JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **102**, 365–384 (1984)

A Diffusion Problem with a Measure as Initial Datum

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Submitted by G.-C. Rota

INTRODUCTION

In the present paper we study the equation

$$u_t = D_{xx}(\varphi(u)) \qquad \text{in the sense of distributions,} \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function, set on a bounded interval with boundary conditions of mixed type and a measure, which needs not be nonnegative, as initial datum.

There is considerable literature about Eq. (1) set in $\mathbb{R}$ or in bounded domains. The reason for this is that Eq. (1) is a model for many physical phenomena, for example, diffusion of a gas through a porous medium and heat conduction with or without interfaces (which corresponds to Eq. (1) with φ strictly increasing if there are not interfaces and monotone increasing in the other case). See, for example, [2, 4, 5, 6, 8, 9], and the references they contain.

Among all these papers, only [8] and [9] consider a measure as initial datum. However, Widder in [12] proved, for the linear equation, that for every nonnegative solution u in $(t > 0)$ there is one and only one measure μ such that

$$\int u(x, t)\, g(x)\, dx \to \int g(x)\, d\mu(x) \qquad (t \to 0) \qquad \forall g \in C_0(\mathbb{R}).$$

These results have been generalized by Aronson (see [1]) to the N-dimensional case and a general linear parabolic equation. Pierre (see [9]) has obtained a similar result for nonnegative solutions of

$$u_t = \Delta\varphi(u).$$

This, and the fact that Eq. (1) with a measure as initial datum is also a model for physical phenomena (see [14]), motivate the present paper.

0022-247X/84 $3.00

We prove a result similar to those of Widder and Pierre but without the restriction that the solution be nonnegative (Theorem 2). We prove existence and uniqueness of a strong solution, that is $u_t \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1))$ (Theorem 1).

We also obtain two comparison theorems. One of them compares the solutions pointwise (Theorem 4) and the other one compares the distribution functions

$$v(x, t) = \int_0^x u(s, t)\, ds$$

(Theorem 3).

The fundamental idea is that the distribution function $v(x, t)$ also satisfies a differential equation and that we can obtain the estimate

$$\text{Total variation of } v(x, t) \text{ on } [0, 1] = V_0^1 v(x, t) \leqslant \int_0^1 d\,|\mu| \qquad \forall t > 0.$$

The existence of a strong solution seems to be new because the only works we know which deal with measures as initial datum ([8, 9]), prove the existence of weak solutions and only for nonnegative measures. As we deal with finite, arbitrary measures, the comparison Theorems 3 and 4 also seem to be new.

Notation

We will denote by u_t or $D_t u$ the partial derivative of the function u with respect to the variable t. Analogously for the other derivatives. And

$$V_0^1 v(x, t)$$

will denote the total variation of $v(x, t)$, as a function of x, on the interval $[0, 1]$.

The results stated above are a consequence of the following theorem proved in a previous paper (see [13]), which establishes

THEOREM 0. *Let* φ, $\psi : \mathbb{R} \to \mathbb{R}$ *be strictly increasing continuous functions, such that* $\psi^{-1} : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous and* φ *satisfies*

$$\exists c > 0 \qquad \textit{such that} \quad |\varphi(x)| \geqslant c\,|x| \quad \textit{when} \quad |x| \to \infty.$$

Suppose $\varphi(0) = \psi(0) = 0$. *Then for every* $F \in L^1(0, 1)$ *there exists one and only one function* $v \in C([0, T]; L^1(0, 1))$ *which satisfies*

(a) $v_t \in L^2_{\text{Loc}}(0, T; L^2(0, 1))$;

(b) $\psi(v) \in H^1(0, 1)$ *in* x *a.e.* t *and* $\psi(v)(0, t) = 0$ *a.e.* t;

(c) $\varphi(D_x(\psi(v))) \in H^1(0, 1)$ *in* x *a.e.* t *and* $\varphi(D_x(\psi(v)))(1, t) = 0$ a.e. t (*in particular* $\varphi(D_x(\psi(v))) \in C([0, 1])$ *in* x *a.e.* t *and therefore* $D_x(\psi(v)) \in C([0, 1])$ *in* x *a.e.* t *and* $D_x(\psi(v))(1, t) = 0$ *a.e.* t);

(d) $\lim_{t\to 0} \int_0^1 |v(x, t) - F(x)|\, dx = 0$;

(e) $v_t = D_x(\varphi(D_x(\psi(v))))$ *a.e.* $(x, t) \in (0, 1) \times (0, T)$.

We will make use of Theorem 0 only when $\psi =$ identity and $F \in L^\infty \subset L^2$. In this case, Theorem 0 is a consequence of the theory of subdifferentials in L^2 (see [16]).

We prove the following theorem.

THEOREM 1. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a strictly increasing continuous function such that* $\varphi^{-1} : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous,* $\varphi(0) = 0$.

