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Existence Results for the p-Laplacian with Nonlinear Boundary Conditions[1]

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In this paper we study the existence of nontrivial solutions for the problem $\Delta_p u = |u|^{p-2}u$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$, with a nonlinear boundary condition given by $|\nabla u|^{p-2}\partial u/\partial\nu = f(u)$ on the boundary of the domain. The proofs are based on variational and topological arguments.

Key Words: p-Laplacian; nonlinear boundary conditions.

1. INTRODUCTION

In this paper we study the existence of nontrivial solutions for the following problem:

$$
\begin{aligned}
\Delta_p u &= |u|^{p-2}u \qquad &&\text{in } \Omega,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(u) &&\text{on } |\Omega.
\end{aligned}
\tag{1.1}
$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, and $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative.

Problems of the form (1.1) appear in a natural way when one considers the Sobelev trace inequality

$$
S^{1/p}\|u\|_{L^q(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}, \qquad 1 \le q \le p^* = \frac{p(N-1)}{N-p}.
$$

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In fact, the extremals (if they exists) are solutions of (1.1) for $f(u) = \lambda|u|^{q-2}u$. See [10] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for $p = 2$ in the subcritical case, $1 < q < \frac{2(N-1)}{N-2}$.

Also, one is led to nonlinear boundary conditions in the study of conformal deformations on Riemannian manifolds with boundary (see, for example, [5, 11, 12]).

The study of existence when the nonlinear term is placed in the equation, that is, when one considers a quasilinear problem of the form $-\Delta_p u = f(u)$ with Dirichlet boundary conditions, has received considerable attention (see, for example, [15, 16, 21], etc.).

However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions, see, for example, [7, 8, 10, 17, 25]. For elliptic systems with nonlinear boundary conditions see [13, 14]. For previous work for the p-Laplacian with nonlinear boundary conditions of different type see [6, 22].

In this work, to obtain solutions of (1.1), we seek to understand critical points of the associated energy functional,

$$\mathscr{F}(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\,dx - \int_{\partial\Omega} F(u)\,d\sigma, \tag{1.2}$$

where $F'(u) = f(u)$ and $d\sigma$ is the measure on the boundary.

In this paper we fix $1 < p < N$ and look for conditions on the nonlinear term $f(u)$ that provide us with the existence of nontrivial solutions of (1.1).

This functional $\mathscr{F}$ is well defined, and C^1 in $W^{1,p}(\Omega)$ if f has a critical or subcritical growth, namely $|f(u)| \leq C(1 + |u|^q)$ with $1 \leq q \leq p^* = \frac{p(N-1)}{N-p}$. Moreover, in the subcritical case $1 < q < p^*$, the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact, while in the critical case $q = p^*$ is only continuous.

First, we deal with a superlinear and subcritical nonlinearity. For simplicity we will consider

$$f(u) = \lambda|u|^{q-2}u, \tag{1.3}$$

where q verifies

$$1 < q < p^* = \frac{p(N-1)}{N-p}.$$

In these cases we prove the following theorems, using standard variational arguments together with the Sobolev trace immersion, which provide the necessary compactness. See [16] for similar results for the p-Laplacian with Dirichlet boundary conditions.

THEOREM 1.1. *Let f satisfy* (1.3) *with $p < q < p^*$; then there exist infinitely many nontrivial solutions of* (1.1) *which are unbounded in $W^{1,p}(\Omega)$.*

THEOREM 1.2. *Let f satisfy* (1.3) *with $1 < q < p$; then there exist infinitely many nontrivial solutions of* (1.1) *which form a compact set in $W^{1,p}(\Omega)$.*

THEOREM 1.3. *Let f satisfy* (1.3) *with $p = q$; then there exists a sequence of eigenvalues λ_n of* (1.1) *such that $\lambda_n \to +\infty$ as $n \to +\infty$.*

In the case $p = q$, the equation and the boundary condition are homogeneous of the same degree, so we are dealing with a nonlinear eigenvalue problem. In the linear case, that is, for $p = 2$, this eigenvalue problem is known as the Steklov problem [2].

Next we consider the critical growth on f. As we have pointed out, in this case the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$ fails, so to recover some sort of compactness, in the spirit of [3], we consider a perturbation of the critical power, that is,

$$f(u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u = |u|^{p(N-1)/(N-p)-2}u + \lambda|u|^{r-2}u. \tag{1.4}$$

Here we use the compensated compactness method introduced in [19, 20] and follow ideas from [15]. We prove the following two theorems.

THEOREM 1.4. *Let f satisfy* (1.4) *with $p < r < p^*$; then there exists a constant $\lambda_0 > 0$ depending on p, r, N, and Ω such that if $\lambda > \lambda_0$, problem* (1.1) *has at least a nontrivial solution in $W^{1,p}(\Omega)$.*

THEOREM 1.5. *Let f satisfy* (1.4) *with $1 < r < p$; then there exists a constant $\lambda_1 > 0$ depending on p, r, N, and Ω such that if $0 < \lambda < \lambda_1$, problem* (1.1) *has infinitely many nontrivial solutions in $W^{1,p}(\Omega)$.*

Next, we deal with supercritical growth on f. More precisely, we study a subcritical perturbation of the supercritical power; that is, we consider

$$f(u) = \lambda|u|^{q-2}u + |u|^{r-2}u, \tag{1.5}$$

with $q \geq p^* > r > p$. In this case, not only does the compactness fail, but the functional $\mathscr{F}$ given in (1.2) is not well defined in $W^{1,p}(\Omega)$, so we have to perform a truncation in the nonlinear term $\lambda|u|^{q-2}u$, following ideas from [4]. For this case we have

THEOREM 1.6. *Let f satisfy* (1.5) *with $q \geq p^* > r > p$; then there exists a constant λ_2 depending on p, q, r, N, and Ω such that if $0 < \lambda < \lambda_2$, problem* (1.1) *has a nontrivial positive solution in $W^{1,p}(\Omega) \cap L^\infty(\partial\Omega)$.*

Finally, we end this article with a nonexistence result for (1.1) in the half-space $\mathbb{R}^N_+ = \{x_1 > 0\}$ that shows that existence may fail when one considers critical or subcritical growth in an unbounded domain. This nonexistence result is a consequence of a Pohozaev-type identity.

THEOREM 1.7. *Let f satisfy* (1.3) *with $q \le p^*$. Let $u \in W^{1,p}(\mathbb{R}^N_+) \cap C^2(\overline{\mathbb{R}^N_+}) \cap L^q(\partial \mathbb{R}^N_+)$ be a nonnegative solution of* (1.1) *such that*

$$|\nabla u(x)|\,|x|^{N/p} \to 0 \qquad \text{as } |x| \to +\infty.$$

Then $u \equiv 0$.

We remark that the decay hypothesis at infinity is necessary, because for $p = 2$ $u(x) = e^{-x_1}$ is a solution of (1.1) for every q.

Throughout the paper, by C we mean a constant that may vary from line to line but remains independent of the relevant quantities.

The rest of the paper is organized as follows. In Sections 2, 3, and 4 we deal with the subcritical case. In Section 2 we prove Theorem 1.1, in Section 3 Theorem 1.2, and in Section 4 Theorem 1.3. Next, in Sections 5 and 6 we consider the critical case. In Section 5 we prove Theorem 1.4, and in Section 6 Theorem 1.5. In Section 7 we deal with the supercritical problem, Theorem 1.6, and finally in Section 8 we prove our nonexistence result, Theorem 1.7.

2. PROOF OF THEOREM 1.1: THE SUBCRITICAL CASE I

In this section we study (1.1) with $f(u) = \lambda |u|^{q-2}u$ with $p < q < p^*$.

Let us begin with the following lemma, which will be helpful in proving the Palais–Smale condition.

LEMMA 2.1. *Let $\phi \in W^{1,p}(\Omega)'$, where $W^{1,p}(\Omega)'$ denotes the dual space of $W^{1,p}(\Omega)$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of*

$$-\Delta_p u + |u|^{p-2}u = \phi. \tag{2.1}$$

Moreover, the operator A_p: $\phi \mapsto u$ is continuous.

Proof. Let us observe that weak solutions $u \in W^{1,p}(\Omega)$ of (2.1) are critical points of the functional

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\,dx - \langle \phi, u\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $W^{1,p}(\Omega)$. Hence, existence and uniqueness are a consequence of the fact that I is a weakly lower semi-continuous, strictly convex functional bounded below.

For the continuous dependence, let us first recall the inequality (cf. [24])

(2.2)
$$\left(|x|^{p-2}x - |y|^{p-2}y, x-y\right) \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2 \\ C_p \dfrac{|x-y|^2}{\left(|x|+|y|\right)^{2-p}} & \text{if } p \leq 2, \end{cases}$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $\mathbb{R}^m$.

