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On geometric characterizations for Monge–Ampère doubling measures

Liliana Forzani [a,1] and Diego Maldonado [b,*,2]

[a] *IMAL-CONICET, Güemes 3450, 3000 Santa Fe, Argentina*
[b] *Departamento de Matemática, FCEyN—Universidad de Buenos Aires, Argentina*

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Abstract

In this article we prove a theorem on the size of the image of sections of a convex function under its normal mapping when the sections satisfy a geometric property. We apply this result to get new geometric characterizations for Monge–Ampère doubling measures.

Keywords: Real Monge–Ampère equation; Monge–Ampère measure; Sections of convex functions; Doubling measures; Spaces of homogeneous type

1. Introduction and main result

Given a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the normal mapping of φ is the set-valued function $\nabla\varphi : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$\nabla\varphi(x_0) = \left\{ p \in \mathbb{R}^n \colon\ \varphi(x) \geqslant \varphi(x_0) + p \cdot (x - x_0),\ \forall x \in \mathbb{R}^n \right\},$$

* Corresponding author.
E-mail addresses: lforzani@intec.unl.edu.ar (L. Forzani), diego@math.unl.edu.ar (D. Maldonado).
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and for $x_0 \in \mathbb{R}^n$, $p \in \nabla\varphi(x_0)$ and $t > 0$, a section of φ at height t is the convex set

$$S_\varphi(x_0, p, t) = \bigl\{x \in \mathbb{R}^n\colon\ \varphi(x) < \varphi(x_0) + p \cdot (x - x_0) + t\bigr\}.$$

The class of sets $E \subset \mathbb{R}^n$ such that $\nabla\varphi(E) = \bigcup_{x\in E} \nabla\varphi(x)$ is Lebesgue measurable is a σ-algebra containing the Borel sets of $\mathbb{R}^n$ and we shall denote such σ-algebra by $\mathcal{A}$. The Monge–Ampère measure associated with φ is defined for $E \in \mathcal{A}$ by

$$\mu(E) = \bigl|\nabla\varphi(E)\bigr|.$$

Throughout this paper the sections $S_\varphi(x, p, t)$ will be assumed to be bounded sets. For a bounded convex set S and a positive λ, the notation λS stands for the λ-dilation of S with respect to its center of mass, that is

$$\lambda S = \bigl\{x^* + \lambda(x - x^*)\colon\ x \in S\bigr\}$$

where x^* is the center of mass of S. We shall study Monge–Ampère measures that satisfy some doubling condition. We say that μ satisfies (DC) if it is doubling with respect to the center of mass on the sections of φ; that is, there exist constants $C > 0$ and $0 < \alpha < 1$ such that for all sections $S_\varphi(x, p, t)$, we have

$$\mu\bigl(S_\varphi(x, p, t)\bigr) \leqslant C\mu\bigl(\alpha S_\varphi(x, p, t)\bigr).$$

On the other hand, we say that μ satisfies (DP) if it is doubling with respect to the parameter on the sections of φ; that is, there exists a constant $C' > 0$ such that for all sections $S_\varphi(x, p, t)$ we have

$$\mu\bigl(S_\varphi(x, p, t)\bigr) \leqslant C'\mu\bigl(S_\varphi(x, p, t/2)\bigr).$$

The study of the properties of Monge–Ampère doubling measures and of the sections associated to convex functions has proved to be useful in the treatment of solutions of the Monge–Ampère equation, its elliptic and parabolic linearizations and its related real analysis, as can be seen in the fundamental papers of Caffarelli [2,3], Caffarelli and Gutiérrez [4,5] and Huang [8]. Also, some of the properties of the sections allow one to work in a more abstract setting by defining a quasi-metric d on $\mathbb{R}^n$ such that $(\mathbb{R}^n, d, \mu)$ becomes a space of homogeneous type. Then the real analysis associated to μ and the sections follows: covering lemmata, types of the Hardy–Littlewood maximal function, Calderón–Zygmund decomposition, BMO, singular integrals, etc. For this framework, we refer to the paper of Aimar et al. [1].