Let μ *be a finite Borel measure on* $[0, 1)$. *Then there exists one and only one function* $u \in L^\infty(0, T; L^1(0, 1))$ *which satisfies*

$$\begin{aligned}
&\text{(a)} \quad u_t \in L^2_{\text{Loc}}(0, T; L^2(0, 1));\\
&\text{(b)} \quad \varphi(u) \in C^1([0, 1]) \text{ in } x \text{ a.e. } t \text{ and } D_x(\varphi(u))(0, t) = 0 \text{ a.e. } t;\\
&\text{(c)} \quad u(1, t) = 0 \text{ a.e. } t;\\
&\text{(d)} \quad u_t = D_{xx}(\varphi(u)) \text{ a.e. } (x, t) \in (0, 1) \times (0, T);\\
&\text{(e)} \quad \lim_{t\to 0} \int_0^x u(s, t)\, ds = \mu([0, x)) \text{ for every } x \in (0, 1) \text{ such that } \mu(\{x\}) = 0.
\end{aligned} \tag{1.1}$$

Proof. (1) *Existence.* Let $v(x, t)$ be the solution of the problem

$$\begin{aligned}
&\text{(a)} \quad v_t = D_x(\varphi(v_x)) \text{ a.e. } (x, t);\\
&\text{(b)} \quad v(0, t) = 0 \text{ a.e. } t;\\
&\text{(c)} \quad v_x(1, t) = 0 \text{ a.e. } t;\\
&\text{(d)} \quad \lim_{t\to 0} \int_0^1 |v(x, t) - F(x)|\, dx = 0,
\end{aligned} \tag{1.2}$$

where $F(x)$ is the distribution function of the measure μ (that is, $F(x) = \mu([0, x))$). We know that $F(0) = 0$, F is left-continuous and of bounded variation; this implies that $F \in L^\infty \subset L^1$ and therefore there is one and only one solution of (1.2) given by Theorem 0.

Let $u(x, t) = v_x(x, t)$. By (1.2c), we have $u(1, t) = 0$ a.e. t, therefore u

satisfies (1.1c). Let us see that u satisfies (1.1a). As φ^{-1} is a strictly increasing Lipschitz continuous function,

$$\int_{t_0}^{t_1}\int_0^1 \left|\frac{v_x(x,t+h)-v_x(x,t)}{h}\right|^2 dx\,dt$$

$$\leqslant c\int_{t_0}^{t_1}\int_0^1 \left(\frac{v_x(x,t+h)-v_x(x,t)}{h}\right) \times \left(\frac{\varphi(v_x(x,t+h))-\varphi(v_x(x,t))}{h}\right) dx\,dt$$

$$= -c\int_{t_0}^{t_1}\int_0^1 \left(\frac{v(x,t+h)-v(x,t)}{h}\right) \times \left(\frac{D_x(\varphi(v_x(x,t+h)))-D_x(\varphi(v_x(x,t)))}{h}\right) dx\,dt$$

$$= -c\int_{t_0}^{t_1}\int_0^1 \left(\frac{v(x,t+h)-v(x,t)}{h}\right)\left(\frac{v_t(x,t+h)-v_t(x,t)}{h}\right) dx\,dt$$

$$= -c\int_{t_0}^{t_1}\int_0^1 \frac{1}{2} D_t\left(\left(\frac{v(x,t+h)-v(x,t)}{h}\right)^2\right) dx\,dt$$

$$= \frac{c}{2}\int_0^1 \frac{v(x,t_0+h)-v(x,t_0)}{h}\,dx - \frac{c}{2}\int_0^1 \frac{v(x,t_1+h)-v(x,t_1)}{h}\,dx.$$

The last member is bounded for almost every $t_0, t_1 \in (0, T]$ and $|h| < \delta(t_0, t_1)$ because $v_t \in L^2_{\text{Loc}}(0, T; L^2(0,1))$. Therefore, for almost every $t_0, t_1 \in (0, T]$,

$$\int_{t_0}^{t_1}\int_0^1 \left|\frac{v_x(x,t+h)-v_x(x,t)}{h}\right|^2 dx\,dt \leqslant c \qquad \text{if} \quad |h| < \delta(t_0, t_1),$$

and we deduce $v_{xt} \in L^2_{\text{Loc}}(0, T; L^2(0,1))$, that is, $u_t \in L^1_{\text{loc}}(0, T; L^2(0,1))$.

Let us prove that u satisfies (1.1b). As $D_x(\varphi(u)) = v_t$, we want to see that

$$\lim_{x\to 0} v_t(x,t) = 0 \text{ a.e. } t.$$

We will prove that

$$\lim_{(x,h)\to(0,0)} \frac{v(x,t+h)-v(x,t)}{h} = 0 \text{ a.e. } t,$$

which implies (1.1b).

In fact, as $v(0, t) = 0$ a.e. t,

$$\frac{v(x, t+h) - v(x, t)}{h} = \int_0^x \frac{v_x(s, t+h) - v_x(s, t)}{h} \, ds.$$

We know that $v_{xt} \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1))$, and therefore

$$\frac{v_x(s, t+h) - v_x(s, t)}{h} \to v_{xt}(s, t) \qquad (h \to 0) \text{ in } L^2(0, 1) \text{ a.e. } t \in (0, T).$$

Therefore, there exists $\delta_1(t)$ a.e. t, such that

$$\int_0^1 \left| \frac{v_x(s, t+h) - v_x(s, t)}{h} \right|^2 ds \leqslant c \qquad \text{if} \quad |h| < \delta_1(t).$$

Then we have

$$\left| \frac{v(x, t+h) - v(x, t)}{h} \right| \leqslant x^{1/2} c^{1/2} < \varepsilon \qquad \text{if} \quad |h| < \delta_1(t), \quad |x| < \frac{\varepsilon^2}{c}.$$

We have that u satisfies (1.1a, b, c, d), because

$$u_t = v_{xt} = v_{tx} = D_{xx}(\varphi(v_x)) = D_{xx}(\varphi(u)).$$

This implies that $D_x(\varphi(u)) \in C([0, 1])$ in x a.e. t, which was stated in (1.1b).