Now, given $\phi_1, \phi_2 \in W^{1,p}(\Omega)'$, let us consider $u_1, u_2 \in W^{1,p}(\Omega)$, the corresponding solutions of problem (2.1). Then, for $i = 1, 2$ we have

$$\int_\Omega \left(|\nabla u_i|^{p-2}\nabla u_i(\nabla u_1 - \nabla u_2) + |u_i|^{p-2}u_i(u_1-u_2) - \phi_i(u_1-u_2)\right)dx = 0.$$

Hence, substracting and using inequality (2.2), we obtain, for $p \geq 2$,

$$\begin{aligned} C_p\int_\Omega |\nabla u_1 - \nabla u_2|^p + |u_1-u_2|^p \, dx &\leq \langle(\phi_1 - \phi_2), (u_1 - u_2)\rangle \\ &\leq \|\phi_1 - \phi_2\|_{W^{1,p}(\Omega)'}\|u_1 - u_2\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Therefore,

$$\|A_p(\phi_1) - A_p(\phi_2)\|_{W^{1,p}(\Omega)} \leq C\left(\|\phi_1 - \phi_2\|_{W^{1,p}(\Omega)'}\right)^{1/(p-1)}.$$

Now, for the case $p \leq 2$, we first observe that

$$\begin{aligned} &\int_\Omega |\nabla(u_1-u_2)|^p \, dx \\ &\quad \leq \left(\int_\Omega \frac{|\nabla(u_1-u_2)|^2}{\left(|\nabla u_1| + |\nabla u_2|\right)^{2-p}}\, dx\right)^{p/2} \left(\int_\Omega \left(|\nabla u_1| + |\nabla u_2|\right)^p dx\right)^{(2-p)/2} \end{aligned}$$

and

$$\int_\Omega |u_1-u_2|^p \, dx \leq \left(\int_\Omega \frac{|u_1-u_2|^2}{\left(|u_1|+|u_2|\right)^{2-p}}\, dx\right)^{p/2} \left(\int_\Omega \left(|u_1|+|u_2|\right)^p dx\right)^{(2-p)/2}.$$

As in the previous case, we get

$$\frac{\|u_1 - u_2\|_{W^{1,p}(\Omega)}}{\left(\|u_1\|_{W^{1,p}(\Omega)} + \|u_2\|_{W^{1,p}(\Omega)}\right)^{2-p}} \leq C\|\phi_1 - \phi_2\|_{W^{1,p}(\Omega)'}. \tag{2.3}$$

Now we observe that

$$\|u_i\|^p_{W^{1,p}(\Omega)} \le \|\phi_i\|_{W^{1,p}(\Omega)'} \|u_i\|_{W^{1,p}(\Omega)}.$$

Hence, (2.3) becomes

$$\begin{aligned} &\|A_p(\phi_1) - A_p(\phi_2)\|_{W^{1,p}(\Omega)} \\ &\qquad \le C\Big(\|\phi_1\|^{1/(p-1)}_{W^{1,p}(\Omega)'} + \|\phi_2\|^{1/(p-1)}_{W^{1,p}(\Omega)'}\Big)^{2-p} \|\phi_1 - \phi_2\|_{W^{1,p}(\Omega)'}, \end{aligned}$$

and the proof is finished. ∎

With this lemma we can verify the Palais–Smale condition for $\mathcal{F}$.

LEMMA 2.2. *The functional $\mathcal{F}$ satisfies the Palais–Smale condition.*

Proof. Let $(u_k)_{k\ge 1} \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence, that is, a sequence such that

$$\mathcal{F}(u_k) \to c \qquad \text{and} \qquad \mathcal{F}'(u_k) \to 0. \tag{2.4}$$

Let us first prove that (2.4) implies that (u_k) is bounded. From (2.4) it follows that there exists a sequence $\varepsilon_k \to 0$ such that

$$|\mathcal{F}'(u_k)w| \le \varepsilon_k \|w\|_{W^{1,p}(\Omega)}, \qquad \forall w \in W^{1,p}(\Omega).$$

Now we have

$$\begin{aligned} c + 1 &\ge \mathcal{F}(u_k) - \frac{1}{q}\mathcal{F}'(u_k)u_k + \frac{1}{q}\mathcal{F}'(u_k)u_k \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\mathcal{F}'(u_k)u_k \\ &\ge \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} - \frac{1}{q}\|u_k\|_{W^{1,p}(\Omega)}\varepsilon_k \\ &\ge \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} - \frac{1}{q}\|u_k\|_{W^{1,p}(\Omega)}; \end{aligned}$$

hence, u_k is bounded in $W^{1,p}(\Omega)$.

By compactness we can assume that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_k \to u$ strongly in $L^q(\partial\Omega)$ and a.e. in $\partial\Omega$. Then, as $p < q < p^*$, it follows that $|u_k|^{q-2}u_k \to |u|^{q-2}u$ in $L^{p^{*\prime}}(\partial\Omega)$ and hence in $W^{1,p}(\Omega)'$. Therefore, according to Lemma 2.1,

$$u_k \to A_p(|u|^{q-2}u), \qquad \text{in } W^{1,p}(\Omega).$$

This completes the proof. ∎

Now we introduce a topological tool, the *genus*, that was introduced in [18], but we will use an equivalent definition due to [9].

Given a Banach space X, we consider the class

$$\Sigma = \{ A \subset X : A \text{ is closed}, \ A = -A \}.$$

Over this class we define the genus, $\gamma\colon \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\bigl\{k \in \mathbb{N}\colon \text{there exists } \varphi \in C(A, \mathbb{R}^k - \{0\}), \\ \varphi(x) = -\varphi(-x)\bigr\}.$$

For the proof of Theorem 1.2, we will use the following theorem, the proof of which can be found in [1].

THEOREM 2.1 [1, Theorem 2.23]. *Let $\mathscr{F}\colon X \to \mathbb{R}$ verify the following*:

(1) *$\mathscr{F} \in C^1(X)$ and even.*

(2) *$\mathscr{F}$ verifies the Palais–Smale condition.*

(3) *There exists a constant $r > 0$ such that $\mathscr{F}(u) > 0$ in $0 < \|u\|_X < r$, and $\mathscr{F}(u) \geq c > 0$ if $\|u\|_X = r$.*

(4) *There exists a closed subspace $E_m \subset X$ of dimension m, and a compact set $A_m \subset E_m$ such that $\mathscr{F} < 0$ on A_m and 0 lies in a bounded component of $E_m - A_m$ in E_m.*

Let B be the unit ball in X. We define

$$\Gamma = \bigl\{ h \in C(X, X) : h(0) = 0, \ h \text{ is an odd homeomorphism} \\ \text{and } \mathscr{F}(h(B)) \geq 0 \bigr\},$$

and

$$\mathscr{K}_m = \bigl\{ K \subset X : K = -K, K \text{ is compact, and } \gamma(K \cap h(\partial B)) \geq m \\ \text{for all } h \in \Gamma \bigr\}.$$

Then,

$$c_m = \inf_{K \in \mathscr{K}_m} \max_{u \in K} \mathscr{F}(u)$$

is a critical value of $\mathscr{F}$, with $0 < c \leq c_m \leq c_{m+1} < \infty$. Moreover, if $c_m = c_{m+1} = \cdots = c_{m+r}$ then $\gamma(K_{c_m}) \geq r + 1$, where $K_{c_m} = \{u \in X : \mathscr{F}'(u) = 0, \mathscr{F}(u) = c_m\}$.

Now we are ready to prove the main result of this section.

Proof of Theorem 1.1. We need to check the hypotheses of Theorem 2.1.

The fact that $\mathcal{F}$ is C^1 is a straightforward adaptation of the results in [23]. The Palais–Smale condition has already been checked in Lemma 2.2.

Let us now check condition (3). From the Sobolev immersion theorem, we obtain

$$\mathcal{F}(u) = \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda}{q}\|u\|^q_{L^q(\partial\Omega)} \geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - C\frac{\lambda}{q}\|u\|^q_{W^{1,p}(\Omega)}$$
$$= g\big(\|u\|_{W^{1,p}(\Omega)}\big),$$

where $g(t) = \frac{1}{p}t^p - C\frac{\lambda}{q}t^q$. As $q > p$, (3) follows for $r = r(C, \lambda, p, q)$ small.

Finally, to verify condition (4), let us consider a sequence of subspaces $E_m \subset W^{1,p}(\Omega)$ of dimension m such that $E_m \subset E_{m+1}$ and $u\mid_{\partial\Omega} \not\equiv 0$ for $u \neq 0$, $u \in E_m$. Hence,

$$\min_{u \in B_m} \int_{\partial\Omega} |u|^q \, d\sigma > 0,$$

where $B_m = \{u \in E_m : \|u\|_{W^{1,p}(\Omega)} = 1\}$. Now we observe that

$$\mathcal{F}(tu) \leq \frac{t^p}{p}\|u\|_{W^{1,p}(\Omega)} - \frac{\lambda t^q}{q} \min_{u \in B_m} \int_{\partial\Omega} |u|^q \, d\sigma < 0$$

for all $u \in B_m$ and $t \geq t_0$. Therefore, 4 follows by taking $A_m = t_0 B_m$. ∎

To see that the critical points of $\mathcal{F}$ that we have found are unbounded in $W^{1,p}(\Omega)$, we need the following result:

LEMMA 2.3. *Let $(c_m) \subset \mathbb{R}$ be the sequence of critical values given by Theorem* 2.1. *Then* $\lim_{m\to\infty} c_m = \infty$.

Proof. Let $M = \{u \in W^{1,p}(\Omega) - \{0\} : \frac{1}{\lambda p}\|u\|^p_{W^{1,p}(\Omega)} \leq \|u\|^q_{L^q(\partial\Omega)}\}$. By the Sobolev trace theorem, there exists a constant $r > 0$ such that

$$r < \|u\|^q_{L^q(\partial\Omega)}, \qquad \forall u \in M. \tag{2.5}$$

Let us define

$$b_m = \sup_{h \in \Gamma} \inf_{\{u \in \partial B \cap E^c_{m-1}\}} \mathcal{F}(h(u)).$$

It is proved in [1] that $b_m \leq c_m$; hence to prove our result it is enough to show that $b_m \to \infty$.