Many geometric conditions for the sections $S_\varphi(x, p, t)$ have been defined in order to explore the behavior of the Monge–Ampère measure. We first list three of them:

(i) the sections satisfy the *shrinking* property if there exist $0 < \tau, \lambda < 1$ such that for all $x \in \mathbb{R}^n$, $p \in \nabla\varphi(x)$ and $t > 0$, it holds that

$$S_\varphi(x, p, \tau t) \subset \lambda S_\varphi(x, p, t);$$

(ii) the sections satisfy the *engulfing* property if there exists a $\theta > 1$ such that if $y \in S_\varphi(x, p, t)$, then

$$S_\varphi(x, p, t) \subset S_\varphi(y, q, \theta t)$$

for all $q \in \nabla\varphi(y)$;

(iii) the sections satisfy the *engulfing** property if there exists a $\theta^* > 1$ such that if $y \in S_\varphi(x, p, t)$, then

$$S_\varphi(y, q, t) \subset S_\varphi(x, p, \theta^* t)$$

for all $q \in \nabla\varphi(y)$.

In [7], Gutiérrez and Huang studied these properties of the sections of a convex function φ when its associated Monge–Ampère measure verifies (DC). Among other results, they proved

Theorem 1 (Gutiérrez and Huang). *The Monge–Ampère measure associated to φ satisfies* (DC) *if and only if the sections of φ satisfy the shrinking property.*

Note that this theorem provides a purely geometric characterization of Monge–Ampère (DC) doubling measures. They also establish

Theorem 2 (Gutiérrez and Huang). *The shrinking property implies the engulfing property of the sections.*

We shall see that the converse to Theorem 2 also holds true. This will be a consequence of our main result, Theorem 3. In order to state it, let us fix some more notation.

We say that $x \in \mathbb{R}^n$ belongs to the ellipsoid $E(y, \rho)$ of center y and radii $\rho = (\rho_1, \dots, \rho_n)$ if

$$\sum_{i=1}^{n} \frac{(x_i - y_i)^2}{\rho_i^2} < 1.$$

For $t > 0$ we write

$$\frac{t}{\rho} = \left(\frac{t}{\rho_1}, \dots, \frac{t}{\rho_n}\right).$$

From now on, we shall omit the subscript φ in the notation of the sections of φ. We will prove

Theorem 3. *If the sections of φ satisfy the engulfing property with constant θ, then there exist constants c and C depending only on θ and the dimension such that, if a section $S(x_0, p, t)$ with center of mass x_0^* and an ellipsoid $E(x_0^*, \rho)$ verify*

$$\alpha_n E(x_0^*, \rho) \subset S(x_0, p, t) \subset E(x_0^*, \rho) \tag{1.1}$$

where α_n is a dimensional constant, then we have

$$cE\left(p, \frac{t}{\rho}\right) \subset \nabla\varphi\bigl(S(x_0, p, t)\bigr) \subset CE\left(p, \frac{t}{\rho}\right). \tag{1.2}$$

Remarks. As we shall see, every constant $c \leqslant 1/4$ works in (1.2). Hypothesis (1.1) in Theorem 3 is always available by means of John's lemma; see Section 3 for its statement.

The paper is organized as follows: Section 2 is devoted to the proof of Theorem 3. In Section 3 we prove that the engulfing property implies the shrinking property and finally, in Section 4, we discuss other conditions on the sections and summarize some related consequences of Theorem 3.

