-1}\|_{L^1(\Omega)}+\|\nabla u\|_{L^1(\Omega)}\big) \tag{2.4}$$

concluding the proof for the case $p=1$.

Now, for any p such that $1<p<\infty$, we use (2.4) for u^p to obtain

$$\begin{aligned}\int_{\Gamma}|u|^p &\leqslant C\left(\int_{\Omega}|u|^p\varphi^{-1}+p\int_{\Omega}|u|^{p-1}|\nabla u|\right)\\ &= C\left(\int_{\Omega}|u|^p\varphi^{-1}+p\int_{\Omega}|u|^{p-1}\varphi^{-\frac{1}{q}}|\nabla u|\varphi^{\frac{1}{q}}\right),\end{aligned}$$

where $q=\frac{p}{p-1}$, and therefore, the proof concludes by using the inequality $ab \leqslant \frac{1}{q}a^q+\frac{1}{p}b^p$ in the last term on the right-hand side. $\square$

Remark 2.1. With an argument analogous to that used in the previous lemma one can prove the following result, which is stronger than (2.1):

$$\left\| u\varphi^{\frac{1}{p}} \right\|_{L^p(\Gamma)} \leqslant C\left(\|u\|_{L^p(\Omega)} + \|\nabla u \varphi\|_{L^p(\Omega)} \right).$$

Existence results for more general data g can be obtained from the previous lemma and embedding theorems. During the rest of this section we will restrict ourselves to the case of power type cusps, for which embedding theorems are well known.

Let $\varphi(x) = x^\alpha$ with $\alpha > 1$. In the next theorem we prove that the restriction of $H^1(\Omega)$ functions are in $L^p(\Gamma)$ under appropriate assumptions on the values of α and p. In the proof we will make use of the inclusion

$$H^1(\Omega) \subset L^r(\Omega) \quad \text{for } 2 \leqslant r \leqslant \frac{2(\alpha+1)}{\alpha-1} \tag{2.5}$$

which is a particular case of the results given in [1].

Theorem 2.1. *Let $u \in H^1(\Omega)$ and $1 \leqslant p \leqslant 2$. If $\alpha < 1 + \frac{2}{p}$ then $u \in L^p(\Gamma)$ and*

$$\|u\|_{L^p(\Gamma)} \leqslant C\|u\|_{H^1(\Omega)}. \tag{2.6}$$

Proof. From Lemma 2.1 we know that

$$\begin{aligned} \|u\|_{L^p(\Gamma)} &\leqslant C\left(\left\| u x^{-\frac{\alpha}{p}} \right\|_{L^p(\Omega)} + \left\| \nabla u x^{\alpha(\frac{p-1}{p})} \right\|_{L^p(\Omega)} \right) \\ &\leqslant C\left(\left\| u x^{-\frac{\alpha}{p}} \right\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right). \end{aligned} \tag{2.7}$$

To bound the first term on the right-hand side of (2.7) we use the Hölder inequality with an exponent q to be chosen below. Then,

$$\int_\Omega |u|^p x^{-\alpha} \leqslant \left(\int_\Omega |u|^{pq} \right)^{\frac{1}{q}} \left(\int_\Omega x^{-\alpha\frac{q}{q-1}} \right)^{\frac{q-1}{q}}.$$

From (2.5), if $\frac{2}{p} \leqslant q \leqslant \frac{2(1+\alpha)}{(\alpha-1)p}$ we have

$$\left(\int_\Omega |u|^{pq} \right)^{\frac{1}{q}} \leqslant C\|u\|_{H^1(\Omega)}^p.$$

On the other hand, $(\int_\Omega x^{-\alpha\frac{q}{q-1}})^{\frac{q-1}{q}}$ is bounded if $q > 1 + \alpha$. So, if $\alpha < 1 + \frac{2}{p}$ we can take q such that $1 + \alpha < q \leqslant \frac{2(1+\alpha)}{(\alpha-1)p}$ and we obtain (2.6). □

Remark 2.2. In particular, it follows from the previous theorem that for $\alpha < 2$ the functions in $H^1(\Omega)$ have traces in $L^2(\Gamma)$, while from Example 2.1 we know that this is not true for $\alpha > 2$. Therefore our result is almost optimal.

