On the Multiplicity of Darlington Realizations of Contractive Matrix-Valued Functions

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We obtain a unique expression for a Darlington realization of a contractive matrix-valued function $S(z)\in \mathcal{S}$, valid for the three following cases: (a) $S(z)$ is inner; (b) $S(z)$ is not inner and $\det[I_n - S^*(1/z)S(z)] \neq 0$ a.e.; (c) $S(z)$ is not inner and $\det[I_n - S^*(1/z)S(z)] = 0$ (z \in D). On the basis of this result we examine the problem of multiplicity of realizations for the case (c).

1. INTRODUCTION AND SUMMARY OF KNOWN RESULTS

We shall denote by $L^2(C^n)$ the class of measurable functions $h(\xi)$ ($\xi = \omega^i$, $0 \leq \omega \leq 2\pi$), with values in $C^n$ such that

$$\|h\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \|h(e^{it})\|^2 \, dt < \infty.$$ 

It consists of the functions whose Fourier series is (in the sense of convergence in the mean)

$$h(\xi) = \sum_k^{\infty} h_k \xi^k, \quad h_k \in C^n \quad \text{and} \quad \|h\|^2 = \sum_k^{\infty} \|h_k\|^2 \quad (\text{cf. [1]}).$$

We shall denote by $L^2_+(C^n)$ the subspace of $L^2(C^n)$ which consists of those functions for which $h_k = 0$, $k < 0$. $H^2(C^n)$ denotes the Hilbert space of
functions \( h(z) = \sum_{k=0}^{\infty} h_k z^k \) \((z = re^{it})\), \( h_k \in C^n\), holomorphic in the unit disk \( D = \{ z; |z| < 1 \}\), such that
\[
\frac{1}{2\pi} \int_0^{2\pi} \| h(re^{it}) \|^2 \, dt \quad (0 < r < 1)
\]
has a bound independent of \( r \).

A function \( u(z) \) holomorphic in \( D \) is called inner if \(|u(z)| < 1 \) \((z \in D)\) and \(|u(\xi)| = 1 \) a.e. A function \( \Phi(z) \), holomorphic in \( D \) is called outer if
\[
\Phi(z) = \chi \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \ln k(t) \, dt \quad (z \in D),
\]
where \( k(t) \geq 0, \ln k(t) \in L^1 \) and \( \chi \) is a complex number of modulus 1. For a function \( w(z) \), meromorphic in \( D \), the characteristic of Nevanlinna is defined by the expression
\[
T(w, r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ w(re^{it}) \, dt
\]
\[
+ \int_0^r \frac{n(t, w) - n(0, w)}{t} \, dt + n(0, w) \ln r,
\]
where
\[
\ln^+ a = \begin{cases} 
\ln a & \text{if } a \geq 1 \\
0 & \text{if } 0 \leq a < 1,
\end{cases}
\]
and \( n(t, w) \) is the number of poles of \( w(z) \), each one with its multiplicity, inside the circle \(|z| < t\).

We say that \( w(z) \) is of bounded characteristic if \( \sup_{|t| < 1} |T(w; r)| < \infty \). According to a theorem of Nevanlinna [2] the class of functions of bounded characteristic coincides with the class of functions that can be written as the ratio of two bounded holomorphic functions \((z \in D)\). Then, these functions are uniquely defined by their boundary values a.e. in the unit circle.

We design with \( N_0 \) the class of functions \( f(z) \) of bounded characteristic \((z \in D)\) that can be written as a product of an inner function and an outer function. For functions of \( N_0 \) the maximum principle holds. A matrix \( S \) is called contractive iff \( I - S^*S \geq 0 \), where \( I \) is the unit matrix and the symbol \(*\) denotes hermitian conjugation. We use \( J \) to design a matrix for which \( J^2 = I \). A matrix \( A \) is called \( J \)-expansive iff \( A^*JA - J \geq 0 \), and \( J \)-unitary iff \( A^*JA - J = 0 \).

We will design by \( \mathcal{F} \) the class of contractive matrix-valued functions, i.e.,
the matrix-valued functions holomorphic in $D$ for which $\|S(z)\| \leq 1$. $S(z) \in S$ is inner if $I - S^*(\xi) S(\xi) = 0$ a.e. We say that a matrix-valued function is of bounded characteristic if all its elements possess that property, and that belongs to the class $N_0$ if all its elements are functions of $N_0$. A matrix-valued functions $S(z) \in S$ belongs to the class $\mathcal{SP}$ if it has the additional property that its boundary values a.e. on the unit circle are, simultaneously, boundary values of a matrix-valued function $\tilde{S}(z)$ meromorphic in $D = \{z; |z| > 1\}$ with elements of bounded characteristic there [3], i.e.,

$$\lim_{|z| \to 1} \tilde{S}(z) = \lim_{|z| \to 1} S(z) \quad \text{a.e.}$$

A meromorphic matrix-valued function $A(z)$ is $J$-expansive ($z \in D$) if it assumes $J$-expanding values at each point of holomorphicity $z$, i.e.,

$$A^*(z) J A(z) - J \geq 0.$$  

and a $J$-expansive matrix-valued function $A(z)$ is $J$-inner if it is $J$-unitary a.e. in the unit circle, i.e.,

$$A^*(\xi) J A(\xi) - J = 0.$$  

An arbitrary $J$-expansive matrix-valued function is of bounded characteristic [3].

Of importance for us is the

**BASIC LEMMA** [3]. Let $A(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$ be a matrix-valued function of bounded characteristic of order $n + m$, with diagonal elements $\alpha(z)$ of order $n$ and $\delta(z)$ of order $m$, where $\det \delta(z) \neq 0$ ($z \in D$), that satisfies the condition

$$A^*(\xi) J A(\xi) - J \geq 0 \quad \text{a.e.},$$

where

$$j = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \quad \text{(1.1)}$$

and

$$a(z) = \alpha(z) - \beta(z) \delta^{-1}(z) \gamma(z) \in N_0;$$

$$b(z) = \beta(z) \delta^{-1}(z) \in N_0;$$

$$c(z) = \delta^{-1}(z) \gamma(z) \in N_0;$$

$$d(z) = \delta^{-1}(z) \in N_0. \quad \text{(1.2)}$$
Then $A(z)$ is $J$-expansive ($z \in D$) and $\|a(z)\| \leq 1$; $\|b(z)\| \leq 1$; $\|c(z)\| \leq 1$; $\|d(z)\| \leq 1$.

We will use also

**Theorem [4].** Let $S(z)$ be a matrix-valued function of order $n$, of the class $\mathcal{F}_\pi$. Then its boundary values a.e. $S(\xi)$ can be represented in the form

$$S(\xi) = U_{2}^{-1}(\xi) S'(\xi) U_{1}(\xi),$$

(1.3)

where $U_{1}(\xi)$ and $U_{2}(\xi)$ are a.e. limiting values of inner matrix-valued functions $U_{1}(z)$ and $U_{2}(z)$, and $S'(\xi)$ has a diagonal form with blocks $S_{1}(\xi)$ of order $k$ and $S_{2}(\xi)$ of order $n-k$, i.e.,

$$S'(\xi) = \text{diag}(S_{1}(\xi), S_{2}(\xi))$$

(1.4)

verifying a.e. the relations $\|S_{1}(\xi)\| < 1$ and $\|S_{2}(\xi)\| = 1$.