We only have to prove that $u \in L^\infty(0, T; L^1(0, 1))$ and satisfies (1.1e). Eq. (1.1e) states that $v(x, t) \to F(x)$ if F is continuous at x. We will prove that $V_0^1 v(x, t) \leqslant V_0^1 F(x)$ for every $t \in (0, T]$. This, and the fact that $v(x, t) \to F(x)$ in $L^1(0, 1)$ imply (1.1e).

As $u(x, t) \in L^1(0, 1)$ for every $t \in (0, T]$,

$$\int_0^1 |u(x, t)| \, dx = V_0^1 v(x, t) \leqslant V_0^1 F(x) = \int_0^1 d|\mu|$$

and we have $u \in L^\infty(0, T; L^1(0, 1))$.

Thus, to conclude the proof we only have to see that $V_0^1 v(x, t) \leqslant V_0^1 F(x)$. This is a version of the saw theorem in [10] for solutions which are not continuous up to the boundary, and is based on the comparison theorem in [3].

Let us first remark that as $v_{xt} \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1))$, we have

(i) $v_t \in L^2_{\mathrm{Loc}}(0, T)$ $\forall x \in (0, 1)$;

(ii) $v_x \in L^\infty_{\mathrm{Loc}}(0, T; L^2(0, 1))$;

and therefore, $v(x,t)$ is continuous in $(0,1)\times(0,T)$. In fact,

$$|v(x+h,t+\delta)-v(x,t)|$$
$$\leqslant \int_x^{x+h} |v_x(s,t+\delta)|\,ds+\int_t^{t+\delta} |v_t(x,\tau)|\,d\tau \leqslant c(h^{1/2}+\delta^{1/2}).$$

Let $0=x_0<x_1<\cdots<x_k=1$ be such that $d_i=v(x_i,t_0)-v(x_{i-1},t_0)$ is always different from 0 and alternating in sign. Let $0<2a<d_i$ for every i. Then there exist $k+1$ points $0=x_0^0<x_1^0<\cdots<x_k^0\leqslant 1$ such that

$$\begin{aligned} &\operatorname{sign}(F(x_i^0)-F(x_{i-1}^0))=\operatorname{sign} d_i, \\ &|d_i|\leqslant |F(x_i^0)-F(x_{i-1}^0)|+2a, \end{aligned}\qquad i=1,\dots,k.$$

From this fact we deduce that

$$\sum_{i=1}^{k} |v(x_i,t_0)-v(x_{i-1},t_0)|\leqslant V_0^1 F(x)+2ak.$$

As $0<a<\frac{1}{2}|d_i|$ is arbitrarily small, we obtain

$$V_0^1 v(x,t_0)\leqslant V_0^1 F(x).$$

Let then $0=x_0<x_1<\cdots<x_k=1$ as above. We define $x_0^0=0$. Let $i\geqslant 1$ and

$$A_i=\{(x,t)\in(0,1)\times(0,t_0)/v(x,t)>v(x_i,t_0)-a\}$$
$$\text{if}\quad (x_i,t_0) \text{ is a high point (HP)},$$

that is,

$$v(x_i,t_0)>\begin{cases} v(x_{i-1},t_0)\\ v(x_{i+1},t_0)\end{cases}$$

$$A_i=\{(x,t)\in(0,1)\times(0,t_0)/v(x,t)<v(x_i,t_0)+a\}$$
$$\text{if}\quad (x_i,t_0) \text{ is a low point (LP)}.$$

Then A_i is an open set. Let H_i be the component of A_i for which $(x_i,t_0)\in \bar{H}_i$. We have $H_i\cap H_{i+1}=\varnothing$, $i=1,\dots,k-1$.

We see that $\operatorname{meas}\{x\in(0,1)/(x,0)\in\partial H_1\}>0$, where ∂H_1 is the boundary of the set H_1. In fact, suppose it doesn't occur, then $v(x,t)$ is the solution in H_1 of the following problem:

(a) $v_t=D_x(\varphi(v_x))$ a.e. in H_1,

(b) if $(\tilde{x},\tilde{t})\in\partial_p H_1$, $\quad 0<\tilde{x}<1, \tilde{t}>\bar{t}$

(i) $\lim_{x \to \tilde{x}} \chi_{H_1}(x, \tilde{t})\, v(x, \tilde{t}) = v(x_1, t_0) - a$

or

(ii) $\lim_{t \to \tilde{t}} \chi_{H_1}(\tilde{x}, t)\, v(\tilde{x}, t) = v(x_1, t_0) - a,$ (1.3)

(c) if $(0, t) \in \partial_p H_1$, $\lim_{x \to 0} \chi_{H_1}(x, t)\, v(x, t) = 0,$

(d) if $(1, t) \in \partial_p H_1$, $\lim_{x \to 1} \chi_{H_1}(x, t)\, v_x(x, t) = 0,$

(e) $\lim_{t \to \bar{t}} \int_0^1 \chi_{H_1}(x, t)\, |v(x, t) - (v(x_1, t_0) - a)|\, dx = 0,$

when (x_1, t_0) is HP and $[\bar{t}, t_0]$ is the projection of H_1 on the interval $[0, T]$.