Now we can give an existence result for problem (1.1) under appropriate assumptions on g and α.

Theorem 2.2. *Let* $1 \leqslant p \leqslant 2$, $g \in L^q(\Gamma_3)$ *with* $q = \frac{p}{p-1}$, *and* $f \in L^2(\Omega)$. *If* $\alpha < 1 + \frac{2}{p}$ *then there exists a unique solution* $u \in V$ *of problem* (1.1).

Proof. Since the bilinear form $a(\cdot,\cdot)$ is coercive and continuous on V, the existence of a unique solution will be a consequence of the Lax–Milgram theorem if we show that the linear functional $L := L_1 + L_2$ is continuous on V.

Since $f \in L^2(\Omega)$, L_1 is continuous and therefore it only remains to prove the continuity of L_2. From Theorem 2.1 we know that $\|u\|_{L^p(\Gamma)} \leqslant C\|u\|_{H^1(\Omega)}$ and so,

$$\left|L_2(u)\right| = \left|\int_{\Gamma_3} gu\right| \leqslant \|g\|_{L^q(\Gamma_3)}\|u\|_{L^p(\Gamma_3)} \leqslant C\|g\|_{L^q(\Gamma_3)}\|u\|_{H^1(\Omega)}$$

and the theorem is proved. □

3. Regularity of the solution

In this section we analyze the regularity of the solution u of problem (1.1). Under appropriate conditions on g we prove, in the next theorem, that $u \in H^2(\Omega)$. In order to obtain this result we will apply the method introduced by Khelif [2,5] which is based in approximating the domain by a sequence of Lipschitz domains.

Theorem 3.1. *Let* $f \in L^2(\Omega)$, *and* g *such that, if* $h(t) := g(t, \varphi(t))$, $h\varphi^{-\frac{1}{2}} \in L^2(0,1)$ *and* $h'(\varphi'')^{-\frac{1}{2}} \in L^2(0,1)$. *Assume also that* $\|\frac{\varphi''\varphi}{(\varphi')^2}\|_{L^\infty(0,1)} < 1$. *Then the problem* (1.1) *has a unique solution* u *belonging to* $H^2(\Omega)$, *and there exists a constant* C *such that*

$$\|u\|_{H^2(\Omega)} \leqslant C\left\{\|f\|_{L^2(\Omega)} + \left\|h\varphi^{-\frac{1}{2}}\right\|_{L^2(0,1)} + \left\|h'(\varphi'')^{-\frac{1}{2}}\right\|_{L^2(0,1)}\right\}. \tag{3.1}$$

Proof. The existence of a unique solution $u \in H^1(\Omega)$ follows from the results of Section 2. Then it only remains to show that $u \in H^2(\Omega)$.

Let $p_n = 1/n$ and define

$$\Omega_n = \left\{(x,y) \in \Omega\colon\ p_n < x < 1\right\},$$
$$\Gamma_1^n = \left\{(x,0)\colon\ p_n \leqslant x \leqslant 1\right\},$$
$$\Gamma_2 = \left\{(1,y)\colon\ 0 \leqslant y \leqslant 1\right\},$$
$$\Gamma_3^n = \left\{\left(x,\varphi(x)\right)\colon\ p_n \leqslant x \leqslant 1\right\},$$

and

$$\Gamma_4^n = \left\{(p_n, y)\colon\ 0 \leqslant y \leqslant \varphi(p_n)\right\},$$

see Fig. 3.

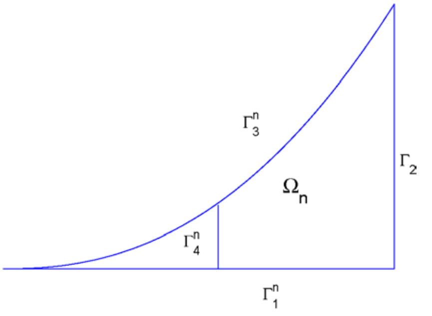

Fig. 3.

