JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 109, 453–462 (1985)

Factorization of Hilbert Port Operators with Poles on the Imaginary Axis*

GRACIELA GNAVI†

Departamento de Fisica, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, (1428) Buenos Aires, Argentina, and Instituto Argentino de Matemática, CONICET

Submitted by G.-C. Rota

A factorization method is given to extract poles located on the imaginary axis for J-biexpansive meromorphic operator-valued functions acting on an infinite-dimensional Hilbert space. Decomposition of a real operator in terms of real factors, applicable to Hilbert ports, is also described, thus generalizing synthesis techniques originally developed for passive n-ports.

Hilbert port operators are a natural extension to infinite-dimensional Hilbert spaces of the operators that characterize electrical networks. In this general context, Hilbert ports are linear systems whose input and output signals belong to an infinite-dimensional Hilbert space [15]. Hilbert ports appear also in scattering theory [10, p. 53] and in nonrelativistic quantum mechanics [13]. In classical network theory ([2], [12]) scattering, chain and transfer operators corresponding to passive n-ports are meromorphic matrix functions, J-expansive in the right half plane Re $p>0$, which satisfy the reality condition $S(\bar{p})=\overline{S(p)}$ (where $\bar{p}$ is the complex conjugate of p and $\bar{S}$ is the matrix whose elements are the complex conjugates of the elements of S). Consequently, passive Hilbert port operators (together with their adjoints) are J-expansive operators in the right half plane, acting on an infinite-dimensional Hilbert space and satisfying the reality condition $S(\bar{p})=\overline{S(p)}$, where $\bar{S}$ is the operator conjugate to S in a Hilbert space where a conjugation is defined.

Synthesis of passive n-ports by product decomposition of their characteristic matrices has been developed in detail (see, for example, [1, 3, 4, 8, 9, 11, 14]). These methods are based on the possibility of detaching poles

* This research was supported in part by grants from Subsecretaría de Ciencia y Tecnologia (SUBCYT) and Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).

† Member of the CONICET, Argentina.

0022-247X/85 $3.00

from a given J-expansive matrix function. In the infinite-dimensional case, factorization of a pole from a given meromorphic J-biexpansive operator-valued function in Re $p>0$ has been studied by Ju. P. Ginzburg [5] and extended by the author [6] to extract an arbitrary number of poles simultaneously (in terms of real factors if the function is real).

In the present article, conditions are given for the extension of these factorization techniques to extract poles located on the imaginary axis. The decomposition, when applied to a passive Hilbert port operator, is performed using real operators of the same class with poles of first order on the imaginary axis.

Let $\mathscr{H}$ be a Hilbert space, P_+ a projector in $\mathscr{H}$, $P_- = I - P_+$, where I denotes the identity operator and $J = P_+ - P_-$. A linear bounded operator U is *J-unitary* iff $U^*JU = J$ and $UJU^* = J$, where U^* is the adjoint of U. A linear bounded operator Y is *J-expansive* iff $Y^*JY \geqslant J$ and it is *J-biexpansive* iff both Y and Y^* are J-expansive. The Cartesian product Hilbert space $\mathscr{H} \times \mathscr{H}$ is denoted by $\mathscr{H}^2$. Given two element of $\mathscr{H}^2$, $x = \langle x_1, x_2 \rangle$ and $y = \langle y_1, y_2 \rangle$, their inner product in $\mathscr{H}^2$ is $(x, y)_2 = (x_1, y_1) + (x_2, y_2)$, where $(\cdot, \cdot)$ stands for the inner product in $\mathscr{H}$.