We have denoted by $\partial_p H_1$ the subset of ∂H_1, $(\tilde{x}, \tilde{t}) \in \partial_p H_1$ if there exists $\varepsilon > 0$ such that

(i) $(\tilde{x} - \varepsilon, \tilde{x}) \times \{\tilde{t}\} \subset H_1$ or $(\tilde{x}, \tilde{x} + \varepsilon) \times \{\tilde{t}\} \subset H_1$

or

(ii) $\{\tilde{x}\} \times (\tilde{t}, \tilde{t} + \varepsilon) \subset H_1,$

which corresponds to conditions (b.i) and (b.ii), respectively.

We remark that it may happen that $\bar{t} = 0$; in this case (1.3e) becomes true because $\chi_{H_1}(x, t) \to 0$ a.e. x as $t \to 0$.

When (x_1, t_0) is LP, $v(x, t)$ is the solution of the problem (1.3a, c, d), and

(b′) if $(\tilde{x}, \tilde{t}) \in \partial_p H_1$, $0 < \tilde{x} < 1$, $\tilde{t} > \bar{t}$

(i) $\lim_{x \to \tilde{x}} \chi_{H_1}(x, \tilde{t})\, v(x, \tilde{t}) = v(x_1, t_0) + a$

or

(ii) $\lim_{t \to \tilde{t}} \chi_{H_1}(\tilde{x}, t)\, v(\tilde{x}, t) = v(x_1, t_0) + a,$

(e′) $\lim_{t \to \bar{t}} \int_0^1 \chi_{H_1}(x, t)\, |v(x, t) - (v(x_1, t_0) + a)|\, dx = 0.$

We make use of the comparison theorem of [3], which applies on every measurable set H_1. We conclude that

if (x_1, t_0) is HP then

$$v(x, t) \leqslant \max\{0, v(x_1, t_0) - a\} \text{ in } H_1, \tag{1}$$

if (x_1, t_0) is LP then

$$v(x, t) \geqslant \min\{0, v(x_1, t_0) + a\} \text{ in } H_1. \tag{2}$$

As $v(0, t) = 0$,

(x_1, t_0) HP implies $v(x_1, t_0) > 2a$, and therefore

$$v(x_1, t_0) - a > 0,$$

(x_1, t_0) LP implies $v(x_1, t_0) < -2a$, and therefore

$$v(x_1, t_0) + a < 0.$$

Then (1) and (2) become

$$\text{if } (x_1, t_0) \text{ is HP then} \qquad v(x, t) \leqslant v(x_1, t_0) - a \text{ in } H_1, \tag{1}$$

$$\text{if } (x_1, t_0) \text{ is LP then} \qquad v(x, t) \geqslant v(x_1, t_0) + a \text{ in } H_1, \tag{2}$$

which is absurd. Therefore, $\text{meas}\{x \in (0, 1)/(x, 0) \in \partial H_1\} > 0$.

As $H_1 \cap H_2 = \varnothing$, there are no points of the form $(0, t)$ on the boundary of H_2. In fact, suppose $(0, t) \in \partial H_2$. Let $\bar{x} \in (0, 1)$ be such that $(\bar{x}, 0) \in \partial H_1$ and let C_1 be a Jordan curve connecting (x_1, t_0) and $(\bar{x}, 0)$, which is contained in H_1. C_1 divides the rectangle $(0, 1) \times (0, t_0)$ into two regions. There also exists a Jordan curve C_2 connecting $(0, t)$ and (x_2, t_0) in H_2 and therefore it must be $C_1 \cap C_2 \neq \varnothing$, absurd.

We will prove that $\text{meas}\{x \in (0, 1)/(x, 0) \in \partial H_2\} > 0$, and therefore inductively deduce that $(0, t) \notin \partial H_i$ $i = 2,..., k$, and $\text{meas}\{x \in (0, 1)/(x, 0) \in \partial H_i\} > 0$, $i = 2,..., k$.

In fact, suppose $\text{meas}\{x \in (0, 1)/(x, 0) \in \partial H_2\} = 0$; then $v(x, t)$ is the solution in H_2 of the following problem:

(a) $v_t = D_x(\varphi(v_x))$ a.e. in H_2,

(b) if $(\tilde{x}, \tilde{t}) \in \partial_p H_2$, $\quad \tilde{x} < 1$, $\tilde{t} > \bar{t}$ (we know that $\tilde{x} > 0$)

$$\text{(i)} \quad \lim_{x \to \tilde{x}} \chi_{H_2}(x, \tilde{t})\, v(x, \tilde{t}) = \begin{cases} v(x_2, t_0) - a & \text{(HP)} \\ v(x_2, t_0) + a & \text{(LP)} \end{cases}$$

or

$$\text{(ii)} \quad \lim_{t \to \tilde{t}} \chi_{H_2}(\tilde{x}, t)\, v(\tilde{x}, t) = \begin{cases} v(x_2, t_0) - a & \text{(HP)} \\ v(x_2, t_0) + a & \text{(LP)} \end{cases}$$

(c) if $(1, t) \in \partial_p H_2$, $\quad \lim_{x \to 1} \chi_{H_2}(x, t)\, v_x(x, t) = 0$,

$$\text{(d)} \quad \lim_{t \to \bar{t}} \int_0^1 \chi_{H_2}(x, t) \left| v(x, t) - \begin{bmatrix} v(x_2, t_0) - a & \text{(HP)} \\ v(x_2, t_0) + a & \text{(LP)} \end{bmatrix} \right| dx = 0.$$

We deduce that

$$v(x,t) \leqslant v(x_2, t_0) - a \text{ in } H_2 \qquad \text{(HP)},$$
$$v(x,t) \geqslant v(x_2, t_0) + a \text{ in } H_2 \qquad \text{(LP)},$$

which is absurd.