We consider the following problem in Ω_n:

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n, \\ u_n = 0 & \text{on } \Gamma_2, \\ \frac{\partial u_n}{\partial \nu} = g & \text{on } \Gamma_3^n, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \Gamma_1^n \cup \Gamma_4^n. \end{cases} \tag{3.2}$$

In what follows the letter C will denote a constant which may depend on φ.

Observe first that the solution u_n satisfies

$$\|u_n\|_{H^1(\Omega_n)} \leqslant C\left\{\|f\|_{L^2(\Omega)} + \left\|h\varphi^{-\frac{1}{2}}\right\|_{L^2(0,1)}\right\} \tag{3.3}$$

with C independent of n. Indeed, this estimate follows by standard arguments using a trace theorem as that given in Remark 2.1 applied on Ω_n. Note that the argument of Lemma 2.1 can be applied to Ω_n providing a constant independent of n.

It is known that the solution of problem (3.2) belongs to $H^{2+\varepsilon}(\Omega_n)$ [2,4], for some positive ε, in particular its first derivatives are continuous. Our goal is to obtain an estimate for $\|u_n\|_{H^2(\Omega_n)}$ valid uniformly in n. Using a method introduced by Khelif [2,5] we will show that

$$\|u_n\|_{H^2(\Omega_n)} \leqslant C\left\{\|f\|_{L^2(\Omega)} + \left\|h\varphi^{-\frac{1}{2}}\right\|_{L^2(0,1)} + \left\|h'(\varphi'')^{-\frac{1}{2}}\right\|_{L^2(0,1)}\right\} \tag{3.4}$$

with C independent of n.

For any ρ and ψ in $H^1(\Omega_n)$ we have

$$\int\limits_{\Omega_n} \rho_x \psi_y = \int\limits_{\Omega_n} \rho_y \psi_x + \int\limits_{\partial\Omega_n} \psi \frac{\partial \rho}{\partial \tau},$$

where τ is the unit tangent vector oriented clockwise. Note that the right-hand side has to be understood in a weak sense, i.e., $\frac{\partial \rho}{\partial \tau} \in H^{-1/2}(\partial\Omega_n)$. Taking

$$\rho = \frac{\partial u_n}{\partial x} \quad \text{and} \quad \psi = \frac{\partial u_n}{\partial y}$$

in the equation given above we obtain

$$\int\limits_{\Omega_n} f^2 = \int\limits_{\Omega_n} (\Delta u_n)^2 = \int\limits_{\Omega_n} (\rho_x + \psi_y)^2 = \int\limits_{\Omega_n} \rho_x^2 + 2\int\limits_{\Omega_n} \rho_x \psi_y + \int\limits_{\Omega_n} \psi_y^2$$

$$= \int_{\Omega_n} \rho_x^2 + 2\int_{\Omega_n} \rho_y \psi_x + \int_{\Omega_n} \psi_y^2 + 2\int_{\partial\Omega_n} \psi \frac{\partial \rho}{\partial \tau}$$

$$= |u_n|_{H^2(\Omega_n)}^2 + 2\int_{\partial\Omega_n} \psi \frac{\partial \rho}{\partial \tau}, \tag{3.5}$$

where $|u_n|_{H^2(\Omega_n)}$ denotes the seminorm of u_n in $H^2(\Omega_n)$.