We will briefly describe the way we have constructed in [4] the matrix-valued functions $U_{1}(\xi)$ and $U_{2}(\xi)$, because of their importance in the development that follows. For each fixed $\xi$ where $T(\xi) = \text{def} S^*(\xi) S(\xi)$ is defined, we arranged the eigenvalues of the nonnegative matrix $I_{n} - T(\xi)$ in nonincreasing order, and considered the orthonormal base of $\mathbb{C}^{n}$ formed by the corresponding eigenvectors $\{|\Phi_{m}(\xi)\}_{m=1}^{n}$. Introducing the notation

$$\Phi_{n}(\xi) = (\Phi_{m_{1}}(\xi), \Phi_{m_{2}}(\xi), ..., \Phi_{m_{n}}(\xi))^{\prime},$$

where the symbol $(\ )'$ denotes transposition, we construct the matrix

$$U_{1}(\xi) = \{\Phi_{m}(\xi)\}_{m=1}^{n},$$

Due to the orthonormality of $\{|\Phi_{m}(\xi)\}_{m=1}^{n}$, $U_{1}(\xi) = I_{n}$ for each fixed $\xi$ in the unit circle except, possibly, a set of zero measure. According to the way we ordered the eigenvalues, the last $(n-k)$ rows of $U_{1}(\xi)$ are the eigenvectors corresponding to the eigenvalue 0. We applied a similar procedure to construct $U_{2}(\xi)$, which $T'(\xi) = \text{def} S(\xi) S^*(\xi)$ playing the role of $T(\xi)$.

Arov has proved in [3] that a necessary and sufficient condition for a matrix-valued function $S(z)$ to be Darlington realizable is that $S(z)$ belongs to the class $\mathcal{F}_\pi$, and considered separately three cases:

(a) $S(z)$ is inner;
(b) $S(z)$ is not inner and $\det[I_{n} - S^*(\xi) S(\xi)] \neq 0$ a.e.;
(c) $S(z)$ is not inner and $\det[I_{n} - S^*(1/z) S(z)] = 0$ ($z \in D$).

Using results from [3, 4] we obtain in this article a unique expression, valid for the three cases for a Darlington realization of a matrix function
This result allows us to examine the problem of multiplicity of realizations not only in Case (b), solved in [3], but also in Case (c) proposed in [3] as an open problem.

2. DARLINGTON REALIZATIONS

By Darlington realization of a matrix-valued function $S(z) \in \mathcal{F}$, of order $n \times m$, we mean the representation of $S(z)$ as the linear fractional transformation [3],

$$S(z) = \begin{bmatrix} a(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{bmatrix}^1,$$

(2.1)

over a constant matrix $c \in \mathcal{F}$, of order $n \times m$, with a $j$-inner matrix of coefficients

$$A(z) = \begin{pmatrix} a(z) \\ \beta(z) \\ \gamma(z) \\ \delta(z) \end{pmatrix}.$$

**Theorem II.1.** Let $S(z) \in \mathcal{F}$ be a matrix-valued function of order $n$. Then $S'(\xi)$, defined by (1.3), can be written as the linear fractional transformation

$$S'(\xi) = \begin{bmatrix} a'(\xi) & \beta'(\xi) \\ \gamma'(\xi) & \delta'(\xi) \end{bmatrix}^1,$$

(2.2)

over a constant matrix $c \in \mathcal{F}$, with a matrix of coefficients

$$A'(\xi) = \begin{pmatrix} a'(\xi) \\ \beta'(\xi) \\ \gamma'(\xi) \\ \delta'(\xi) \end{pmatrix}$$

that is $j$-unitary a.e.

**Proof.** As an illustration we describe briefly the steps of the proof, which consist of

1. making a convenient selection of $c$ (formula (2.5)) and
2. carry out the demonstration of the thesis with the particular choice of $A'(\xi)$ specified in (2.6).

Let us observe that, since $S(z) \in \mathcal{F}$, the nonnegative matrix-valued functions

$$F_1(\xi) \overset{\text{def}}{=} I_k - S^*_1(\xi) S_1(\xi) \quad \text{a.e.},$$

$$F_1'(\xi) \overset{\text{def}}{=} I_k - S_1(\xi) S^*_1(\xi) \quad \text{a.e.}$$
are boundary values of matrix-valued functions $F_1(z)$ and $F_1'(z)$ of bounded characteristic in $D$. Hence there exist [5, 6] solutions $\theta(\xi)$ and $\psi(\xi)$ of the factorization problems

\begin{align*}
F_1(\xi) &= \theta^*(\xi) \theta(\xi) \quad \text{a.e.,} \\
F_1'(\xi) &= \psi^*(\xi) \psi(\xi) \quad \text{a.e.}
\end{align*}

(2.3) \hspace{1cm} (2.4)

that are a.e. boundary values of bounded holomorphic functions $\theta(z)$ and $\psi(z)$. These solutions are uniquely defined by the following normalization conditions: $\det \theta(z)$ and $\det \psi(z)$ are outer functions, $\theta(0) \geq 0$ and $\psi(0) \geq 0$. We will construct the representation (2.2) with

$$
e \begin{pmatrix} 0_k & 0 \\ 0 & I_{n-k} \end{pmatrix},$$

(2.5)

defined for each value of $\xi$ where $S(\xi)$ is defined [4]; and the matrix of coefficients $A'(\xi)$ with elements

\begin{align*}
\alpha'(\xi) &= \{ \text{diag}(\psi^*(\xi), 0_{n-k}) + \varepsilon \left[ U_2^{-1}(\xi) + U_1^{-1}(\xi) S^*(\xi) \right] \}^{-1} + \frac{1}{2} U_2(\xi) \varepsilon, \\
\beta'(\xi) &= S'(\xi) \{ \text{diag}(\theta(\xi), 0_{n-k}) + \varepsilon \left[ U_1^{-1}(\xi) + U_2^{-1}(\xi) S'(\xi) \right] \}^{-1} - \frac{1}{2} U_2(\xi) \varepsilon, \\
\gamma'(\xi) &= S^*(\xi) \{ \text{diag}(\psi^*(\xi), 0_{n-k}) \\
&\quad + \varepsilon \left[ U_2^{-1}(\xi) + U_1^{-1}(\xi) S^*(\xi) \right] \}^{-1} - \frac{1}{2} U_2(\xi) \varepsilon, \\
\delta'(\xi) &= \{ \text{diag}(\theta(\xi), 0_{n-k}) + \varepsilon \left[ U_1^{-1}(\xi) + U_2^{-1}(\xi) S'(\xi) \right] \}^{-1} + \frac{1}{2} U_1(\xi) \varepsilon,
\end{align*}