We will prove that

$$\text{meas}\left\{ x \in (0,1)/(x,0) \in \partial H_i \text{ and } F(x) \begin{bmatrix} > v(x_i, t_0) - a & \text{(HP)} \\ < v(x_i, t_0) + a & \text{(LP)} \end{bmatrix} \right\} > 0.$$

In fact, $v(x,t)$ is the solution in H_i, $i = 1,\ldots, k$; of the following problem:

(a) $v_t = D_x(\varphi(v_x))$ a.e. in H_i,

(b) if $(\tilde{x}, \tilde{t}) \in \partial_p H_i$, $\quad 0 < \tilde{x} < 1, \tilde{t} > \bar{t} = 0$

(i) $$\lim_{x \to \tilde{x}} \chi_{H_i}(x, \tilde{t})\, v(x, \tilde{t}) = \begin{cases} v(x_i, t_0) - a & \text{(HP)} \\ v(x_i, t_0) + a & \text{(LP)} \end{cases}$$

or

(ii) $$\lim_{t \to \tilde{t}} \chi_{H_i}(\tilde{x}, t)\, v(\tilde{x}, t) = \begin{cases} v(x_i, t_0) - a & \text{(HP)} \\ v(x_i, t_0) + a & \text{(LP)} \end{cases}$$

(c) if $(1, t) \in \partial_p H_i$, $\quad \lim_{x \to 1} \chi_{H_i}(x,t)\, v_x(x,t) = 0,$

(d) $$\lim_{t \to 0} \int_0^1 \chi_{H_i}(x,t)\, |v(x,t) - F(x)|\, dx = 0.$$

When $i = 2,\ldots, k$, $(0, t) \notin \partial_p H_i$. When $i = 1$, $v(x,t)$ satisfies

(e) if $(0, t) \in \partial_p H_1$, $\quad \lim_{x \to 0} \chi_{H_1}(x,t)\, v(x,t) = 0.$

Therefore, suppose

$$F(x) = \begin{cases} \leqslant v(x_i, t_0) - a & \text{(HP)} \\ \geqslant v(x_i, t_0) + a & \text{(LP)} \end{cases}$$

a.e. in $\{x \in (0,1)/(x,0) \in \partial H_i\}$, we may one more time apply the comparison theorem of [3] to conclude that

when $i = 2,\ldots, k$,

$$v(x,t) \begin{cases} \leqslant v(x_i, t_0) - a & \text{(HP)} & \text{a.e. in } H_i \\ \geqslant v(x_i, t_0) + a & \text{(LP)} & \text{a.e. in } H_i \end{cases} \qquad \text{absurd;}$$

when $i=1$,

$$v(x,t)\begin{cases}\leqslant \max\{0, v(x_1,t_0)-a\} & \text{(HP)} \quad \text{a.e. in } H_1\\ \geqslant \min\{0, v(x_1,t_0)+a\} & \text{(LP)} \quad \text{a.e. in } H_1\end{cases} \qquad \text{absurd.}$$

Let therefore $x_i^0 \in (0,1)$ be such that $(x_i^0, 0) \in \partial H_i$ and

$$F(x_i^0)\begin{cases}>v(x_i,t_0)-a & \text{(HP)}\\ <v(x_i,t_0)+a & \text{(LP)}\end{cases}.$$

Then we have

(1) if (x_i, t_0) is (HP), then $\quad F(x_i^0) > \begin{cases}F(x_{i-1}^0)\\ F(x_{i+1}^0)\end{cases}.$

In fact,

$$F(x_i^0) > v(x_i,t_0)-a > \begin{cases}v(x_{i-1},t_0)+2a-a\\ v(x_{i+1},t_0)+2a-a\end{cases}$$
$$= \begin{cases}v(x_{i-1},t_0)+a > F(x_{i-1}^0)\\ v(x_{i+1},t_0)+a > F(x_{i+1}^0)\end{cases},$$

(2) $\quad |F(x_i^0)-F(x_{i-1}^0)| \geqslant |v(x_i,t_0)-v(x_{i-1},t_0)|-2a;$

in fact, suppose (x_i, t_0) HP,

$$|F(x_i^0)-F(x_{i-1}^0)| = F(x_i^0)-F(x_{i-1}^0)$$
$$F(x_i^0) > v(x_i,t_0)-a$$
$$F(x_{i-1}^0) < v(x_{i-1},t_0)+a$$

therefore,

$$F(x_i^0)-F(x_{i-1}^0) > v(x_i,t_0)-v(x_{i-1},t_0)-2a$$
$$= |v(x_i,t_0)-v(x_{i-1},t_0)|-2a.$$

It only remains to see that $0=x_0^0<x_1^0<\cdots<x_k^0\leqslant 1$. We will prove it inductively.

$x_0^0=0$ by definition, $x_1^0>0$ because we can choose it in such a way since the set from where we choose it is of positive measure. Let us see that $x_2^0>x_1^0$. In fact, $x_2>x_1$ and $(x_2^0,0)\in\partial H_2$, we deduce that $x_2^0>x_1^0$ in the same way as we have proved that $(0,t)\notin\partial H_2$.

In the same way it can be proved that $x_{i+1}^0>x_i^0$, $i=2,\ldots,k-1$.