2. Proof of Theorem 3

Once we are given an ellipsoid $E(x_0^*, \rho)$ such that (1.1) holds, we first assume that $\varphi(x_0) = 0$ and $p = 0$, so we have φ non-negative in $\mathbb{R}^n$ and $x \in S(x_0, 0, t)$ if and only if $\varphi(x) < t$. Let us begin with the inclusion

$$\nabla\varphi\bigl(S(x_0, p, t)\bigr) \subset CE\left(p, \frac{t}{\rho}\right). \tag{2.1}$$

If $q \in \nabla\varphi(S(x_0, 0, t))$, $q \in \nabla\varphi(x)$ for some $x \in S(x_0, 0, t)$ and by the engulfing property,

$$S(x_0, 0, t) \subset S(x, q, \theta t). \tag{2.2}$$

We define a couple of auxiliary vectors w and h by

$$w = \frac{1}{(\sum_{j=1}^n q_j^2 \rho_j^2)^{1/2}} \bigl(q_1\rho_1^2, \ldots, q_n\rho_n^2\bigr), \tag{2.3}$$

$$h = \beta(x_0^* - \gamma_n w) + (1-\beta)x \tag{2.4}$$

where γ_n and $\beta \in (0,1)$ are constants to fix in a while. We claim that if $h \in S(x_0, 0, t)$ then (2.1) holds. Indeed, if $h \in S(x_0, 0, t)$, by (2.2) we have $h \in S(x, q, \theta t)$, that is

$$\begin{aligned}\theta t &> \varphi(h) - \varphi(x) - q \cdot (h - x) = \varphi(h) - \varphi(x) - q \cdot \bigl(\beta x_0^* - \beta\gamma_n w - \beta x\bigr)\\ &= \varphi(h) - \varphi(x) - \beta q \cdot x_0^* + \beta\gamma_n q \cdot w + \beta q \cdot x\end{aligned}$$

which yields

$$\begin{aligned}\beta\gamma_n q \cdot w &< \theta t + \varphi(x) - \varphi(h) + \beta q \cdot x_0^* - \beta q \cdot x\\ &\leqslant \theta t + \varphi(x) + \beta q \cdot (x_0^* - x) \leqslant \theta t + \varphi(x) + \beta\bigl(\varphi(x_0^*) - \varphi(x)\bigr)\\ &= \theta t + \beta\varphi(x_0^*) + (1-\beta)\varphi(x) \leqslant \theta t + \beta t + (1-\beta)t = (\theta + 1)t\end{aligned}$$

where we have used that φ is non-negative and convex, and the fact that $x, x_0 \in S(x_0, 0, t)$. We summarize these inequalities in the following:

$$\beta\gamma_n q \cdot w < (\theta+1)t. \tag{2.5}$$

On the other hand, we have

$$\beta\gamma_n q \cdot w = \beta\gamma_n \frac{\sum_{j=1}^n q_j^2\rho_j^2}{(\sum_{j=1}^n q_j^2\rho_j^2)^{1/2}} = \beta\gamma_n \left(\sum_{j=1}^n q_j^2\rho_j^2\right)^{1/2}. \tag{2.6}$$

By linking (2.5) and (2.6), we finally obtain

$$\sum_{j=1}^n q_j^2\rho_j^2 < \left(\frac{\theta+1}{\beta\gamma_n}\right)^2 t^2 \tag{2.7}$$

and this is the inclusion (2.1) with $C = (\theta+1)/(\beta\gamma_n)$. To prove that $h \in S(x_0, 0, t)$, note that by (2.4) it is enough to show

$$x_0^* - \gamma_n w \in S(x_0, 0, t),$$

since x already belongs to the convex set $S(x_0, 0, t)$. In fact, we shall see that it is possible to choose γ_n such that

$$x_0^* - \gamma_n w \in \alpha_n E(x_0^*, \rho). \tag{2.8}$$

We write

$$\sum_{i=1}^n \left[\frac{1}{\rho_i}(x_{0,i}^* - \gamma_n w_i - x_{0,i}^*)\right]^2 = \frac{1}{\sum_{i=1}^n q_i^2\rho_i^2} \sum_{i=1}^n \frac{\gamma_n^2}{\rho_i^2} q_i^2\rho_i^4 = \gamma_n^2, \tag{2.9}$$