(2.6)

where $U_1(\xi)$ and $U_2(\xi)$ are the unitary matrix-valued functions a.e. involved in (1.3) and defined in [4]. With this choice of $\varepsilon$ and $A'(\xi)$, (2.2) holds. In fact

$$S'(\xi)[\gamma'(\xi) \varepsilon + \delta'(\xi)] - [\alpha'(\xi) \varepsilon + \beta'(\xi)]=\text{diag}(S^*(\xi) S(\xi) - I_k, 0_{n-k}) \{ \text{diag}(\psi^*(\xi), 0_{n-k}) \\
&\quad + \varepsilon \left[ U_2^{-1}(\xi) + U_1^{-1}(\xi) S^*(\xi) \right] \}^{-1} \varepsilon = 0 \quad \text{a.e.}
$$

To finish the proof we will show that

$$A'*(\xi) f A'(\xi) - j = 0 \quad \text{a.e.}$$
or, equivalently, that

\[\begin{align*}
\gamma'(\xi) & \gamma'(\xi) = I_n \quad \text{a.e.,} \\
\delta'(\xi) & \delta'(\xi) - \beta'(\xi) \beta'(\xi) = I_n \quad \text{a.e.,} \\
\alpha'(\xi) & \beta'(\xi) - \gamma'(\xi) \delta'(\xi) = 0 \quad \text{a.e.} \quad (2.7)
\end{align*}\]

Substituting (2.6) into (2.7) we get

\[\begin{align*}
\alpha'(\xi) & \alpha'(\xi) - \gamma'(\xi) \gamma'(\xi) \\
& = \text{diag}(\psi^{-1}(\xi)[I_k - S_1(\xi) S_1^*(\xi)] \psi^{-1}(\xi), 0_{n-k}) \\
& + \frac{1}{2} \{ [U_2(\xi) + S'(\xi) U_1(\xi)]^{-1} \text{diag}(\psi(\xi), 0, n-k) + \varepsilon \}^{-1} \\
& + \frac{1}{2} \{ \text{diag}(\psi(\xi), 0, n-k)[U_2^{-1}(\xi) + U_1^{-1}(\xi) S'^*(\xi)]^{-1} + \varepsilon \}^{-1} - I_n \quad \text{a.e.}
\end{align*}\]

\[\begin{align*}
\delta'(\xi) & \delta'(\xi) - \beta'(\xi) \beta'(\xi) \\
& = \text{diag}(\theta^{-1}(\xi)[I_k - S_1^*(\xi) S_1(\xi)] \theta^{-1}(\xi), 0_{n-k}) \\
& + \frac{1}{2} \{ \text{diag}(\theta(\xi), 0, n-k)[U_1^{-1}(\xi) + U_2^{-1}(\xi) S'(\xi)]^{-1} + \varepsilon \}^{-1} + \varepsilon \\
& = I_n \quad \text{a.e.}
\end{align*}\]

\[\begin{align*}
\alpha'(\xi) & \beta'(\xi) - \gamma'(\xi) \delta'(\xi) \\
& = \frac{1}{2} \varepsilon \{ \text{diag}(\theta(\xi), 0, n-k)[U_2^{-1}(\xi) + U_1^{-1}(\xi) S'(\xi)]^{-1} + \varepsilon \}^{-1} \\
& - \frac{1}{2} \{ [U_2(\xi) + S'(\xi) U_1(\xi)]^{-1} \text{diag}(\psi(\xi), 0, n-k) + \varepsilon \}^{-1} = 0 \quad \text{a.e.}
\end{align*}\]

This completes the proof of Theorem II.1.

Using this theorem and (1.3) we obtain

\[\begin{align*}
S(\xi) & = U_2^{-1}(\xi)[\alpha'(\xi) \varepsilon + \beta'(\xi)] [\gamma'(\xi) \varepsilon + \delta'(\xi)]^{-1} U_1^{-1}(\xi) \\
& = [U_2^{-1}(\xi) \alpha'(\xi) \varepsilon + U_2^{-1}(\xi) \beta'(\xi)] [U_1^{-1}(\xi) \gamma'(\xi) \varepsilon \\
& + U_1^{-1}(\xi) \delta'(\xi)]^{-1} \quad \text{a.e.}
\end{align*}\]

This is a linear fractional transformation of \(S(\xi)\) over the same constant matrix \(\varepsilon\), given by (2.5), with a matrix of coefficients \(A(\xi) = U(\xi) A'(\xi)\), where

\[U(\xi) = \text{diag}(U_2^{-1}(\xi), U_1^{-1}(\xi)). \quad (2.8)\]

Note that \(A(\xi)\) verifies

\[A(\xi) = \lim_{|z| \to 1} A(z) \quad \text{a.e.},\]
where
\[ A(z) = U(z) A'(z) \] (2.9)

is a matrix-valued function of bounded characteristic in \( D \). This is due to the construction of \( U_1(\xi) \), \( U_2(\xi) \) (cf. [4]) and \( A'(\xi) \). We shall prove now

**Theorem II.2.** Let \( S(z) \in \mathcal{S}_\pi \) be a matrix-valued functions of order \( n \). Then the representation
\[
S(z) = [\alpha(z)e + \beta(z)][\gamma(z)e + \delta(z)]^{-1}
\] (2.10)

with \( e \) specified by (2.5) and \( A(z) = (\alpha(z) \beta(z) \gamma(z) \delta(z)) \) given by (2.9) is a Darlington realization of \( S(z) \).

**Proof.** To prove the thesis we will show that \( A(\xi) \) is \( j \)-unitary a.e., and \( A(z) \) is \( j \)-expansive \( (z \in D) \). Since \( U_1(\xi) \) and \( U_2(\xi) \) are unitary a.e. by construction, using (2.7), (2.8), and (2.9) we have
\[
A^*(\xi) A'(\xi) - j = U^*(\xi) A'^*(\xi) A'(\xi) U(\xi) - j
\]
\[
= U^*(\xi) jU(\xi) - j = 0 \quad \text{a.e.} \quad (2.11)
\]

In view of this result, we will verify that the elements of \( A(\xi) \) are a.e. boundary values of matrix-valued functions that satisfy the hypotheses of the Basic Lemma [3].