Now, $b_{m+1} \geq \inf_{u \in \partial B \cap E^c_m} \mathcal{F}(h(u))$ for all $h \in \Gamma$. We will construct $\tilde{h}_m \in \Gamma$ such that $\lim_{m\to\infty} \inf_{u \in \partial B \cap E^c_m} \mathcal{F}(\tilde{h}_m(u)) = \infty$. First, let us define the sequence

$$d_m = \inf\big\{\|u\|_{W^{1,p}(\Omega)} : u \in M \cap E^c_m\big\}$$

and observe that $d_m \to \infty$. In fact, if not, there exists a sequence $u_m \in M \cap E_m^c$ such that $u_m \rightharpoonup 0$ weakly in $W^{1,p}(\Omega)$ and therefore $u_m \to 0$ in $L^q(\partial\Omega)$, a contradiction of (2.5).

Next, let us consider $h_m(u) = R^{-1} d_m u$, where $R > 1$ is to be fixed. From h_m we will construct $\tilde{h}_m$.

Given $u \in W^{1,p}(\Omega)$ such that $u|_{\partial\Omega} \not\equiv 0$, pick $\beta = \beta(u)$ such that

$$\frac{1}{\lambda p} \| \beta u \|_{W^{1,p}(\Omega)}^p = \| \beta u \|_{L^q(\partial\Omega)}^q,$$

so $\beta u \in M$.

If we consider $g(t) = \mathscr{F}(tu)$ with $u|_{\partial\Omega} \not\equiv 0$, it is easy to see that g is increasing in $[0, \beta(u)]$, so g achieves its maximum on that interval for $t = \beta(u)$.

Take $u_0 \in E_m^c \cap B$ such that $u_0|_{\partial\Omega} \not\equiv 0$; then for $R > 1$,

$$R^{-1} d_m \le d_m \le \| \beta u_0 \|_{W^{1,p}(\Omega)} = \beta(u_0).$$

This inequality implies that for every $R > 1$ and for every $u_0 \in E_m^c \cap B$ such that $u_0|_{\partial\Omega} \not\equiv 0$, it holds that

$$\mathscr{F}\big(h_m(u_0)\big) = \mathscr{F}\big(R^{-1} d_m u_0\big) \ge 0.$$

As $h_m(0) = 0$, it follows that

$$h_m(E_m^c \cap B) \subset \big\{ u \in W^{1,p}(\Omega) : \mathscr{F}(u) \ge 0 \big\};$$

therefore, $h_m|_{E_m^c}$ satisfies the requirements needed to belong to Γ, so it is natural to try to extend h_m to $W^{1,p}(\Omega)$ so it belongs to Γ.

Given $\varepsilon > 0$, consider $Z_\varepsilon = d_m R^{-1}(E_m^c \cap B) + \varepsilon(E_m \cap B)$. Let us see that for ε small, $Z_\varepsilon \subset M^c$. If not, there exists a sequence $\varepsilon_j \to 0$ and a sequence $(u_j) \subset M$ such that $u_j \in Z_{\varepsilon_j}$. In particular, u_j is bounded in $W^{1,p}(\Omega)$, so we can assume that

$$u_j \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega),$$
$$u_j \to u \qquad \text{in } L^q(\partial\Omega).$$

Moreover, as $u_j \in M$ it follows that $u|_{\partial\Omega} \not\equiv 0$. On the other hand, as $\| \cdot \|_{W^{1,p}(\Omega)}$ is weakly lower semi-continuous, we have that $u \in M$, and, as $\varepsilon_j \to 0$, $u \in d_m R^{-1}(E_m^c \cap B)$, a contradiction.

So we have proved that there exists $\varepsilon_0 > 0$ such that $Z_{\varepsilon_0} \subset M^c$. This fact allows us to define

$$\tilde{h}_m(u) = \begin{cases} h_m(u) = d_m R^{-1} u & \text{if } u \in E_m^c, \\ \varepsilon_0 u & \text{if } u \in E_m. \end{cases}$$

Now, if $u \in E_m \cap B$ we have

$$\tilde{h}_m(u) = \varepsilon_0 u \in Z_{\varepsilon_0} \subset M^c;$$

then

$$\begin{aligned}
&\mathscr{F}\big(\tilde{h}_m(u)\big)\\
&\quad = \mathscr{F}(\varepsilon_0 u) = \frac{1}{p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda}{q}\|\varepsilon_0 u\|^q_{L^q(\partial\Omega)}\\
&\quad = \frac{\lambda}{q}\left(\frac{q-1}{\lambda p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} + \left(\frac{1}{\lambda p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} - \|\varepsilon_0 u\|^q_{L^q(\partial\Omega)}\right)\right) \geq 0.
\end{aligned}$$

That is, given $u \in B$, if we decompose $u = u_1 + u_2$ with $u_1 \in E_m^c$ and $u_2 \in E_m \cap B$, we obtain $\tilde{h}_m(u) = \tilde{h}_m(u_1) + \tilde{h}_m(u_2) = d_m R^{-1} u_1 + \varepsilon_0 u_2 \in Z_{\varepsilon_0} \subset M^c$, from which it follows that $\mathscr{F}(\tilde{h}_m(u)) \geq 0$ and hence $\tilde{h}_m \in \Gamma$.

Finally, we need to prove that $\mathscr{F}(\tilde{h}_m(u)) \to \infty$ as $m \to \infty$ for $u \in \partial B \cap E_m^c$, but this follows from the facts that $d_m \to \infty$, that $d_m \leq \beta(u)$ for $u \in B \cap E_m^c$, and that we can choose R large enough.

If $u \in \partial B \cap E_m^c$, $\tilde{h}_m(u) = d_m R^{-1} u$ and

$$\begin{aligned}
\mathscr{F}\big(\tilde{h}_m(u)\big) &= \frac{(d_m R^{-1})^p}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda (d_m R^{-1})^q}{q}\|u\|^q_{L^q(\partial\Omega)}\\
&= (d_m R^{-1})^p\left(\frac{1}{p} - \frac{\lambda}{q}(d_m R^{-1})^{q-p}\|u\|^q_{L^q(\partial\Omega)}\right)\\
&\geq (d_m R^{-1})^p\left(\frac{1}{p} - \frac{\lambda}{q}(\beta(u) R^{-1})^{q-p}\|u\|^q_{L^q(\partial\Omega)}\right)\\
&= (d_m R^{-1})^p\left(\frac{1}{p} - \frac{R^{p-q}}{pq}\right)
\end{aligned}$$

As $q > p$ we conclude that if R is large enough, then $\mathscr{F}(\tilde{h}_m(u)) \to +\infty$. ∎

3. PROOF OF THEOREM 1.2: THE SUBCRITICAL CASE II

Now we deal with $f(u) = \lambda |u|^{q-2} u$ in the case $1 < q < p$. In this case, we look for nonpositive critical values of $\mathscr{F}$.

We begin with the following lemma.

LEMMA 3.1. *For every $n \in \mathbb{N}$ there exists a constant $\varepsilon > 0$ such that*

$$\gamma(\mathscr{F}^{-\varepsilon}) \geq n,$$

where $\mathscr{F}^c = \{u \in W^{1,p}(\Omega) : \mathscr{F}(u) \leq c\}$.

Proof. Let $E_n \subset W^{1,p}(\Omega)$ be an n-dimensional subspace such that $u|_{\partial\Omega} \not\equiv 0$ for all $u \in E_n$, $u \neq 0$ (cf. Section 2).

Hence we have, for $u \in E_n$, $\|u\|_{W^{1,p}(\Omega)} = 1$,

$$\mathscr{F}(tu) = \frac{t^p}{p} - \frac{\lambda t^q}{q}\int_{\partial\Omega} |u|^q \, d\sigma \leq \frac{t^p}{p} - a_n \frac{\lambda t^q}{q}, \tag{3.1}$$

where $a_n = \inf\{\int_{\partial\Omega}|u|^q \, d\sigma : u \in E_n, \|u\|_{W^{1,p}(\Omega)} = 1\}$. Observe that $a_n > 0$ because E_n is finite dimensional. As $q < p$ we obtain from (3.1) that there exists positive constants ρ and ε such that

$$\mathscr{F}(\rho u) < -\varepsilon \qquad \text{for } u \in E_n, \|u\|_{W^{1,p}(\Omega)} = \rho.$$

Therefore, if we set $S_{\rho,n} = \{u \in E_n : \|u\|_{W^{1,p}(\Omega)} = \rho\}$, we have that $S_{\rho,n} \subset \mathscr{F}^{-\varepsilon}$. Hence by the monotonicity of the genus,

$$\gamma(\mathscr{F}^{-\varepsilon}) \geq \gamma(S_{\rho,n}) = n,$$

as we wanted to show. ∎

LEMMA 3.2. *The functional $\mathscr{F}$ is bounded below and verifies the Palais–Smale condition.*

Proof. First, by the Sobolev-trace inequality, we have

$$\mathscr{F}(u) \geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - C\frac{\lambda}{q}\|u\|^q_{W^{1,p}(\Omega)} \equiv h(\|u\|_{W^{1,p}(\Omega)}),$$

where $h(t) = \frac{1}{p}t^p - C\frac{\lambda}{q}t^q$. As $h(t)$ is bounded below we conclude that $\mathscr{F}$ is bounded below.