Let S_J be the class of operators $S(p)$ holomorphic in the open right half plane (Re $p>0$) except for a set of isolated points and equal to a J-biexpansive operator at each point of holomorphism. The class M_J of *meromorphic* operators in the right half plane is composed of operators $S(p) \in S_J$ such that $S^{-1}(p) \in S_{-J}$. We extend the definition of $S(p)$ to the left half plane (Re $p<0$) in the following way

$$S(p) = JS^{*-1}(-\bar{p})J, \qquad \text{Re } p < 0.$$

The following symmetry principle is derived from the one established by Ju. P. Ginzburg for operator-valued functions that take J-unitary values on an arc of the unit circle: if the operator $A(p)$ is holomorphic in a domain D symmetric with respect to the imaginary axis and takes J-unitary values on a segment of the imaginary axis contained in D, the operator $A^{-1}(p)$ exists and is bounded for $p \in D$ and $A^{-1}(p) = JA^*(-\bar{p})J$, $p \in D$. If p_0 is a pole of order n of $A(p)$, then $A^{-1}(p)$ has a pole of the same order at $-\bar{p}_0$. This principle follows from the fact that the operators $A^*(-\bar{p})\,JA(p)—J$ and $A(p)\,JA^*(-\bar{p}) - J$ are holomorphic in D, equal to zero on a segment contained in D and, therefore, equal to zero for $p \in D$. When $S(p)$ satisfies this principle, its extension into the left half plane coincides with the corresponding analytic continuation of the operator. Note, also, that if $S(p)$ has a pole on the imaginary axis, $S^{-1}(p)$ has a pole of the same order at that point.

The following theorem establishes important properties of operators belonging to the class S_J with poles on the imaginary axis.

THEOREM 1. *Let $S(p) \in S_J$ be holomorphic and J-unitary in two segments of the imaginary axis with the exception of the points $p_1 = i\omega_1$ and $p_2 = i\omega_2$, interior to these segments, where it has poles. Also, let $S(p) = (p - i\omega_k)^{-n_k} C_k + (p - i\omega_k)^{-n_k+1} B_k + \cdots$ be the Laurent expansion in a neighborhood of the poles. Then, the following properties hold.*

(I) $C_k J C_k^* = 0$ $(k = 1, 2)$;

(II) $C_k J B_k^* \geqslant 0$ $(k = 1, 2)$;

(III)

$$\begin{pmatrix} C_1 J B_1^* & C_1 J C_2^* / i(\omega_1 - \omega_2) & C_1/(i\omega_1 - p) \\ C_2 J C_1^* / i(\omega_2 - \omega_1) & C_2 J B_2^* & C_2/(i\omega_2 - p) \\ C_1^*/(-i\omega_1 - \bar{p}) & C_2^*/(-i\omega_2 - \bar{p}) & (S^*(p) J S(p) - J)/(p + \bar{p}) \end{pmatrix} \geqslant 0.$$

Proof. Let $p = i\omega$. Then, in a neighborhood of p_k $(k = 1, 2)$

$$\begin{aligned} S(p) J S^*(p) = J = {} & |\omega - \omega_k|^{-2n_k} C_k J C_k^* \\ & + |\omega - \omega_k|^{-2n_k} i(\omega - \omega_k)(B_k J C_k^* - C_k J B_k^*) \\ & + o(|\omega - \omega_k|^{-2n_k + 1}), \end{aligned}$$

therefore

$$\lim_{p \to i\omega_k} |p - i\omega_k|^{2n_k} (S(p) J S^*(p)) = C_k J C_k^* = 0$$

and

$$\lim_{p \to i\omega_k} |p - i\omega_k|^{2n_k - 1} (S(p) J S^*(p)) = i(B_k J C_k^* - C_k J B_k^*) = 0.$$

On the other hand, if $\operatorname{Re} p > 0$,

$$\begin{aligned} S(p) J S^*(p) - J = {} & |p - i\omega_k|^{-2n_k} (C_k J B_k^* (\bar{p} + i\omega_k) + B_k J C_k^* (p - i\omega_k)) \\ & + o(|p - i\omega_k|^{-2n_k}) \geqslant 0, \end{aligned}$$

which shows that $C_k J B_k^* \geqslant 0$. This completes the proof of properties (I) and (II).