Let us consider the function
\[
d(z) = \left\{ \begin{array}{ll}
\{ \text{diag}(\theta(z), 0_{n-k}) \} U_1(z) + \varepsilon[I_n + S(z)] \\
\times \{ I_n + \varepsilon[I_n + S(z)] \}^{-1} \end{array} \right. \quad (z \in D). \quad (2.12)
\]

Assuming that the unity is not an eigenvalue of \(-S(z)\), the function \( \{ I_n + \varepsilon[I_n + S(z)] \}^{-1} \) exists [3], and taking into account that \( \theta(z) \), \( U_1(z) \), and \( S(z) \) are bounded and holomorphic \( (z \in D) \), we conclude that
\( d(z) \in N_0 \). Note that
\[
d^{-1}(\xi) = \delta(\xi) = \lim_{|z| \to 1} \delta(z) \quad \text{a.e.}
\]

From (2.11) we know that
\[
\alpha^*(\xi) \alpha(\xi) - \gamma^*(\xi) \gamma(\xi) = I_n \quad \text{a.e.,}
\]
\[
\gamma^*(\xi) \delta(\xi) - \alpha^*(\xi) \beta(\xi) = 0 \quad \text{a.e.}
\]
Then, using (1.2) we obtain

\[ \alpha^{* -1}(\xi) = a(\zeta) \quad \text{a.e.} \]

This relationship leads us to examine if \( a(\zeta) \) is the boundary value of a matrix-valued function \( u(\zeta) \) satisfying the hypothesis of the Basic Lemma. The expression (2.6) of \( \alpha'(<) \) may be used to derive a formal expression for \( u(z) \),

\[ u(z) = 2\{2I_n + [I_n + S(z)]e\}^{-1} \{U_2^{-1}(z) \text{diag}(\psi(z), O_{n-k}) + [I_n + S(z)]e\}. \quad (2.13) \]

Admitting, without restriction, that unity is not an eigenvalue of \( S(\zeta) \) \((z \in D)\), it can be seen that \( 2I_n + [I_n + S(z)]e \) exists and its elements are functions of \( N_0 \) (cf. [3]).

Consider now the second factor in (2.13). The matrix-valued function \( U_2^{-1}(z) \) is, by its construction, of bounded characteristic in \( D \). We can select an inner scalar function \( h_1(z) \) being common denominator of all the elements of \( U_2^{-1}(z) \), and construct a function \( \psi(\zeta) = \psi_0(\zeta) h_1(\zeta) \), where \( \psi_0(\zeta) \) is a solution of (2.4) uniquely defined by the normalization conditions. Therefore \( \psi(\zeta) \) is also solution of (2.4). Furthermore,

\[ \psi(\zeta) = \lim_{|z| \to 1} \psi(z) \quad \text{a.e.}, \]

where \( \psi(z) = \psi_0(z) h_1(z) \) is bounded and holomorphic in \( D \). With this choice of \( \psi(z) \) the elements of \( U_2^{-1}(z) \text{diag}(\psi(z), O_{n-k}) \) are scalar functions of \( N_0 \). Since \( [I_n + S(z)] \) is holomorphic and bounded \((z \in D)\), the above conclusions allow us to affirm that \( a(z) \in N_0 \).

Consider now the element

\[ b(\zeta) = \beta(\zeta) \delta^{-1}(\zeta) = \beta(\zeta) d(\zeta) \quad \text{a.e.} \]

Using (2.6) and (2.12), \( b(z) \) may be written

\[ b(z) = \{S(z) - \frac{1}{2}e[I_n + S(z)]\} \{I_n + e[I_n + S(z)]\}^{-1} \quad (z \in D). \]

When we examined the element \( d(z) \), we have shown that \( I_n + e[I_n + S(z)] \) is invertible. This fact, together with property of boundedness and holomorphicity of \( S(z) \), implies that \( b(z) \in N_0 \).

Let us now consider the block

\[ c(\zeta) = \delta^{-1}(\zeta) \gamma(\zeta) = d(\zeta) U_1^{-1}(\zeta) \text{diag}(S^*_1(\zeta), S^*_2(\zeta)) \]

\[ \times \{\text{diag}(\psi^*(\zeta), O_{n-k}) + \text{diag}(O_k, I_{n-k})[U_2^{-1}(\zeta) + U_1^{-1}(\zeta)] \times \text{diag}(S^*_1(\zeta), S^*_2(\zeta))\}^{-1} - \frac{1}{2}d(\zeta) U_1^{-1}(\zeta) \text{diag}(O_k, I_{n-k}). \]
Since we have proved that \( d(\xi) \in N_0 \), it follows immediately that the second term at the right also satisfies that condition. Hence it is left to show that the first term belongs to the class \( N_0 \). We denote this term by \( c'(\xi) \).

Replacing \( d(\xi) \) in it and using the equality

\[
\{ I_n + \frac{1}{2} \text{diag}(O_k, I_{n-k})[I_n + S(\xi)] \}^{-1} = I_n - \{ I_n + \frac{1}{2} \text{diag}(O_k, I_{n-k})[I_n + S(\xi)] \}^{-1} \frac{1}{2} \text{diag}(O_k, I_{n-k}) \times [I_n + S(\xi)]
\]

we obtain

\[
c'(\xi) = c_1(\xi) - c_2(\xi), \tag{2.14}
\]

where

\[
c_1(\xi) = \{ \text{diag}(\theta(\xi), O_{n-k}) U_1(\xi) + \text{diag}(O_k, I_{n-k})[I_n + S(\xi)] \}
\times U_1^{-1}(\xi) \text{diag}(S_1^*(\xi), S_2^*(\xi)) \{ \text{diag}(\psi^*(\xi), O_{n-k}) \\
+ \text{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi) \text{diag}(S_1^*(\xi), S_2^*(\xi))] \}^{-1}
\]

and

\[
c_2(\xi) = \frac{1}{2} d(\xi) \text{diag}(O_k, I_{n-k})[I_n + S(\xi)] U_1^{-1}(\xi) \text{diag}(S_1^*(\xi), S_2^*(\xi))
\times \{ \text{diag}(\psi^*(\xi), O_{n-k}) + \text{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi) \\
\times \text{diag}(S_1^*(\xi), S_2^*(\xi))] \}^{-1}.
\]

The term \( c_1(\xi) \) can be rewritten in the form

\[
c_1(\xi) = c_3(\xi) + 2c_2(\xi), \tag{2.15}
\]

where

\[
c_3(\xi) = \text{diag}(\theta(\xi), S_1^*(\xi), O_{n-k}) \{ \text{diag}(\psi^*(\xi), O_{n-k}) \\
+ \text{diag}(O_k, I_{n-k})[U_2^{-1}(\xi) + U_1^{-1}(\xi) \text{diag}(S_1^*(\xi), S_2^*(\xi))] \}^{-1}.
\]

Observe that the factor between brackets is, due to its construction, a triangular matrix-valued function, with its right superior block equal to zero. Therefore

\[
c_3(\xi) = \text{diag}(\theta(\xi), S_1^*(\xi) \psi^{-1}(\xi), O_{n-k}). \tag{2.16}
\]
Replacing (2.16) and (2.15) in (2.14) we obtain

\[ c'(\xi) = \text{diag}(\theta(\xi) S^*_1(\xi) \psi^{-1}(\xi), O_{n-k}) + c_2(\xi). \]