Now to prove the Palais–Smale condition, let $u_j \in W^{1,p}(\Omega)$ be a Palais–Smale sequence. As $c = \lim_{j\to\infty}\mathscr{F}(u_j)$, using that $\mathscr{F}'(u_j) = \varepsilon_j \to 0$ in $W^{1,p}(\Omega)'$, we have that, for j large enough,

$$\begin{aligned} c - 1 &\leq \left(\frac{1}{p} - \frac{1}{q}\right)\|u_j\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\langle \varepsilon_j, u_j\rangle \\ &\leq \left(\frac{1}{p} - \frac{1}{q}\right)\|u_j\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|\varepsilon_j\|_{(W^{1,p}(\Omega))'}\|u_j\|_{W^{1,p}(\Omega)} \\ &\leq \left(\frac{1}{p} - \frac{1}{q}\right)\|u_j\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|u_j\|_{W^{1,p}(\Omega)}, \end{aligned}$$

from which it follows that $\|u_j\|_{W^{1,p}(\Omega)} \leq C$ (recall that $p > q$).

Therefore, for a subsequence,

$$\begin{aligned} u_j &\rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega), \\ u_j &\to u \qquad \text{in } L^q(\partial\Omega), \end{aligned}$$

and the result follows as in Lemma 2.2. ∎

Finally, the following two theorems give us the proof of Theorem 1.2.

THEOREM 3.1. *Let*

$$\Sigma = \{A \subset W^{1,p}(\Omega) - \{0\} : A \text{ is closed}, A = -A\},$$
$$\Sigma_k = \{A \subset \Sigma : \gamma(A) \geq k\},$$

where γ stands for the genus.

Then

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \mathcal{F}(u)$$

is a negative critical value of $\mathcal{F}$, and, moreover, if $c = c_k = \cdots = c_{k+r}$, then $\gamma(K_c) \geq r + 1$, where $K_c = \{u \in W^{1,p}(\Omega) : \mathcal{F}(u) = c, \mathcal{F}'(u) = 0\}$.

Proof. According to Lemma 3.1, for every $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\mathcal{F}^{-\varepsilon}) \geq k$. As $\mathcal{F}$ is even and continuous it follows that $\mathcal{F}^{-\varepsilon} \in \Sigma_k$; therefore $c_k \leq -\varepsilon < 0$. Moreover, by Lemma 3.2, $\mathcal{F}$ is bounded below, so $c_k > -\infty$. Let us now see that c_k is in fact a critical value for $\mathcal{F}$. To this end let us suppose that $c = c_k = \cdots = c_{k+r}$. As $\mathcal{F}$ is even it follows that K_c is symmetric. The Palais–Smale condition implies that K_c is compact; therefore if $\gamma(K_c) \leq r$ by the continuity property of the genus (see [23]) there exists a neighborhood of K_c, $N_\delta(K_c) = \{v \in W^{1,p}(\Omega) : d(v, K_c) \leq \delta\}$, such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \leq r$.

By the usual deformation argument, we get

$$\eta\left(1, \mathcal{F}^{c+\varepsilon/2} - N_\delta(K_c)\right) \subset \mathcal{F}^{c-\varepsilon/2}.$$

On the other hand, by the definition of c_{k+r} there exists $A \subset \Sigma_{k+r}$ such that $A \subset \mathcal{F}^{c+\varepsilon/2}$. Hence

$$\eta\left(1, A - N_\delta(K_c)\right) \subset \mathcal{F}^{c-\varepsilon/2}. \tag{3.2}$$

Now by the monotonicity of the genus (see [23]), we have

$$\gamma\left(\overline{A - N_\delta(K_c)}\right) \geq \gamma(A) - \gamma\left(N_\delta(K_c)\right) \geq k.$$

As $\eta(1, \cdot)$ is an odd homeomorphism it follows that (see [23])

$$\gamma\left(\eta\left(1, \overline{A - N_\delta(K_c)}\right)\right) \geq \gamma\left(\overline{A - N_\delta(K_c)}\right) \geq k.$$

But as $\eta(1, \overline{A - N_\delta(K_c)}) \in \Sigma_k$, then

$$\sup_{u \in \eta(1, \overline{A - N_\delta(K_c)})} \mathcal{F}(u) \geq c = c_k,$$

a contradiction of (3.2). ∎

We end the section by showing that the critical points of $\mathscr{F}$ are a compact set of $W^{1,p}(\Omega)$.

THEOREM 3.2. *The set* $K = \{u \in W^{1,p}(\Omega) : \mathscr{F}'(u) = 0\}$ *is compact in* $W^{1,p}(\Omega)$.

Proof. As $\mathscr{F}$ is C^1 it is immediate that K is closed. Let u_j be a sequence in K. We have that

$$0 = F'(u_j)u_j = \|u_j\|^p_{W^{1,p}(\Omega)} - \lambda \int_{\partial\Omega} |u_j|^q \, d\sigma \geq \|u_j\|^p_{W^{1,p}(\Omega)} - C\lambda \|u_j\|^q_{W^{1,p}(\Omega)}.$$

As $1 < q < p$, we conclude that u_j is bounded in $W^{1,p}(\Omega)$. Now we can use the Palais–Smale condition to extract a convergent subsequence. ∎

4. PROOF OF THEOREM 1.3: A NONLINEAR EIGENVALUE PROBLEM

In this section we deal with $f(u) = \lambda |u|^{p-2}u$, which is a nonlinear eigenvalue problem.

Let us consider $M_\alpha = \{u \in W^{1,p}(\Omega) : \|u\|^p_{W^{1,p}(\Omega)} = p\alpha\}$ and

$$\varphi(u) = \frac{1}{p} \int_{\partial\Omega} |u|^p \, d\sigma.$$

With a minimax technique we are looking for critical points of φ restricted to the manifold M_α.

Let us define ρ: $W^{1,p}(\Omega) - \{0\} \to (0, +\infty)$ by

$$\rho(u) = \left(\frac{p\alpha}{\|u\|^p_{W^{1,p}(\Omega)}} \right)^{1/p}.$$

This function ρ is even and bounded away from the origin and verifies that $\rho(u)u \in M_\alpha$ if $u \neq 0$. Moreover, we have that the derivative of ρ is given by

$$\langle \rho'(u), v \rangle = -(p\alpha)^{1/p} \|u\|^{-(p+1)}_{W^{1,p}(\Omega)} \times \left(\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv \, dx \right). \tag{4.1}$$

We observe that ρ' is odd and uniformly continuous over bounded sets away from the origin. It is straightforward to check, from (4.1), that $\langle \rho'(u), v \rangle = 0$ if and only if $\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv \, dx = 0$.

As $p > 1$, it follows that $W^{1,p}(\Omega)$ is a reflexive uniformly convex Banach space, so given $\varphi \in W^{1,p}(\Omega)'$, there exists a unique element in $W^{1,p}(\Omega)$, that we will denote by $J(\varphi)$ such that

$$\langle \varphi, J(\varphi) \rangle = \|\varphi\|^2_{W^{1,p}(\Omega)'},$$

$$\|J(\varphi)\|_{W^{1,p}(\Omega)} = \|\varphi\|_{W^{1,p}(\Omega)'}.$$

Therefore we define $J: W^{1,p}(\Omega)' \to W^{1,p}(\Omega)$ as the duality mapping which is odd and uniformly continuous over bounded sets.

Let us now define

$$\langle Pu; v \rangle = \frac{\int_{\partial\Omega} |u|^p \, d\sigma}{\|u\|^p_{W^{1,p}(\Omega)}} \left(\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv \, dx - \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} v \, d\sigma \right),$$

$$\langle Du; v \rangle = \int_{\partial\Omega} |u|^{p-2} uv \, d\sigma - \langle Pu; v \rangle,$$

and

$$Tu = J(Du) - Au,$$

where A is given by

$$A = \frac{\langle \rho'(u); J(Du) \rangle \langle Pu + Du; u \rangle + \langle Pu; J(Du) \rangle}{(\langle \rho'(u); u \rangle + 1) \langle Pu + Du; u \rangle}.$$

This application, T, is uniformly continuous and odd. Moreover, it is bounded in M_α, so there exist constants τ_0, $\gamma_0 > 0$ such that, for every $\tau \in [-\tau_0, \tau_0]$ and every $u \in M_\alpha$ it holds that

$$\|u + \tau Tu\|_{W^{1,p}(\Omega)} \geq \gamma_0 > 0.$$

Now, we are able to define the flow,

$$H(u, \tau) = \rho(u + \tau Tu)(u + \tau Tu),$$

so we obtain a well-defined application, H, which is odd in u and uniformly continuous and verifies $H(u, 0) = u$.

The main property of H is that it defines trajectories in M_α along which the functional φ is increasing.

LEMMA 4.1. *There exists an application $r(u, \tau)$ such that $r(u, \tau) \to 0$ as $\tau \to 0$ uniformly in $u \in M_\alpha$ and*

$$\varphi(H(u, \tau)) = \varphi(u) + \int_0^\tau \|Du\|^2_{W^{1,p}(\Omega)'} + r(u, s) \, ds$$

for every $u \in M_\alpha$, $\tau \in [-\tau_0, \tau_0]$.

