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Solutions to a stationary nonlinear Black–Scholes type equation

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Abstract

We study by topological methods a nonlinear differential equation generalizing the Black–Scholes formula for an option pricing model with stochastic volatility. We prove the existence of at least a solution of the stationary Dirichlet problem applying an upper and lower solutions method. Moreover, we construct a solution by an iterative procedure.

1. Introduction

In this work we study a nonlinear differential equation arising in an option pricing model. From the Black–Scholes model, if volatility is stochastic, the following PDE on the variables σ and S is obtained [3]:

$$\mathcal{L}f - \frac{1}{2}\rho\sigma^2 V f_\sigma = rf - rSf_S, \tag{1}$$

where $\mathcal{L}$ is the operator given by

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}V^2\sigma^2\frac{\partial^2}{\partial\sigma^2} + \rho\sigma^2 SV\frac{\partial^2}{\partial S\partial\sigma}.$$

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We study the stationary case for a generalization of (1) under Dirichlet conditions.

Applying a Newton-type iteration [2] we prove under appropriate conditions the existence of a solution in the Sobolev space $H^2(\Omega)$ which is obtained recursively. In Section 4 we obtain a solution under different assumptions, applying an upper and lower solutions method.

Our main interest is a better understanding of Black–Scholes type equations.

2. Black–Scholes type differential equations

The Black–Scholes equation for pricing options has been studied by many authors (see, for example, [1,4,6,8,9]).

In particular, stochastic volatility models are proposed: specifically, we shall consider as in [3] the following processes

$$dS_t = S_t \sigma_t \, dZ_t + S_t \mu \, dt,$$
$$d\sigma_t = V \sigma_t \, dW_t + \alpha \sigma_t \, dt,$$

where Z_t and W_t are two standard Brownian motions with correlation coefficient ρ, formally $E(dZ_t, dW_t) = \rho \, dt$. If $f(S, \sigma, t)$ is the price of an option depending on the price of the asset S, then by Ito's lemma [7], it holds

$$df(S, \sigma, t) = f_S \, dS + f_\sigma \, d\sigma + \mathcal{L} f \, dt.$$

Under an appropriate choice of the portfolio the stochastic term of the equation vanishes (for details, see [3]).

3. Stationary solutions to a nonlinear Black–Scholes type equation

We study the following stationary Dirichlet problem:

$$\text{(1a)} \quad \begin{cases} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho \sigma^2 V S \frac{\partial^2 f}{\partial S \partial \sigma} - \frac{1}{2}\rho \sigma^2 V \frac{\partial f}{\partial \sigma} \\ \quad = r g(f) f - r S \frac{\partial f}{\partial S} \quad \text{in } \Omega_0, \\ f = h_0 \quad \text{on } \partial \Omega_0, \end{cases}$$

with $g \in C^2(\mathbb{R})$, $h_0 \in H^2(\Omega_0)$, $\overline{\Omega}_0 \subset (0, a) \times (0, b)$ with $C^{1,1}$ boundary.

In this section we shall apply an iterative method in order to solve (1a).

Let us introduce the change of variables Φ given by $y = \log S$, $x = \sigma / V$; adding a parameter $\lambda \in [0, 1]$ into (1a) we obtain the following problem for $u(x, y) = f(S, \sigma)$ in the domain $\Omega = \Phi(\Omega_0)$:

$$\text{(1b)}_\lambda \quad \begin{cases} \Delta u + 2\rho \frac{\partial^2 u}{\partial x \partial y} = \lambda \left(\rho \frac{\partial u}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right) \frac{\partial u}{\partial y} + \frac{2 r g(u)}{x^2 V^2} u \right) \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial \Omega. \end{cases}$$

For simplicity, we define

$$F\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = \rho \frac{\partial u}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right)\frac{\partial u}{\partial y} + \frac{2rg(u)}{x^2 V^2} u$$

and the linear operator

$$Lu = \Delta u + 2\rho \frac{\partial^2 u}{\partial x \partial y}.$$

We remark that L is strictly elliptic for $\rho < 1$.