The matrix-valued function defined by (2.16) is the boundary value a.e. of the matrix-valued function

\[ c_3(z) = \text{diag}(\theta(z) S^*_1(1/z) \psi^{-1}(1/z), O_{n-k}) \quad (z \in D) \]

of bounded characteristic in D. Let \( b_2(z) \) be an inner scalar function that is the common denominator of \( S^*_3(1/z) \psi^{-1}(1/z). \) If \( \theta_0(\xi) \) is a solution of the factorization problem (2.3) satisfying the normalization conditions, then \( \theta(\xi) = b_2(\xi) \theta_0(\xi) \) is also a solution of that problem. Moreover, \( \theta(\xi) \) verifies

\[ \theta(\xi) = \lim_{|z| \to 1} \theta(z) \quad \text{a.e.,} \]

where \( \theta(z) \) is an holomorphic and bounded matrix-valued function \((z \in D).\)

Therefore we can conclude that \( c_3(\xi) \in N_0.\) To prove that \( c(\xi) \in N_0,\) it is only left to show that \( c_2(\xi) \in N_0.\) By virtue of (1.3) the term \( c_2(\xi) \) can be written as follows:

\[
c_2(\xi) = \tfrac{1}{2} \text{d}(\xi) \begin{bmatrix} U_1^{-1}(\xi) \text{diag}(S^*_1(\xi), S^*_2(\xi)) + U_2^{-1}(\xi) \\
\times \text{diag}(S_1(\xi) S^*_1(\xi), I_{n-k}) + U_2^{-1}(\xi) - U_2^{-1}(\xi) \end{bmatrix}
\]

\[
\times \text{diag}(\psi^*(\xi), O_{n-k})
\]

\[
+ \text{diag}(O_{k}, I_{n-k}) \begin{bmatrix} U_2^{-1}(\xi) + U_1^{-1}(\xi) \text{diag}(S^*_1(\xi), S^*_2(\xi)) \end{bmatrix}^{-1}
\]

\[
= \tfrac{1}{2} \text{d}(\xi) \begin{bmatrix} U_1^{-1}(\xi) \text{diag}(S_1(\xi) S^*_1(\xi), I_{n-k}) + U_1^{-1}(\xi) \end{bmatrix}
\]

\[
\times \text{diag}(\psi^*(\xi), O_{n-k})
\]

\[
+ \text{diag}(O_{k}, I_{n-k}) \begin{bmatrix} U_2^{-1}(\xi) + U_1^{-1}(\xi) \text{diag}(S^*_1(\xi), S^*_2(\xi)) \end{bmatrix}^{-1}
\]

\[
= \tfrac{1}{2} \text{d}(\xi) \begin{bmatrix} S_1(\xi) S^*_1(\xi) - I_{k} \end{bmatrix} \psi^{-1}(\xi), O_{n-k})
\]

\[
+ \tfrac{1}{2} \text{d}(\xi) \begin{bmatrix} S_1(\xi) S^*_1(\xi) - I_{k} \end{bmatrix} \psi^{-1}(\xi), O_{n-k})
\]

\[
\times \text{diag}(\psi^*(\xi), O_{n-k}) + \text{diag}(O_{k}, I_{n-k}) \begin{bmatrix} U_2^{-1}(\xi) + U_1^{-1}(\xi) \end{bmatrix}
\]

\[
\times \text{diag}(S^*_1(\xi), S^*_2(\xi)) \end{bmatrix}^{-1}
\]

\[
= \text{diag}(S^*_1(\xi), S^*_2(\xi)) \end{bmatrix}^{-1}
\]

Without loss of generality we can suppose that the unity is not eigenvalue of \( S(z) \), and, using the same argument we have mentioned when we consider the block \( b(z) \), we can affirm that \([I + S(\xi)]\) is invertible, therefore \([U_2^{-1}(\xi) + U_1^{-1}(\xi) \text{diag}(S^*_1(\xi), S^*_2(\xi))]\) is also invertible. Taking this into
account and recalling that $\psi(\xi)$ is solution of the factorization problem (2.4) we arrive at the conclusion that

$$c_2(\xi) = \frac{1}{2}d(\xi) \ U_2^{-1}(\xi) \ \text{diag}(\psi(\xi), O_{n-k}) + \frac{1}{2}d(\xi) \ \text{diag}(O_k, I_{n-k})$$

$$\times \{ \text{diag}(\psi^*(\xi), O_{n-k}) [U_2^{-1}(\xi) + U_1^{-1}(\xi)] \times \text{diag}(O_k, I_{n-k}) \}^{-1} \text{diag}(O_k, I_{n-k})$$

$$= \frac{1}{2}d(\xi) \ \text{diag}(O_k, I_{n-k}) \ U_2^{-1}(\xi) \ \text{diag}(\psi(\xi), O_{n-k}) + \frac{1}{2}d(\xi)$$

$$\times \text{diag}(O_k, I_{n-k})$$

Recalling that with the convenient construction of $(\psi(\xi)$ we have set when the block $a(z)$ was examined, the elements of $U_2^{-1}(\xi) \ \text{diag}(\psi(z), O_{n-k})$ are bounded and holomorphic scalar functions ($z \in D$), we arrive to the conclusions that $c_2(\xi) \in N_0$. Therefore, $c(\xi) \in N_0$. Hence the matrix of coefficients $A(z)$, defined by (2.9) is $j$-inner and the proof that (2.10) is a Darlington realization is finished.

3. SET OF REALIZATIONS

The Darlington realization of a matrix-valued function $S(z) \in \mathcal{S}\pi$ is not unique [3]. It is important for a practical viewpoint (synthesis of an $n$-port with specified scattering matrix) to describe all the possible realizations of $S(z)$. An analogous problem has been cited by Cauer [6], in the case of reactance matrices, the equivalence problem.

Let us consider an arbitrary realization of $S(z) \in \mathcal{S}\pi$,

$$S(z) = [\alpha(z)\varepsilon + \beta(z)] [\gamma(z)\varepsilon + \delta(z)]^{-1}$$

(3.1)

over a constant matrix $\varepsilon \ (\in \mathcal{S})$ with a $j$-expansive matrix of coefficients

$$A(z) = \begin{pmatrix} \alpha(z) \\ \gamma(z) \end{pmatrix} \begin{pmatrix} \beta(z) \\ \delta(z) \end{pmatrix}$$

since $\varepsilon \in \mathcal{S}$, the matrices

$$F = \ln - \varepsilon^*\varepsilon, \quad F' = \ln - \varepsilon\varepsilon^*,$$

are nonnegative, and there exist unitary matrices $V_1$ and $V_2$ diagonalizing them. If we denote by $r$ the dimension of the range of $F$, and introduce the matrix

$$\varepsilon' \overset{\text{def}}{=} V_2 \in V_1^{-1},$$

(3.2)
we know (cf. 4, Theorem III.1]) that $\varepsilon'$ may be written in a diagonal form, i.e.,
\[\varepsilon' = \text{diag}(\varepsilon_1, \varepsilon_2),\]
with blocks $\varepsilon_1$ of order $r$ and $\varepsilon_2$ of order $n-r$ satisfying $\|\varepsilon_1\| < 1$ and $\|\varepsilon_2\| = 1$. Note that
\[F_1 = I_r - \varepsilon_1^* \varepsilon_1,\]
\[F'_1 = I_r - \varepsilon_1 \varepsilon_1^*,\]
are positive definite matrices.