To simplify notation we introduce the one variable functions

$$v(t) := \frac{\partial u_n}{\partial x}\big(t, \varphi(t)\big) \quad \text{and} \quad w(t) := \frac{\partial u_n}{\partial y}\big(t, \varphi(t)\big).$$

Then, the boundary conditions imply

$$\begin{cases} \frac{\partial u_n}{\partial y} = 0 & \text{on } \Gamma_1^n \cup \Gamma_2, \\ w = v\varphi' + h\sqrt{1 + (\varphi')^2} & \text{on } \Gamma_3^n, \\ \frac{\partial u_n}{\partial x} = 0 & \text{on } \Gamma_4^n. \end{cases}$$

Therefore, (3.5) becomes

$$|u_n|_{H^2(\Omega_n)}^2 = \int_{\Omega_n} f^2 - 2\int_{p_n}^{1} w(t)v'(t)\,dt, \tag{3.6}$$

and so, we have to bound the last term on the right-hand side.

From the boundary condition on Γ_3^n we have

$$\int_{p_n}^{1} w(t)v'(t)\,dt = \int_{p_n}^{1} w(t)\left(\frac{w(t)}{\varphi'(t)}\right)' dt - \int_{p_n}^{1} w(t)\left(\frac{h(t)\sqrt{1 + (\varphi'(t))^2}}{\varphi'(t)}\right)' dt$$

$$= I + II. \tag{3.7}$$

For the first term we have

$$I = \int_{p_n}^{1} w(t)w'(t)\frac{1}{\varphi'(t)}\,dt - \int_{p_n}^{1} w(t)^2 \frac{\varphi''(t)}{\varphi'(t)^2}\,dt$$

$$= \frac{1}{2}\int_{p_n}^{1} \big(w(t)^2\big)' \frac{1}{\varphi'(t)}\,dt - \int_{p_n}^{1} w(t)^2 \frac{\varphi''(t)}{\varphi'(t)^2}\,dt.$$

Now, since $\frac{\partial u_n}{\partial y}$ is continuous, it follows from the boundary condition on Γ_2 that $w(1) = 0$. Therefore, integrating by parts, we obtain for the first term in the right-hand side of the last equation,

$$\frac{1}{2}\int_{p_n}^{1} \big(w(t)^2\big)' \frac{1}{\varphi'(t)}\,dt = -\frac{1}{2}w(p_n)^2 \frac{1}{\varphi'(p_n)} + \frac{1}{2}\int_{p_n}^{1} w(t)^2 \frac{\varphi''(t)}{\varphi'(t)^2}\,dt$$

and then,

$$I = -\frac{1}{2} w(p_n)^2 \frac{1}{\varphi'(p_n)} - \frac{1}{2} \int_{p_n}^{1} w(t)^2 \frac{\varphi''(t)}{\varphi'(t)^2}\, dt. \tag{3.8}$$

Using now the boundary condition on Γ_4^n and the fact that $\frac{\partial u_n}{\partial x}$ is continuous, it follows that $v(p_n) = 0$ and so, from the boundary condition on Γ_3^n we obtain

$$w(p_n) = h(p_n)\sqrt{1 + \varphi'(p_n)^2}.$$

Therefore, replacing in (3.8) we have

$$I = -\frac{1}{2} h^2(p_n)\bigl(1 + \varphi'(p_n)^2\bigr) \frac{1}{\varphi'(p_n)} - \frac{1}{2} \int_{p_n}^{1} w^2(t) \frac{\varphi''(t)}{\varphi'(t)^2}\, dt. \tag{3.9}$$

To bound the first term on the right-hand side we observe that, for any $s \in (0, 1)$,

$$\begin{aligned}
\bigl|h(s) - h(0)\bigr| &= \left|\int_0^s h'(t)\right| \leqslant \bigl\|h'(\varphi'')^{-\frac{1}{2}}\bigr\|_{L^2(0,1)} \left(\int_0^s \varphi''(t)\, dt\right)^{\frac{1}{2}} \\
&= \bigl\|h'(\varphi'')^{-\frac{1}{2}}\bigr\|_{L^2(0,1)} \varphi'(s)^{\frac{1}{2}}.
\end{aligned}$$

In particular h is continuous at 0 and consequently, since $h\varphi^{-\frac{1}{2}} \in L^2(0, 1)$, it follows that $h(0) = 0$ (recall that $0 < \varphi(t) < t$ for all t small enough).

