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Factorization of Reciprocal Hilbert Port Operators*

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Factorization of J-biexpansive operator-valued functions on an infinite-dimensional Hilbert space is performed, to extract terms with an arbitrary number of poles of first order on the left and right half planes. Special emphasis is given to the case of reciprocal poles, which has an application to the synthesis of reciprocal passive linear systems as an extension of the classical synthesis of reciprocal n-ports by factor decomposition.

J-expansive operators play an important role in modern network theory [2, 12, 14]. Scattering, chain and transfer matrices of electrical passive n-ports are J-expansive matrix functions in the right half plane, $\operatorname{Re} p > 0$. Decomposition of these matrices into products of J-expansive matrices with a simpler structure, called elementary operators, can be applied to circuit synthesis (see, e.g., [1, 3, 8, 9, 11, 15]). The only singularities of these elementary operators are simple poles, originally belonging to the given J-expansive operator (or its inverse, eventually), and transfered to the elementary factor by the extraction method.

The idea of an electrical network has a natural extension to that of a Hilbert port, which is a linear system whose input and output signals belong to an infinite-dimensional Hilbert space [16]. This concept has a wide variety of applications, including scattering theory and non relativistic quantum mechanics [10, 13]. A passive Hilbert port, in particular, is a system described by an operator J-expansive in the right half plane, together with its adjoint. A large number of physical systems that can be described by means of a J-expansive operator function possess two additional properties: reality and reciprocity. Reality originates from the fact that the output signals must be real for all real inputs. On the other

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hand, reciprocity is connected to the reversibility exhibited by a wide class of physical systems. The mathematical definitions of these properties are given in the next section.

The present paper describes factorizations of passive Hilbert port operators with special emphasis on reciprocal systems. These factorizations generalize results obtained by Efimov and Potapov [3], Ginzburg [4], and this author [6] concerning extraction of poles located in $\operatorname{Re} p \neq 0$ and, in particular, can be applied to decompose a reciprocal operator into a product of elementary reciprocal operators, preserving the reality condition if desired. The problem of factorizing passive Hilbert ports with poles on the imaginary axis requires additional properties to be imposed on the given operator and has been studied in [7].

Let $\mathscr{H}$ be a Hilbert space, P_+ a projector in $\mathscr{H}$, $P_- = I - P_+$, where I denotes the identity operator. Let $J = P_+ - P_-$. A linear bounded operator U is *J-unitary* iff $U^*JU = J$, $UJU^* = J$, where U^* denotes the adjoint of U. A linear bounded operator S is *J-expansive* iff $S^*JS \geqslant J$. It is *J-biexpansive* if both S and S^* are J-expansive. The symbol $\mathscr{H}^m$ denotes the Cartesian product Hilbert space $\mathscr{H}^m = \mathscr{H} \times \mathscr{H} \times \cdots \times \mathscr{H}$ (m times). Given $x = \langle x_1, x_2,..., x_m \rangle$ and $y = \langle y_1, y_2,..., y_m \rangle \in \mathscr{H}^m$, the inner product in $\mathscr{H}^m$ is $(x, y)_m = (x_1, y_1) + (x_2, y_2) + \cdots + (x_m, y_m)$, where $(\cdot, \cdot)$ denotes the inner product in $\mathscr{H}$.

Let S_J be the class of operators $S(p)$ holomorphic in the open right half plane ($\operatorname{Re} p > 0$), except for a set of isolated points, that are equal to a J-biexpansive operator at each point of holomorphism. Let M_J be the class of operators $S(p) \in S_J$ such that $S^{-1}(p) \in S_{-J}$. This M_J is the class of J-biexpansive operators *meromorphic* in the right half plane. The operator $E(p) \in S_J$ is an *elementary* operator iff it is J-unitary on the imaginary axis and has poles of first order only.

We extend the definition of $S(p)$ to the left half plane taking

$$S^{-1}(p) = JS^*(-\bar{p})\, J, \qquad \operatorname{Re} p < 0. \tag{1}$$

Consequently, $S(p) \in M_J$ is $-J$-biexpansive in $\operatorname{Re} p < 0$. Moreover, if $S(p)$ is holomorphic in a domain symmetric with respect to the imaginary axis and takes J-unitary values on a segment of the imaginary axis contained in this domain, then this extension coincides with the analytic continuation of $S(p)$ into the left half plane.

If a conjugation is defined in $\mathscr{H}$, an operator $S(p)$ is *real* iff $S(\bar{p}) = \overline{S(p)}$. The operator $R(p)$ is *reciprocal* iff $R(-\bar{p}) = J'\overline{R(p)}J'$, where $J' = J'^*$, $J'^2 = I$, $JJ' = -J'J$, J and J' real operators. Given that $R(p)$ has a pole at $p = p_1$, then reciprocity implies that it has a pole of the same order at $p = -\bar{p}_1$. Henceforth, p_1 and $-\bar{p}_1$ will be called *reciprocal poles*.