Consider now the linear fractional transformation
\[\varepsilon' = [\alpha' \varepsilon_0 + \beta' \varepsilon_0 + \delta']^{-1}\]
specified by
\[\varepsilon_0 = \text{diag}(O_r, I_{n-r})\]
and the $j$-unitary matrix
\[U_\varepsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\]
where
\[\alpha = \text{diag}(F_1^{-1/2}, \varepsilon_2), \quad \beta = \text{diag}(F_1^{-1/2}, O_{n-r}),\]
\[\gamma = \text{diag}(\varepsilon_1^* F_1^{-1/2}, O_{n-r}), \quad \delta = \text{diag}(F_1^{-1/2}, I_{n-r}).\]
Using this construction we conclude that
\[\varepsilon = [\alpha \varepsilon_0 + \beta \varepsilon_0 + \delta]^{-1},\]
where
\[U_\varepsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \text{diag}(V_2^{-1}, V_1^{-1}) \times U_\varepsilon,\]
is $j$-unitary. This is due to (3.2) and the fact that $V_1$ and $V_2$ are unitary. Substituting (3.6) into (3.1), the resultant expression for $S(z)$ is
\[S(z) = [\alpha_0(z) \varepsilon_0 + \beta_0(z)] [\gamma_0(z) \varepsilon_0 + \delta_0(z)]^{-1} \quad (z \in D)\]
with a matrix of coefficients
\[A_0(z) = A(z) \times U_\varepsilon.\]
Using (1.2) together with (3.7) we obtain a linear fractional transformation of

\[ S'(\xi) = \left[ \alpha'(\xi) e_0 + \beta'(\xi) \right] \left[ \gamma'(\xi) e_0 + \delta'(\xi) \right]^{-1} \text{ a.e.,} \]  

\[ (3.10) \]

with a matrix of coefficients, expressed in terms of \( A_0, A'(\xi) = \text{diag}(U_2(\xi), U_1(\xi)) A_0(\xi). \) \( A'(\xi) \) is \( j \)-unitary a.e. because the unitarity of \( U_1(\xi) \) and \( U_2(\xi) \), and the \( j \)-unitarity of \( A_0(\xi). \) For a matrix-valued function \( S(z) \in \mathcal{H}, \) we define [4]

\[ N_s = \{ h(\xi) \in L^2_+(C^n), \quad T(\xi) h(\xi) = h(\xi) \text{ a.e.} \}, \]  

\[ (3.11) \]

where \( T(\xi) = S^*(\xi) S(\xi), \) we have proved in [4] that \( N_s \) is a closed linear manifold and, consequently, that

\[ L^2_+(C^n) = N_s \oplus N_{s \perp}, \]

where the symbol \( \perp \) denotes orthogonal complement. For each value of \( \xi \) where \( T(\xi) \) is denoted we set [4],

\[ N_\xi = \{ h \in \mathbb{C}^n, \quad T(\xi) h = h \}, \]

\[ N'_\xi = \{ h' \in \mathbb{C}^n, \quad T'(\xi) h' = h' \}, \]

where \( T'(\xi) = S(\xi) S^*(\xi) \) a.e. These relations imply

\[ C^n = N_\xi \oplus N_{\xi \perp}, \]

\[ C^n = N'_\xi \oplus N'_{\xi \perp}, \]

and we known from [4] that the subspaces \( N_\xi \) have the same dimension for almost every \( \xi \) in the unit circle.

Let us consider a vector-valued function \( h(\xi) \in N_s. \) From definition (3.11) it follows that

\[ [I_n - T(\xi)] h(\xi) = 0 \quad \text{a.e.} \]

This relationship, together with (1.3), may be used to derive

\[ U_1^{-1}(\xi) [I_n - S'^*(\xi) S'(\xi)] U_1(\xi) h(\xi) = 0 \quad \text{a.e.} \]

\[ (3.12) \]

Taking into account the particular way of constructing \( U_1(\xi) \) and the fact that \( h(\xi) \in N_s, \) we conclude that

\[ x(\xi) = U_1(\xi) h(\xi) = (0, \ldots, 0, x_{k+1}(\xi), \ldots, x_n(\xi))^t, \]

\[ (3.13) \]
Since \( U_1(\xi) \) is unitary a.e. from (3.12) it follows that
\[
[I_n - S^{*}(\xi) S'(\xi)] x(\xi) = 0 \quad \text{a.e.}
\]
Recalling that the dimension of the range of \( F \) is \( r \), we suppose without loss of generality, that \( \dim N_\xi = n - k \), and introduce the notation
\[
w'(\xi) \overset{\text{def}}{=} \begin{bmatrix} \gamma'(\xi) \epsilon_0 + \delta'(\xi) \end{bmatrix} \begin{bmatrix} w'_{11}(\xi) & w'_{12}(\xi) \\ w'_{21}(\xi) & w'_{22}(\xi) \end{bmatrix}, \tag{3.14}
\]
where \( w'_{11}(\xi) \) is a block of \( k \times r \) (\( k \) columns and \( r \) rows), \( w'_{12}(\xi) \) of \( (n-k) \times r \), \( w'_{21}(\xi) \) of \( k \times (n-r) \) and \( w'_{22}(\xi) \) of \( (n-k) \times (n-r) \). Therefore, the expression (3.13) may be written using (3.4) in the alternative form
\[
w^{*}(\xi) \text{diag}(I_r, 0_n, I_r) w'(\xi) x(\xi) = 0 \quad \text{a.e.} \tag{3.15}
\]
This relationship, together with (3.13) and (3.14), ensures that \( w'_{12}(\xi) = 0 \) a.e. Using for \( \gamma'(\xi) \) and \( \delta'(\xi) \) a notation consistent with (3.14), it is easy now to express the blocks of \( w'(\xi) \) in terms of the blocks of \( \gamma'(\xi) \) and \( \delta'(\xi) \), i.e.,
\[
\gamma'(\xi) = \begin{bmatrix} \gamma'_{11}(\xi) & \gamma'_{12}(\xi) \\ \gamma'_{21}(\xi) & \gamma'_{22}(\xi) \end{bmatrix}, \quad \delta'(\xi) = \begin{bmatrix} \delta'_{11}(\xi) & \delta'_{12}(\xi) \\ \delta'_{21}(\xi) & \delta'_{22}(\xi) \end{bmatrix},
\]
where the blocks \( \gamma'_{11}(\xi) \) and \( \delta'_{11}(\xi) \) are of \( r \times k \); \( \gamma'_{21}(\xi) \) and \( \delta'_{21}(\xi) \) of \( r \times (n-k) \); \( \gamma'_{12}(\xi) \) and \( \delta'_{12}(\xi) \) of \( (n-r) \times k \); and \( \gamma'_{22}(\xi) \) and \( \delta'_{22}(\xi) \) of \( (n-k) \times (n-r) \); and
\[
w'(\xi) = \begin{bmatrix} \begin{bmatrix} \delta'_{11}^{-1}(\xi) \\ 0 \end{bmatrix} & \delta'_{12}(\xi) \delta'_{11}^{-1}(\xi) + \delta'_{22}(\xi) \\ \begin{bmatrix} \gamma'_{22}(\xi) + \delta'_{22}(\xi) \end{bmatrix} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \delta'_{11}^{-1}(\xi) \\ 0 \end{bmatrix}, \tag{3.16}
\]
On examining the expressions (3.16), (3.15), and (1.4) it is found that
\[
I_k - S^{*}(\xi) S(\xi) = \delta_{11}^{-1}(\xi) \delta_{11}(\xi) \tag{3.17}
\]
we know [5] that there exists a solution \( \theta_0(\xi) \) of the factorization problem (3.17), verifying
\[
\theta_0(\xi) = \lim_{|z| \to 1} \theta_0(z),
\]
where \( \theta_0(z) \) is bounded and holomorphic in \( D \), uniquely defined among the infinite set of solutions by the normalization conditions \( \theta_0(0) > 0 \) and \( \det \theta_0(z) \) is an outer function. We also know [5] that any solution of (3.17) satisfies \( \theta(\xi) = V(\xi) \theta_0(\xi) \) a.e., where \( V(\xi) \) is a unitary matrix valued function a.e. This fact and (3.17) imply
\[
\delta_{11}'(\xi) = \theta^{-1}(\xi) \quad \text{a.e.,} \tag{3.18}
\]
where $\theta(\xi)$ is the boundary value a.e. of a bounded holomorphic matrix-valued function $(z \in D)$ and solution of the factorization problem (3.17).

