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Dirichlet and Periodic-Type Boundary Value Problems for Painlevé II

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It is established that, under certain conditions, the Dirichlet problem on a bounded interval for the Painlevé II equation is uniquely solvable and solutions are constructed in an iterative manner. Moreover, conditions for the existence of periodic solutions are set down.

1. INTRODUCTION

The Painlevé II equation [1] arose originally in work by Painlevé, Gambier, and Fuchs on canonical forms for second-order ODEs whose solutions do not admit movable singularities. The considerable interest in Painlevé equations in recent times is due, in large measure, to the celebrated Painlevé conjecture in soliton theory of Ablowitz and Segur [2] concerning the admittance of symmetry reduction to a Painlevé equation as a test for integrability. In this connection, the Painlevé II equation arises, in particular, as a symmetry reduction not only of the KdV and mKdV equations but also of the nonlinear Schrödinger equation [3]. In addition, the Painlevé II equation arises directly as a physical model describing the electric field in both electrolytes [4–6] and semiconductors

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[7]. The application of a Bäcklund transformation for Painlevé II in the context of steady electrolysis has recently been discussed by Rogers et al. in [8]. Whereas there is an extensive literature on initial value problems for the Painlevé II equation, the literature on two-point boundary value problems for this equation is relatively sparse. Hastings and McLeod [9] investigated a boundary value problem on $(-\infty, \infty)$ for Painlevé II which arises in plasma physics in the work of DeBoer and Ludford [10]. Boundary value problems for Painlevé II on $(0, \infty)$ were considered by Holmes and Spence [11]. Thompson [12] investigated two-point boundary value problems in two-ion electrodiffusion for a Painlevé II-type equation. Therein, the boundary conditions involved vanishing derivatives at the end points. Here, attention is concentrated on the Dirichlet problem for Painlevé II on a bounded interval, and conditions are established for its unique solvability. An iterative procedure for solution construction is described. To conclude, conditions are established for the solvability of a class of periodic boundary value problems for Painlevé II.

2. UNIQUE SOLVABILITY OF THE DIRICHLET PROBLEM

Here, we study certain boundary value problems for the Painlevé II$^{\pm}$ equation

$$P^{\pm}: \qquad \frac{d^2Y}{dz^2} = 2Y^3 \pm zY + C$$

on the bounded interval $[\underline{a}, \bar{a}]$ of the real line. Let us consider the usual Sobolev spaces $H^m(I)$, where $I = (\underline{a}, \bar{a})$ and $S_\sigma^{\pm}\colon H^2(I) \to L^2(I)$ are the semilinear operators given by

$$S_\sigma^{\pm} Y = \frac{d^2Y}{dz^2} - \sigma g^{\pm}(z, Y),$$

where $g^{\pm}(z, Y) = 2Y^3 \pm zY$ and $0 \le \sigma \le 1$. It will be assumed throughout that

$$\underline{a} > -\left(\frac{\pi}{\bar{a} - \underline{a}}\right)^2 \qquad \text{for } P^+$$

and

$$\bar{a} < \left(\frac{\pi}{\bar{a} - \underline{a}}\right)^2 \qquad \text{for } P^-.$$

We denote $\delta_\sigma^+ = (\frac{\pi}{\bar{a} - \underline{a}})^2 + \sigma \min\{\underline{a}, 0\}$ and $\delta_\sigma^- = (\frac{\pi}{\bar{a} - \underline{a}})^2 - \sigma \max\{\bar{a}, 0\}$. In the sequel we shall require the following classical result:

THE LERAY–SCHAUDER FIXED-POINT THEOREM. *Let K be a compact mapping of a Banach space E into itself and suppose there exists a constant M such that $\|x\|_E \leq M$ for all $x \in E$ and $\sigma \in [0, 1]$ satisfying $x = \sigma K x$. Then K has a fixed point.*

To apply the above result to our boundary value problem, we shall need the following "a priori" bounds for the operators $S_\sigma^\pm$:

LEMMA 1. *Let $Y_1, Y_2 \in H^2(I)$ with $Y_1 = Y_2$ on ∂I. Then*

$$\left\| S_\sigma^\pm Y_1 - S_\sigma^\pm Y_2 \right\|_2 \geq \delta_\sigma^\pm \|Y_1 - Y_2\|_2$$

and

$$\left\| S_\sigma^\pm Y_1 - S_\sigma^\pm Y_2 \right\|_2 \geq \frac{\delta_\sigma^\pm (\bar{a} - \underline{a})}{\pi} \left\| \frac{dY_1}{dz} - \frac{dY_2}{dz} \right\|_2 .$$

Proof. We have that

$$\begin{aligned} &\left\| S_\sigma^\pm Y_1 - S_\sigma^\pm Y_2 \right\|_2 \|Y_1 - Y_2\|_2 \\ &\quad \geq -\int_I \left(S_\sigma^\pm Y_1 - S_\sigma^\pm Y_2 \right)(Y_1 - Y_2) \\ &\quad = \int_I \left(\frac{dY_1}{dz} - \frac{dY_2}{dz} \right)^2 + \sigma \int_I 2\left(Y_1^3 - Y_2^3\right)(Y_1 - Y_2) \\ &\qquad \pm \sigma \int_I z(Y_1 - Y_2)^2 . \end{aligned}$$

Since $(Y_1^3 - Y_2^3)(Y_1 - Y_2) \geq 0$, for S_σ^+ it is seen that

$$\left\| S_\sigma^\pm Y_1 - S_\sigma^\pm Y_2 \right\|_2 \|Y_1 - Y_2\|_2 \geq \int_I \left(\frac{dY_1}{dz} - \frac{dY_2}{dz} \right)^2 + \sigma \underline{a} \|Y_1 - Y_2\|_2^2 .$$

From Poincaré's inequality it is seen that $\frac{\pi}{\bar{a} - \underline{a}} \|Y_1 - Y_2\|_2 \leq \|dY_1/dz - dY_2/dz\|_2$, and the result follows. The proof for S_σ^- is analogous. ∎

We now apply the Leray–Schauder Theorem to derive the following result:

THEOREM 1. *The Dirichlet problem*

$$\mathbb{D}: \begin{cases} \dfrac{d^2 Y}{dz^2} = 2Y^3 \pm zY + C, & z \in I = (\underline{a}, \bar{a}) \\ Y(\underline{a}) = \underline{y}, & Y(\bar{a}) = \bar{y} \end{cases}$$

is uniquely solvable in $H^2(I)$ *for any* $C \in L^2(I)$ *and arbitrary boundary data* $\underline{y},\ \bar{y} \in \mathbb{R}$.

Proof. For every $Y \in H^1(I)$ let

$$G_Y^{\pm}(z) = \int_{\underline{a}}^{z} \left[C(s) + g^{\pm}(s, Y(s)) \right] ds,$$

$$\xi(Y) = \frac{\bar{y} - \underline{y} - \int_{\underline{a}}^{\bar{a}} G_Y^{\pm}(z)\, dz}{\bar{a} - \underline{a}}$$

and define the operator K via

$$KY(z) = \underline{y} + \xi(Y)(z - \underline{a}) + \int_{\underline{a}}^{z} G_Y^{\pm}(s)\, ds.$$

From the imbedding $H^1(I) \hookrightarrow C(\bar{I})$ we deduce that $K\colon H^1(I) \to H^1(I)$ is well defined and continuous. Moreover, it follows by construction that

$$\frac{d^2(KY)}{dz^2}(z) = C(z) + g^{\pm}(z, Y), \qquad KY|_{\partial I} = \varphi,$$

where

$$\phi = \left(\frac{\bar{y} - \underline{y}}{\bar{a} - \underline{a}} \right) z + \frac{\underline{y}\bar{a} - \bar{y}\underline{a}}{\bar{a} - \underline{a}}.$$