We shall make extensive use of the following:

Fundamental Inequality.

Let the operator $S(p) \in S_J$ have poles at the points p_k, $\operatorname{Re} p_k = \sigma_k \neq 0$ $(k = 1, 2, ..., m)$ with a Laurent expansion

$$S(p) = (p - p_k)^{-n_k} C_k + (p - p_k)^{-n_k + 1} B_k + \cdots .$$

Then

$$\begin{bmatrix} \frac{S(\lambda_1) JS^*(\lambda_1) - J}{\lambda_1 + \bar{\lambda}_1} & \cdots & \frac{S(\lambda_1) JS^*(\lambda_m) - J}{\lambda_1 + \bar{\lambda}_m} & \frac{S(\lambda_1) - S(p)}{\lambda_1 - p} \\ \vdots & & & \\ \frac{S(\lambda_m) JS^*(\lambda_1) - J}{\lambda_m + \bar{\lambda}_1} & \cdots & \frac{S(\lambda_m) JS^*(\lambda_m) - J}{\lambda_m + \bar{\lambda}_m} & \frac{S(\lambda_m) - S(p)}{\lambda_m - p} \\ \frac{S^*(\lambda_1) - S^*(p)}{\bar{\lambda}_1 - \bar{p}} & \cdots & \frac{S^*(\lambda_m) - S^*(p)}{\bar{\lambda}_m - \bar{p}} & \frac{S^*(p) JS(p) - J}{p + \bar{p}} \end{bmatrix} \geqslant 0. \tag{2}$$

A proof of this assertion is given in [5]. It is a generalization of a similar inequality for $m = 1$ due to Ginzburg [4].

We shall now proceed to prove two lemmas stating properties of Laurent coefficients of poles of operators $S(p) \in S_J$ with special emphasis on reciprocal poles.

LEMMA 1. *Let the operator $S(p) \in S_J$ have poles at the points $p_1, p_2 = -\bar{p}_1$ $(\operatorname{Re} p_1 = \sigma_1 \neq 0)$ and let $S(p) = (p - p_j)^{-n_j} C_j + (p - p_j)^{-n_j + 1} B_j + \cdots$ $(j = 1, 2)$ be the Laurent expansion in the neighborhood of these points. Then, the following properties hold.*

(i) $C_2 J C_1^* = 0$;

(ii) $C_2 J B_1^* = B_2 J C_1^*$;

(iii)

$$\begin{pmatrix} C_1 J C_1^* / 2\sigma_1 & C_1 J B_2^* & C_1 / (p_1 - p) \\ B_2 J C_1^* & C_2 J C_2^* / 2\sigma_2 & C_2 / (p_2 - p) \\ C_1^* / (\bar{p}_1 - \bar{p}) & C_2^* / (\bar{p}_2 - \bar{p}) & (S^*(p) JS(p) - J) / (p + \bar{p}) \end{pmatrix} \geqslant 0.$$

Proof. In the neighborhood of p_1 we have

$$S(p) = (p - p_1)^{-n_1} C_1 + (p - p_1)^{-n_1 + 1} B_1 + \cdots .$$

From (1), we see that

$$S^{-1}(p) = (-1)^{n_1} (p - p_2)^{-n_1} J C_1^* J + (-1)^{n_1 - 1} (p - p_2)^{-n_1 + 1} J B_1^* J + \cdots ,$$

in the neighborhood of p_2. Therefore, the identity $S(p)S^{-1}(p)=I$ implies that (i) and (ii) are valid.

Inequality (iii) is derived from (2) for $m=2$ by multiplying the matrix on the left by

$$N=\operatorname{diag}\{(\lambda_1-p_1)^{n_1}, (\lambda_2-p_2)^{n_2}\},$$

and on the right by N^* and taking limits for $\lambda_j \to p_j$ ($j=1, 2$).