We start at a solution u_0 of $(1b)_{\lambda_0}$ and construct recursively a solution of $(1b)_{\lambda_0+\varepsilon}$ for some step ε. Thus, we have solutions for $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$, and if ε can be chosen uniformly the procedure gives a solution of problem $(1b)_1$.

In order to define a convergent sequence we apply Newton's method: let $\psi : H^2(\Omega) \to L^2(\Omega)$ be given by

$$\psi(u) = Lu - (\lambda_0 + \varepsilon)F(x, u, \nabla u)$$

and define

$$u_{n+1} = u_n - \left[D\psi(u_n)\right]^{-1}\left(\psi(u_n)\right).$$

Under appropriate conditions the differential $D\psi(u)$ given by

$$\begin{aligned} D\psi(u)(\varphi) &= L\varphi - (\lambda_0 + \varepsilon)\left(\frac{\partial F}{\partial u}(x, u, \nabla u)\varphi + \frac{\partial F}{\partial u_x}(x, u, \nabla u)\frac{\partial \varphi}{\partial x} \right. \\ &\qquad \left. + \frac{\partial F}{\partial u_y}(x, u, \nabla u)\frac{\partial \varphi}{\partial y}\right) \\ &= L\varphi - (\lambda_0 + \varepsilon)\left(\frac{2r[g(u)u]'}{x^2 V^2}\varphi + \rho \frac{\partial \varphi}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right)\frac{\partial \varphi}{\partial y}\right) \end{aligned}$$

is invertible. Hence, the sequence is well defined and converges quadratically to a zero of ψ.

We remark that if $z = [D\psi(u_n)]^{-1}(\psi(u_n))$ then

$$\begin{aligned} &Lz - (\lambda_0 + \varepsilon)\left(\frac{2r[g(u_n)u_n]'}{x^2 V^2} z + \rho \frac{\partial z}{\partial x} + \left(1 - \frac{2r}{x^2 V^2}\right)\frac{\partial z}{\partial y}\right) \\ &\quad = \psi(u_n) = Lu_n - F(x, u_n, \nabla u_n). \end{aligned}$$

Then $u_{n+1} = u_n - z$ is the unique solution of the linear problem

$$\begin{aligned} Lu_{n+1} = (\lambda_0 + \varepsilon)\Bigg[&\rho\left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right) + \left(1 - \frac{2r}{x^2 V^2}\right)\left(\frac{\partial u_{n+1}}{\partial y} - \frac{\partial u_n}{\partial y}\right) \\ &+ \frac{2r(g(u_n) + u_n g'(u_n))}{x^2 V^2}(u_{n+1} - u_n) + F\left(x, u_n, \frac{\partial u_n}{\partial x}, \frac{\partial u_n}{\partial y}\right)\Bigg] \end{aligned}$$

with the boundary condition

$$u_{n+1} = h \quad \text{on } \partial\Omega.$$

If we assume that $g(u)u$ is nondecreasing with respect to u, then the following lemma shows that $\{u_n\}$ is well defined.

Lemma 1. *Let $s \in C(\overline{\Omega})$ and $L_s : H^2(\Omega) \to L^2(\Omega)$ be the linear operator given by*

$$L_s z = Lz - \lambda\left[\rho\frac{\partial z}{\partial x} + \left(1 - \frac{2r}{x^2V^2}\right)\frac{\partial z}{\partial y} + sz\right]$$

with $s \geqslant 0$ and $0 \leqslant \lambda \leqslant 1$. Then $L_s|_{H_0^1(\Omega)}$ is invertible and onto. Moreover, there exists a constant $c > 0$ depending only on $\|s\|_\infty$ such that $\|z\|_{2,2} \leqslant c\|L_s z\|_2$ for any $z \in H^2 \cap H_0^1(\Omega)$.

