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A Fixed Point Operator for a Nonlinear Boundary Value Problem[1]

P. Amster and M. C. Mariani

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina
E-mail: pamster, mcmarian@dm.uba.ar

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We study a semilinear second order equation with a nonlinear boundary condition for the axial deformation of a nonlinear elastic beam in the presence of friction. Under appropriate conditions we define a fixed point operator in order to obtain solutions for this equation.

Key Words: nonlinear BVP; fixed point methods.

1. INTRODUCTION

We study the semilinear second order ODE

$$u'' + g(t, u, u') = 0, \qquad u'(0) = -f(u(0)), \qquad u'(\pi) = f(u(\pi)) \qquad (1)$$

with g: $[0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ and f: $\mathbb{R} \to \mathbb{R}$ continuous.

In 1995 Rebelo and Sanchez [3] considered problem (1) for $g = g(t, u)$ satisfying a sign condition or either an increasing condition with respect to u, and $f \in C(\mathbb{R}, \mathbb{R})$ continuous and strictly increasing, which may be regarded as a mathematical model for the axial deformation of a nonlinear elastic beam, with two nonlinear elastic springs acting at the extremities according to the law $u'(0) = -f(u(0))$, $u'(\pi) = f(u(\pi))$, and the total force exerted by the nonlinear spring undergoing the displacement u given by $g(t, u)$. Moreover, (1) is the second order analogue of a fourth order problem for the deflection of a beam resting on elastic bearings, considered

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among other authors by Grossinho and Ma (see [1, 2, 4] for asymmetric boundary conditions):

$$u'''' + g(t,u) = 0,$$
$$u''(0) = u''(\pi) = 0,$$
$$u'''(0) = -f(u(0)), \qquad u'''(\pi) = f(u(\pi))$$

We assume that the force g depends also on the velocity u', adding a dissipative effect to the model. Under appropiate conditions we find solutions of (1) by the use of fixed point methods.

2. A FINITE DIMENSIONAL FIXED POINT OPERATOR FOR (1)

In this section we define a fixed point operator θ_1: $\mathbb{R}^2 \to \mathbb{R}^2$ in order to obtain solutions of (1). More precisely, we set $\theta_1 = \theta(1,\cdot)$ for some θ: $[0,1]\times\mathbb{R}^2 \to \mathbb{R}^2$ and establish conditions for the solvability of (1) by the use of the homotopy invariance of the Brouwer degree. First we state the unique solvability of the associated two-point BVP under the conditions

$$\begin{aligned}(2.1)\qquad &[g(t,u,x) - g(t,v,y)](u-v)\\ &\quad\le c_1(u-v)^2 + c_2|(u-v)(x-y)| + c_3(x-y)^2\\ &\qquad + [r(t)(u-v) + \psi(u-v)](x-y),\end{aligned}$$

for some $\psi \in L^1_{loc}(\mathbb{R})$, $r \in H^1(0,\pi)$, $r' \ge 0$ a.e., and $c_i \in \mathbb{R}$ such that $c_1 + c_2 + c_3 = 1 - \delta < 1$, and

$$(2.2)\qquad |g(t,u,x)| \le g_0(u) + k|x|.$$

THEOREM 2.1. *Let g satisfy* (2.1) *and* (2.2). *Then the Dirichlet problem*

$$u'' + g(t,u,u') = 0 \qquad \text{in } (0,\pi)$$
$$u(0) = u_0, \qquad u(\pi) = u_\pi$$

is uniquely solvable in $H^2(0,\pi)$ for arbitrary boundary data u_0, $u_\pi \in \mathbb{R}$.