An extension of inequality (iii) for m poles can be derived with the help of the following operators on $\mathscr{H}^m$, defined in terms of $m \times m$ matrices of operators on $\mathscr{H}$.

$$T=(T_{jk}) \qquad (j, k=1, 2,..., m),$$

$$T_{jk}=\begin{cases} C_j J/(p_j+\bar{p}_k) & \text{when } p_j \neq -\bar{p}_k, \\ B_j J & \text{when } p_j = -\bar{p}_k, \end{cases}$$

$$Z=\operatorname{diag}\{C_1, C_2,..., C_m\},$$

$$A=TZ^*,$$

$$L(p)=\operatorname{diag}\{(p_1-p)^{-1}I, (p_2-p)^{-1}I,..., (p_m-p)^{-1}I\},$$

$$W(p)=(S^*(p)JS(p)-J)/(p+\bar{p}).$$

LEMMA 2. *If the operator* $S(p) \in S_J$ *has poles at the points* p_j, $\operatorname{Re} p_j=\sigma_j \neq 0$ ($j=1, 2,..., m$) *and its Laurent expansion in the neighborhood of these points is* $S(p)=(p-p_j)^{-n_j}C_j+(p-p_j)^{-n_j+1}B_j+\cdots$, *then given* $f \in \mathscr{H}^m$ *and* $h=\langle h_1, h_1,..., h_1\rangle \in \mathscr{H}^m$, *the following inequality holds*:

$$(Af, f)_m (W(p)h_1, h_1) \geqslant |(L^*(p)Z^*f, h)_m|^2. \tag{3}$$

Taking the proof of Lemma 1 into account, this inequality is a direct consequence of (2).

The possibility of extracting elementary factors with an arbitrary number of poles (including pairs of reciprocal poles if desired), from a given operator $S(p) \in M_J$ is based on the following two theorems.

THEOREM 1. *Let* T, Z, *and* A *be the bounded linear operators defined previously and let* $\mathscr{L}=\ker Z+\operatorname{rge} T^*$. *Then,* $\mathscr{L}$ *is dense in* $\mathscr{H}^m$.

Proof. To simplify notation, we shall take p_1 and $p_2=-\bar{p}_1$ as the only pair of reciprocal poles. An extension of the proof to the case of several pairs of reciprocal poles is obvious. Let $u \perp \mathscr{L}$, $u=\langle u_1, u_2,..., u_m\rangle \in \mathscr{H}^m$.

Then $u \in \ker T \cap \operatorname{cl} \operatorname{rge} Z^*$ (where cl rge denotes the closure of the range). Thus,

$$C_j J \sum_{k \neq n} (p_j + \bar{p}_k)^{-1} u_k + B_j J u_n = 0 \qquad (j, n) = (1, 2) \text{ or } (2, 1), \tag{4}$$

and

$$C_j J \sum_k (p_j + \bar{p}_k)^{-1} u_k = 0 \qquad j \neq 1, 2. \tag{5}$$

Given that $u \in \operatorname{cl} \operatorname{rge} Z^*$, there exists a sequence $\{x_n\}$, $x_n = \langle x_{n1}, x_{n2}, ..., x_{nm} \rangle$, such that $Z^* x_n \to u$. Since $A \geqslant 0$ by Lemma 2, we have

$$\begin{aligned}(Ax_n, x_n)_m = &\sum_{j \neq 1,2} \sum_k (C_j J C_k^* x_{nk}, x_{nj})/(p_j + \bar{p}_k) \\ &+ \sum_{k \neq 2} (C_1 J C_k^* x_{nk}, x_{n1})/(p_1 + \bar{p}_k) + (B_1 J C_2^* x_{n2}, x_{n1}) \\ &+ \sum_{k \neq 1} (C_2 J C_k^* x_{nk}, x_{n2})/(p_2 + \bar{p}_k) + (B_2 J C_1^* x_{n1}, x_{n2}) \geqslant 0. \end{aligned} \tag{6}$$

Taking limits for $C_k^* x_{nk} \to u_k$, $k \neq 1$, and using (4) we get

$$\begin{aligned}&\sum_{j \neq 1,2} \sum_k (Ju_k, u_j)/(p_j + \bar{p}_k) + \sum_{k \neq 1,2} (Ju_k, C_1^* x_{n1})/(p_1 + \bar{p}_k) \\ &\quad + (JC_1^* x_{n1}, C_1^* x_{n1})/(p_1 + \bar{p}_1) + \sum_{k \neq 1} (Ju_k, u_2)/(p_2 + \bar{p}_k) \\ &\quad - \sum_{k \neq 2} \{(Ju_k, C_1^* x_{n1})/(p_1 + \bar{p}_k) + (C_1^* x_{n1}, Ju_k)/(\bar{p}_1 + p_k)\} \geqslant 0.\end{aligned}$$

Now, from (5) we obtain

$$\sum_k (Ju_k, u_j)/(p_j + \bar{p}_k) = 0, \qquad j \neq 1, 2.$$

Therefore, for $C_1^* x_{n1} \to u_1$, we have

$$\sum_{k \neq 1} (Ju_k, u_2)/(p_2 + \bar{p}_k) - \sum_{k \neq 2} (Ju_1, u_k)/(p_k + \bar{p}_1) \geqslant 0.$$

Starting again from (6) and taking limits for $C_k^* x_{nk} \to u_k$, $k \neq 2$, using (4) and then letting $C_2^* x_{n2} \to u_2$, we obtain the opposite inequality. This implies that in both cases $(Ax_n, x_n)_m \to 0$, so that, by Lemma 2, we have $(u, L(p)h)_m = 0$, with $h = \langle h_1, h_1, ..., h_1 \rangle$. Since h_1 is arbitrary and the factors $(p - p_k)^{-1}$ are linearly independent, then $u = 0$. Therefore $\mathscr{L}$ is dense in $\mathscr{H}^m$.