The proof is finished.

(2) *Uniqueness.* Suppose there exist two functions u_1 and u_2 in $L^\infty(0, T; L^1(0, 1))$ which satisfy (1.1a, b, c, d, e). As $D_t u_i \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1))$, $i = 1, 2$, it is easy to see that both functions

$$v_i(x, t) = \int_0^x u_i(s, t)\, ds$$

satisfy (1.2a, b, c).

We know that

$$\lim_{t \to 0} v_i(x, t) = \lim_{t \to 0} \int_0^x u_i(s, t)\, ds = \mu([0, x)) = F(x)$$

if F is continuous at $x \in (0, 1)$. As $u_i \in L^\infty(0, T; L^1(0, 1))$, $v_i \in L^\infty((0, 1) \times (0, T))$ and therefore

$$\lim_{t \to 0} v_i(x, t) = F(x) \text{ in } L^1(0, 1),$$

That is v_i satisfies (1.2d) for $i = 1, 2$. By Theorem 0 we know that (1.2) has a unique solution, therefore $v_1(x, t) = v_2(x, t)$ a.e. $(x, t) \in (0, 1) \times (0, T)$, and we deduce

$$u_1(x, t) = D_x v_1(x, t) = D_x v_2(x, t) = u_2(x, t) \text{ a.e.}$$

The proof is finished.

COROLLARY. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the hypothesis of Theorem* 1. *Let μ be a finite Borel measure on* $[0, 1)$. *Then there exists one and only one function $u(x, t)$ that satisfies,*

$$\begin{aligned}
&\text{(a)} \quad u_t \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1)),\\
&\text{(b)} \quad \varphi(u) \in C^1([0, 1]) \textit{ in } x, \textit{ a.e. } t \textit{ and } D_x(\varphi(u))(0, t) = 0 \textit{ a.e. } t,\\
&\text{(c)} \quad u(1, t) = 0 \textit{ a.e. } t,\\
&\text{(d)} \quad u_t = D_{xx}(\varphi(u)) \textit{ a.e. } (x, t) \in (0, 1) \times (0, T), \qquad\qquad (1.4)\\
&\text{(e)} \quad u(x, t) \rightharpoonup \mu (t \to 0), \textit{ that is,}\\
&\int_0^1 u(x, t)\, g(x)\, dx \to \int_0^1 g(x)\, d\mu(x), \qquad \textit{for every} \quad g \in C([0, 1]).
\end{aligned}$$

Proof. (1) *Existence.* Let $u(x, t)$ be the solution of (1.1) obtained in Theorem 1. We know that if

$$v(x, t) = \int_0^x u(s, t)\, ds$$

then, if $F(x) = \mu([0, x))$,

$$V_0^1 v(x, t) \leqslant V_0^1 F(x) \qquad \text{for every} \quad t > 0$$

$$\text{and } v(x, t) \to F(x) \qquad \text{if } F \text{ is continuous at } x.$$

We will prove that $v(1, t) \to F(1)$ and then applying Helly's first theorem we will deduce that $u(x, t) \rightharpoonup \mu (t \to 0)$, and this will finish the proof of the existence.

We prove the following lemma and then we continue with the proof of uniqueness.

LEMMA. *Let F_n, F be functions of bounded variation such that $F_n(0) = F(0) = 0$ and $V_0^1 F_n \leqslant V_0^1 F$ for every $n \in \mathbb{N}$. Suppose that F is left-continuous and $F_n(x) \to F(x)$ if F is continuous at $x \in (0, 1)$. Then $F_n(1) \to F(1)$.*

Proof of the Lemma. Let $\varepsilon > 0$; there exist points of continuity of F, $0 < x_1 < \cdots < x_N < 1$ such that

$$\sum_{i=1}^{N} |F(x_i) - F(x_{i-1})| > V_0^1 F - \varepsilon$$

and x_N can be chosen arbitrarily close to 1.

This election may be done in the following way. One can choose N points of continuity of F, $x_1, ..., x_N$ such that if we put $x_0 = 0$,

$$\sum_{i=1}^{N} |F(x_i) - F(x_{i-1})| + |F(1) - F(x_N)| > V_0^1 F - \varepsilon/2.$$

As F is left-continuous at $x = 1$, we have $|F(x) - F(1)| < \varepsilon/2$ if $1 - \delta < x < 1$ for some $\delta > 0$. We choose the point x_N of the partition on the interval $(1 - \delta, 1)$, and we have what we wanted.

As $F_n(x_i) \to F(x_i)$ for $i = 0, ..., N$, we have

$$\sum_{i=1}^{N} |F_n(x_i) - F_n(x_{i-1})| > V_0^1 F - 2\varepsilon \qquad \text{if} \quad n \geqslant n_0(\varepsilon).$$

And on the other hand, as $V_0^1 F_n \leqslant V_0^1 F$ for every n,

$$|F_n(1) - F_n(x_N)| + \sum_{i=1}^{N} |F_n(x_i) - F_n(x_{i-1})| \leqslant V_0^1 F.$$

Therefore,

$$|F_n(1)-F_n(x_N)|+V_0^1F-2\varepsilon<|F_n(1)-F_n(x_N)| + \sum_{i=1}^{N} |F_n(x_i)-F_n(x_{i-1})| \leqslant V_0^1F \qquad \text{if} \quad n \geqslant n_0(\varepsilon).$$