On the other hand, for $Y \in B_R = \{Y \in H^1(I)\colon \|Y\|_{1,2} \le R\}$ we have that $\|Y\|_\infty \le cR$ for some constant c, and then

$$\left\| g^{\pm}(\cdot, Y) \right\|_\infty \le \bar{c}, \qquad \left\| G_Y^{\pm} \right\|_\infty \le \bar{c},$$

for some constant $\bar{c}$ depending on R. Hence, $K(B_R)$ is bounded in $H^2(I)$, and the compactness of the imbedding $H^2(I) \hookrightarrow H^1(I)$ implies that K is compact. Let $\sigma \in [0, 1]$ and assume that $Y = \sigma KY$. Then $d^2Y/dz^2 = \sigma(C + g^{\pm}(z, Y))$ and $Y|_{\partial I} = \sigma\varphi$. By Lemma 1

$$\|Y - \sigma\varphi\|_{1,2} \le c_\sigma \left\| S_\sigma^{\pm} Y - S_\sigma^{\pm}(\sigma\varphi) \right\|_2 = c_\sigma \left\| \sigma C - S_\sigma^{\pm}(\sigma\varphi) \right\|_2$$

for some constant c_σ. Since $\{c_\sigma\}_{0 \le \sigma \le 1}$ is bounded, the set $\{Y\colon Y = \sigma KY\}$ is uniformly bounded in $H^1(I)$, and by the Leray–Schauder Theorem, K has a fixed point corresponding to a solution of the Dirichlet problem $\mathbb{D}$. ∎

3. SOLUTIONS OF THE DIRICHLET PROBLEM VIA ITERATION

Here, $P^{\pm}$ is embedded in a one-parameter family of equations to show that a solution of the boundary value problem $\mathbb{D}$ can be obtained by construction via a continuation-type procedure. Specifically, we consider a parameter-dependent version of $P^{\pm}$ as follows:

$$P_\lambda^{\pm} \qquad \frac{d^2Y}{dz^2} = \lambda(2Y^3 \pm zY + C).$$

Starting at a solution corresponding to a value λ_0 of the parameter λ, we construct a solution for $\lambda_0 + \varepsilon$ as the limit of a recursive sequence in the Sobolev space $H^1(I)$. We remark that every term of this sequence is obtained as a solution of a *linear* Dirichlet problem. Let Y_0 be a solution of $P_{\lambda_0}^{\pm}$ with $0 \le \lambda_0 < 1$ and with Dirichlet conditions $Y_0(\underline{a}) = \underline{y}$, $Y_0(\bar{a}) = \bar{y}$. We consider the sequence of boundary value problems

$$\frac{d^2Y_{n+1}}{dz^2} = (\lambda_0 + \varepsilon)\left[\left(6Y_n^2 \pm z\right)Y_{n+1} - 4Y_n^3 + C\right],$$
$$Y_{n+1}(\underline{a}) = \underline{y}, \qquad Y_{n+1}(\bar{a}) = \bar{y}$$

for some $z \in I$ and $\varepsilon \le 1 - \lambda_0$ to be determined. By classical results, $\{Y_n\}$ is well defined. For simplicity, we introduce the following notation:

$$c_\varepsilon^{\pm}(R) = \frac{12(\lambda_0 + \varepsilon)}{\delta_{\lambda_0+\varepsilon}^{\pm}}\left(\|Y_0\|_2 + \frac{\bar{a} - \underline{a}}{\pi}R\right),$$

$$A_\varepsilon^{\pm}(R) = \frac{\varepsilon\pi}{\delta_{\lambda_0+\varepsilon}^{\pm}(\bar{a} - \underline{a})}\|2Y_0^3 \pm zY_0 + C\|_2.$$

We remark that $\delta_{\lambda_0+\varepsilon}^{\pm} \ge \delta > 0$ for some positive constant δ, proving that $c_\varepsilon^{\pm}(R)$ is bounded and $A_\varepsilon^{\pm}(R) \to 0$ as $\varepsilon \to 0$ for any fixed R. The following result may be established:

THEOREM 2. *Choose $R > 0$ and $\varepsilon > 0$ such that*

$$A_\varepsilon^{\pm}(R)\left(\frac{1}{R} + c_\varepsilon^{\pm}(R)\right) < 1.$$

Then the sequence $\{Y_n\}$ converges in $B_R(Y_0) \subset H^1(I)$ to a solution of $(P_{\lambda_0+\varepsilon}^{\pm})$ satisfying the Dirichlet boundary conditions.

Proof. First we establish that $\|dY_n/dz - dY_0/dz\|_2 \leq R$ for every n: indeed, by definition we have

$$L_0(Y_1 - Y_0) := \frac{d^2(Y_1 - Y_0)}{dz^2} - (\lambda_0 + \varepsilon)\big(6Y_0^2 \pm z\big)(Y_1 - Y_0)$$
$$= \varepsilon\big(2Y_0^3 \pm zY_0 + C\big),$$

and then, after the manner of Lemma 1, we obtain

$$\left\|\frac{d(Y_1 - Y_0)}{dz}\right\|_2 \leq \frac{\pi}{\delta^{\pm}_{\lambda_0+\varepsilon}(\bar{a} - \underline{a})}\|L_0(Y_1 - Y_0)\|_2$$
$$\leq \frac{\varepsilon\pi}{\delta^{\pm}_{\lambda_0+\varepsilon}(\bar{a} - \underline{a})}\big\|2Y_0^3 \pm zY_0 + C\big\|_2 = A^{\pm}_{\varepsilon}(R) < R.$$

More generally, we define $L_n^{\pm}$ as the linear operator given by

$$L_n^{\pm}T = \frac{d^2T}{dz^2} - (\lambda_0 + \varepsilon)\big(6Y_n^2 \pm z\big)T,$$

and then, for $T_n = Y_{n+1} - Y_n$, we have

$$L_n^{\pm}T_n = (\lambda_0 + \varepsilon)$$
$$\times\big[2\big(Y_n^3 - Y_{n-1}^3\big) \pm z(Y_n - Y_{n-1}) - \big(6Y_{n-1}^2 \pm z\big)(Y_n - Y_{n-1})\big]$$
$$= 2(\lambda_0 + \varepsilon)(2Y_{n-1} + Y_n)T_{n-1}^2.$$

Moreover, since $T_n(\underline{a}) = T_n(\bar{a}) = 0$ it is seen that

$$\left\|\frac{dT_n}{dz}\right\|_2 \leq \frac{\pi}{\delta^{\pm}_{\lambda_0+\varepsilon}(\bar{a} - \underline{a})}\|L_n^{\pm}T_n\|_2 = \frac{2(\lambda_0 + \varepsilon)\pi}{\delta^{\pm}_{\lambda_0+\varepsilon}(\bar{a} - \underline{a})}\big\|(2Y_{n-1} + Y_n)T_{n-1}^2\big\|_2$$

for $n \geq 1$. Assume that $\|dY_n/dz - dY_0/dz\|_2 \leq R$ for every $n \leq N$; then

$$\|Y_n - Y_0\|_2 \leq \frac{\bar{a} - \underline{a}}{\pi}\left\|\frac{dY_n}{dz} - \frac{dY_0}{dz}\right\|_2 \leq \frac{\bar{a} - \underline{a}}{\pi}R$$

and $\|2Y_{n-1} + Y_n\|_2 \leq 3(\|Y_0\|_2 + \frac{\bar{a} - \underline{a}}{\pi}R)$. As

$$\big|T_{n-1}^2(t)\big| = 2\left|\int_{\underline{a}}^{t} T_{n-1}\frac{dT_{n-1}}{dz}\right| \leq \frac{2(\bar{a} - \underline{a})}{\pi}\left\|\frac{dT_{n-1}}{dz}\right\|_2^2,$$

we conclude that

$$\left\|\frac{dT_n}{dz}\right\|_2 \leq \frac{12(\lambda_0+\varepsilon)}{\delta^{\pm}_{\lambda_0+\varepsilon}}\left(\|Y_0\|_2 + \frac{\bar{a}-\underline{a}}{\pi}R\right)\left\|\frac{dT_{n-1}}{dz}\right\|_2^2 = c_\varepsilon^{\pm}(R)\left\|\frac{dT_{n-1}}{dz}\right\|_2^2,$$