In the absence of reciprocal poles, proof of Theorem 1 is straightforward and is a direct generalization of [6, Lemma 2, Part I].

THEOREM 2. *The equation $AX = Z$, where A and Z are the operators defined previously, has a solution in a subspace $\mathscr{L}$ dense in $\mathscr{H}^m$. Furthermore, the operator Z^*X is bounded on $\mathscr{L}$ and can accordingly be extended to all of $\mathscr{H}^m$ as a bounded nonnegative operator.*

Proof. The proof of the boundedness of Z^*X will be carried out by induction.

For $m = 1$, $A = C_1JC_1^*$ and the equation is $C_1JC_1^*X = C_1$. The operator X is defined on $\mathscr{L} = \ker C_1 + \operatorname{rge} JC_1^*$, which is dense in $\mathscr{H}$ according to Theorem 1. Therefore $Xy = (C_1JC_1^*)^{[-1]}C_1y$, $y \in \mathscr{L}$, where $A^{[-1]}$ denotes the inverse of the operator defined by the hermitian operator A on its range. From (3) we know that

$$(C_1JC_1^*f, f)\,(W(p)h_1, h_1) \geqslant |p_1 - p|^{-2}|(C_1^*f, h_1)|^2.$$

Let $f = Xy$, $h_1 = y$, and let p_0 be a point of holomorphism of $W(p)$. Then,

$$(C_1y, Xy)\,\|W(p_0)\|\,(y, y) \geqslant |p_1 - p_0|^{-2}|(C_1^*Xy, y)|^2,$$

so that

$$0 \leqslant (C_1^*Xy, y)/(y, y) \leqslant |p_1 - p_0|^2\|W(p_0)\|.$$

Now, assume that the proposition is valid for $m-1$ poles, that is, that the operator $Z_{m-1}^*X_1 = Z_{m-1}^*A_1^{[-1]}Z_{m-1}$ is defined and bounded on a subspace $\mathscr{L}_{m-1}$. Here A_1 denotes the operator A corresponding to $m-1$ poles and $Z_{m-1} = \operatorname{diag}\{C_1, C_2, ..., C_{m-1}\}$. Since $\mathscr{L}_{m-1}$ is dense in $\mathscr{H}^{m-1}$, by Theorem 1, $Z_{m-1}^*X_1$ can be extended to all of $\mathscr{H}^{m-1}$ as a bounded operator.

The operator A on $\mathscr{H}^m$ will be expressed as $A = \begin{pmatrix} A_1 & A_{12} \\ A_{12}^* & A_2 \end{pmatrix}$, where $A_2 = C_mJC_m^*/2\sigma_m$ and the definition of A_{12} follows directly from that of A, $A_{12}\colon \mathscr{H} \to \mathscr{H}^{m-1}$. The operator equation $AX = Z$ can be written in terms of A_1, A_{12}, and A_2 through the following set of equations:

$$A_1X_{11} + A_{12}X_{21} = Z_{m-1}, \tag{7}$$

$$A_1X_{12} + A_{12}X_{22} = 0, \tag{8}$$

$$A_2X_{22} + A_{12}^*X_{12} = C_m, \tag{9}$$

$$A_2X_{21} + A_{12}^*X_{11} = 0. \tag{10}$$

According to Theorem 1, X is defined on $\mathscr{L} = \ker Z + \operatorname{rge} T^*$ which is dense in $\mathscr{H}^m$. Then, $A_1 X_{12} = -A_{12} X_{22}$ on a subspace $\mathscr{L}'$ dense in $\mathscr{H}$, so that

$$(A_2 - A_{12}^* A_1^{[-1]} A_{12}) X_{22} v = C_m v, \qquad v \in \mathscr{L}'. \tag{11}$$

Since $A = ZT^*$, there exists a bounded operator F such that $A_{12} = Z_{m-1} F$ and therefore, $A_{12}^* A_1^{[-1]} A_{12} = F^* Z_{m-1}^* A_1^{[-1]} Z_{m-1} F$. Thus, the inductive hypothesis implies the boundedness of $A_{12}^* A_1^{[-1]} A_{12}$.