Moreover,

$$\frac{h^2(p_n)}{\varphi'(p_n)} \leqslant \bigl\|h'(\varphi'')^{-\frac{1}{2}}\bigr\|^2_{L^2(0,1)}$$

and so, we obtain from (3.9),

$$|I| \leqslant C\bigl\|h'(\varphi'')^{-\frac{1}{2}}\bigr\|^2_{L^2(0,1)} + \frac{1}{2} \int_{p_n}^{1} w^2(t) \frac{\varphi''(t)}{\varphi'(t)^2}\, dt. \tag{3.10}$$

Let us now estimate the second term on the right-hand side of (3.7). A simple computation shows that

$$II = -\int_{p_n}^{1} \frac{w(t)h(t)\varphi''(t)}{\varphi'(t)^2\sqrt{1 + \varphi'(t)^2}}\, dt + \int_{p_n}^{1} \frac{w(t)h'(t)\sqrt{1 + \varphi'(t)^2}}{\varphi'(t)}\, dt = III + IV.$$

Using the arithmetic–geometric inequality $ab \leqslant \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ valid for all $\epsilon > 0$, we have

$$|III| \leqslant \frac{\epsilon}{2} \int_{p_n}^{1} \frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\, dt + \frac{1}{2\epsilon} \int_{p_n}^{1} \frac{h(t)^2\varphi''(t)}{\varphi'(t)^2(1 + \varphi'(t)^2)}\, dt$$

$$\leqslant \frac{\epsilon}{2}\int_{p_n}^{1} \frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\,dt + \frac{1}{2\epsilon}\left\|h\varphi^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)},$$

while, on the other hand, we have

$$\begin{aligned}|IV| &\leqslant \frac{\epsilon}{2}\int_{p_n}^{1} \frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\,dt + \frac{1}{2\epsilon}\int_{p_n}^{1}\frac{h'(t)^2(1+\varphi'(t)^2)}{\varphi''(t)}\,dt \\ &\leqslant \frac{\epsilon}{2}\int_{p_n}^{1} \frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\,dt + \frac{C}{2\epsilon}\left\|h'(\varphi'')^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}.\end{aligned}$$

So

$$\begin{aligned}|II| &\leqslant \epsilon\int_{p_n}^{1} \frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\,dt + \frac{1}{2\epsilon}\left\|h\varphi^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)} \\ &\quad + \frac{C}{2\epsilon}\left\|h'(\varphi'')^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}. \end{aligned} \tag{3.11}$$

Therefore, using the estimates (3.10) and (3.11), we obtain from (3.7),

$$\begin{aligned}\left|\int_{p_n}^{1} w(t)v'(t)\,dt\right| &\leqslant \left(\frac{1}{2}+\epsilon\right)\int_{p_n}^{1}\frac{w(t)^2\varphi''(t)}{\varphi'(t)^2}\,dt \\ &\quad + \frac{1}{2\epsilon}\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)}\left\|h\varphi^{-\frac{1}{2}}\right\|^2_{L^2(0,1)} \\ &\quad + C\left(1+\frac{1}{2\epsilon}\right)\left\|h'(\varphi'')^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}. \end{aligned} \tag{3.12}$$

But, from the boundary condition on Γ_1^n we know that $\frac{\partial u_n}{\partial y}(t,0)=0$ and therefore,

$$w^2(t) = \left|\frac{\partial u_n}{\partial y}\big(t,\varphi(t)\big)\right|^2 = \left(\int_0^{\varphi(t)}\frac{\partial^2 u_n}{\partial y^2}(t,y)\,dy\right)^2 \leqslant \varphi(t)\int_0^{\varphi(t)}\left|\frac{\partial^2 u_n}{\partial y^2}(t,y)\right|^2 dy$$

and consequently,

$$\int_{p_n}^{1}\frac{w^2(t)}{\varphi(t)}\,dt \leqslant \left\|\frac{\partial^2 u_n}{\partial y^2}(t,y)\right\|^2_{L^2(\Omega_n)}.$$