Proof. An elementary computation gives us

$$\varphi(H(u,\tau)) = \varphi(u) + \int_0^\tau \left\langle \varphi'(H(u,s)); \frac{\partial H}{\partial s}(u,s) \right\rangle ds$$

$$= \varphi(u) + \int_0^\tau \|Du\|^2_{W^{1,p}(\Omega)'} + \left\langle \varphi'(H(u,s)); \frac{\partial H}{\partial s}(u,s) \right\rangle$$

$$- \langle Du; J(Du) \rangle ds.$$

Hence, if we define $r(u,\tau) = \langle \varphi'(H(u,s)); \frac{\partial}{\partial s} H(u,s)\rangle - \langle Du; J(Du)\rangle$, by our choice of A it holds that $r(u,0) = 0$, and the result follows as T (and therefore H) is bounded in M_α. ▮

Now we are ready to prove the Deformation Lemma needed to apply the mini-max technique.

LEMMA 4.2. *Given $\beta > 0$, we denote $\varphi_\beta = \{u \in M_\alpha : \varphi(u) > \beta\}$. Let $\beta > 0$ be fixed, and suppose that there exists a relatively open set $U \subset M_\alpha$ and positive constants $\delta < \rho$ such that*

$$\|Du\|_{W^{1,p}(\Omega)'} \geq \delta, \qquad \text{if } u \in V_\rho = \{u \in M_\alpha : u \notin U, \text{ and } |\varphi(u) - \beta| \leq \rho\}.$$

Then, there exists an $\varepsilon > 0$ and a continuous, odd operator H_ε such that

$$H_\varepsilon(\varphi_{\beta-\varepsilon} - U) \subset \varphi_{\beta+\varepsilon}.$$

Proof. First, we take $\tau_1 > 0$ such that $|r(u,\tau)| \leq \frac{1}{2}\delta^2$ for all $u \in M_\alpha$, $\tau \in [-\tau_1, \tau_1]$.

By Lemma 4.1 we have that $\varphi(H(u,\tau)) \geq \varphi(u) + \frac{1}{2}\delta^2\tau$ for every $u \in V_\rho$ and $0 < \tau < \tau_1$.

Let $\varepsilon = \min\{\rho, \frac{1}{4}\delta^2\tau_1\}$, and from the definition of V_ρ, if $u \in V_\rho \cap \varphi_{\beta-\varepsilon}$, we obtain

$$\varphi(H(u,\tau_1)) \geq \varphi(u) + 2\varepsilon \geq \beta + \varepsilon.$$

Again by Lemma 4.1, given $u \in V_\rho$, we have that $\varphi(H(u,\tau))$ is strictly increasing for τ small, and hence we can define

$$t_\varepsilon(u) = \min\{\tau \geq 0 : \varphi(H(u,\tau)) = \beta + \varepsilon\}.$$

This $t_\varepsilon(u)$ is well defined and continuous and verifies $0 < t_\varepsilon(u) \leq \tau_1$. Now, we choose H_ε as

$$H_\varepsilon(u) = \begin{cases} H(u, t_\varepsilon(u)) & \text{if } u \in V_\varepsilon, \\ u & \text{if } u \in \varphi_{\beta-\varepsilon} - (U \cup V_\varepsilon). \end{cases}$$

Finally it is straightforward to check that H_ε satisfies all of our requirements. ∎

Now we prove the Palais–Smale condition for the functional φ on M_α.

LEMMA 4.3. *Let $\beta > 0$ and $(u_j) \subset M_\alpha$ be a Palais–Smale sequence on M_α above level β, that is,*

$$\varphi(u_j) \geq \beta, \qquad Du_j \to 0.$$

Then there exists a subsequence that converges strongly in $W^{1,p}(\Omega)$.

Proof. As M_α is bounded, we can assume that $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Also, as φ is compact, we can assume that $\varphi(u_j) \to \varphi(u)$ and hence $\varphi(u) \geq \beta$ and

$$\mu_j \equiv \frac{\int_{\partial\Omega} |u_j|^p \, d\sigma}{\|u_j\|^p_{W^{1,p}(\Omega)}} \to \mu \equiv \frac{\int_{\partial\Omega} |u|^p \, d\sigma}{\alpha p};$$

therefore $u \not\equiv 0$ and $\varphi'(u) \neq 0$.

Now, as φ' is compact and $Du_j \to 0$ we have

$$0 = \lim_j Du_j = \lim_j \varphi'(u_j) - Pu_j = \varphi'(u) - \mu \lim_j P_0 u_j,$$

where

$$\langle P_0 u_j ; v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv \, dx - \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} v \, d\sigma.$$

Therefore $P_0 u_j \to \mu^{-1} \varphi'(u)$, and the result follows from applying Lemma 2.1 as $A_p = P_0^{-1}$. ∎

THEOREM 4.1. *Let $C_k = \{C \subset M_\alpha : C$ is compact, symmetric and $\gamma(C) \leq k\}$ and let*

$$\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u). \tag{4.2}$$

Then $\beta_k > 0$ and there exists $u_k \in M_\alpha$ such that $\varphi(u_k) = \beta_k$ and u_k is a weak solution of (1.1) *with $\lambda_k = \alpha/\beta_k$.*

Proof. First, let us see that $\beta_k > 0$. It is immediate that $\gamma(M_\alpha) = +\infty$; hence β_k is well defined in the sense that for every k, $C_k \neq \emptyset$. As we can choose a set $C \in C_k$ with the property $u|_{\partial\Omega} \not\equiv 0$ if $u \in C$, we conclude that $\beta_k = \sup_{C \in C_k} \min_{u \in C} \varphi(u) > 0$.

Now, for a fixed k let us prove the existence of the solution u_k. First, let us see that there exists a sequence $(u_j) \in M_\alpha$ such that $\varphi(u_j) \to \beta_k$ and

$Du_j \to 0$. To see this fact, assume that it is false; then there exists positive constants δ and ρ such that

$$\|Du\| \geq \delta, \qquad \text{if } u \in M_\alpha \text{ and } |\varphi(u) - \beta_k| \leq \rho.$$

We can assume that $\delta < \beta_k$. By the deformation Lemma 4.2 there exists a constant $\varepsilon > 0$ and a continuous and odd H_ε such that $H_\varepsilon(\varphi_{\beta_k - \varepsilon}) \subset \varphi_{\beta_k + \varepsilon}$. By the definition of β_k there exists $C_\varepsilon \in C_k$ such that $\varphi(u) \geq \beta_k - \varepsilon$ for every $u \in C_\varepsilon$, then $\varphi(u) \geq \beta_k + \varepsilon$ for every $u \in H_\varepsilon(C_\varepsilon)$. But we have that $\gamma(H_\varepsilon(C_\varepsilon)) \geq k$, a contradiction of the definition of β_k. So we have proved that there exits a sequence $(u_j) \in M_\alpha$ such that $\varphi(u_j) \to \beta_k$ and $Du_j \to 0$. From Lemma 4.3 we can extract a converging subsequence $u_j \to u_k$ that gives us the desired solution that must verify, by continuity of φ, $\varphi(u_k) = \beta_k$. ∎

This theorem proves the existence of nontrivial solutions for (1.1), but we can prove the following

THEOREM 4.2. *Let $K_j = \{u \in M_\alpha; \varphi(u) = \beta_j, Du = 0\}$. If $\beta_j = \beta_{j+1} = \cdots = \beta_{j+r}$, then $\gamma(K_j) \geq r + 1$.*

Proof. The proof is analogous to that of Theorem 3.1. ∎

In this way we have proved the existence of infinitely many solutions. The next theorem gives us the existence of infinitely many eigenvalues.

THEOREM 4.3. *Let β_k be as in* (4.2); *then*

$$\lim_k \beta_k = 0,$$

and therefore

$$\lim_k \lambda_k = +\infty.$$

Proof. Let E_j be a sequence of subspaces of $W^{1,p}(\Omega)$, such that $E_i \subset E_{i+1}$, $\overline{\cup E_i} = W^{1,p}(\Omega)$ and $\dim(E_i) = i$. Let E_i^c be the topological complementary of E_i.

Let

$$\tilde{\beta}_k = \sup_{C \in C_k} \min_{u \in C \cap E_{k-1}^c} \varphi(u).$$

$\tilde{\beta}_k$ is well defined and $\tilde{\beta}_k \geq \beta > 0$. Let us prove that $\lim_k \tilde{\beta}_k = 0$. Assume, by contradiction, that there exists a constant $\kappa > 0$ such that $\tilde{\beta}_k > \kappa > 0$ for all k. Then for every k there exists C_k such that

$$\tilde{\beta}_k > \min_{u \in C_k \cap E_{k-1}^c} \varphi(u) > \kappa.$$

Hence there exists $u_k \in C_k \cap E_{k-1}^c$ such that

$$\tilde{\beta}_k > \varphi(u_k) > \kappa.$$

As M_α is bounded, we can assume, taking a subsequence if necessary, that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_k \to u$ strongly in $L^p(\partial\Omega)$. Hence $\varphi(u) \geq \kappa > 0$, but this is a contradiction of the fact that $u \equiv 0$ because $u_k \in E_{k-1}^c$. ∎

5. PROOF OF THEOREM 1.4: THE CRITICAL CASE I

In this section we study the critical case with a perturbation. We consider $f(u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u$ with $p < r < p^*$.