Let us consider inequality (3) with $f = \langle -A_1^{[-1]} Z_{m-1} u_n, y \rangle$, $u_n \in \mathscr{L}_{m-1}$, $y \in \mathscr{H}$. Taking limits for $u_n \to Fy$, we obtain

$$\begin{aligned}((A_2 - A_{12}^* A_1^{[-1]} A_{12}) y, y)\,(W(p) h_1, h_1)\\ \geqslant |(-L_{m-1}^*(p) Z_{m-1}^* A_1^{[-1]} A_{12} y, h)_{m-1} + (\bar{p}_m - \bar{p})^{-1} (C_m^* y, h_1)|^2,\end{aligned}$$

where $L_{m-1}(p) = \operatorname{diag}\{(p_1 - p)^{-1} I, (p_2 - p)^{-1} I, \ldots, (p_{m-1} - p)^{-1} I\}$ and $h = \langle h_1, h_1, \ldots, h_1 \rangle \in \mathscr{H}^{m-1}$. Now, for $y = X_{22} v$, $v \in \mathscr{L}'$ and $h_1 = v$ we have, taking (11) into account,

$$\begin{aligned}&(C_m v, X_{22} v)\,(W(p) v, v)\,|p_m - p|^2 \\ &\quad\geqslant |(\bar{p}_m - \bar{p})\,(-L_{m-1}^*(p) Z_{m-1}^* A_1^{[-1]} A_{12} X_{22} v, h)_{m-1} + (C_m^* X_{22} v, v)|^2.\end{aligned} \tag{12}$$

Consider now a circle Γ: $|p - p_m| = \rho$, such that $S(p)$ is holomorphic on Γ. Let $\sup_{p \in \Gamma} \|W(p)\| = \gamma$. Take $p_a, p_b \in \Gamma$ such that $p_a - p_m = -(p_b - p_m)$. Then from (12) and defining $\phi(p) = (L_{m-1}^*(p) Z_{m-1}^* A_1^{[-1]} A_{12} X_{22} v, h)_{m-1}$, we have

$$\begin{aligned}\rho^2 \gamma(v, v)\,(v, C_m^* X_{22} v) \geqslant (C_m^* X_{22} v, v)^2 + \rho^2 |\phi(p_a)|^2\\ + 2(C_m^* X_{22} v, v)\operatorname{Re}\{(\bar{p}_a - \bar{p}_m)\,\phi(p_a)\} \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\rho^2 \gamma(v, v)\,(v, C_m^* X_{22} v) \geqslant (C_m^* X_{22} v, v)^2 + \rho^2 |\phi(p_b)|^2\\ + 2(C_m^* X_{22} v, v)\operatorname{Re}\{-(\bar{p}_a - \bar{p}_m)\,\phi(p_b)\} \geqslant 0.\end{aligned}$$

Given v, we have the following alternatives:

(i) $\operatorname{Re}\{(\bar{p}_a - \bar{p}_m)\,\phi(p_a)\} \geqslant 0$ and/or $\operatorname{Re}\{-(\bar{p}_a - \bar{p}_m)\,\phi(p_b)\} \geqslant 0$. In this case

$$\rho^2 \gamma(v, v)\,(v, C_m^* X_{22} v) \geqslant (C_m^* X_{22} v, v)^2. \tag{13}$$

(ii) Both $\operatorname{Re}\{(\bar{p}_a - \bar{p}_m)\,\phi(p_a)\} < 0$ and $\operatorname{Re}\{-(\bar{p}_a - \bar{p}_m)\,\phi(p_b)\} < 0$.

Since $\phi(p)$ is a continuous function of p on Γ, there exists $p_0 \in \Gamma$ such that Re $\{(\bar{p}_a - \bar{p}_m)\, \phi(p_0)\} = 0$. This implies that inequality (13) is also valid in this case.

We can write (13) as

$$0 \leqslant (C_m^* X_{22} v, v)/(v, v) \leqslant \rho^2 \gamma. \tag{14}$$

Thus $C_m^* X_{22}$ is bounded on a dense subspace $\mathscr{L}'$ of $\mathscr{H}$.

Consider now the relationship

$$Z_{m-1}^* X_{12} v = -Z_{m-1}^* A_1^{[-1]} A_{12} X_{22} v, \qquad v \in \mathscr{L}', \tag{15}$$

which is a consequence of (8). We will show that $Z_{m-1}^* X_{12}$ is a bounded operator.