The proof of the third statement is based on the following inequality for operators of the class S_J (see [6, p. 11]).

$$\begin{pmatrix} \dfrac{S(\lambda_1) J S^*(\lambda_1) - J}{\lambda_1 + \bar{\lambda}_1} & \dfrac{S(\lambda_1) J S^*(\lambda_2) - J}{\lambda_1 + \bar{\lambda}_2} & \dfrac{S(\lambda_1) - S(p)}{\lambda_1 - p} \\ \dfrac{S(\lambda_2) J S^*(\lambda_1) - J}{\lambda_2 + \bar{\lambda}_1} & \dfrac{S(\lambda_2) J S^*(\lambda_2) - J}{\lambda_2 + \bar{\lambda}_2} & \dfrac{S(\lambda_2) - S(p)}{\lambda_2 - p} \\ \dfrac{S^*(\lambda_1) - S^*(p)}{\bar{\lambda}_1 - \bar{p}} & \dfrac{S^*(\lambda_2) - S^*(p)}{\bar{\lambda}_2 - \bar{p}} & \dfrac{S^*(p) J S(p) - J}{p + \bar{p}} \end{pmatrix} \geqslant 0,$$

where Re λ_k and Re p are different from zero. Multiplying this inequality on the left by $N = \operatorname{diag}\{(\lambda_1 - i\omega_1)^{n_1} I, (\lambda_2 - i\omega_2)^{n_2} I, I\}$, on the right by N^* and taking limits for Re $\lambda_k \to 0$, the desired expression is obtained.

The matrix inequality of Theorem 1 can be expressed more concisely by defining the operators

$$T = \begin{pmatrix} B_1 J & -i(\omega_1 - \omega_2)^{-1} C_1 J \\ i(\omega_1 - \omega_2)^{-1} C_2 J & B_2 J \end{pmatrix}, \tag{1}$$

$$Z = \operatorname{diag}\{C_1, C_2\}, \tag{2}$$

$$V = TZ^*, \qquad W'(p) = \operatorname{diag}\{W(p)/2, W(p)/2\},$$

$$L(p) = \operatorname{diag}\{(i\omega_1 - p)^{-1} I, (i\omega_2 - p)^{-1} I\}.$$

Since $B_k J C_k^* = C_k J B_k^*$, then $V = TZ^* = ZT^*$. Taking $f = \langle f_1, f_2 \rangle$ and $h = \langle h_1, h_1 \rangle$, then

$$(Vf, f)_2 + (W'(p)h, h)_2 + 2\,\mathrm{Re}(Z^* f, L(p)h)_2 \geq 0. \tag{3}$$

Therefore, $(Vf, f)_2 \geq 0$, $(W'(p)h, h)_2 \geq 0$ and

$$(Vf, f)_2 (W'(p)h, h)_2 \geq |(Z^* f, L(p)h)_2|^2. \tag{4}$$

A modified version of inequality (4) is given in the next lemma.

LEMMA 1. *Let $S(p) \in S_J$ have poles at the points $p_1 = i\omega_1$ and $p_2 = i\omega_2$ of the imaginary axis. Then, for all g, $u \in \mathscr{H}^2$, there exists $a > 0$ such that*

$$a(u, u)_2 (Vg, g)_2 \geq |(Z^* g, u)_2|^2, \tag{5}$$

where a is independent of g and u.