To prove our existence result, since we have lost the compactness in the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\partial\Omega)$, we can no longer expect the Palais–Smale condition to hold. In any case, we can prove a *local Palais–Smale condition* that will hold for $\mathcal{F}(u)$ below a certain value of energy.

The technical result used here, the concentrated compactness method, is mainly due to [19, 20].

Let u_j be a bounded sequence in $W^{1,p}(\Omega)$; then there exists a subsequence that we still denote u_j, such that

$$u_j \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega),$$

$$u_j \to u \qquad \text{strongly in } L^r(\partial\Omega), \quad 1 \leq r < p^*,$$

$$|\nabla u_j|^p \rightharpoonup d\mu, \qquad |(u_j|_{\partial\Omega})|^{p^*} \rightharpoonup d\eta,$$

weakly-* in the sense of measures. We observe that $d\eta$ is a measure supported on $\partial\Omega$.

If we consider $\phi \in C^\infty(\overline{\Omega})$, from the Sobolev trace inequality we obtain, passing to the limit,

$$\left(\int_{\partial\Omega} |\phi|^{p^*}\, d\eta\right)^{1/p^*} S^{1/p} \leq \left(\int_\Omega |\phi|^p\, d\mu + \int_\Omega |u|^p |\nabla\phi|^p\, dx + \int_\Omega |\phi u|^p\, dx\right)^{1/p}, \tag{5.1}$$

where S is the best constant in the Sobolev trace embedding theorem.

From (5.1), we observe that, if $u = 0$ we get a reverse Holder-type inequality (but it involves one integral over $\partial\Omega$ and one over Ω) between the two measures μ and η.

Now we state the following lemma due to [19, 20].

LEMMA 5.1. *Let u_j be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit u such that*

$$|\nabla u_j|^p \rightharpoonup d\mu \qquad \text{and} \qquad |(u_j\,|_{\partial\Omega})|^{p^*} \rightharpoonup d\eta,$$

weakly- in the sense of measures. Then there exists $x_1, \ldots, x_l \in \partial\Omega$ such that*

(1) $d\eta = |u|^{p^*} + \sum_{j=1}^{l} \eta_j \delta_{x_j}$, $\eta_j > 0$,

(2) $d\mu \geq |\nabla u|^p + \sum_{j=1}^{l} \mu_j \delta_{x_j}$, $\mu_j > 0$,

(3) $(\eta_j)^{p/p^*} \leq \mu_j / S$

Next, we use Lemma 5.1 to prove a local Palais–Smale condition.

LEMMA 5.2. *Let $u_j \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence for $\mathcal{F}$, with energy level c. If $c < (1/p - 1/p^*)S^{p^*/(p^*-p)}$, where S is the best constant in the Sobolev trace inequality, then there exists a subsequence u_{j_k} that converges strongly in $W^{1,p}(\Omega)$.*

Proof. From the fact that u_j is a Palais–Smale sequence it follows that u_j is bounded in $W^{1,p}(\Omega)$ (see Lemma 2.2). By Lemma 5.1 there exists a subsequence, that we still denote u_j, such that

$$u_j \rightharpoonup u \qquad \text{weakly in } W^{1,p}(\Omega),$$

$$u_j \to u \qquad \text{in } L^r(\partial\Omega), \quad 1 < r < p^*, \quad \text{and} \quad \text{a.e. in } \partial\Omega,$$

$$|\nabla u_j|^p \rightharpoonup d\mu \geq |\nabla u|^p + \sum_{k=1}^{l} \mu_k \delta_{x_k}, \tag{5.2}$$

$$|u_j\,|_{\partial\Omega}|^{p^*} \rightharpoonup d\eta = |u\,|_{\partial\Omega}|^{p^*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.$$

Let $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\phi \equiv 1 \text{ in } B(x_k, \varepsilon), \qquad \phi \equiv 0 \text{ in } B(x_k, 2\varepsilon)^c, \qquad |\nabla\phi| \leq \frac{2}{\varepsilon},$$

where x_k belongs to the support of $d\eta$.

Consider $\{u_j \phi\}$. Obviously this sequence is bounded in $W^{1,p}(\Omega)$. As $\mathcal{F}'(u_j) \to 0$ in $W^{1,p}(\Omega)'$, we obtain that

$$\lim_{j\to\infty} \langle \mathcal{F}'(u_j); \phi u_j \rangle = 0.$$

By (5.2) we obtain

$$\lim_{j\to\infty} \int_\Omega |\nabla u_j|^{p-2} \nabla u_j \nabla\phi u_j \, dx$$

$$= \int_{\partial\Omega} \phi \, d\eta + \lambda \int_{\partial\Omega} |u|^r \phi \, d\sigma - \int_\Omega \phi \, d\mu - \int_\Omega |u|^p \phi \, dx.$$

Now, by Hölder inequality and weak convergence, we obtain

$$
\begin{aligned}
0 &\le \lim_{j\to\infty}\left|\int_\Omega |\nabla u_j|^{p-2}\nabla u_j \nabla\phi u_j\, dx\right| \\
&\le \lim_{j\to\infty}\left(\int_\Omega |\nabla u_j|^p\, dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla\phi|^p|u_j|^p\, dx\right)^{1/p} \\
&\le C\left(\int_{B(x_k,2\varepsilon)\cap\Omega} |\nabla\phi|^p|u|^p\, dx\right)^{1/p} \\
&\le C\left(\int_{B(x_k,2\varepsilon)\cap\Omega} |\nabla\phi|^N\, dx\right)^{1/N}\left(\int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{pN/(N-p)}\, dx\right)^{(N-p)/pN} \\
&\le C\left(\int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{pN/(N-p)}\, dx\right)^{(N-p)/pN} \to 0 \qquad \text{as } \varepsilon\to 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
(5.3)\qquad &\lim_{\varepsilon\to 0}\left[\int_{\partial\Omega}\phi\, d\eta + \lambda\int_{\partial\Omega}|u|^r\phi\, d\sigma - \int_\Omega \phi\, d\mu - \int_\Omega |u|^p\phi\, dx\right] \\
&\quad = \eta_k - \mu_k = 0.
\end{aligned}
$$

By Lemma 5.1 we have that $(\eta_k)^{p/p^*}S \le \mu_k$; therefore by (5.3) we obtain

$$(\eta_k)^{p/p^*}S \le \eta_k.$$

Then, either $\eta_k = 0$ or

$$\eta_k \ge S^{p^*/(p^*-p)}. \tag{5.4}$$

If (5.4) does indeed occur for some k_0, then, from the fact that u_j is a Palais–Smale sequence, we obtain

$$
\begin{aligned}
(5.5)\qquad c &= \lim_{j\to\infty}\mathscr{F}(u_j) = \lim_{j\to\infty}\mathscr{F}(u_j) - \frac{1}{p}\langle \mathscr{F}'(u_j); u_j\rangle \\
&\ge \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_{\partial\Omega}|u|^{p^*}\, d\sigma + \left(\frac{1}{p}-\frac{1}{p^*}\right)S^{p^*/(p^*-p)} \\
&\quad + \lambda\left(\frac{1}{p}-\frac{1}{r}\right)\int_{\partial\Omega}|u|^r\, d\sigma \\
&\ge \left(\frac{1}{p}-\frac{1}{p^*}\right)S^{p^*/(p^*-p)}.
\end{aligned}
$$

As $c < (1/p - 1/p^*)S^{p^*/(p^*-p)}$, it follows that $\int_{\partial\Omega}|u_j|^{p^*}\,d\sigma \to \int_{\partial\Omega}|u|^{p^*}\,d\sigma$, and therefore $u_j \to u$ in $L^{p^*}(\partial\Omega)$. Now the proof is finished with the continuity of the operator A_p. ∎

Proof of Theorem 1.4. In view of the previous result, we look for critical values below level c. For that purpose, we want to use the Mountain Pass Lemma. Hence we have to check the following conditions:

1) There exists constants $R, r > 0$ such that if $\|u\|_{W^{1,p}(\Omega)} = R$, then $\mathscr{F}(u) > r$.

2) There exists $v_0 \in W^{1,p}(\Omega)$ such that $\|v_0\|_{W^{1,p}(\Omega)} > R$ and $\mathscr{F}(v_0) < r$.

Let us first check 1). By the Sobolev trace theorem we have

$$\begin{aligned}\mathscr{F}(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{1}{p^*}\int_{\partial\Omega}|u|^{p^*}\,d\sigma - \frac{\lambda}{r}\int_{\partial\Omega}|u|^r\,d\sigma \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{1}{p^*}S^{p^*}\|u\|^{p^*}_{W^{1,p}(\Omega)} - \frac{\lambda}{r}C\|u\|^r_{W^{1,p}(\Omega)}.\end{aligned}$$

Let

$$g(t) = \frac{1}{p}t^p - \frac{1}{p^*}S^{p^*}t^{p^*} - \frac{\lambda}{r}Ct^r.$$

It is easy to check that $g(R) > r$ for some $R, r > 0$.