We know that, since $A \geqslant 0$, then

$$(A_1 x, x)_{m-1} (A_2 y, y) \geqslant |(A_{12} y, x)_{m-1}|^2. \tag{16}$$

Since $Z_{m-1}^* A_1^{[-1]} Z_{m-1}$ is bounded by hypothesis, let κ be the norm of this operator. Taking $x = A_1^{[-1]} Z_{m-1} u$, $u \in \mathscr{L}_{m-1}$ and $y = X_{22} v$, $v \in \mathscr{L}'$, in (16) and using (14) and (15) we get

$$\begin{aligned}(Z_{m-1}^* A_1^{[-1]} Z_{m-1} u, u)_{m-1} (J C_m^* X_{22} v, C_m^* X_{22} v)/2\sigma_m \\ \geqslant |(A_{12} X_{22} v, A_1^{[-1]} Z_{m-1} u)_{m-1}|^2 = |(Z_{m-1}^* X_{12} v, u)_{m-1}|^2,\end{aligned}$$

and, therefore,

$$\kappa (u, u)_{m-1} \rho^4 \gamma^2 (v, v)/2|\sigma_m| \geqslant |(Z_{m-1}^* X_{12} v, u)_{m-1}|^2,$$

for all $u \in \mathscr{H}^{m-1}$.

Now, let $u = Z_{m-1}^* X_{12} v$, $v \in \mathscr{L}'$. Then,

$$(Z_{m-1}^* X_{12} v, Z_{m-1}^* X_{12} v)_{m-1}/(v, v) \leqslant \kappa \rho^4 \gamma^2 / 2|\sigma_m|.$$

Therefore, $Z_{m-1}^* X_{12}$ is bounded on $\mathscr{L}'$, a dense subspace of $\mathscr{H}$.

The same reasoning can be applied to the rest of the poles, taking $A_2 = C_k J C_k^*/2\sigma_k$, $k = 1, 2, \ldots, m-1$. This leads to the conclusion that $Z^* X$ is a bounded operator on $\mathscr{L}$.

The previous result shows that there exists a bounded operator Y on $\mathscr{H}^m$, which is the extension of $Z^* X$. That is,

$$Yy = Z^* X y, \qquad y \in \mathscr{L}. \tag{17}$$

Since $A \geqslant 0$, then

$$0 \leqslant (AXy, Xy)_m = (Zy, Xy)_m = (y, Z^* X y)_m = (y, Yy)_m.$$

Therefore, $Y \geqslant 0$.
This completes the proof of Theorem 2.

Let $Y=(Y_{jk})$, $j, k=1, 2,..., m$. From the definition (17) of Y we find that $TY=TZ^*X=AX=Z$ on $\mathscr{L}$, so that

$$C_j\delta_{jk}=\sum_n T_{jn}Y_{nk},$$

where δ_{jk} is the Kroenecker delta, and

$$C_j=\sum_{n,k} T_{jn}Y_{nk}. \tag{18}$$

It is not difficult to show (see [6, p. 18] for a similar proof) that

$$\sum_{n,q} Y_{jq}JY_{nk}=(\bar{p}_j+p_k)\,Y_{jk} \qquad (j, k=1, 2,..., m).$$

If we define the operators

$$P_k=J\sum_n Y_{nk} \qquad (k=1, 2,..., m), \tag{19}$$

then, the following identities hold

$$P_j^*JP_k=\sum_{q,n} Y_{jq}JY_{nk}=(\bar{p}_j+p_k)\,Y_{jk}. \tag{20}$$

Therefore, if p_s and p_t are a pair of reciprocal poles (i.e., $p_s=-\bar{p}_t$),

$$P_s^*JP_t=0. \tag{21}$$

A consequence of these identities is the fact that the operator

$$E(p)=I+\sum_k (p-p_k)^{-1}P_k,$$

is J-biexpansive in $\operatorname{Re} p>0$, J-unitary on $\operatorname{Re} p=0$ and $-J$-biexpansive in $\operatorname{Re} p<0$. Thus $E(p)$ is an elementary operator.
Since $E^{-1}(p)=JE^*(-\bar{p})J$, then

$$\begin{aligned} C_jE^{-1}(p)&=C_jJ\left(I-\sum_k (p+\bar{p}_k)^{-1}P_k^*\right)J \\ &=C_j-\sum_{k,n} (p+\bar{p}_k)^{-1}C_jJY_{kn}, \end{aligned}$$

where the sum over k extends over poles $p_k\neq -\bar{p}_j$ only, since $C_jJC_k^*=0$ for reciprocal poles so that $C_jJY_{kn}=0$ for all n.