so that we just have to take $\gamma_n < \alpha_n$ to get (2.8). This completes the proof of inclusion (2.1), with any $\beta \in (0, 1)$. Now, we face the inclusion

$$\frac{1}{4}E\left(p, \frac{t}{\rho}\right) \subset \nabla\varphi\big(S(x_0, 0, t)\big), \tag{2.10}$$

always in the case $\varphi(x_0) = 0$. Let us take $q \in \nabla\varphi(x)$ with $x \notin S(x_0, 0, t)$ and consider $\bar{x} \in \partial S(x_0, 0, t)$ such that $x_0, \bar{x}$ and x are aligned. Then we have

$$x - x_0 = k(\bar{x} - x_0) \tag{2.11}$$

for some $k > 1$. Since φ is convex, for $q \in \nabla\varphi(x)$ and $\bar{q} \in \nabla\varphi(\bar{x})$ it holds that

$$0 \leqslant (q - \bar{q}) \cdot (x - \bar{x}) = (q - \bar{q}) \cdot \big[(x - x_0) + (x_0 - \bar{x})\big].$$

Replacing $(x - x_0)$ in this equation by its expression in (2.11) we get

$$q \cdot (\bar{x} - x_0) \geqslant \bar{q} \cdot (\bar{x} - x_0).$$

We also have $q' \cdot (x' - x_0) \geqslant \varphi(x')$ for all $x' \in \mathbb{R}^n$ and $q' \in \nabla\varphi(x')$. In particular, applying this to $\bar{x}$, we get that $\bar{q} \cdot (\bar{x} - x_0) \geqslant \varphi(\bar{x}) = t$. We combine all this in the following inequalities

$$\left(\sum_i q_i^2\frac{\rho_i^2}{t^2}\right)^{1/2}\left(\sum_i\frac{(\bar{x}_i-x_{0,i})^2}{\rho_i^2}\right)^{1/2}\geqslant\sum_i q_i\frac{1}{t}(\bar{x}_i-x_{0,i})$$
$$=\frac{1}{t}q\cdot(\bar{x}-x_0)\geqslant\frac{1}{t}\bar{q}\cdot(\bar{x}-x_0)\geqslant 1.$$

Now, if we found an $\varepsilon>0$ such that

$$\sum_i\frac{(\bar{x}_i-x_{0,i})^2}{\rho_i^2}<\frac{1}{\varepsilon} \tag{2.12}$$

we would have $\sum_i q_i^2\rho_i^2/t^2>\varepsilon$, which implies $q\notin\varepsilon E(0,t/\rho)$. We will now find and estimate the size of such an ε. Since $\bar{x}\in\partial S(x_0,0,t)$, we have $\bar{x}\in\overline{E(x_0^*,\rho)}$; that is,

$$\sum_i\frac{(\bar{x}_i-x_{0,i}^*)^2}{\rho_i^2}\leqslant 1, \tag{2.13}$$

and we write

$$\sum_i\frac{(\bar{x}_i-x_{0,i})^2}{\rho_i^2}=\sum_i\frac{(\bar{x}_i-x_{0,i}^*+x_{0,i}^*-x_{0,i})^2}{\rho_i^2}$$
$$=\sum_i\frac{(\bar{x}_i-x_{0,i}^*)^2}{\rho_i^2}+2\sum_i\frac{(\bar{x}_i-x_{0,i}^*)(x_{0,i}^*-x_{0,i})}{\rho_i^2}+\sum_i\frac{(x_{0,i}^*-x_{0,i})^2}{\rho_i^2}$$
$$\leqslant 1+2\left(\sum_i\frac{(\bar{x}_i-x_{0,i}^*)^2}{\rho_i^2}\right)^{1/2}\left(\sum_i\frac{(x_{0,i}^*-x_{0,i})^2}{\rho_i^2}\right)^{1/2}$$
$$+\sum_i\frac{(x_{0,i}^*-x_{0,i})^2}{\rho_i^2}$$
$$<1+2+1=4,$$

where we have used that $x_0\in S(x_0,0,t)\subset E(x_0^*,\rho)$ and (2.13). If we now take $\varepsilon=1/4$ inclusion (2.10) follows.