2) is immedate, as for a fixed $w \in W^{1,p}(\Omega)$ with $w\,|_{\partial\Omega} \not\equiv 0$ we have

$$\lim_{t\to\infty}\mathscr{F}(tw) = -\infty.$$

Now the candidate for critical value according to the Mountain Pass Theorem is

$$c = \inf_{\phi\in\mathscr{C}}\ \sup_{t\in[0,1]}\mathscr{F}(\phi(t)), \tag{5.6}$$

where $\mathscr{C} = \{\phi : [0,1] \to W^{1,p}(\Omega);$ continuous and $\phi(0) = 0,\ \phi(1) = v_0\}$. The problem is to show that $c < (1/p - 1/p^*)S^{p^*/(p^*-p)}$ to apply the local Palais–Smale condition.

We fix $w \in W^{1,p}(\Omega)$ with $\|w\|_{L^{p^*}(\partial\Omega)} = 1$ and define $h(t) = \mathscr{F}(tw)$. We want to study the maximum of h. As $\lim_{t\to\infty} h(t) = -\infty$ it follows that there exists a $t_\lambda > 0$ such that $\sup_{t>0}\mathscr{F}(tw) = h(t_\lambda)$. Differentiating, we obtain

$$0 = h'(t_\lambda) = t_\lambda^{p-1}\|w\|^p_{W^{1,p}(\Omega)} - t_\lambda^{p^*-1} - t_\lambda^{r-1}\lambda\|w\|^r_{L^r(\partial\Omega)}, \tag{5.7}$$

from which it follows that

$$\|w\|^{p}_{W^{1,p}(\Omega)} = t_\lambda^{p^*-p} + t_\lambda^{r-p}\lambda\|w\|^{r}_{L^r(\partial\Omega)}.$$

Hence $t_\lambda \le \|w\|^{p/(p^*-p)}_{W^{1,p}(\Omega)}$; then from (5.7), as $t_\lambda^{p^*-r} + \lambda\|w\|^r_{L^r(\partial\Omega)} \to \infty$ as $\lambda \to \infty$, we obtain that

$$\lim_{\lambda\to\infty} t_\lambda = 0. \tag{5.8}$$

On the other hand, it is easy to check that if $\lambda > \tilde{\lambda}$ it must be $\mathscr{F}(t_{\tilde{\lambda}}w) \ge \mathscr{F}(t_\lambda w)$, so by (5.8) we get

$$\lim_{\lambda\to\infty} \mathscr{F}(t_\lambda w) = 0.$$

But this identity means that there exists a constant $\lambda_0 > 0$ such that if $\lambda \ge \lambda_0$, then

$$\sup_{t\ge 0} \mathscr{F}(tw) < \left(\frac{1}{p} - \frac{1}{p^*}\right) S^{p^*/(p^*-p)},$$

and the proof is finished if we choose $v_0 = t_0 w$ with t_0 large to have $\mathscr{F}(t_0 w) < 0$. ∎

6. PROOF OF THEOREM 1.5: THE CRITICAL CASE II

In this section we deal with problem (1.4) when $1 < q < p$; that is, we are considering $f(u) = |u|^{p^*-2}u + \lambda|u|^{r-2}u$. Applying a mini-max technique, we will show the existence of infinitely many nontrivial critical points of the associated functional $\mathscr{F}$ when λ is small enough.

We begin, as in the previous section, by using Lemma 5.1 to prove a local Palais–Smale condition.

LEMMA 6.1. *Let $(u_j) \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence for $\mathscr{F}$, with energy level c. If $c < (1/p - 1/p^*)S^{p^*/(p^*-p)} - K\lambda^{p^*/(p^*-r)}$, where K depends only on p, r, N, and $|\partial\Omega|$, then there exists a subsequence (u_{j_k}) that converges strongly in $W^{1,p}(\Omega)$.*

Proof. From the fact that u_j is a Palais–Smale sequence it follows that u_j is bounded in $W^{1,p}(\Omega)$ (see Lemma 2.2 and Lemma 5.2).

Now the proof follows exactly as in Lemma 5.2 until we get to

$$c \ge \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\partial\Omega} |u|^{p^*}\, d\sigma + \left(\frac{1}{p} - \frac{1}{p^*}\right) S^{p^*/(p^*-p)}$$
$$+ \lambda\left(\frac{1}{p} - \frac{1}{r}\right)\int_{\partial\Omega} |u|^r\, d\sigma,$$

where u is the weak limit of u_j in $W^{1,p}(\Omega)$.

Now applying Hölder inequality, we find

$$c \geq \left(\frac{1}{p} - \frac{1}{p^*}\right) S^{p^*/(p^*-p)} + \left(\frac{1}{p} - \frac{1}{p^*}\right) \|u\|_{L^{p^*}(\partial\Omega)}^{p^*} + \lambda\left(\frac{1}{p} - \frac{1}{r}\right) |\partial\Omega|^{1-r/p^*} \|u\|_{L^{p^*}(\partial\Omega)}^{r},$$

where $|\partial\Omega|$ is the $N-1$-dimensional Hausdorff measure of $\partial\Omega$. Now, let $f(x) = c_1 x^{p^*} - \lambda c_2 x^r$. This function reaches its absolute minimum at $x_0 = (\lambda c_2 r / p^* c_1)^{1/(p^*-r)}$; that is,

$$f(x) \geq f(x_0) = -K\lambda^{p^*/(p^*-r)},$$

where $K = K(p, q, N, |\partial\Omega|)$.

Hence $c \geq (1/p - 1/p^*) S^{p^*/(p^*-p)} - K\lambda^{p^*/(p^*-r)}$, which contradicts our hypothesis. Therefore,

$$\lim_{j\to\infty} \int_{\partial\Omega} |u_j|^{p^*} \, d\sigma = \int_{\partial\Omega} |u|^{p^*} \, d\sigma,$$

and the rest of the proof is as that of Lemma 5.2. ∎

We now observe, using the Sobolev trace theorem, that

$$\mathscr{F}(u) \geq \frac{1}{p}\|u\|_{W^{1,p}(\Omega)}^{p} - c_1\|u\|_{W^{1,p}(\Omega)}^{p^*} - \lambda c_2 \|u\|_{W^{1,p}(\Omega)}^{r} = j\big(\|u\|_{W^{1,p}(\Omega)}\big),$$

where $j(x) = \frac{1}{p}x^p - c_1 x^{p^*} - \lambda c_2 x^r$. As j attains a local but not a global minimum (j is not bounded below), we have to perform some sort of truncation. To this end let x_0, x_1 be such that $m < x_0 < M < x_1$, where m is the local minimum of j and M is the local maximum and $j(x_1) > j(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau(x)$ such that $\tau(x) = 1$ if $x \leq x_0$, $\tau(x) = 0$ if $x \geq x_1$, and $0 \leq \tau(x) \leq 1$. Finally, let $\varphi(u) = \tau(\|u\|_{W^{1,p}(\Omega)})$ and define the truncated functional as follows:

$$\tilde{\mathscr{F}}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \, dx - \frac{1}{p^*}\int_{\partial\Omega} |u|^{p^*}\varphi(u)\, d\sigma - \frac{\lambda}{r}\int_{\partial\Omega} |u|^r \, d\sigma.$$

As above, $\tilde{\mathscr{F}}(u) \geq \tilde{j}(\|u\|_{W^{1,p}(\Omega)})$, where $\tilde{j}(x) = \frac{1}{p}x^p - c_1 x^{p^*}\tau(x) - \lambda c_2 x^r$. We observe that if $x \leq x_0$ then $\tilde{j}(x) = j(x)$, and if $x \geq x_1$ then $\tilde{j}(x) = \frac{1}{p}x^p - \lambda c_2 x^r$.

Now we state a lemma that contains the main properties of $\tilde{\mathscr{F}}$.

Lemma 6.2. *$\tilde{\mathscr{F}}$ is C^1; if $\tilde{\mathscr{F}}(u) \leq 0$ then $\|u\|_{W^{1,p}(\Omega)} < x_0$ and $\mathscr{F}(v) = \tilde{\mathscr{F}}(v)$ for every v close enough to u. Moreover, there exists $\lambda_1 > 0$ such that if $0 < \lambda < \lambda_1$ then $\tilde{\mathscr{F}}$ satisfies a local Palais–Smale condition for $c \leq 0$.*

Proof. We only have to check the local Palais–Smale condition. Observe that every Palais–Smale sequence for $\tilde{\mathcal{F}}$ with energy level $c \leq 0$ must be bounded; therefore by Lemma 6.1 if λ verifies $0 < (1/p - 1/p^*)S^{p^*/(p^*-p)} - K\lambda^{p^*/(p^*-r)}$, then there exists a convergent subsequence. ∎

The following lemma gives the final ingredients needed in the proof of Theorem 1.3.

LEMMA 6.3. *For every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that*

$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \geq n,$$

where $\tilde{\mathcal{F}}^{-\varepsilon} = \{u, \tilde{\mathcal{F}}(u) \leq -\varepsilon\}$.

Proof. The proof is analogous to that of Lemma 3.1. ∎

Finally, we are ready to prove the main result of this section.

Proof of Theorem 1.5. The proof is analogous to that of Theorem 1.2; here we use Lemma 6.1 and Lemma 6.3 instead of Lemma 3.2 and Lemma 3.1, respectively, to work with the functional $\tilde{\mathcal{F}}$ and Lemma 6.2 to conclude on $\mathcal{F}$. ∎

7. PROOF OF THEOREM 1.6: THE SUPERCRITICAL CASE

In this section we will consider a nonlinearity f of the form

$$f(u) = \lambda|u|^{q-2}u + |u|^{r-2}u,$$

where $q \geq p^* > r > p$. In this case the functional $\mathcal{F}$ is not well defined in $W^{1,p}(\Omega)$, so to apply variational arguments we perform a truncation on the supercritical term, find a solution of the truncated problem, and finally show that this solution lies below the truncation level, so it is a solution of our original problem.