Proof. Let us consider $A = [0,1]\times B_M$, where $B_M \subset H^1(0,\pi)$ is the open ball of radius M centered at 0, and T: $\overline{A} \to H^1(0,\pi)$ is given by $T(\sigma,\bar u) = u$, with u the unique solution of the linear problem

$$u'' + \sigma g(t,\bar u,\bar u') = 0 \qquad \text{in } (0,\pi)$$
$$u(0) = u_0, \qquad u(\pi) = u_\pi.$$

It is immediate that T is compact and that $T_0 = T(0, \cdot)$ is constant. Moreover, (2.1) implies that $\|u - v\|_{1,2} \leq c\|(u-v)'' + \sigma(g(t,u,u') - g(t,v,v'))\|_2$ for any $u, v \in H^1(0,\pi)$ such that $u = v$ in ∂I. Then for $T(\sigma, \bar{u}) = \bar{u}$ it holds that

$$\|\bar{u} - \varphi\|_{1,2} \leq c\sigma\|g(\cdot, \varphi, \varphi')\|_2,$$

where φ is the line through the points $(0, u_0)$ and (π, u_π). Hence, choosing M large enough we conclude that $T(\sigma, \bar{u}) \neq \bar{u}$ for any $\bar{u} \in \partial B_M$. Moreover, by definition of the Leray–Schauder degree we have that

$$deg_{LS}(I - T_0, B_M, 0) = deg_B(I - T_0|_X, B_M \cap X, 0),$$

where $X = span\{T_0\}$, and hence $deg_{LS}(I - T_0, B_M, 0) = 1$ for $M > \|T_0\|_{1,2}$. By homotopy invariance, we conclude that $deg_{LS}(I - T_1, B_M, 0) = 1$, proving that $T(1, u) = u$ for some $u \in B_M$. ∎

We will also make use of the following

THEOREM 2.2. *Let g satisfy* (2.1) *and* (2.2) *and consider the sets*

$$S_g = \{u \in H^2(0,\pi) : u'' + g(t,u,u') = 0\},$$
$$P_1 = \{\varphi(t) = x + ty : x, y \in \mathbb{R}\},$$

and the trace mapping Tr: $S_g \to P_1$ *given by* $Tr(u)(t) = \frac{u(\pi)-u(0)}{\pi}t + u(0)$.
Then Tr is an homeomorphism for the H^2*-norm.*

Proof. From the previous theorem, Tr is bijective, and its continuity is obvious. Conversely, if $\varphi_n \to \varphi$ let $u_n = Tr^{-1}(\varphi_n)$, $u = Tr^{-1}(\varphi)$ and for $w_n = u_n - u$ we have

$$\begin{aligned}
0 &= \int_0^\pi [w_n'' + (g(t,u,u') - g(t,u_n,u_n'))]w_n \\
&\leq [w_n w_n']_0^\pi + \int_0^\pi c_1(w_n)^2 + c_2 \mid w_n w_n'| \\
&\quad + \int_0^\pi (c_3 - 1)(w_n')^2 + [\Psi(w_n)]_0^\pi + \int_0^\pi r w_n w_n' \\
&\leq -\delta \int_0^\pi (w_n')^2 + [w_n w_n']_0^\pi + [\Psi(w_n)]_0^\pi + \left[r\frac{w_n^2}{2}\right]_0^\pi
\end{aligned}$$

with $\Psi(u) = \int_0^u \psi(t)\,dt$.

As $w_n(0) = \varphi_n(0) - \varphi(0) \to 0$, $w_n(\pi) = \varphi_n(\pi) - \varphi(\pi) \to 0$ and being w_n bounded in $H^1(0,\pi)$, it suffices to prove that $|(w_n'(0), w_n'(\pi))|$ is bounded. From the inequality

$$\|w_n''\|_2 \leq \|g(\cdot, u_n, u_n') - g(\cdot, u, u')\|_2$$

it is easy to conclude that w_n is bounded in $H^2(0, \pi)$. Hence, $\|w_n'\|_\infty$ is bounded and the result follows. ∎