We introduce now the notation

$$
\alpha'(\xi) = \begin{pmatrix}
\alpha_{11}'(\xi) & \alpha_{12}'(\xi) \\
\alpha_{21}'(\xi) & \alpha_{22}'(\xi)
\end{pmatrix}, \quad \beta'(\xi) = \begin{pmatrix}
\beta_{11}'(\xi) & \beta_{12}'(\xi) \\
\beta_{21}'(\xi) & \beta_{22}'(\xi)
\end{pmatrix},
$$

and consider again a vector-valued function $h(\xi) \in N_s$. We know [4] that for each fixed value of $\xi$ except, possibly, a set of zero measure it holds that

$$S(\xi) h(\xi) = h'(\xi) \in N'_s$$

which may be written in the equivalent form

$$S'(\xi) x(\xi) = \left[ \alpha'(\xi) \xi_0 + \beta'(\xi) \right] w'(\xi) x(\xi) = x'(\xi), \quad (3.19)$$

where $x(\xi)$ is defined by (3.13) and

$$x'(\xi) = U_2(\xi) h'(\xi) = (0, \ldots, 0, x_{k+1}'(\xi), \ldots, x_n'(\xi))'. \quad (3.20)$$

To derive this last expression we use analogous arguments to those we mentioned in obtaining (3.13). A vector-valued function $g(\xi) \in N_{s+1}$ verifies the following expression [4] for almost every fixed $\xi$ on the unit circle

$$S(\xi) g(\xi) = g'(\xi) \in N'_{s+1},$$

which may be written in the alternative form

$$S'(\xi) y(\xi) = y'(\xi), \quad (3.21)$$

where

$$y(\xi) = U_1(\xi) g(\xi) = (y_1(\xi), \ldots, y_k(\xi), 0, \ldots, 0)', \quad (3.22)$$

$$y'(\xi) = U_2(\xi) g'(\xi) = (y_1'(\xi), \ldots, y_k'(\xi), 0, \ldots, 0)'.$$

From (3.19), (3.20), (3.21), and (3.22) we conclude, after a simple calculation, that

$$[\alpha_{22}'(\xi) + \beta_{22}'(\xi)] [\gamma_{22}'(\xi) + \delta_{22}'(\xi)]^{-1} = S_2(\xi) \quad \text{a.e.,} \quad (3.23)$$

$$\beta_{11}'(\xi) \delta_{11}'^{-1}(\xi) = S_1(\xi) \quad \text{a.e.,} \quad (3.24)$$

$$\alpha_{12}'(\xi) + \beta_{12}'(\xi) = 0 \quad \text{a.e.,} \quad (3.25)$$

$$S_2(\xi) \delta_{21}'(\xi) \delta_{11}'^{-1}(\xi) - \beta_{21}'(\xi) \delta_{11}'^{-1}(\xi) = 0 \quad \text{a.e.,}$$

$$\alpha_{21}'(\xi) + \beta_{21}'(\xi) = 0 \quad \text{a.e.,}$$

$$S_2(\xi) \delta_{21}'(\xi) \delta_{11}'^{-1}(\xi) - \beta_{21}'(\xi) \delta_{11}'^{-1}(\xi) = 0 \quad \text{a.e.,}$$
where the last equality may be immediately rewritten as

\[ \beta_{21}(\xi) = S_{21}(\xi) \delta_{21}(\xi) \quad \text{a.e.} \tag{3.26} \]

The fact that \( A'(\xi) \) is \( j \)-unitary a.e. is equivalent to the following system for the blocks of \( A'(\xi) \),

\[
\begin{align*}
\alpha^*(\xi) \alpha'(\xi) - \gamma^*(\xi) \gamma'(\xi) &= I_n \quad \text{a.e.,} \\
\delta^*(\xi) \delta'(\xi) - \beta^*(\xi) \beta'(\xi) &= I_n \quad \text{a.e.,} \\
\alpha^*(\xi) \beta'(\xi) - \gamma^*(\xi) \delta'(\xi) &= 0 \quad \text{a.e.}
\end{align*}
\]

Let us develop these equations using the notation we introduce for the block of \( A'(\xi) \),

\[
\begin{align*}
\alpha'_{11}(\xi) \alpha'_{11}(\xi) + \alpha'_{21}(\xi) \alpha'_{21}(\xi) - \gamma'_{11}(\xi) \gamma'_{11}(\xi) - \gamma'_{21}(\xi) \gamma'_{21}(\xi) &= I_k \quad \text{a.e.,} \\
\alpha'_{22}(\xi) + \alpha'_{21}(\xi) \alpha'_{12}(\xi) - \gamma'_{22}(\xi) \gamma'_{22}(\xi) - \gamma'_{12}(\xi) \gamma'_{12}(\xi) &= I_{n-k} \quad \text{a.e.,} \\
\alpha'_{12}(\xi) + \alpha'_{21}(\xi) \alpha'_{12}(\xi) - \gamma'_{12}(\xi) \gamma'_{12}(\xi) - \gamma'_{22}(\xi) \gamma'_{22}(\xi) &= 0 \quad \text{a.e.}
\end{align*}
\]

From (3.28) and (3.30), we get

\[ \gamma'_{21}(\xi) = S_{21}(\xi) \alpha'_{21}(\xi) \quad \text{a.e.} \tag{3.31} \]

and replacing the above equation, together with (3.24) and (3.26), in (3.29) we find that

\[ \gamma'_{11}(\xi) = S_{11}(\xi) \alpha'_{11}(\xi) \quad \text{a.e.} \tag{3.32} \]
Let us consider now the factorization problem

\[ I_k - S_1(\xi) S_1^*(\xi) = \psi(\xi) \psi^*(\xi) \quad \text{a.e.} \]