Thus, using (18),

$$C_j E^{-1}(p_j) = \begin{cases} 0 & \text{if } p_j \neq -\bar{p}_k,\ k = 1, 2, \ldots, m, \\ B_j J P_k^* J & \text{if } p_j = -\bar{p}_k \text{ for some value of } k. \end{cases} \tag{22}$$

With the help of the properties obtained previously the factorization procedure is established in the following

THEOREM 3. *Let the operator* $S(p) \in M_J$ *have poles at the points* p_j ($\operatorname{Re} p_j = \sigma_j \neq 0$) *and a Laurent expansion in the neighborhood of these points of the form*

$$S(p) = (p - p_j)^{-n_j} C_j + (p - p_j)^{-n_j + 1} B_j + \cdots \qquad (j = 1, 2, \ldots, m).$$

If we define the operator $E(p) = I + \sum_j (p - p_j)^{-1} P_j$, *where the* P_j*'s are given by* (19), *then,*

(I) $E(p) \in M_J$ *and is J-unitary on the imaginary axis;*

(II) $S(p) = S_1(p)\, E(p)$, *with* $S_1(p) \in M_J$;

(III) $S_1(p)$ *has a pole of order* $n_j - 1$ *at the point* $p_j (j = 1, 2, \ldots, m)$;

(IV) *If* $p_j = -\bar{p}_k$ *for some* k $(k = 1, 2, \ldots, m)$ *then* $S^{-1}(p)$ *has a pole of order* n_k *at the point* p_j *and* $S_1^{-1}(p)$ *has a pole of order* $n_k - 1$ *at the point* p_j;

(V) *If* $p_j \neq -\bar{p}_k$ $(k = 1, 2, \ldots, m)$ *and* p_j *is a pole of order* $m_j > 0$ *of the operator* $S^{-1}(p)$, *the order of the pole of* $S_1^{-1}(p)$ *at* p_j *is* m_j.

A detailed proof of this theorem will not be carried out: Propositions (I)–(V) are essentially contained in Eqs. (18)–(22) and can be derived, mutatis mutandis, from [6, Theorem 2]; property (IV) is a consequence of the definition of $S(p)$ in $\operatorname{Re} p < 0$ in terms of its values in $\operatorname{Re} p > 0$.

Remark 1. Since $S(p)$ is J-biexpansive, a similar factorization theorem can be established to extract elementary factors on the left, i.e., to obtain a decomposition of the form $S(p) = \tilde{E}(p)\, \tilde{S}_1(p)$.

Remark 2. The operator $S^{-1}(p)$ is $-J$-biexpansive. Therefore, an extraction of its poles can be carried out in the form $S^{-1}(p) = F^{-1}(p)\, S_1^{-1}(p)$, with both factors belonging to M_{-J} and $F^{-1}(p)$ elementary. Equivalently, $S(p) = S_1(p)\, F(p)$, where $F(p) \in M_J$ is elementary and has poles which are reciprocal to the poles of $F^{-1}(p)$,

$$F(p) = J F^{*-1}(-\bar{p}) J.$$

The previous results have an interesting application to the problem of synthesizing *reciprocal* Hilbert ports. This reciprocity condition is satisfied by a large variety of physical systems and it is desirable to preserve it in a product decomposition, i.e., to carry out the factorization in terms of elementary operators that are also reciprocal.

Let $R(p)\in S_J$ be reciprocal with a pole at the point p_1 ($\operatorname{Re} p_1 \neq 0$). Then, in the neighborhood of p_1,

$$R(p)\,(p-p_1)^{-n}C_1+(p-p_1)^{-n+1}B_1+\cdots.$$

The reciprocity property, $R(-\bar{p})=J'\overline{R(p)}J'$ indicates that

$$R(-\bar{p})=(\bar{p}-\bar{p}_1)^{-n}J'\bar{C}_1J'+(\bar{p}-\bar{p}_1)^{-n+1}J'\bar{B}_1J'+\cdots,$$

so that, in the neighborhood of $-\bar{p}_1$,

$$R(p)=(p+\bar{p}_1)^{-n}(-1)^nJ'\bar{C}_1J'+(p+\bar{p}_1)^{-n+1}(-1)^{n-1}J'\bar{B}_1J'+\cdots.$$