Proof. Use of inequality (3) taking $h_1 = (i\omega_1 - p) u_1 \in \mathscr{H}$, $f = tg$, $g = \langle g_1, g_2 \rangle \in \mathscr{H}^2$, $t \in \mathbf{R}$, gives

$$t^2 (Vg, g)_2 + |i\omega_1 - p|^2 (W(p) u_1, u_1) \\ + 2t\,\mathrm{Re}\{(C_1 u_1, g_1) + (i\omega_1 - p)(i\omega_2 - p)^{-1} (C_2 u_1, g_2)\} \geq 0.$$

Assume that p_0, p_0' and p_0'' are points of holomorphism of $S(p)$ such that $p_1 - p_0 = i(\omega_2 - \omega_1)/2 + \sigma_0$, $p_1 - p_0' = -i(\omega_2 - \omega_1)/2 + \sigma_0'$, $p_2 - p_0'' = -i(\omega_2 - \omega_1)/2 + \sigma_0$, with $\sigma_0 \neq 0$ and $\sigma_0' \neq 0$. It is not difficult to show that σ_0 and σ_0' can always be chosen so that $(p_1 - p_0')/(p_2 - p_0') = -k(p_1 - p_0)/(p_2 - p_0)$, $k > 0$. In this case, it is readily seen that

$$t^2 (Vg, g)_2 + ([|i\omega_1 - p_0|^2 W(p_0) + |i\omega_1 - p_0'|^2 W(p_0')]\, u_1, u_1) \\ - 2t|(C_1 u_1, g_1)| \geq 0.$$

Defining $m = \max(|p_1 - p_0|, |p_1 - p_0'|)$ it follows that

$$t^2(Vg, g)_2 + m^2(\| W(p_0)\| + \| W(p_0')\|)(u_1, u_1) - 2t|(C_1 u_1, g_1)| \geq 0.$$

Taking now $h_1 = (i\omega_2 - p)\, u_2$, a similar argument using p_0' and p_0'' leads to the inequality

$$t^2(Vg, g)_2 + t|(Zu, g)_2| + (a/4)(u, u)_2 \geq 0,$$

where $a = 2m^2\{\| W(p_0)\| + W\|(p_0')\| + \| W(p_0'')\|\}$.

The thesis follows directly from the last inequality.

The following lemma states the conditions required to carry out the desired factorization procedure.

LEMMA 2. *If the linear, bounded operators T and Z defined in* (1) *and* (2) *satisfy the condition* $\ker T \cap \operatorname{cl} \operatorname{rge} Z^* = \mathbf{0}$ (*where* $\mathbf{0}$ *is the null vector*, $\ker T$ *the kernel of T and* $\operatorname{cl} \operatorname{rge} Z^*$ *the closure of the range of Z^**), *the operator equation $ZT^*Q = Z$ has a solution Q on a subspace dense in $\mathscr{H}^2$ and the extension of the operator Z^*Q to all of $\mathscr{H}^2$ is a nonnegative bounded operator.*

Proof. The operator Q is defined on the subspace $\mathscr{L} = \ker Z + \operatorname{rge} T^*$. If $x \perp \mathscr{L}$, then $x \in \ker T \cap \operatorname{cl} \operatorname{rge} Z^*$, so by hypothesis $x = \mathbf{0}$. Thus, $\mathscr{L}$ is dense in $\mathscr{H}$. Taking $g = Qu$, $u \in \mathscr{L}$, in (4), then

$$a(u, u)_2(VQu, Qu)_2 \geq |(Z^*Qu, u)_2|^2.$$

Since $VQ = Z$ and $V \geq 0$,

$$0 \leq (Z^*Qu, u)_2/(u, u)_2 \leq a.$$

Therefore, Z^*Q is bounded and non-negative on $\mathscr{L}$ and it can be extended to all of $\mathscr{H}^2$. If R denotes this extension, then $0 \leq R \leq a$.

Remark. When considering the extraction of a pole located in $\operatorname{Re} p \neq 0$ (see [7]) the condition $\ker T \cap \operatorname{cl} \operatorname{rge} Z^* = \mathbf{0}$ is a consequence of inequality (5), so that $\mathscr{L}$ is always dense in $\mathscr{H}^2$. For poles located on the imaginary axis this is no longer true and so it must be imposed as an additional requirement.