Assuming that g satisfies (2.1) and (2.2), we may define a fixed point operator for (1) in the following way: for fixed $(x, y) \in \mathbb{R}^2$ we consider $\varphi(t) = \frac{y-x}{\pi} t + x$ and $u_{x,y}$ the unique solution of the problem

$$\begin{aligned} &u'' + g(t, u, u') = 0 && \text{in } (0, \pi) \\ &u(0) = x, \qquad u(\pi) = y. \end{aligned}$$

Then

$$u_{x,y}(t) = \varphi(t) - \int_0^\pi G(t,s) g(s, u_{x,y}, u'_{x,y})\, ds,$$

where G is the Green function

$$G(t,s) = \begin{cases} \dfrac{t(s-\pi)}{\pi} & \text{if } s \geq t \\ \dfrac{(t-\pi)s}{\pi} & \text{if } s \leq t. \end{cases}$$

Then

$$u'_{x,y}(0) = \frac{y-x}{\pi} - \int_0^\pi \frac{s-\pi}{\pi} g(s, u_{x,y}, u'_{x,y})\, ds$$
$$u'_{x,y}(\pi) = \frac{y-x}{\pi} - \int_0^\pi \frac{s}{\pi} g(s, u_{x,y}, u'_{x,y})\, ds$$

and any solution of (1) will be given by $u_{x,y}$, where x, y satisfy

$$\frac{y-x}{\pi} - \int_0^\pi \frac{s-\pi}{\pi} g(s, u_{x,y}, u'_{x,y})\, ds = -f(x),$$
$$\frac{y-x}{\pi} - \int_0^\pi \frac{s}{\pi} g(s, u_{x,y}, u'_{x,y})\, ds = f(y).$$

Thus, it suffices to find the zeroes of the mapping $\theta_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\theta_1(x, y) = \left(y - x - \int_0^\pi (s-\pi) g(s, u_{x,y}, u'_{x,y})\, ds,\ y - x - \int_0^\pi s g(s, u_{x,y}, u'_{x,y})\, ds \right) + \pi(f(x), -f(y)).$$

Continuity of θ_1 follows immediately from Theorem 2.2. This allows us to prove the following result:

THEOREM 2.3. *Let $A \subset [0,1] \times \mathbb{R}^2$ be open and bounded such that*

$$A_\sigma = \{(x, y) \in \mathbb{R}^2 : (\sigma, x, y) \in A\}$$

is not empty for every σ, and assume that if u is a solution of the problem

$$u'' + \sigma g(t, u, u') = 0, \qquad u'(0) = -f(u(0)), \qquad u'(\pi) = f(u(\pi))$$

then $(u(0), u(\pi)) \notin \partial A_\sigma$. Moreover, let us suppose that

$$deg_B(\tilde{f}, A_0, 0) \neq 0,$$

where $\tilde{f}$ is given by

$$\tilde{f}(x, y) = (y - x + \pi f(x), y - x - \pi f(y)).$$

Then problem (1) *admits at least one solution u with* $(u(0), u(\pi)) \in A_1$.

Proof. We extend the definition of θ_1 in the following way: for $\sigma \in [0, 1]$ we define $\theta_\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\theta_\sigma(x, y) = \left(y - x - \int_0^\pi \sigma(s - \pi) g(s, u^\sigma_{x,y}, (u^\sigma_{x,y})')\, ds, y - x \right.$$
$$\left. - \int_0^\pi \sigma s g(s, u^\sigma_{x,y}, (u^\sigma_{x,y})')\, ds \right) + \pi(f(x), -f(y)),$$

where $u^\sigma_{x,y}$ is the unique solution of the problem

$$u'' + \sigma g(t, u, u') = 0 \qquad \text{in } (0, \pi)$$
$$u(0) = x, \qquad u(\pi) = y.$$

Hence, θ_σ does not vanish on ∂A_σ, and as $\tilde{f} = \theta_0$, the result follows from the homotopy invariance of the Brouwer degree. ∎