Thus, $R(p)$ has a pole of the same order at $p_2=-\bar{p}_1$, with Laurent coefficients $C_2=(-1)^nJ'\bar{C}_1J'$ and $B_2=(-1)^{n-1}J'\bar{B}_1J'$. Also, the validity of (1) and reciprocity imply

$$R^{-1}(p)=JR(-\bar{p})J=JJ'\overline{R(p)}J'J,$$

so that the poles of $R(p)$ and $R^{-1}(p)$ coincide and $R(p)\in M_J$. In this circumstance, the elementary operator $E(p)=I+(p-p_1)^{-1}P_1+(p-p_2)^{-1}P_2$ given by Theorem 3 is reciprocal too. To see this, it suffices to note that the operator Y defined in (17) has the properties $Y_{22}=J'\bar{Y}_{11}J'$ and $Y_{12}=J'\bar{Y}_{21}J'$ when $R(p)$ is reciprocal. Therefore

$$P_2=J(Y_{12}+Y_{22})=JJ'(\bar{Y}_{21}+\bar{Y}_{11})J'=-J'J(\bar{Y}_{21}+\bar{Y}_{11})J'=-J'\bar{P}_1J'.$$

Then,

$$\begin{aligned}E(-\bar{p})&=I-(\bar{p}+p_1)^{-1}P_1-(\bar{p}-\bar{p}_1)^{-1}P_2\\&=I+(\bar{p}+p_1)^{-1}J'\bar{P}_2J'+(\bar{p}-\bar{p}_1)^{-1}J'\bar{P}_1J'\\&=J'\overline{E(p)}J'.\end{aligned}$$

Let $R(p)=R_1(p)\,E(p)$. We have, then,

$$R_1(-\bar{p})=R(-\bar{p})E^{-1}(-\bar{p})=J'\overline{R(p)}J'J'\overline{E^{-1}(p)}J'=J'\overline{R_1(p)}J',$$

so that $R_1(p)$ is reciprocal.

The previous results may be condensed in the following

THEOREM 4. *Let $R(p)\in S_J$ be reciprocal and have a pole at the point p_1 ($\operatorname{Re} p_1\neq 0$). There exists a product decomposition of the form*

$R(p)=R_1(p)E(p)$, $E(p)=I+(p-p_1)^{-1}P_1-(p+\bar{p}_1)^{-1}J'\bar{P}_1J'$, satisfying properties (I)–(IV) *of Theorem* 3 *with $R_1(p)$ and $E(p)$ reciprocal.*

Another important property of physical systems is reality. It is then desirable to study the possibility of extracting elementary factors that are real *and* reciprocal. This is expressed in the following

THEOREM 5. *Let $Q(p)\in S_J$ be reciprocal and real, with a pole at the point p_1* ($\operatorname{Re} p_1=\sigma_1\neq 0$). *There exists a factorization of the form $Q(p)=Q_1(p)E(p)$ satisfying conditions* (I)–(IV) *of Theorem* 3, *with $Q_1(p)$ and $E(p)$ real and reciprocal. If* $\operatorname{Im} p_1\neq 0$, $E(p)=I+(p-p_1)^{-1}P_1+(p-\bar{p}_1)^{-1}\bar{P}_1-(p+\bar{p}_1)^{-1}J'\bar{P}_1J'-(p+p_1)^{-1}J'P_1J'$. If $\operatorname{Im} p_1=0$, $E(p)=I+(p-\sigma_1)^{-1}P_1-(p+\sigma_1)^{-1}J'P_1J'$, *with P_1 real.*

Proof. If $Q(p)$ has a pole of order n at p_1 ($\operatorname{Im} p_1\neq 0$), then it also has poles of the same order at $\bar{p}_1$, $-\bar{p}_1$ and $-p_1$ with Laurent coefficients $\bar{C}_1$, $(-1)^nJ'\bar{C}_1J'$, $(-1)^nJ'C_1J'$ and $\bar{B}_1$, $(-1)^{n-1}J'\bar{B}_1J'$, $(-1)^{n-1}J'B_1J'$ respectively, due to reality and reciprocity. By Theorem 3, it is then possible to extract a factor of the form

$$E(p)=I+(p-p_1)^{-1}P_1+(p-\bar{p}_1)^{-1}P_2+(p+\bar{p}_1)^{-1}P_3+(p+p_1)^{-1}P_4.$$

Let $Y=(Y_{jk})$, $j,k=1,2,3,4$, be the operator defined in (17). The following relationships arise due to reality and reciprocity.

$$Y_{11}=\bar{Y}_{22}=J'\bar{Y}_{33}J'=J'Y_{44}J',$$
$$Y_{12}=\bar{Y}_{21}=J'\bar{Y}_{34}J'=J'Y_{43}J',$$

with similar identities for the other Y_{jk}'s.

Thus we see that

$$P_1=\bar{P}_2=-J'\bar{P}_3J'=-J'P_4J',$$

which establishes the reality and reciprocity of $E(p)$. The reciprocity and reality of $Q_1(p)$ is now straightforward.

If $\operatorname{Im} p_1=0$, then C_1 is real and so $P_1=\bar{P}_1$. Extraction of the pair of poles at σ_1 and $-\sigma_1$ gives the desired result.

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