Hence,

$$\left\|\frac{dT_n}{dz}\right\|_2 \leq \left(c_\varepsilon^{\pm}(R)\left\|\frac{dT_0}{dz}\right\|_2\right)^{2^n-1}\left\|\frac{dT_0}{dz}\right\|_2 \leq \left(c_\varepsilon^{\pm}(R)A_\varepsilon^{\pm}(R)\right)^{2^n-1}\left\|\frac{dT_0}{dz}\right\|_2$$

and

$$\left\|\frac{dY_{N+1}}{dz} - \frac{dY_0}{dz}\right\|_2 \leq \sum_{n=0}^{N}\left\|\frac{dT_j}{dz}\right\|_2 \leq \left\|\frac{dT_0}{dz}\right\|_2 \sum_{n=0}^{N}\left(c_\varepsilon^{\pm}(R)A_\varepsilon^{\pm}(R)\right)^{2^n-1}.$$

By hypothesis, $c_\varepsilon^{\pm}(R)A_\varepsilon^{\pm}(R) < 1 - \frac{1}{R}A_\varepsilon^{\pm}(R)$, and then

$$\left\|\frac{dY_{N+1}}{dz} - \frac{dY_0}{dz}\right\|_2 \leq \left\|\frac{dT_0}{dz}\right\|_2 \frac{1}{1 - c_\varepsilon^{\pm}(R)A_\varepsilon^{\pm}(R)} < R.$$

Our result is now established by induction. Furthermore,

$$\left\|\frac{dY_{n+1}}{dz} - \frac{dY_m}{dz}\right\|_2 \leq \left\|\frac{dT_0}{dz}\right\|_2 \sum_{j=m}^{n}\left(c_\varepsilon^{\pm}(R)A_\varepsilon^{\pm}(R)\right)^{2^j-1},$$

proving that $\{Y_n\}$ is a Cauchy sequence for the H^1 norm. Let $Y = \lim_{n\to\infty} Y_n$; then $Y_n \to Y$ uniformly. Since

$$\frac{d^2Y_n}{dz^2} \to (\lambda_0+\varepsilon)\left[(6Y^2 \pm z)Y - 4Y^3 + C\right]$$
$$= (\lambda_0+\varepsilon)(2Y^3 \pm zY + C)$$

it is clear that $Y \in H^2(I)$ is a solution of $P^{\pm}_{\lambda_0+\varepsilon}$ satisfying the stated boundary conditions. ∎

Remark. $P_0^{\pm}$ is trivially solvable, and its unique solution is $\phi = (\frac{\bar{y}-\underline{y}}{\bar{a}-\underline{a}})z + \frac{\underline{y}\bar{a}-\bar{y}\underline{a}}{\bar{a}-\underline{a}}$. On the other hand, from Lemma 1 we deduce that

$$\|Y_0 - \phi\|_{1,2} \leq c\left\|S^{\pm}_{\lambda_0}Y_0 - S^{\pm}_{\lambda_0}\phi\right\|_2 \leq \bar{c}$$

for some fixed constant $\bar{c}$, proving that the choice of the step ε can be considered as independent of λ_0. This implies the existence of a sequence

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = 1,$$

where the solutions Y_j of $P^{\pm}_{\lambda_j}$ are constructed as in Theorem 2, and Y_N is the unique solution of the original problem.