3. SOLUTIONS FOR INVERTIBLE f

In this section we establish a fixed point operator in $H^1(0, \pi)$ for solutions of (1) when f is invertible. Indeed, we may define the compact operator $K\colon C^1[0, \pi] \to C^1[0, \pi]$ by

$$Ku(t) = \varphi_u(t) - \int_0^\pi G(t, s) g(s, u, u')\, ds,$$

where

$$\varphi_u(t) = \frac{f^{-1}(u'(\pi)) - f^{-1}(-u'(0))}{\pi} t + f^{-1}(-u'(0)).$$

As before, it is clear that any solution of (1) may be regarded as a fixed point of K. Moreover,

$$\|Ku\|_\infty \leq \|\varphi_u\|_\infty + \sup_{0 \leq t \leq \pi} \{\|G(t, \cdot)\|_1\} \|g(\cdot, u, u')\|_\infty$$

and

$$\|(Ku)'\|_\infty \le \|\varphi_u'\|_\infty + \sup_{0\le t\le\pi}\left\{\left\|\frac{\partial G}{\partial t}(t,\cdot)\right\|_1\right\}\|g(\cdot,u,u')\|_\infty$$

and a simple computation shows that

$$\sup_{0\le t\le\pi}\{\|G(t,\cdot)\|_1\} = \frac{\pi^2}{8}, \qquad \sup_{0\le t\le\pi}\left\{\left\|\frac{\partial G}{\partial t}(t,\cdot)\right\|_1\right\} = \frac{\pi}{2}$$

Then we have:

THEOREM 3.1. *Let us assume that f^{-1} and g satisfy the linear-growth conditions*

$$|f^{-1}(x)| \le k_f|x| + l_f, \qquad |g(t,x,y)| \le k_g|(x,y)| + l_g$$

for some positive constants k_f, k_g, l_f, l_g. Then (1) *admits a solution for* $(k_g, k_f) \in \mathscr{C}^\circ$, *where $\mathscr{C} \subset \mathbb{R}^2$ is the convex hull of the points* $(0,0)$, $(0,1)$, $(\frac{2}{\pi},0)$, $(4(\pi-2)/\pi^2, (4-\pi)/2)$.

Proof. From the growth conditions we obtain

$$\|g(t,u,u')\|_\infty \le k_g\|u\|_{1,\infty} + l_g$$

and

$$\|\varphi_u\|_\infty = \max\{|f^{-1}(-u'(0))|, |f^{-1}(u'(\pi))|\} \le k_f\|u\|_{1,\infty} + l_f,$$

$$\|\varphi_u'\|_\infty = \frac{|f^{-1}(u'(\pi)) - f^{-1}(-u'(0))|}{\pi} \le \frac{2}{\pi}(k_f\|u\|_{1,\infty} + l_f).$$

Hence

$$\|Ku\|_\infty \le \left(k_f + k_g\frac{\pi^2}{8}\right)\|u\|_{1,\infty} + c$$

and

$$\|(Ku)'\|_\infty \le \left(\frac{2}{\pi}k_f + k_g\frac{\pi}{2}\right)\|u\|_{1,\infty} + c$$

for some constant c. Taking R large enough, we conclude that $K(B_R(0)) \subset B_R(0)$ and by the Schauder theorem K has a fixed point. ∎

4. AN ITERATIVE PROCEDURE FOR (1)

In this section we will embed problem (1) in a family of problems

$$(1)_\lambda \begin{cases} u''(t) + \lambda g(t,u,u') = 0 \\ u(0) = f^{-1}(-u'(0)), \qquad u(\pi) = f^{-1}(u'(\pi)). \end{cases}$$

Starting at a solution u_0 for λ_0 we define recursively a sequence which converges in $C^1([0,\pi])$ to a solution of $(1)_{\lambda_0+\varepsilon}$ for some step ε. We recall the following well-known result from the theory of linear operators:

LEMMA 4.1. *Let L be the linear operator given by $Lu = u'' + r(t)u' + s(t)u$, with $r, s \in L^\infty(0,\pi)$, $s \le 0$. Then there exists a constant c depending only on $\|r\|_\infty$ and $\|s\|_\infty$ such that*

$$\|u\|_{2,2} \le c\|Lu\|_2$$

for any $u \in H^2 \cap H_0^1(0,\pi)$. Moreover, the problem

$$Lu = \psi$$
$$u|_{\partial I} = \varphi$$

is uniquely solvable in $H^2(0,\pi)$ for any $\psi \in L^2(0,\pi)$ and any boundary Dirichlet data φ.

Let u_0 be a solution of $(1)_{\lambda_0}$ and consider the sequence $\{u_n\} \subset H^2(0,\pi)$ given by the linear problems

$$\frac{u''_{n+1}}{\lambda_0+\varepsilon} + \frac{\partial g}{\partial u'}(t, u_n, u'_n)(u_{n+1}-u_n)' + \frac{\partial g}{\partial u}(t, u_n, u'_n)(u_{n+1}-u_n) + g(t, u_n, u'_n) = 0$$
$$u_{n+1}(0) = f^{-1}(-u'_n(0)), \qquad u_{n+1}(\pi) = f^{-1}(u'_n(\pi)).$$

As a basic assumption, we will suppose that g is C^2 with respect to u, u', and $\frac{\partial g}{\partial u}(t, u_0(t), u'_0(t)) \le 0$.

We remark that if $\{u_n\}$ is well defined and $u_n \to u$ for the C^1-norm, then u is a solution of $(1)_{\lambda_0+\varepsilon}$. Moreover, if $\frac{\partial g}{\partial u}(t, u_n(t), u'_n(t)) \le 0$, from Lemma 4.1 we conclude that u_{n+1} is well defined. In this case, for $z_n = u_{n+1} - u_n$ we have

$$L_n z_n := z''_n + (\lambda_0+\varepsilon)[r_n(t)z'_n + s_n(t)z_n] = -(\lambda_0+\varepsilon)R_n$$

with $r_n(t) = (\partial g/\partial u')(t, u_n, u'_n)$, $s_n(t) = \frac{\partial g}{\partial u}(t, u_n, u'_n)$, and R_n the Taylor remainder

$$R_n(t) = \frac{1}{2}\left[\frac{\partial^2 g}{\partial u^2}(t,\xi)z_{n-1}^2 + 2\frac{\partial^2 g}{\partial u\,\partial u'}(t,\xi)z_{n-1}z'_{n-1} + \frac{\partial^2 g}{\partial (u')^2}(t,\xi)(z'_{n-1})^2\right]$$

for some mean value $\xi \in L^\infty((0,\pi), \mathbb{R}^2)$. Writing

$$\varphi_n(t) = m_n t + f^{-1}(-u'_n(0)) - f^{-1}(-u'_{n-1}(0)),$$

where the slope m_n is given by

$$m_n = \frac{1}{\pi}\left[f^{-1}(u_n'(\pi)) - f^{-1}(u_{n-1}'(\pi)) - f^{-1}(-u_n'(0)) + f^{-1}(-u_{n-1}'(0))\right],$$

we obtain from Lemma 4.1 and the imbedding $H^2(0,\pi) \hookrightarrow C^1([0,\pi])$

$$\|z_n - \varphi_n\|_{1,\infty} \le c_n \|L_n(z_n - \varphi_n)\|_2 \le c_n(\lambda_0 + \varepsilon)(\|R_n\|_2 + \|r_n\varphi_n' + s_n\varphi_n\|_2)$$

for some constant c_n depending only on $\|r_n\|_\infty$ and $\|s_n\|_\infty$.