The next step is to derive properties of the operator R which will be used in the construction of the factor that is to be extracted from the operator $S(p)$. Let us define the operators in $\mathscr{H}^2$, $J_2 = \operatorname{diag}\{J, J\}$, $D = \operatorname{diag}\{i\omega_1 I, i\omega_2 I\}$, $G = (G)_{kj} = J$.

For $u \in \mathscr{L}$, $ZGZ^*Qu = (DV - VD)\, Qu$. Noting that Q^* is defined on the range of V, we have

$$Q^*Z(GZ^*Q - D)u = -Q^*VDQu = -Z^*DQu = -DZ^*Qu,$$

and, therefore

$$R(GR-D)=-DR. \tag{6}$$

If we express the operator R in matrix form, $R=(R)_{kj}$, equation $Z=TR$ can be expressed as

$$C_1=B_1J(R_{11}+R_{12})+C_1J(R_{21}+R_{22})/i(\omega_1-\omega_2), \tag{7}$$

$$C_2=B_2J(R_{22}+R_{21})+C_2J(R_{12}+R_{11})/i(\omega_2-\omega_1). \tag{8}$$

Let us define the operators

$$P_j=J(R_{1j}+R_{2j}) \qquad (j=1,2). \tag{9}$$

From these definitions, taking (6) into account, we obtain the identities

$$P_k^*JP_j=(R_{k1}+R_{k2})\,J(R_{1j}+R_{2j})=i(\omega_j-\omega_k)\,R_{kj}, \tag{10}$$

whence we may conclude that

$$P_1^*JP_1=0=P_2^*JP_2. \tag{11}$$

These results are used to establish

THEOREM 2. *The operator* $E(p)=I+(p-i\omega_1)^{-1}P_1+(p-i\omega_2)^{-1}P_2$, *where* P_1 *and* P_2 *are defined in* (9), *is J-biexpansive in* $\operatorname{Re} p>0$ *and J-unitary on* $\operatorname{Re} p=0$.

Proof. If we define $a_k=(p-i\omega_k)^{-1}$ $(k=1,2)$, using (9), (10), and (11), we then get

$$\begin{aligned} E^*(p)\,JE(p)-J &\\ &=(a_1+a_1^*)\,R_{11}+(a_2+a_2^*)\,R_{22}+(a_1+a_2^*+a_1a_2^*i(\omega_1-\omega_2))\,R_{21}\\ &\quad+(a_1^*+a_2+a_1^*a_2i(\omega_2-\omega_1))\,R_{12}\\ &=(p+\bar{p})\,(I\ \ I)\,L^*(p)\,RL(p)\,(I\ \ I)'. \end{aligned} \tag{12}$$

In expression (12), $(I\ \ I)'$ denotes the transpose of the matrix $(I\ \ I)$. Therefore,

$$E^*(p)\,JE(p)-J\geqslant 0, \qquad \text{for} \quad \operatorname{Re} p>0, \tag{13}$$

$$=0, \qquad \text{for} \quad \operatorname{Re} p=0, \tag{14}$$

$$\leqslant 0, \qquad \text{for} \quad \operatorname{Re} p<0. \tag{15}$$

From (9) and (10) it may be easily verified that $JE^*(-\bar{p})\,JE(p)=I$.

On the other hand, the boundedness of P_1 and P_2 indicates that, for p large enough, $E^{-1}(p)$ exists and is equal to $JE^*(\bar{p})J$. This implies that $P_1 JP_1^* = P_2 JP_2^* = 0$ and $P_1 - JP_1^*J = -P_2 + JP_2^*J = (P_2 JP_1^* + P_1 JP_2^*)\, J/i(\omega_1 - \omega_2) = 0$.

From these identities we see that

$$E(p)\, JE^*(p) = J, \qquad \text{for} \quad \operatorname{Re} p = 0. \tag{16}$$

Using (15) together with the symmetry principle, we obtain the inequality

$$J - E(p)\, JE^*(p) \leqslant 0, \qquad \text{for} \quad \operatorname{Re} p > 0. \tag{17}$$

The J-biexpansivity of $E(p)$ in $\operatorname{Re} p > 0$ is stated in (13) and (17) and the J-unitarity on the imaginary axis in (14) and (16).