Then,

$$|F_n(1)-F_n(x_N)| \leqslant 2\varepsilon \qquad \text{if} \quad n \geqslant n_0(\varepsilon).$$

As $F_n(x_N) \to F(x_N)$ $(n \to \infty)$

$$|(\limsup F_n(1)) - F(x_N)| \leqslant 2\varepsilon,$$
$$|(\liminf F_n(1)) - F(x_N)| \leqslant 2\varepsilon.$$

As $x_N \in (1-\delta, 1)$ can be chosen arbitrarily close to 1, and $F(x) \to F(1)$ when $x \nearrow 1$,

$$|(\limsup F_n(1)) - F(1)| \leqslant 2\varepsilon,$$
$$|(\liminf F_n(1)) - F(1)| \leqslant 2\varepsilon.$$

As ε is arbitrary

$$\lim_{n\to\infty} F_n(1) = F(1).$$

The proof is finished.

We continue with the proof of the corollary.

(2) *Uniqueness.* Let u_1 and u_2 be two solutions of (1.4). As $u_i(x,t) \rightharpoonup \mu (t \to 0)$, there exist $\delta > 0$ and $c > 0$ such that $V_0^1 v_i(x,t) \leqslant c$ if $0 < t < \delta$, where

$$v_i(x,t) = \int_0^x u_i(s,t)\, ds.$$

Therefore $\int_0^1 |u_i(x,t)|\, dx = V_0^1 v_i(x,t) \leqslant c$ if $0 < t < \delta$.

Let us see that $\int_0^1 |u_i(x,t)|\, dx \leqslant \int_0^1 |u_i(x,t_0)|\, dx$ if $t > t_0$.

In fact, let $S(t)$ be the semigroup associated to the m-accretive operator $-D_{xx}(\varphi(u))$ with the corresponding boundary conditions (see [13]). We prove that

$$u_i(x,t) = S(t-t_0)\, u_i(x,t_0) \qquad \text{if} \quad t > t_0,$$

and this implies what we have stated above.

Therefore $u_i \in L^\infty(0, T; L^1(0, 1))$. As $u_i(x, t) \rightharpoonup \mu$ $(t \to 0)$,

$$\int_0^x u_i(s, t)\, ds \to \mu([0, x)) \qquad \text{if} \quad \mu(\{x\}) = 0.$$

Therefore, by Theorem 1, $u_1(x, t) = u_2(x, t)$ a.e. $(x, t) \in (0, 1) \times (0, T)$.

Let us prove that if u is solution of (1.4), then

$$u(x, t) = S(t - t_0)\, u(x, t_0) \qquad \text{for} \quad t > t_0 > 0.$$

In fact, as $u_t \in L^1_{\text{Loc}}(0, T; L^1(0, 1))$,

$$u(x, t) - u(x, t_0) = \int_{t_0}^t u_t(x, z)\, dz \text{ a.e. } x \in (0, 1),$$

therefore

$$\int_0^1 |u(x, t) - u(x, t_0)|\, dx \leqslant \int_0^1 \int_{t_0}^t |u_t(x, z)|\, dz\, dx$$

and we deduce

$$\lim_{t \searrow t_0} \int_0^1 |u(x, t) - u(x, t_0)|\, dx = 0.$$

Then $u(x, t)$ is a solution in $(0, 1) \times (t_0, T)$ of

$$\begin{aligned} u_t &= D_{xx}(\varphi(u)), \\ \lim_{x \to 0} D_x(\varphi(u))(x, t) &= 0 \text{ a.e. } t, \\ u(1, t) &= 0, \\ \lim_{t \searrow t_0} \int_0^1 |u(x, t) - u(x, t_0)|\, dx &= 0. \end{aligned}$$

By uniqueness we deduce that $u(x, t) = S(t - t_0)\, u(x, t_0)$ (see [13]). The proof is finished.

We prove now a theorem which states the existence of a weak limit (in the sense of measures) for every solution of the equation with the corresponding boundary conditions in $(0, 1) \times (0, T)$.

THEOREM 2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and there exists a constant $c > 0$ with $|\varphi(p)| \geqslant c\,|p|$ for $|p| \to \infty$. Let $u(x, t) \in L^\infty(0, T; L^1(0, 1))$ be a solution of*

(a) $u_t \in L^1_{\mathrm{Loc}}(0, T; L^1(0, 1))$,

(b) $\varphi(u) \in C^1([0, 1])$ *in* x, *a.e.* t *and* $D_x(\varphi(u))(0, t) = 0$ *a.e.* t,

(c) $u(1, t) = 0$ *a.e.* t,

(d) $u_t = D_{xx}(\varphi(u))$ *a.e.* $(x, t) \in (0, 1) \times (0, T)$. (2.1)

Then, there exists one and only one finite Borel measure μ *such that*

$$u(x, t) \rightharpoonup \mu (t \to 0).$$

If $u \geqslant 0$, *then* $\mu \geqslant 0$. *If* $u \leqslant 0$, *then* $\mu \leqslant 0$.

Proof. As $u \in L^\infty(0, T; L^1(0, 1))$, there exist a sequence (t_n) with $t_n \to 0$ and a finite Borel measure μ such that

$$u(x, t_n) \rightharpoonup \mu (n \to \infty).$$

We will prove that $u(x, t) \rightharpoonup \mu$ $(t \to 0)$.

Let F be the distribution function of μ and let $v(x, t) = \int_0^x u(s, t)\, ds$. Then, we have $v(x, t_n) \to F(x)$ if F is continuous at $x \in (0, 1)$.