Proof of Theorem 1.6. We follow ideas from [4]. Let us consider a truncation of $|u|^{q-2}u$,

$$h(u) = \begin{cases} 0 & u < 0, \\ u^{q-1} & 0 \leq u < K, \\ K^{q-r}u^{r-1} & u \geq K. \end{cases}$$

Then h verifies $h(u) \leq K^{q-r}u^{r-1}$.

So we consider the truncated problem

$$
(7.1) \qquad \begin{cases} \Delta_p u = u^{p-1} & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda h(u) + u^{r-1} & \text{on } \partial\Omega, \end{cases}
$$

and we look at a positive nontrivial solution of (7.1) that satisfies $u \le K$. Such a solution will be a nontrivial positive solution of (1.1).

To this end, we consider the truncated functional

$$
(7.2) \quad \mathcal{F}_\lambda(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)\,dx - \lambda \int_{\partial\Omega} H(u)\,d\sigma - \int_{\partial\Omega} \frac{|u|^r}{r}\,d\sigma,
$$

where $H(u)$ verifies $H'(u) = h(u)$.

By the results of Section 2 there exists a Mountain Pass solution $u = u_\lambda$ for (7.1) that is a critical point of $\mathcal{F}_\lambda$ with energy level c_λ. One can easily check that this least energy solution u is positive. Moreover, the energy level c_λ is a decreasing function of λ, so we have that $\mathcal{F}_\lambda(u) = c_\lambda \le c_0$. Now using (7.2), (7.1), and that $H(u) \le \frac{1}{r} h(u)u$, we have that

$$
\begin{aligned}
c_0 \ge \mathcal{F}_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \lambda\int_{\partial\Omega} H(u)\,d\sigma - \int_{\partial\Omega}\frac{|u|^r}{r}\,d\sigma \\
&\ge \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,dx - \frac{1}{r}\left(\lambda\int_{\partial\Omega} h(u)u\,d\sigma + \int_{\partial\Omega}|u|^r\,d\sigma\right) \\
&= \left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega |\nabla u|^p + |u|^p\,dx.
\end{aligned}
$$

So, as $r > p$ we obtain

$$
\|u\|_{W^{1,p}(\Omega)} \le C = C(c_0, p, r).
$$

Now by the Sobolev trace inequality we get

$$
(7.3) \qquad \|u\|_{L^s(\partial\Omega)} \le S^{-1/p}\|u\|_{W^{1,p}(\Omega)} \le C = C(c_0, p, r, s, \Omega).
$$

Let us define

$$
u_L(x) = \begin{cases} u(x) & u(x) \le L, \\ L & u(x) > L. \end{cases}
$$

Multiplying the equation (7.1) by $u_L^{p\beta}u$, we get

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\big(u_L^{p\beta}u\big)dx + \int_\Omega u^p u_L^{p\beta}\,dx \\
&\qquad = \lambda\int_{\partial\Omega} h(u)u u_L^{p\beta}\,d\sigma + \int_{\partial\Omega} u^r u_L^{p\beta}\,d\sigma.
\end{aligned}
$$

Therefore, using that $h(u)u \leq K^{q-r}u^r$ and the definition of u_L, we obtain

$$\int_\Omega |\nabla u|^p u_L^{p\beta}\,dx + \int_\Omega u^p u_L^{p\beta}\,dx \leq (\lambda K^{q-r} + 1)\int_{\partial\Omega} u^r u_L^{p\beta}\,d\sigma.$$

Now we set $w_L = uu_L^\beta$. Then, we obtain

$$\begin{aligned}
\|w_L\|_{W^{1,p}(\Omega)}^p &= \int_\Omega |\nabla w_L|^p + |w_L|^p\,dx \\
&\leq C\left(\int_\Omega |\nabla u|^p u_L^{p\beta}\,dx + \int_\Omega u^p \beta^p u_L^{p(\beta-1)}|\nabla u_L|^p\,dx + \int_\Omega u^p u_L^{p\beta}\,dx\right) \\
&\leq C\left(\int_\Omega |\nabla u|^p u_L^{p\beta}\,dx + \int_\Omega u^p u_L^{p\beta}\,dx\right) \\
&\leq C(\lambda K^{q-r} + 1)\int_{\partial\Omega} u^r u_L^{p\beta}\,d\sigma.
\end{aligned}$$

Therefore, by Holder and Sobolev trace inequalities, we get

$$\begin{aligned}
\|w_L\|_{L^{p^*}(\partial\Omega)}^p \leq S^{-1}\|w_L\|_{W^{1,p}(\Omega)}^p &\leq C(\lambda K^{q-r} + 1)\int_{\partial\Omega} u^r u_L^{p\beta}\,d\sigma \\
&\leq C(\lambda K^{q-r} + 1)\left(\int_{\partial\Omega} u^{p^*}\,d\sigma\right)^{(r-p)/p^*}\left(\int_{\partial\Omega} w_L^{\alpha^*}\,d\sigma\right)^{p/\alpha^*},
\end{aligned}$$

where $\alpha^* = pp^*/(p^* - r + p) < p^*$. So by (7.3),

$$\begin{aligned}
\|w_L\|_{L^{p^*}(\partial\Omega)}^p &\leq C(\lambda K^{q-r} + 1)\|u\|_{L^{p^*}(\partial\Omega)}^{r-p}\|w_L\|_{L^{\alpha^*}(\partial\Omega)}^p \\
&\leq C(\lambda K^{q-r} + 1)\|w_L\|_{L^{\alpha^*}(\partial\Omega)}^p.
\end{aligned}$$

Now if $u^{\beta+1} \in L^{\alpha^*}(\partial\Omega)$, by the dominated convergence theorem and Fatou's lemma we get

$$\|u^{\beta+1}\|_{L^{p^*}(\partial\Omega)}^p \leq C(\lambda K^{q-r} + 1)\|u^{\beta+1}\|_{L^{\alpha^*}(\partial\Omega)}^p,$$

that is,

$$\|u\|_{L^{p^*(\beta+1)}(\partial\Omega)} \leq C(\lambda K^{q-r} + 1)^{1/p(\beta+1)}\|u\|_{L^{\alpha^*(\beta+1)}(\partial\Omega)}.$$

Let $\kappa = p^*/\alpha^*$. Iterating the last inequality, we have

$$\|u\|_{L^{\kappa^j \alpha^*}(\partial\Omega)} \leq C(\lambda K^{q-r} + 1)^\theta \|u\|_{L^{\alpha^*}(\partial\Omega)},$$

where $\theta = \frac{1}{p}\sum_{j=1}^\infty \kappa^{-j}$. Again using (7.3), we get

$$\|u\|_{L^\infty(\partial\Omega)} \leq C(\lambda K^{q-r} + 1)^\theta.$$

Hence, if $K_0 > C$, for every $K \geq K_0$ there exists $\lambda(K)$ such that if $\lambda < \lambda(K)$ then

$$\|u\|_{L^\infty(\partial\Omega)} \leq K.$$

This finishes the proof. ∎

8. PROOF OF THEOREM 1.7: A NONEXISTENCE RESULT

In this section we prove a nonexistence result for positive regular decaying solutions for (1.1) in $\mathbb{R}^N_+ = \{x_1 > 0\}$. That is, we are dealing with a positive regular solution of

$$\begin{cases} \Delta_p u = u^{p-1} & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = u^{q-1} & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{8.1}$$

that satisfies the hypotheses of Theorem 1.7.

We observe that in the special case $p = 2$ there exists a solution if we drop the decaying assumption, namely $u(x) = e^{-x_1}$ is a solution for every q.

Proof of Theorem 1.7. First we multiply the equation in (8.1) by u and integrate by parts to obtain

$$\int_{\mathbb{R}^N_+} (|\nabla u|^p + u^p)\, dx - \int_{\partial\mathbb{R}^N_+} u^q \, dx' = 0. \tag{8.2}$$

Note that our decaying and integrability assumptions on u justify all of the integrations by parts made along this proof.

Now we multiply by $x\nabla u$ and integrate by parts to obtain

$$-\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla(x\nabla u)\, dx + \int_{\partial\mathbb{R}^N_+} u^{q-1} x\nabla u \, dx' = \frac{1}{p}\int_{\mathbb{R}^N_+} x\nabla u^p \, dx.$$

Hence further integrations by parts give us

$$\left(-1 + \frac{N}{p}\right)\int_{\mathbb{R}^N_+} |\nabla u|^p \, dx - \frac{N-1}{q}\int_{\partial\mathbb{R}^N_+} u^q \, dx' = \frac{N}{p}\int_{\mathbb{R}^N_+} u^p \, dx.$$

Using (8.2), we arrive at

$$\left(-1 + \frac{N}{p} - \frac{N-1}{q}\right)\int_{\partial\mathbb{R}^N_+} u^q \, dx' = \left(-1 + \frac{2N}{p}\right)\int_{\mathbb{R}^N_+} u^p \, dx > 0.$$

Therefore, if u is not identically zero, we must have

$$-1 + \frac{N}{p} - \frac{N-1}{q} > 0,$$

that is,

$$q > p^* = \frac{p(N-1)}{N-p},$$

as we wanted to show. ∎

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