The main result is the following.

THEOREM 3. *Let* $S(p) \in M_J$ *be holomorphic and J-unitary in two segments of the imaginary axis except at the points* $p_1 = i\omega_1$ *and* $p_2 = i\omega_2$, *interior to these segments, where it has poles. Given the operators T and Z defined in* (1) *and* (2) *in terms of the coefficients of the Laurent expansion in the neighborhood of* p_1 *and* p_2, *let* $\ker T \cap \operatorname{cl} \operatorname{rge} Z^* = \mathbf{0}$. *If* $E(p) = I + (p - i\omega_1)^{-1} P_1 + (p - i\omega_2)^{-1} P_2$, *where* P_1 *and* P_2 *are defined in* (9), *then*

(I) $E(p) \in M_J$ *and* $E(i\omega)$ *is J-unitary*;

(II) $S(p) = S_1(p)\, E(p)$, *where* $S_1(p) \in M_J$;

(III) *If* $S(p)$ *has a pole of order* n_k *at* $p = i\omega_k$ *then* $S_1(p)$ *has a pole of order* $n_k - 1$ *at that point* $(k = 1, 2)$;

(IV) $S_1^{-1}(p)$ *has a pole of the same order as* $S_1(p)$ *at* $p = i\omega_k$ $(k = 1, 2)$.

Proof. Property (I) is established in Theorem 2.

(II) Given $u \in \mathscr{L}$, $h = \langle h_1, h_1 \rangle$ and $f = Qu$, Eq. (4) can be expressed as follows.

$$(Ru, u)_2 (W'(p)h, h)_2 \geqslant |(Ru, L(p)\, h)_2|^2.$$

Since $\mathscr{L}$ is dense in $\mathscr{H}^2$ and the operators are bounded, the preceding inequality is valid for all $u \in \mathscr{H}^2$. Taking $u = L(p)h$, then

$$(W'(p)h, h)_2 \geqslant (L^*(p)\, RL(p)h, h)_2 \geqslant 0.$$

Using (12) and the definition of $W'(p)$ we get, for $\operatorname{Re} p \neq 0$, $(p + \bar{p})^{-1}(S^*(p)\, JS(p) - J) \geqslant (p + \bar{p})^{-1}(E^*(p)\, JE(p) - J) \geqslant 0$.

Now, $S(p)=S_1(p)\,E(p)$, therefore

$$E^*(p)\,S_1^*(p)\,JS_1(p)\,E(p)\geqslant E^*(p)\,JE(p),\qquad \operatorname{Re} p>0,$$

and, since $E(p)$ is invertible,

$$S_1^*(p)\,JS_1(p)\geqslant J,\qquad \operatorname{Re} p>0.$$

The operator $S^{-1}(p)\in S_{-J}$, thus

$$\begin{aligned}S_1(p)\,JS_1^*(p)-J\\ &=S(p)\{E^{-1}(p)\,JE^{-1*}(p)-S^{-1}(p)\,JS^{-1*}(p)\}\,S^*(p)\geqslant 0,\end{aligned}$$

for $\operatorname{Re} p>0$. This shows that $S_1(p)\in M_J$.

(III) Given $u\in\mathscr{L}=\operatorname{rge} T^*+\ker Z$, then $ZJ_2Ru=ZJ_2Z^*Qu=0$, since $C_kJC_k^*=0$ $(k=1,2)$. Therefore

$$C_1JR_{11}=C_1JR_{12}=C_2JR_{21}=C_2JR_{22}=0, \tag{18}$$

and, consequently, using (9) we get

$$C_kE^{-1}(p)=C_k\{I-(p-i\omega_j)^{-1}JP_j^*J\}\qquad (k\neq j).$$

From (7) and (8), for $k\neq j$, we have

$$C_kE^{-1}(i\omega_k)=C_k-C_kJ(R_{j1}+R_{j2})/i(\omega_k-\omega_j)=B_kJP_k^*J. \tag{19}$$