4. THE PERIODIC BOUNDARY VALUE PROBLEM FOR $P^{\pm}$

In this section, we study the existence of solutions of the periodic boundary value problem for $P^{\pm}$, namely,

$$\mathbb{P}: \begin{cases} \dfrac{d^2Y}{dz^2} = 2Y^3 \pm zY + C, & z \in I \\ Y(\underline{a}) = Y(\bar{a}), & \dfrac{dY}{dz}(\underline{a}) = \dfrac{dY}{dz}(\bar{a}). \end{cases}$$

In this connection, define Y_s as the unique solution of the Dirichlet problem

$$\begin{cases} \dfrac{d^2Y}{dz^2} = 2Y^3 \pm zY + C, & z \in I \\ Y(\underline{a}) = Y(\bar{a}) = s \end{cases}$$

for fixed $s \in \mathbb{R}$. By Theorem 1, the mapping ψ given by $\psi(s) = Y_s$ is well defined. Furthermore,

LEMMA 2. *$\psi: \mathbb{R} \to H^1(I)$ is continuous.*

Proof. We have that

$$\begin{aligned} 0 &= -\int_{\underline{a}}^{\bar{a}} \left(S_1^{\pm} Y_s - S_1^{\pm} Y_{s_0}\right)\left(Y_s - Y_{s_0}\right) \\ &\geq -\left(Y_s - Y_{s_0}\right)\left(\frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right)\Bigg|_{\underline{a}}^{\bar{a}} \\ &\quad + \int_{\underline{a}}^{\bar{a}} \left(\frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right)^2 \pm \int_{\underline{a}}^{\bar{a}} z\left(Y_s - Y_{s_0}\right)^2. \end{aligned}$$

Hence, using that

$$\begin{aligned}\|Y_s - Y_{s_0}\|_2 &\le \big\| Y_s - Y_{s_0} - (s - s_0)\big\|_2 + |s - s_0|(\bar{a} - \underline{a})^{1/2} \\ &\le \frac{\bar{a} - \underline{a}}{\pi}\left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2 + |s - s_0|(\bar{a} - \underline{a})^{1/2},\end{aligned}$$

we obtain, for P^+,

$$\begin{aligned}&(s - s_0)\left(\frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right)\Big|_{\underline{a}}^{\bar{a}} \\ &\ge \left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2^2 \pm \int_{\underline{a}}^{\bar{a}} z\big(Y_s - Y_{s_0}\big)^2 \\ &\ge \left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2^2 + \min\{\underline{a}, 0\}\|Y_s - Y_{s_0}\|_2^2 \\ &\ge \frac{\delta_1^+(\bar{a} - \underline{a})^2}{\pi^2}\left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2^2 \\ &\quad + \min\{\underline{a}, 0\}\left(2\frac{\bar{a} - \underline{a}}{\pi}\left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2 |s - s_0|(\bar{a} - \underline{a})^{1/2} \right. \\ &\qquad\qquad\left. + (s - s_0)^2(\bar{a} - \underline{a})\right).\end{aligned}$$

In the same way we obtain, for P^-,

$$\begin{aligned}&(s - s_0)\left(\frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right)\Big|_{\underline{a}}^{\bar{a}} \\ &\ge \frac{\delta_1^-(\bar{a} - \underline{a})^2}{\pi^2}\left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2^2 \\ &\quad - \max\{\bar{a}, 0\}\left(2\frac{\bar{a} - \underline{a}}{\pi}\left\| \frac{dY_s}{dz} - \frac{dY_{s_0}}{dz}\right\|_2 |s - s_0|(\bar{a} - \underline{a})^{1/2} \right. \\ &\qquad\qquad\left. + (s - s_0)^2(\bar{a} - \underline{a})\right).\end{aligned}$$