To complete the proof we need to deal with the general case on $\varphi(x_0)$ and $p\in\nabla\varphi(x_0)$. Given $x_0\in\mathbb{R}^n$ and $p\in\nabla\varphi(x_0)$ we define the convex function ψ with $\psi(x_0)=0$ by

$$\psi(x)=\varphi(x)-\varphi(x_0)-p\cdot(x-x_0).$$

Then

$$S_\psi(x_0,0,t)=S_\varphi(x_0,p,t),$$
$$\nabla\psi\big(S_\psi(x_0,0,t)\big)+p=\nabla\varphi\big(S_\varphi(x_0,p,t)\big),$$

and we finish the proof of Theorem 3, by applying the known facts to ψ. □

Now we just take Lebesgue measure in (1.1) and (1.2) to get the following

Corollary 4. *If the sections of φ satisfy the engulfing property, then there exist positive constants C_1, C_2, depending only on the engulfing constant and the dimension, such that for every section $S = S(x, p, t)$ it holds that*

$$C_1 t^n \leqslant |S|\mu(S) \leqslant C_2 t^n, \tag{2.14}$$

where μ is the Monge–Ampère measure associated to φ and $|\cdot|$ is the Lebesgue measure.

3. Equivalence between engulfing and shrinking properties

When the sections $S(x, p, t)$ and the measure μ satisfy (2.14) with constants independent of x and p we will write

$$|S|\mu(S) \simeq t^n.$$

Our next goal is to prove the following

Theorem 5. *If the Monge–Ampère measure associated to φ satisfies*

$$|S|\mu(S) \simeq t^n \tag{3.1}$$

for every section $S = S(x, p, t)$, then the sections of φ satisfy the shrinking property.

Before proving Theorem 5, we need to introduce some concepts about normalization. Given an affine transformation $Tx = Ax + b$ where A is an $n \times n$ invertible real matrix and $b \in \mathbb{R}^n$, we set $\psi_s(y) = (1/s)\varphi(T^{-1}y)$, φ being a convex function. For $E \in \mathcal{A}$, $\mu(E) = |\nabla\varphi(E)|$ and $\bar{\mu}(E) = |\nabla\psi_s(E)|$ are the Monge–Ampère measures associated to φ and ψ_s respectively. We have

$$\frac{1}{s}(A^{-1})^t\big(\nabla\varphi(E)\big) = \nabla\psi_s(TE)$$

and

$$\frac{1}{s^n}|\det T^{-1}|\mu(E) = \bar{\mu}(TE).$$

The sections of φ and ψ_s are related as follows:

$$T\big(S_\varphi(x, p, t)\big) = S_{\psi_s}\left(Tx, \frac{1}{s}(A^{-1})^t p, \frac{t}{s}\right). \tag{3.2}$$

Now, let us mention a lemma due to Fritz John.

Lemma 6. *Let S be a bounded convex set in $\mathbb{R}^n$ with non-empty interior, and let E be the ellipsoid of minimum volume containing S whose center is the center of* *mass of S. Then, there exists a constant α_n depending only on the dimension such that*

$$\alpha_n E \subset S \subset E.$$

Since E is an ellipsoid, there is an affine transformation T such that $T(E) = B(0,1)$ and

$$B(0,\alpha_n) \subset T(S) \subset B(0,1).$$

The set $T(S)$ is called a normalization of S and T is called an affine transformation that normalizes S. We say that a convex set S is normalized when its center of mass is 0 and $B(0,\alpha_n) \subset S \subset B(0,1)$. Finally, one can also check that if T normalizes $S_\varphi(x,p,t)$ then $T(\alpha S_\varphi(x,p,t)) = \alpha T(S_\varphi(x,p,t))$. This implies that if μ verifies doubling conditions (DC) or (DP) on the sections of φ then $\bar{\mu}$ (as defined above) verifies the same conditions on the sections of ψ_s and with the same constants. We shall use the following estimate:

Theorem 7 (Aleksandrov). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded and convex set, and $u \in C(\bar{\Omega})$, u convex and $u|_{\partial\Omega} = 0$. Then there exists a constant c_n depending only on dimension n such that*

$$\left|u(x)\right|^n \leqslant c_n \left(\operatorname{diam}(\Omega)\right)^{n-1} \operatorname{dist}(x,\partial\Omega)\mu(\Omega)$$

for all $x \in \Omega$. Where μ is the Monge–Ampère measure associated to u.

Proof of Theorem 5. Let us focus our attention in the case that the constants in (2.14) depend only on n and θ. Following the lines of the proof of Theorem 2.1 in [7], we shall show that there exists a constant $0 < \beta(n,\theta) \leqslant 1$ depending only on n and θ (the engulfing constant) such that $S_\varphi(x_0,p,\tau t) \subset \lambda S_\varphi(x_0,p,t)$ holds for every τ and λ with $0 < \tau < 1$ and $1 - \beta(n,\theta)(1-\tau)^n \leqslant \lambda < 1$.

Given a section $S_\varphi(x_0,p,t)$, let T be an affine transformation which normalizes it, that is,

$$B(0,\alpha_n) \subset T\left(S_\varphi(x_0,p,t)\right) \subset B(0,1).$$

We denote by x_0^* the center of mass of $S_\varphi(x_0,p,t)$, and define $\psi(y) = \varphi(T^{-1}y)$. For $\lambda \in (0,1)$, because of (3.2), we have that

$$T\left(S_\varphi(x_0,p,\lambda t)\right) = S_\psi(Tx_0,q,\lambda t),$$

where $q = (A^{-1})^t p$. Since the center of mass of $S_\psi(Tx_0,q,\lambda t)$ is $Tx_0^* = 0$,

$$\begin{aligned} T\left(\lambda S_\varphi(x_0,p,t)\right) &= T\left\{x_0^* + \lambda(x - x_0^*)\colon x \in S_\varphi(x_0,p,t)\right\} \\ &= \left\{\lambda Tx\colon x \in S_\varphi(x_0,p,t)\right\} = \lambda S_\psi(Tx_0,q,t). \end{aligned}$$

If we set $\psi^*(y) = \psi(y) - \psi(Tx_0) - q \cdot (y - Tx_0) - t$, then $\nabla\psi^* = \nabla\psi - q$ (this implies that the Monge–Ampère measures associated to ψ^* and ψ are the same) and

$$\psi^*|_{\partial S_\psi(Tx_0,q,t)} = 0.$$

If $y \in S_\psi(Tx_0, q, t) - \lambda S_\psi(Tx_0, q, t)$ we have $\operatorname{dist}(y, \partial S_\psi(Tx_0, q, t)) \leqslant 1 - \lambda$ and, by Theorem 7,

$$\begin{aligned} \left|\psi^*(y)\right|^n &\leqslant c_n \operatorname{dist}\bigl(y, \partial S_\psi(Tx_0, q, t)\bigr)\bigl(\operatorname{diam}\bigl(S_\psi(Tx_0, q, t)\bigr)\bigr)^{n-1} \\ &\quad \times \mu_\psi\bigl(S_\psi(Tx_0, q, t)\bigr) \\ &\leqslant c_n(1-\lambda)\bigl(\operatorname{diam}\bigl(S_\psi(Tx_0, q, t)\bigr)\bigr)^{n-1}\mu_\psi\bigl(S_\psi(Tx_0, q, t)\bigr). \end{aligned}$$