In a neighborhood of $p=i\omega_k$

$$\begin{aligned}S_1(p)&=S(p)\,E^{-1}(p)\\ &=(p-i\omega_k)^{-n_k-1}C_kJP_k^*J+(p-i\omega_k)^{-n_k}\\ &\quad\times\{C_k-B_kJP_k^*J-C_kJP_j^*J/(p-i\omega_j)\}\\ &\quad+o((p-i\omega_k)^{-n_k})\qquad (j\neq k).\end{aligned}$$

Using (7), (8), (18) and (19), we obtain

$$\lim_{p\to i\omega_k}(p-i\omega_k)^{n_k}S_1(p)=0.$$

(IV) This property is a direct consequence of the symmetry principle.

Consider, now, a conjugation operation defined in $\mathscr{H}$ such that $J=\bar{J}$ (here $\bar{A}$ denotes the operator conjugate to A). An operator $S(p)$ is *real* iff $S(\bar{p})=\overline{S(p)}$. If a real operator $S_R(p)$ has a pole of order n at $p_0=i\omega_0$, then it has a pole of the same order at $\bar{p}_0=-i\omega_0$ since in the neighborhood of p_0

$$S_R(p)=(p-i\omega_0)^{-n}C+(p-i\omega_0)^{-n+1}B+\cdots$$

and so

$$S_R(p)=\overline{S_R(\bar{p})}=(p+i\omega_0)^{-n}\bar{C}+(p+i\omega_0)^{-n+1}\bar{B}+\cdots,$$

in the neighborhood of $\bar{p}_0$.

Therefore, under the conditions of validity of Theorem 3, a factor of the form

$$E_R(p)=I+(p-i\omega_0)^{-1}P+(p+i\omega_0)^{-1}\bar{P}$$

can be extracted from $S_R(p)$ giving

$$S_R(p)=S_{R1}(p)\,E_R(p), \qquad \text{with} \quad S_{R1}(p)\in MR_J,$$

where MR_J denotes the class of real meromorphic operators.

This can be readily seen noting that the operators defining the factorization are $C_1=C$, $C_2=\bar{C}$, $B_1=B$ and $B_2=\bar{B}$, so that $R_{22}=\bar{R}_{11}$ and $R_{21}=\bar{R}_{12}$, and also

$$P_2=J(R_{12}+R_{22})=J(\bar{R}_{21}+\bar{R}_{11})=\bar{P}_1.$$

Since Hilbert port operators belong to the class MR_J, this decomposition is achieved in terms of operators which represent Hilbert ports with simpler structure. It can be interpreted as an extension of the classical method of synthesis of passive n-ports with poles on the boundary of the right half plane by factor decomposition (see [3, 8, 14]).

The extraction of a single pole located on the imaginary axis can be treated using a special case of Theorem 1. Taking $f_2=0$ in (4),

$$(C_1JB_1^*f,f)(W(p)g,g)\,|i\omega_1-p|^2\geq|(C_1^*f,g)|^2. \tag{20}$$

From (20), it is straightforward to check the assertions of the next lemma.

LEMMA 3. *The operator equation* $C_1JB_1^*X=C_1$ *has solution* X *on a subspace* $\mathscr{S}$ *dense in* $\mathscr{H}$ *iff* $\ker B_1J\cap \operatorname{cl}\operatorname{rge} C_1^*=\mathbf{0}$. *The operator* C_1^*X *is bounded on* $\mathscr{S}$.

Again, in contrast with the problem of extraction of poles located in $\operatorname{Re} p\neq 0$, inequality (20) does not guarantee that $\ker B_1J\cap\operatorname{cl}\operatorname{rge} C_1^*=\mathbf{0}$, a property which is erroneously derived in [5, Theorem 6].