Therefore, replacing in (3.12) we obtain

$$2\left|\int_{p_n}^{1} w(t)v'(t)\,dt\right| \leqslant (1+2\epsilon)\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)}\left\|\frac{\partial^2 u_n}{\partial y^2}(t,y)\right\|^2_{L^2(\Omega_n)}$$

$$+\frac{1}{\epsilon}\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)}\left\|h\varphi^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}$$
$$+C\left(2+\frac{1}{\epsilon}\right)\left\|h'(\varphi'')^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}. \tag{3.13}$$

Hence, using this estimate in (3.6), we conclude that

$$|u_n|^2_{H^2(\Omega_n)} \leqslant \|f\|^2_{L^2(\Omega)} + C\left(2+\frac{1}{\epsilon}\right)\left\{\left\|h\varphi^{-\frac{1}{2}}\right\|^2_{L^2(0,1)} + \left\|h'(\varphi'')^{-\frac{1}{2}}\right\|^2_{L^2(0,1)}\right\}$$
$$+(1+2\epsilon)\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)}\left\|\frac{\partial^2 u_n}{\partial y^2}(t,y)\right\|^2_{L^2(\Omega_n)}, \tag{3.14}$$

where we have used that

$$\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)} < 1.$$

From this fact, we also observe that $\epsilon > 0$ may be chosen in such a way that

$$(1+2\epsilon)\left\|\frac{\varphi''\varphi}{(\varphi')^2}\right\|_{L^\infty(0,1)} < 1.$$

So, recalling now (3.3), we obtain (3.4).

Now, using a standard argument and the Rellich theorem, one can show that there is a subsequence, that for simplicity we continue calling u_n, such that, for each Ω_k, u_n is defined on Ω_k for n large enough and converges weakly in $H^2(\Omega_k)$ and strongly in $H^1(\Omega_k)$. Moreover, if we call u the limit function, it follows from (3.4) and the weak convergence in H^2, that u satisfies the estimate (3.1). So, it remains only to show that u is the solution of (1.1). Therefore we have to see that

$$\int_\Omega \nabla u\cdot\nabla v = \int_\Omega fv + \int_{\Gamma_3} gv \quad \forall v \in V.$$

It is enough to show that, given $v \in V$,

$$\int_{\Omega_k} \nabla u\cdot\nabla v - \int_\Omega fv - \int_{\Gamma_3} gv \to 0$$

when $k \to \infty$. Moreover, by density, we can assume that $v \in W^{1,\infty}(\Omega)\cap V$. For $n \geqslant k$, we have

$$\int_{\Omega_k} \nabla u\cdot\nabla v - \int_\Omega fv - \int_{\Gamma_3} gv$$
$$= \int_{\Omega_k} (\nabla u - \nabla u_n)\cdot\nabla v + \int_{\Omega_k} \nabla u_n\cdot\nabla v - \int_\Omega fv - \int_{\Gamma_3} gv$$
$$= \int_{\Omega_k} (\nabla u - \nabla u_n)\cdot\nabla v + \int_{\Omega_n} \nabla u_n\cdot\nabla v - \int_{\Omega_n\setminus\Omega_k} \nabla u_n\cdot\nabla v - \int_\Omega fv - \int_{\Gamma_3} gv$$

$$= \int_{\Omega_k} (\nabla u - \nabla u_n) \cdot \nabla v - \int_{\Omega \setminus \Omega_n} f v - \int_{\Gamma_3 \setminus \Gamma_3^n} g v - \int_{\Omega_n \setminus \Omega_k} \nabla u_n \cdot \nabla v, \tag{3.15}$$

where we have used that u_n is the solution of problem (3.2). But,

$$\left| \int_{\Omega_n \setminus \Omega_k} \nabla u_n \cdot \nabla v \right| \leqslant \|u_n\|_{H^1(\Omega_n)} \|v\|_{W^{1,\infty}(\Omega)} |\Omega_n \setminus \Omega_k|^{\frac{1}{2}}$$