In order to estimate $\mu_\psi(S_\psi(Tx_0, q, t))$, we use (2.14) for μ_ψ to get

$$\mu\bigl(S_\psi(Tx_0, q, t)\bigr) \leqslant C(n, \theta)t^n,$$

where $C = C(n, \theta)$, since $S_\varphi(x_0, p, t)$ is already normalized by T, that is,

$$B(0, \alpha_n) \subset T\bigl(S_\varphi(x_0, p, t)\bigr) = S_\psi(Tx_0, q, t) \subset B(0, 1).$$

Then we have that

$$\left|\psi^*(y)\right|^n \leqslant c(n, \theta)(1-\lambda)t^n$$

which yields

$$\psi^*(y) \geqslant -c(n, \theta)^{-1/n}(1-\lambda)^{1/n}t,$$

and using the definition of ψ^*,

$$\psi(y) - \psi(Tx_0) - q \cdot (y - Tx_0) \geqslant \bigl(1 - c(n, \theta)^{-1/n}(1-\lambda)^{1/n}\bigr)t \geqslant \tau t,$$

for all $\tau \leqslant 1 - c(n, \theta)^{-1/n}(1-\lambda)^{1/n}$; that is to say, $1 - \beta(n, \theta)(1-\tau)^n \leqslant \lambda < 1$ for some $\beta(n, \theta) \in (0, 1]$. Then,

$$S_\psi(Tx_0, q, \tau t) \subset \lambda S_\psi(Tx_0, q, t),$$

and applying T^{-1} to both sides we finally obtain

$$S_\varphi(x_0, p, \tau t) \subset \lambda S_\varphi(x_0, p, t),$$

the shrinking property. Therefore, the converse to Theorem 2 is established and the engulfing and shrinking properties are equivalent. □

4. A survey of results via Theorem 5 and comments

In this section we mention some results and link them by applying Theorem 5 in order to provide a unified view for the behavior of Monge–Ampère (DC) doubling measures in terms of the properties of the sections. But before that, we add some more notation. By John's lemma, there exist positive dimensional constants C_1, C_2 such that for every $S(x, p, t)$ section of φ there exists an ellipsoid $E(x^*, \rho)$ verifying $C_1 E(x^*, \rho) \subset S(x, p, t) \subset C_2 E(x^*, \rho)$. We will denote this by writing $S(x, p, t) \simeq E(x^*, \rho)$. Analogously, if there exist positive constants c_1, c_2, independent of x, p, t and s such that $c_1 E(p, s) \subset \nabla\varphi(S(x, p, t)) \subset c_2 E(p, s)$, for every section $S(x, p, t)$ and $s > 0$, then we will write $\nabla\varphi(S(x, p, t)) \simeq E(p, s)$.

Caffarelli in [2] (and later Gutiérrez and Huang in [7]) proved that if the sections of φ are bounded and its associated Monge–Ampère measure satisfies (DC) then φ is strictly convex. On the other hand, in [3] Caffarelli proved that if the Monge–Ampère measure associated to φ satisfies (DC) with φ strictly convex, then φ is $C^{1,\alpha}$ on compact sets of $\mathbb{R}^n$ with α depending on the local Lipschitz constant for φ. See also the recent book [6] for a comprehensive exposition of these results. Of course, in the case that the convex function φ has derivatives at x_0, if $p \in \nabla\varphi(x_0)$ then $p = \nabla\varphi(x_0)$, the gradient of φ at x_0. In this section, we shall assume φ to be regular which allows to omit the p in the notation for the sections. Because of the comments above and the properties that will be required of the sections, we shall see that this assumption is not actually a restriction. Then, the sections of φ may be denoted by $S(x, t)$ for $x \in \mathbb{R}^n$ and $t > 0$.