Let $s \to s_0$. Then $\|dY_s/dz - dY_{s_0}/dz\|_2 \to 0$, provided that $\|dY_s/dz - dY_{s_0}/dz\|_2$ and $(dY_s/dz - dY_{s_0}/dz)|_{\underline{a}}^{\bar{a}}$ are bounded. Since

$$\|Y_s - s\|_{1,2} \le c_1\big\| S_1^{\pm} Y_s - S_1^{\pm} s\big\|_2 = c_1\big\| C - (2s^3 \pm zs)\big\|_2,$$

we conclude that $\|dY_s/dz\|_2$ is bounded. Moreover,

$$\left\|\frac{d^2Y_s}{dz^2}\right\|_2 = \|2Y_s^3 \pm zY_s + C\|_2,$$

which establishes that Y_s is bounded in $H^2(I) \hookrightarrow C^1(\bar{I})$. This implies the boundedness of $\|dY_s/dz - dY_{s_0}/dz\|_2$ and $(dY_s/dz - dY_{s_0}/dz)|_{\underline{a}}^{\bar{a}}$, and so completes the proof. ∎

To consider the solvability of the periodic boundary value problem, we observe that Y is a solution of $\mathbb{P}$ if and only if $Y = Y_s$ for some s such that $\int_{\underline{a}}^{\bar{a}} d^2Y_s/dz^2 = 0$. Thus, we may define the mapping $\theta\colon \mathbb{R} \to \mathbb{R}$ given by

$$\theta(s) = \int_{\underline{a}}^{\bar{a}} 2Y_s^3 \pm zY_s + C.$$

Continuity of θ follows immediately from Lemma 2 and the imbedding $H^1(I) \hookrightarrow C(\bar{I})$.

THEOREM 3. *The periodic boundary value problem $\mathbb{P}$ is solvable for any $C \in L^\infty(I)$. Furthermore, there exist $s_{\text{inf}}, s_{\text{sup}} \in \mathbb{R}$ such that any solution of $\mathbb{P}$ belongs to the compact arc $\{Y_s\colon s_{\text{inf}} \le s \le s_{\text{sup}}\}$.*

Proof. It suffices to establish the existence of $s_{\text{inf}}, s_{\text{sup}}$ such that

$$\theta(s_-) < 0 < \theta(s_+)$$

for any $s_- < s_{\text{inf}}, s_+ > s_{\text{sup}}$. Let $s \gg 0$ and consider z_0 such that $Y_s(z_0)$ is maximum. Note that if we define

$$\phi_z(Y) = 2Y^3 \pm zY + C(z),$$

then considering Y large we may assume that ϕ_z is increasing for any z. Hence, if $Y_s(z_0) > s$ we obtain that

$$\frac{d^2Y_s}{dz^2}(z) > 2s^3 \pm zs + C(z) > 0$$

a.e. in a neighborhood of z_0, a contradiction. Thus, $Y_s \le s$, and then

$$0 \le \frac{dY_s}{dz}(a) - \frac{dY_s}{dz}(\bar{a}) = \theta(s).$$

The proof for s_{inf} is analogous. ∎

Remark. It is straightforward to compute sufficient values of s_{inf} and s_{sup} explicitly in each case. For a general formulation, we may define

$$\gamma^{\pm} = \begin{cases} -\infty & \text{if } 6s^2 \pm z \ge 0 \text{ for every } s \in \mathbb{R},\ z \in I \\ \sup\{s \in \mathbb{R}\colon 6s^2 \pm z < 0 \text{ for some } z \in I\} & \text{otherwise} \end{cases}$$

and note that ϕ_z is increasing for $Y > \gamma^{\pm}$ and $Y < -\gamma^{\pm}$. Hence, it suffices to take

$$s_{\sup} = \max\{\gamma^{\pm}, \sup\{s \in \mathbb{R}: 2s^3 \pm zs + C(z) \leq 0 \text{ for some } z \in I\}\}$$

and

$$s_{\inf} = \min\{-\gamma^{\pm}, \inf\{s \in \mathbb{R}: 2s^3 \pm zs + C(z) \geq 0 \text{ for some } z \in I\}\}.$$

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