Proof. By classical results [5], the linear problem

$$L_s z = \varphi \quad \text{in } \Omega, \qquad z|_{\partial\Omega} = 0$$

is uniquely solvable in $H^2(\Omega)$ for any $\varphi \in L^2(\Omega)$. Assume the existence of $s_n \geqslant 0$ and $u_n \in H^2 \cap H_0^1(\Omega)$ such that

$$\|s_n\|_\infty \leqslant M, \qquad \|u_n\|_{2,2} = 1, \qquad \|L_{s_n} u_n\|_2 \to 0.$$

As $\int_\Omega L_{s_n} u_n \cdot u_n \to 0$, we obtain

$$-\int_\Omega Lu_n.u_n + \lambda\left[\rho\int_\Omega \frac{\partial u_n}{\partial x}u_n + \int_\Omega \left(1 - \frac{2r}{x^2V^2}\right)\frac{\partial u_n}{\partial y}u_n + \int_\Omega s_n u_n^2\right] \to 0.$$

By ellipticity, $-\int_\Omega Lu_n \cdot u_n \geqslant k\|u_n\|_{1,2}^2$ for some positive constant k. Moreover, if we define the fields

$$F_1(x, y) = \left(u_n^2, 0\right), \qquad F_2(x, y) = \left(1 - \frac{2r}{x^2V^2}\right)\left(0, u_n^2\right),$$

we see that

$$\int_\Omega \frac{\partial u_n}{\partial x}u_n = \frac{1}{2}\int_\Omega \operatorname{div} F_1 = \frac{1}{2}\int_{\partial\Omega} F_1 \cdot \nu \, dS = 0$$

and

$$\int_\Omega \left(1 - \frac{2r}{x^2V^2}\right)\frac{\partial u_n}{\partial y}u_n = \frac{1}{2}\int_\Omega \operatorname{div} F_2 = \frac{1}{2}\int_{\partial\Omega} F_2 \cdot \nu \, dS = 0.$$

As $s_n \geqslant 0$, we deduce that $\|u_n\|_{1,2} \to 0$. This implies that $\|Lu_n\|_2 \to 0$, which contradicts the invertibility of L. □

Theorem 2. *Let us assume that $g(u)u$ is nondecreasing with respect to u. Then there exists ε such that $\{u_n\}$ converges for the norm $\|\cdot\|_{2,2}$ to a solution of* $(1\mathrm{b})_{\lambda_0+\varepsilon}$.

Proof. Let $z_n = u_{n+1} - u_n$, and consider

$$s_n(x, y) = \frac{2r(g(u_n) + u_n g'(u_n))}{x^2 V^2} \geqslant 0.$$

Then for $\lambda = \lambda_0 + \varepsilon$ it holds

$$\begin{aligned} L_{s_n} z_n &= (\lambda_0 + \varepsilon) F(x, u_n, \nabla u_n) - L u_n \\ &= (\lambda_0 + \varepsilon)\big[F(x, u_n, \nabla u_n) - F(x, u_{n-1}, \nabla u_{n-1}) \\ &\qquad - DF(x, u_{n-1}, \nabla u_{n-1})(x, z_{n-1}, \nabla z_{n-1})\big] \\ &= (\lambda_0 + \varepsilon)\frac{r}{x^2 V^2}\big(\xi_n g''(\xi_n) + 2g'(\xi_n)\big) z_{n-1}^2 \end{aligned}$$

for some mean value $\xi_n(x, y)$ between u_n and u_{n-1}. If $\|u_n - u_0\|_{2,2} \leqslant R$ for some constant R and any $n \leqslant N$, then there exists a constant K such that

$$\big\|g(u_n) + u_n g'(u_n)\big\|_\infty \leqslant K, \qquad \big\|\xi_n g''(\xi_n) + 2g'(\xi_n)\big\|_\infty \leqslant K.$$

By the previous lemma, we have that

$$\|z_n\|_{2,2} \leqslant c\|L_{s_n} z_n\|_2 \leqslant c(\lambda_0 + \varepsilon)\frac{r}{x^2 V^2} K \big\|z_{n-1}^2\big\|_2 \leqslant c_1 \|z_{n-1}\|_{2,2}^2$$

for some constant c_1. Inductively,

$$\|z_n\|_{2,2} \leqslant \big(c_1 \|z_0\|_{2,2}\big)^{2^n - 1} \|z_0\|_{2,2}$$

and hence

$$\|u_{N+1} - u_0\|_{2,2} \leqslant \sum_{j=0}^{N} T^{2^j - 1} \|z_0\|_{2,2}$$

for $T = c_1 \|z_0\|_{2,2}$. As

$$L_{s_0} z_0 = \varepsilon F(x, u_0, \nabla u_0),$$

we may choose ε such that $\|z_0\|_{2,2} < 1/c_1$. Hence, $T < 1$ and

$$\|u_{N+1} - u_0\|_{2,2} \leqslant \varepsilon \big\|F(x, u_0, \nabla u_0)\big\|_2 \frac{1}{1 - T}$$