and, since $\|u_n\|_{H^1(\Omega_n)}$ are uniformly bounded, the last term on the right-hand side of (3.15) can be made smaller than any positive constant by taking k large enough. Then, the proof concludes by using that, for k fixed,

$$\int_{\Omega_k} (\nabla u - \nabla u_n) \cdot \nabla v \to 0$$

when $n \to \infty$. □

Observe that the domains with power type cusps, i.e., $\varphi(t) = t^\alpha$, $\alpha > 1$, are in the class considered here. In fact,

$$\frac{\varphi'' \varphi}{(\varphi')^2} = \frac{\alpha - 1}{\alpha}.$$

In what follows we will show that the hypothesis $h\varphi^{-\frac{1}{2}} \in L^2(0,1)$ assumed in the previous theorem is not too restrictive and cannot be substantially relaxed. With this goal we consider $\varphi(t) = t^\alpha$, $\alpha > 1$. In this case, the hypothesis is $ht^{-\frac{\alpha}{2}} \in L^2(0,1)$ and we will prove that, if the solution of problem (1.1) belongs to $H^2(\Omega)$ then, $ht^{-\frac{\alpha(r-1)}{r}} \in L^r(0,1)$ for any $r < 2$. In particular, if h is continuous at $t = 0$, it follows that $h(0) = 0$.

We will show in the next lemma that, for $u \in H^2(\Omega)$, $\frac{\partial u}{\partial \nu}$ is the restriction to Γ_3 of a function in $W^{1,r}(\Omega)$, for $r < 2$. Then, the result will follow by using again the results of [6].

Lemma 3.1. *Let $u \in H^2(\Omega)$, and consider $v = \eta \cdot \nabla u$, where*

$$\eta(x,y) := \frac{1}{\sqrt{x^2 + \alpha^2 y^2}} (-\alpha y, x).$$

Then,

(i) $v = \frac{\partial u}{\partial \nu}$ *on* Γ_3,
(ii) $v = -\frac{\partial u}{\partial \nu}$ *on* Γ_1,
(iii) $v \in W^{1,r}(\Omega)$ *for* $r < 2$.

Proof. The first two assertions follow immediately from the fact that $\eta(x,y)$ agrees with the outward normal on Γ_3 and with the inward normal on Γ_1.

To prove (iii), let us call

$$a(x,y) := \frac{\alpha y}{\sqrt{x^2 + \alpha^2 y^2}} \quad \text{and} \quad b(x,y) := \frac{x}{\sqrt{x^2 + \alpha^2 y^2}}.$$

Then, we have

$$v = -a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y}. \tag{3.16}$$

Since a and b are bounded functions, we have that $v \in L^2(\Omega)$. Therefore we have to show that the first derivatives of v are in $L^r(\Omega)$ for any $r < 2$. Now, a straightforward computation yields

$$\frac{\partial a}{\partial y} = \frac{\alpha x^2}{(x^2 + \alpha^2 y^2)^{\frac{3}{2}}}, \qquad \frac{\partial b}{\partial x} = \frac{\alpha^2 y^2}{(x^2 + \alpha^2 y^2)^{\frac{3}{2}}}$$

and

$$\frac{\partial a}{\partial x} = \frac{-\alpha x y}{(x^2 + \alpha^2 y^2)^{\frac{3}{2}}}, \qquad \frac{\partial b}{\partial y} = \frac{-\alpha^2 x y}{(x^2 + \alpha^2 y^2)^{\frac{3}{2}}}.$$

Integrating these expressions over Ω one can easily check that

$$\int_\Omega \left|\frac{\partial a}{\partial y}\right|^s \leqslant C \int_0^1 x^{\alpha - s}\, dx,$$

$$\int_\Omega \left|\frac{\partial b}{\partial x}\right|^s \leqslant C \int_0^1 x^{2s\alpha - 3s + \alpha}\, dx,$$

$$\int_\Omega \left|\frac{\partial a}{\partial x}\right|^s \leqslant C \int_0^1 x^{s(\alpha+1) - 3s + \alpha}\, dx,$$

and

$$\int_\Omega \left|\frac{\partial b}{\partial y}\right|^s \leqslant C \int_0^1 x^{s(\alpha+1) - 3s + \alpha}\, dx.$$