Thus, if $u_{n-1}, u_n \in B_R(u_0)$ we have that

$$\|z_n\|_{1,\infty} \le (1+\bar{c})\|\varphi_n\|_{1,\infty} + c\|z_{n-1}\|_{1,\infty}^2,$$

where the constants $\bar{c}$, c can be choosen depending only on R. Furthermore, if f^{-1} is Lipschitz with constant k_f then $\|\varphi_n\|_{1,\infty} \le k_f\|z_{n-1}\|_{1,\infty}$. Hence,

$$\|z_n\|_{1,\infty} \le b\|z_{n-1}\|_{1,\infty} + c\|z_{n-1}\|_{1,\infty}^2$$

for $b = (1+\bar{c})k_f$. Moreover,

$$z_0'' + (\lambda_0 + \varepsilon)\left[\frac{\partial g}{\partial u'}(t, u_0, u_0')z_0' + \frac{\partial g}{\partial u}(t, u_0, u_0')z_0\right] = -\varepsilon g(t, u_0, u_0')$$

and as $z_0|_{\partial I} = 0$, we obtain

$$\|z_0\|_{1,\infty} \le \varepsilon c_0 \|g(\cdot, u_0, u_0')\|_2.$$

Thus we have:

THEOREM 4.2. *With the previous notations, let us assume that u_0 is a solution of $(1)_{\lambda_0}$, and*

(i) $\frac{\partial g}{\partial u}(t, x, y) \le 0$ *for* $(x, y) \in K_R = B_R(u_0([0,\pi]) \times u_0'([0,\pi]))$

(ii) f^{-1} *is Lipschitz on* K_R *with constant*

$$k_f < \frac{1}{(1+\bar{c})}.$$

Then the sequence $\{u_n\}$ *is well defined and converges in* $B_R(u_0) \subset C^1([0,T])$ *for any step* ε *such that*

$$\varepsilon c_0 \|g(\cdot, u_0, u_0')\|_2 < \frac{R(1-b)}{(1+cR)}. \tag{4.1}$$

Proof. From the previous computations and (4.1), it follows that $\|z_0\|_{1,\infty} \leq R$ or, equivalently, $u_1 \in B_R(u_0)$. This proves that u_2 is well defined. Moreover, as

$$\varepsilon c_0 \|g(\cdot, u_0, u_0')\|_2 < R[1 - (b + c\varepsilon c_0 \|g(\cdot, u_0, u_0')\|_2)]$$

we conclude that $b + c\|z_0\|_{1,\infty} < 1$ and $\|z_1\|_{1,\infty} \leq (b + c\|z_0\|_{1,\infty})\|z_0\|_{1,\infty} < R$. Inductively, we see that the sequence $\{u_n\}$ is well defined, and

$$\|z_n\|_{1,\infty} \leq (b + c\|z_0\|_{1,\infty})^n \|z_0\|_{1,\infty}.$$

Hence

$$\sum_{n=0}^{\infty} \|z_n\|_{1,\infty} \leq \frac{\|z_0\|_{1,\infty}}{1 - (b + c\|z_0\|_{1,\infty})} \leq R,$$

which implies that $\{u_n\}$ is a Cauchy sequence with $\lim_{n\to\infty} u_n = u \in B_R(u_0)$. ∎

Furthermore, if g and its first and second order derivatives with respect to u, u' are bounded in $[0, \pi] \times \mathbb{R}^2$, with $\frac{\partial g}{\partial u} \leq 0$ the step ε can be choosen independently of u_0. As $(1)_0$ is trivially solvable, we have:

COROLLARY 4.3. *Let us assume that*

(i) *g, ∂g, and $\partial^2 g$ are bounded, and $\frac{\partial g}{\partial u} \leq 0$ in $[0, \pi] \times \mathbb{R}^2$.*

(ii) *f^{-1} is Lipschitz with constant k_f small enough.*

Then there exists a sequence $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n = 1$ such that a solution of $(1)_{\lambda_j}$ can be constructed recursively.

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