Thus, taking ε small we may assume that $\|u_n - u_0\|_{2,2} \leqslant R$ for any n, and the previous computations imply that $\{u_n\}$ is a Cauchy sequence. This completes the proof. □

The following theorem shows that under an extra assumption the step ε may be chosen uniformly.

Theorem 3. *Let us assume that*

$$0 \leqslant \frac{d[g(u)u]}{du} \leqslant M$$

for some constant M. Then the step ε of Theorem 2 may be chosen independent of u_0. Hence, there exists a sequence

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = 1,$$

where the solutions u_j of $(1\mathrm{b})_{\lambda_j}$ are constructed as in Theorem 2, and u_N is the unique solution of the original problem.

Proof. It suffices to prove the existence of a constant C such that if u satisfies $(1\mathrm{b})_\lambda$ for some λ, then $\|u\|_{2,2} \leqslant C$: Indeed, in that case in the proof of Theorem 2 we have that $\|u_0\|_{2,2} \leqslant C$, and by the imbedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ we conclude that the constant K can be considered such that

$$\left\|g(u) + ug'(u)\right\|_\infty \leqslant K, \qquad \left\|ug''(u) + 2g'(u)\right\|_\infty \leqslant K$$

for any $u \in B_{C+R}(u_0) \subset H^2(\Omega)$.

If u satisfies $(1\mathrm{b})_\lambda$, with the notation of Lemma 1 there exists a mean value ξ such that $L_s u = 0$ for $s = 2r^2/x^2V^2(\xi g'(\xi) + g(\xi))$, with $\|s\|_\infty \leqslant M$. As

$$\|u - h\|_{2,2} \leqslant c\|L_s h\|_{2,2} \leqslant C$$

for some constant C independent of u, our claim is proved. □

4. An upper and lower solutions method for (1a)

In this section we obtain solutions of (1a) by an upper and lower solutions method. As before, we shall consider the equivalent problem $(1\mathrm{b})_1$. Our main result is the following:

Theorem 4. *Let us assume that there exists a nonnegative constant α such that*:

(i) $h(x, y) \leqslant \alpha$ *for* $(x, y) \in \partial\Omega$,
(ii) $g(\alpha) \geqslant 0$.

Then the problem $(1\mathrm{b})_1$ admits a solution $u \in H^2(\Omega)$ with $0 \leqslant u \leqslant \alpha$.

Proof. Let

$$M = \sup_{0 \leqslant u \leqslant \alpha} g(u) + ug'(u),$$

and choose a positive constant s satisfying

$$s > \frac{2r}{x^2V^2}M$$

for any x such that $(x, y) \in \overline{\Omega}$ for some y. Hence the function

$$\psi(x, u) := \frac{2rg(u)}{x^2V^2}u - su$$

is strictly decreasing with respect to u for $u \in [0, \alpha]$. We define a sequence $\{u_n\}$ in the following way: set $u_0 \equiv \alpha$, and consider u_{n+1} as the unique solution (given by Lemma 1) of the linear problem

$$\begin{cases} Lu - su = \frac{2rg(u_n)}{x^2V^2}u_n - su_n & \text{in } \Omega, \\ u|_{\partial\Omega} = h, \end{cases}$$

where

$$Lu = \Delta u + 2\rho\frac{\partial u^2}{\partial x\partial y} - \rho\frac{\partial u}{\partial x} - \left(1 - \frac{2r}{x^2V^2}\right)\frac{\partial u}{\partial y}.$$

We claim that:

(i) $0 \leqslant u_n \leqslant \alpha$ for every n.
(ii) For any $(x, y) \in \Omega$ the sequence $\{u_n(x, y)\}$ is nonincreasing.