Therefore,

$$\frac{\partial a}{\partial y} \in L^s(\Omega) \quad \text{if } s < \alpha + 1, \tag{3.17}$$

$$\frac{\partial b}{\partial x} \in L^s(\Omega), \quad \begin{cases} \forall s & \text{if } \alpha \geqslant \frac{3}{2}, \\ s < \frac{1+\alpha}{3-2\alpha} & \text{if } \alpha < \frac{3}{2}, \end{cases} \tag{3.18}$$

$$\frac{\partial b}{\partial y}, \frac{\partial a}{\partial x} \in L^s(\Omega), \quad \begin{cases} \forall s & \text{if } \alpha \geqslant 2, \\ s < \frac{1+\alpha}{2-\alpha} & \text{if } \alpha < 2. \end{cases} \tag{3.19}$$

Now, let w be any of the first derivatives of u. Then, in view of (3.16), in order to prove (iii) it is enough to see that, for $r < 2$, $\frac{\partial a}{\partial x} w$, $\frac{\partial a}{\partial y} w$, $\frac{\partial b}{\partial x} w$, $\frac{\partial b}{\partial y} w \in L^r(\Omega)$, and this is the aim of the rest of the proof. We will make use of the imbedding theorem (2.5).

First choose $p = \frac{2(\alpha+1)}{2(\alpha+1) - r(\alpha-1)}$. Since $w \in H^1(\Omega)$, it follows from (2.5) that $w \in L^{rq}(\Omega)$, where $q = \frac{2(\alpha+1)}{r(\alpha-1)}$ is the dual exponent of p. On the other hand, since $r < 2$, we have $rp < \alpha + 1$ and so, we obtain from (3.17) that $\frac{\partial a}{\partial y} \in L^{rp}(\Omega)$. Then, applying the Hölder inequality we obtain that $\frac{\partial a}{\partial y} w \in L^r(\Omega)$.

In a similar way, using (3.18), (3.19), and again (2.5), we can prove that $\frac{\partial a}{\partial x} w$, $\frac{\partial b}{\partial y} w$, $\frac{\partial b}{\partial x} w \in L^2(\Omega)$ choosing now $p = \frac{(\alpha+1)}{2}$ and $q = \frac{\alpha+1}{\alpha-1}$.

Therefore, taking derivatives in the expression (3.16) we obtain $\frac{\partial v}{\partial x} \in L^2(\Omega)$ and $\frac{\partial v}{\partial y} \in L^r(\Omega)$, for $r < 2$, concluding the proof. $\square$

In [6], the authors characterize the traces of $W^{1,r}$ for general cuspidal domains. Applying their results for our case it follows in particular that for $v \in W^{1,r}(\Omega)$ (see [6, p. 108]),

$$\int_0^1 \frac{|v(t, t^\alpha) - v(t, 0)|^r}{t^{\alpha(r-1)}}\, dt \leqslant C \|v\|_{W^{1,r}(\Omega)}. \tag{3.20}$$

From this estimate and our previous lemma we can easily obtain the following corollary.

Corollary 3.1. *Let u be the solution of problem* (1.1) *and $h(t) := g(t, t^\alpha)$. If $u \in H^2(\Omega)$ then,*

$$\int_0^1 \frac{|h(t)|^r}{t^{\alpha(r-1)}} < \infty \quad \textit{for any } r < 2. \tag{3.21}$$

Proof. Let v defined from u as in Lemma 3.1. Then, we know from that lemma that $v \in W^{1,r}(\Omega)$. Therefore, (3.21) follows immediately from (3.20) and the fact that $v = 0$ on Γ_1. $\square$

Acknowledgments

We thank Julián Fernández Bonder and Noemí Wolanski for helpful comments.

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