From among the infinite set of solutions we uniquely define a function \( \psi_0(\xi) \), which is the boundary value of a bounded and holomorphic matrix-valued function \( \psi_0(z) \) \( (z \in D) \) satisfying the normalization conditions \( \psi_0(0) > 0 \) and \( \det \psi(z) \) is an outer function. Any solution of this problem is obtained by means of

\[ \psi(\xi) = \psi_0(\xi) V_2(\xi), \]

where \( V_2(\xi) \) is an isometric matrix-valued function [5]. Substituting (3.31) and (3.32) into (3.27) we obtain

\[ \alpha_{11}(\xi) = \psi^{-1}*(\xi) \quad \text{a.e.} \]
\[ \gamma_{11}(\xi) = S_1^*(\xi) \psi^{-1}*(\xi) \quad \text{a.e.,} \]

and recalling that the functions \( \theta(\xi) \) and \( \psi(\xi) \) may be expressed in terms of the solutions \( \theta_0(\xi) \) and \( \psi_0(\xi) \), i.e.,

\[ \theta(\xi) = V_1(\xi) \theta_0(\xi), \quad (3.33) \]
\[ \psi(\xi) = \psi_0(\xi) V_2(\xi), \quad (3.34) \]

using (3.18) and (3.24) we arrive at the following results:

\[ \delta_{11}(\xi) = \theta_0^{-1}(\xi) V_1^{-1}(\xi), \]
\[ \beta_{11}(\xi) = S_1(\xi) \theta_0^{-1}(\xi) V_1^{-1}(\xi), \]
\[ \alpha_{11}(\xi) = \psi^{-1}*(\xi) V_2(\xi), \]
\[ \gamma_{11}(\xi) = S_1^*(\xi) \psi^*^{-1}(\xi) V_2(\xi), \]

where \( V_1(\xi) \) and \( V_2(\xi) \) are unitary matrix-valued functions a.e. We know also from (3.16) that

\[ \gamma'_{12}(\xi) = -\delta_{12}(\xi) \quad \text{a.e.} \]

Finally, by virtue of the above results, we get the following expression for the matrix-valued function \( A'(\xi) \),

\[
A'(\xi) = \begin{pmatrix}
\psi_0^{-1}(\xi) V_2(\xi) & \alpha_{12}(\xi) & S_1(\xi) \theta_0^{-1}(\xi) V_1^{-1}(\xi) & -\alpha'_{12}(\xi) \\
\alpha'_{21}(\xi) & \alpha'_{22}(\xi) & S_2(\xi) \delta'_{21}(\xi) & \beta_{22}(\xi) \\
S_1^*(\xi) \psi_0^{-1}(\xi) V_2(\xi) & \gamma_{12}(\xi) & \theta_0^{-1}(\xi) V_1^{-1}(\xi) & -\gamma'_{12}(\xi) \\
S_1^*(\xi) \alpha_{21}(\xi) & \gamma_{22}(\xi) & \delta'_{21}(\xi) & \delta'_{22}(\xi)
\end{pmatrix}
\]
and, recalling that $S(z) \in \mathcal{F}$, 

$$
A'(z) = 
\begin{pmatrix}
\psi_{0}^{-1}(1/z) V_{2}(z) & \alpha_{12}(z) & S_{1}(z) \theta_{0}^{-1}(z) V_{1}^{-1}(z) & -\alpha_{12}'(z) \\
\alpha_{21}'(z) & \alpha_{22}(z) & \delta_{21}(z) & \beta_{22}'(z) \\
S_{1}^{*}(1/z) \psi_{0}^{*}^{-1}(1/z) V_{2}(z) & \gamma_{12}'(z) & \theta_{0}^{-1}(z) V_{1}^{-1}(z) & -\gamma_{12}'(z) \\
S_{2}^{*}(1/z) \alpha_{21}'(z) & \gamma_{22}(z) & \delta_{21}(z) & \delta_{22}'(z)
\end{pmatrix}
$$

($z \in D$). \ (3.35)

The relationships (3.8) and (3.9) allows us to conclude that the matrix of coefficients of any Darlington realization of a matrix-valued function $S(z) \in \mathcal{F}$, may be written in the form

$$
A(z) = \text{diag}(U_{1}^{-1}(z), U_{2}^{-1}(z)) A'(z) U_{1}^{-1}, \quad (3.36)
$$

where $A'(z)$ is specified by (3.3) and $U_{1}$ by (3.5); and verify the hypothesis of the Basic Lemma [3]. The formula (3.36) is a solution to the problem of multiplicity of Darlington realizations. Varying $V_{1}(\xi)$ and $V_{2}(\xi)$ \ [3] we obtain all the possible realizations of $S(z)$.

To complete this development it is convenient to show that (3.36) contains, as a particular case, formula (4.14) obtained by Arov in [3]. When the condition $I - T(\xi) > 0$ a.e. is satisfied, dim $N_{s} = 0$ and (3.35) is reduced to

$$
A'(z) = \begin{pmatrix}
\psi_{0}^{*}^{-1}(1/z) & S_{1}(z) \theta_{0}^{-1}(z) & V_{2}(z) & 0 \\
S_{1}^{*}(1/z) \psi_{0}^{*}^{-1}(1/z) & \theta_{0}^{-1}(z) & 0 & V_{1}^{-1}(z)
\end{pmatrix}
$$

and, in addition, it holds that

$$
S(\xi) = U_{2}^{-1}(\xi) S_{1}(\xi) U_{1}(\xi) \quad \text{a.e.}
$$

The above relationship, together with

$$
I_{n} - T(\xi) = U_{1}^{-1}(\xi) \theta_{0}^{*}(\xi) \theta_{0}(\xi) U_{1}(\xi) \quad \text{a.e.,}
$$

$$
I_{n} - T'(\xi) = U_{2}^{-1} \psi_{0}(\xi) \psi_{0}^{*}(\xi) U_{2}(\xi) \quad \text{a.e.}
$$

Introducing the notation

$$
P(\xi) \overset{\text{def}}{=} \theta_{0}(\xi) U_{1}(\xi),
$$

$$
\Omega(\xi) \overset{\text{def}}{=} U_{2}^{-1}(\xi) \psi_{0}(\xi),
$$

and further
we conclude that

\[
A(z) = \begin{pmatrix} Q^{-1}(1/\bar{z}) & S(z) & P^{-1}(z) & V_2(z) & 0 \\ S^*(1/\bar{z}) & \Omega^{-1}(1/\bar{z}) & P^{-1}(z) & 0 & V_1^{-1}(z) \end{pmatrix} U_e^{-1}
\]

which coincides with formula (4.14) from [3] obtained by Arov as a description of the set of realizations of \( S(z) \) when \( I_n - T(\xi) > 0 \) a.e.

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