In order to prove claims (i) and (ii) we proceed by induction: assume, for example, that $u_1(x_0, y_0) > \alpha$ for some $(x_0, y_0) \in \overline{\Omega}$. As $u_1|_{\partial\Omega} = h \leqslant \alpha$, we deduce that $(x_0, y_0) \in \Omega$ and we may assume that (x_0, y_0) is a maximum. As $\nabla u_1(x_0, y_0) = 0$, we have that

$$\left.\left(\Delta u_1 + 2\rho\frac{\partial^2 u_1}{\partial x\partial y} - su_1\right)\right|_{(x_0,y_0)} = \frac{2rg(\alpha)}{x^2V^2}\alpha - s\alpha \geqslant -s\alpha.$$

Hence

$$\left.\left(\Delta u_1 + 2\rho\frac{\partial^2 u_1}{\partial x\partial y}\right)\right|_{(x_0,y_0)} = s\big[u_1(x_0, y_0) - \alpha\big] > 0,$$

which contradicts the maximum principle. On the other hand, as ψ is nonincreasing we have that $Lu_1 - su_1 = \psi(x, \alpha) \leqslant \psi(x, 0) \leqslant 0$, and being $u_1|_{\partial\Omega} = h \geqslant 0$ we obtain by the minimum principle that $u_1 \geqslant 0$.

Next, we assume as inductive hypothesis that $0 \leqslant u_n \leqslant u_{n-1}$. As before, if $[u_{n+1} - u_n](x_0, y_0) > 0$ is maximum, then

$$\begin{aligned}(Lu_{n+1} - su_{n+1})|_{(x_0,y_0)} &= \left.\left(\frac{2rg(u_n)}{x^2V^2}u_n - su_n\right)\right|_{(x_0,y_0)} \\ &\geqslant \left.\left(\frac{2rg(u_{n-1})}{x^2V^2}u_{n-1} - su_{n-1}\right)\right|_{(x_0,y_0)} = (Lu_n - su_n)|_{(x_0,y_0)}.\end{aligned}$$

Hence, as $\nabla u_{n+1}(x_0, y_0) = \nabla u_n(x_0, y_0)$, we conclude that

$$\left(\Delta(u_{n+1}-u_n)+2\rho\frac{\partial^2(u_{n+1}-u_n)}{\partial x\partial y}\right)\bigg|_{(x_0,y_0)}$$
$$=s\big[u_{n+1}(x_0,y_0)-u_n(x_0,y_0)\big]>0,$$

a contradiction. The inequality $u_{n+1}\geqslant 0$ follows in the same way as before.

Hence, there exists a function $u:\overline{\Omega}\to\mathbb{R}$ such that $u_n(x,y)\to u(x,y)$ for every (x,y). By Lemma 1, there exists $H\in H^2(\Omega)$ such that

$$LH-sH=0,\qquad H|_{\partial\Omega}=h.$$

Moreover,

$$\|u_{n+1}-H\|_{2,2}\leqslant c\big\|Lu_{n+1}-su_{n+1}-(LH-sH)\big\|_2=c\big\|\psi(\cdot,u_n)\big\|_2$$

and $|\psi(\cdot,u_n)|\leqslant K$ for a constant K independent of n. Hence, the sequence $\{u_n\}$ is bounded in $H^2(\Omega)$. Fix p such that $2<p<\infty$, and suppose that $u_n\not\longrightarrow u$ in $W^{1,p}(\Omega)$. Then there exists a subsequence $\{u_{n_j}\}$ with

$$\|u_{n_j}-u\|_{1,p}\geqslant\varepsilon$$

for some $\varepsilon>0$. By the compactness of the imbedding $H^2(\Omega)\hookrightarrow W^{1,p}(\Omega)$ the sequence $\{u_{n_j}\}$ admits a subsequence that converges in $W^{1,p}(\Omega)$ to some v with $\|v-u\|_{1,p}\geqslant\varepsilon$, a contradiction since $u_n\to u$ pointwise. Thus, taking limit in the equality

$$Lu_{n+1}-su_{n+1}=\frac{2rg(u_n)}{x^2V^2}u_n-su_n,$$

we easily conclude that

$$Lu=\frac{2rg(u)}{x^2V^2}u$$

and the proof is complete. □

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