We list some more possible properties for the family of all sections:

(A) There exist constants K_1, K_2, K_3 and ϵ_1, ϵ_2 all positive, with the following property: given two sections $S(x_0, t_0), S(x, t)$ with $t \leqslant t_0$ such that

$$S(x_0, t_0) \cap S(x, t) \neq \emptyset,$$

and given T an affine transformation that normalizes $S(x_0, t_0)$, there exists $z \in B(0, K_3)$ depending on $S(x_0, t_0)$ and $S(x, t)$, such that

$$B\left(z, K_2\left(\frac{t}{t_0}\right)^{\epsilon_2}\right) \subset T\big(S(x, t)\big) \subset B\left(z, K_1\left(\frac{t}{t_0}\right)^{\epsilon_1}\right)$$

and

$$Tx \in B\left(z, \frac{1}{2}K_2\left(\frac{t}{t_0}\right)^{\epsilon_2}\right).$$

(B) There exists a constant $\delta > 0$ such that given a section $S(x, t)$ and $y \notin S(x, t)$, if T is an affine transformation that normalizes $S(x, t)$, then

$$B\big(T(y), \epsilon^{\delta}\big) \cap T\big(S\big(x, (1-\epsilon)t\big)\big) = \emptyset,$$

for any $0 < \epsilon < 1$.

(C) $\bigcap_{t>0} S(x, t) = \{x\}$ and $\bigcup_{t>0} S(x, t) = \mathbb{R}^n$.

These geometric properties allowed Caffarelli and Gutiérrez to develop in [4] a real analysis theory related to the Monge–Ampère equation by proving a Besicovitch type covering lemma for the sections of φ if its associated Monge–Ampère measure satisfies (DP) (see [8] for the parabolic case). We stress the importance of these geometric conditions (A), (B) and (C), since they also characterize (DC), as we shall prove in the next theorem. So, since (DC) implies (DP), we note that the doubling conditions in Caffarelli–Gutiérrez theory can be taken for granted once we assume the geometric properties above. We are now in position to prove the following characterizations for (DC).

Theorem 8. *Let* $S(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, *be the sections of a strictly convex function* φ *with* $\varphi \in C^1(\mathbb{R}^n)$. *Then the following are equivalent*:

(i) *The Monge–Ampère measure associated to* φ *satisfies* (DC).
(ii) *The sections satisfy properties* (A) *and* (B).
(iii) *The sections satisfy the engulfing property.*
(iv) *If* $S(x,p,t) \simeq E(x^*,\rho)$ *then*

$$\nabla\varphi\big(S(x,p,t)\big) \simeq E\left(p, \frac{t}{\rho}\right).$$

(v) *The Monge–Ampère measure satisfies*

$$|S|\mu(S) \simeq t^n,$$

for all sections $S = S(x,t)$.
(vi) *The sections satisfy the shrinking property.*

Proof. We will follow the order (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i). By Corollary 2.2 in [7] one knows that (DC) implies property (B), and in the same paper Gutiérrez and Huang prove (Theorem 2.3) that this doubling condition implies property (A). Moreover, in this property one can take $z = Tx$ and $\epsilon_2 = 1$. On the other hand, in [1], Aimar et al. showed that, if the family $S(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, satisfies (A), (B) and (C), then it also satisfies both the engulfing and the engulfing* properties with constants θ and θ^*, respectively, that depend only on δ, K_1 and ϵ_1 (of course, property (C) is satisfied by the sections of any strictly convex function). Now, by Theorem 3 we know that the engulfing property implies that $\nabla\varphi(S(x,p,t)) \simeq E(p,t/\rho)$ if $S(x,p,t) \simeq E(x^*,\rho)$ and, by Corollary 4, this last condition implies $|S|\mu(S) \simeq t^n$ for every section $S = S(x,t)$. The next step is to use Theorem 5 to get that $|S|\mu(S) \simeq t^n$ implies the shrinking property of the sections and finally, by virtue of the equivalence between shrinking property and (DC) (Theorem 1), we complete the proof. □

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