The factorization theorem for a single pole can be stated as follows.

THEOREM 4. *Let the operator* $S(p)\in M_J$ *be holomorphic and J-unitary on a segment of the imaginary axis with the exception of the interior point* $p_0=i\omega_0$ *in whose neighborhood a Laurent expansion* $S(p)=$

$(p-i\omega_0)^{-n}C+(p-i\omega_0)^{-n+1}B+\cdots$ *is valid, such that* $\ker BJ \cap \operatorname{cl}\operatorname{rge} C^* = \mathbf{0}$. *Given the operator* $E'(p)=I+(p-i\omega_0)^{-1}P'$, *where* P' *is the extension to all of* $\mathscr{H}$ *of the bounded operator* JC^*X *defined on a subspace dense in* $\mathscr{H}$ *and* X *is the solution to the operator equation* $CJB^*X=C$, *the following properties hold.*

(I) $P'J=JP'\geqslant 0$ *and* $P'^2=0$;

(II) $E'(p)\in M_J$ *and* $E'(i\omega)$ *is J-unitary*;

(III) $S(p)=S_1'(p)\,E'(p)$, *where* $S_1'(p)\in M_J$;

(IV) $S_1'(p)$ *and its inverse have a pole of order* $n-1$ *at* $p=i\omega_0$.

A procedure similar to the one developed to extract two poles simultaneously can be used to derive these properties.

References

1. V. Belevitch, Factorization of scattering matrices with applications to passive network synthesis, *Phillips Res. Rep.* **18** (1963), 275–317.
2. V. Belevitch, "Classical Network Theory," Holden–Day, San Francisco, 1968.
3. A. V. Efimov and V. P. Potapov, *J*-expansive matrix functions and their role in the theory of electrical circuits, *Uspekhi Mat. Nauk* **28** (1973), 65–130. [In Russian]
4. A. Fettweiss, Factorization methods for transfer matrices of lossless two-ports, *IEEE Trans. Circuit Theory* **CT-17** (1970), 86–94.
5. Ju. P. Ginzburg, A maximum principle for *J*-contractive operator functions and some of its consequences, *Izv. Vysh. Uchebn. Zaved. Mat.* **32** (1963), 45–53. [In Russian]
6. G. Gnavi, "Factorización de operadores característicos de puertos de Hilbert," Trabajos de Matemática No. 5, pp. 1–47, Instituto Argentino de Matemática, 1975.
7. G. Gnavi, Factorization of *J*-expansive meromorphic operator-valued functions, *Adv. in Appl. Math.* **2** (1981), 13–23.
8. A. González Domínguez, On the factorization of scattering matrices, chain matrices and transfer matrices, *Ann. Mat. Pura Appl.* (IV) **103** (1975), 161–186.
9. I. V. Kovalishina, The multiplicative structure of analytical reactance matrix functions, *Izv. Akad. Nauk Armjan. SSSR* **1** (1966), 138–146. [In Russian]
10. P. D. Lax and R. S. Phillips, "Scattering Theory," Academic Press, New York, 1967.
11. M. S. Livschitz and M. Sh. Flekser, Realization of a reactive fourpole as a chain of simple fourpoles, *Dokl. Akad. Nauk SSSR* **135** (1960), 542–544. [In Russian]
12. R. Saeks, "Generalized Networks," Holt, Rinehart & Winston, New York, 1972.
13. J. S. Toll, Two, three and fourpoint functions in quantum field theory, Paper SMR 2/7 *in* "Proceedings, Seminar on High Energy Physics and Elementary Particles, May–June 1965, Int. Centre for Theoretical Physics," pp. 1–43.
14. D. C. Youla, "Cascade Synthesis of Passive *n*-Ports," Tech. Rept. RADC-TDR-64-332, Rome Air Development Center, Rome, N. Y., 1964.
15. A. H. Zemanian, "Realizability Theory for Continuous Linear Systems," Academic Press, New York/London, 1972.