Let us observe that if $u \geqslant 0$ then v is nondecreasing and therefore F is nondecreasing. This implies that $\mu \geqslant 0$. Analogously if $u \leqslant 0$, then $\mu \leqslant 0$.

It can be easy proved that $v_t \in L^1_{\mathrm{Loc}}(0, T; L^1(0, 1))$ and $v(x, t)$ is a solution of

$$\begin{aligned} v_t &= D_x(\varphi(v_x)) \text{ a.e.},\\ v(0, t) &= 0 \text{ a.e. } t,\\ v_x(1, t) &= 0 \text{ a.e. } t,\\ v(x, t_n) &\to F(x) \text{ a.e.} \qquad (n \to \infty). \end{aligned}$$

As $v \in L^\infty((0, 1) \times (0, T))$ we have $v(x, t_n) \to F(x)$ in $L^1(0, 1)$. We prove that $v(x, t) \to F(x)$ in $L^1(0, 1)$ $(t \to 0)$. In fact, let $w(x, t)$ be the solution of the problem

$$\begin{aligned} w_t &= D_x(\varphi(w_x)) \text{ a.e.},\\ w(0, t) &= 0 \text{ a.e. } t,\\ w_x(1, t) &= 0 \text{ a.e. } t,\\ w(x, t) &\to F(x) \qquad \text{in} \quad L^1(0, 1) \qquad (t \to 0). \end{aligned}$$

with $w_t \in L^2_{\mathrm{Loc}}(0, T; L^2(0, 1))$, given by Theorem 0. Then v and w are two solutions of the problem: differential equation + boundary conditions + the following initial condition

$$w(x, t_n) \to F(x) \quad \text{in} \quad L^1(0, 1) \quad (n \to \infty)$$

with $t_n \to 0$ and $w_t, v_t \in L^1_{\text{Loc}}(0, T; L^1(0, 1))$.

By the uniqueness of the solution of this problem (see the proof of the comparison theorem in [3]), we get

$$v(x, t) = w(x, t) \text{ a.e.}$$

and therefore $v(x, t) \to F(x)$ in $L^1(0, 1)$ $(t \to 0)$.

As was proved in Theorem 1, $V_0^1 v(x, t) \leqslant V_0^1 F$ for every $t > 0$ and $v(x, t) \to F(x)$ $(t \to 0)$ if F is continuous at x.

Again as in the proof of the Corollary we deduce that $v(1, t) \to F(1)$ $(t \to 0)$ and therefore

$$u(x, t) \rightharpoonup \mu \qquad (t \to 0).$$

The uniqueness is a consequence of the uniqueness of the weak limit of measures. The theorem is proved.

We will now prove a comparison theorem between the distribution functions of two solutions in terms of the distribution functions of the initial measures.

THEOREM 3. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function. Let u_1 and u_2 be solutions of* (2.1) *with*

$$u_i(x, t) \rightharpoonup \mu_i \qquad (t \to 0), \quad i = 1, 2.$$

Suppose that the distributions functions of the initial measures, F_1 and F_2 satisfy

$$F_1(x) \leqslant F_2(x) \qquad a.e. \quad x \in (0, 1).$$

Then

$$\int_0^x u_1(s, t)\, ds \leqslant \int_0^x u_2(s, t)\, ds \qquad a.e. \quad (x, t) \in (0, 1) \times (0, T).$$

Proof. Let $v_i(x, t) = \int_0^x u_i(s, t)\, ds$. Then $D_t v_i \in L^1_{\text{Loc}}(0, T; L^1(0, 1))$, $V_0^1 v_i(x, t) \leqslant c$ $\forall t$ (see the proof of the corollary) and v_i satisfies

$$\begin{aligned}
&\text{(a)} \quad D_t v_i = D_x(\varphi(D_x v_i)) \text{ a.e.},\\
&\text{(b)} \quad v_i(0, t) = 0 \text{ a.e. } t,\\
&\text{(c)} \quad D_x v_i(1, t) = 0 \text{ a.e. } t,\\
&\text{(d)} \quad v_i(x, t) \to F_i(x) \text{ a.e. } x.
\end{aligned} \tag{3.1}$$

As $v_i(0, t) = 0$, $|v_i(x, t)| \leqslant c$ a.e. and therefore $v_i(x, t) \to F_i(x)$ in $L^1(0, 1)$ $(t \to 0)$, $i = 1, 2$.

As $F_1(x) \leqslant F_2(x)$ a.e. x, we deduce that

$$v_1(x, t) \leqslant v_2(x, t) \qquad \text{a.e.}$$

(see [13]). The theorem is proved.

This result has been proved by J. L. Vásquez (see [11]) in the case $\mu_i = \delta_{x_i}$, the measure of mass concentrated at the point x_i, or $\mu_i \in L^1(\mathbb{R})$ and nonnegative. He uses this result to estimate the free boundary of a solution with initial datum of compact support.

THEOREM 4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function such that $\varphi(0) = 0$ and there exists a constant $c > 0$ such that $|\varphi(p)| \geqslant c\,|p|$, $|p| \to \infty$. Let u_1 and u_2 be two solutions of* (2.1) *with*

$$u_i(x, t) \rightharpoonup \mu_i, \qquad i = 1, 2.$$

Suppose $\mu_1 \leqslant \mu_2$, then

$$u_1(x, t) \leqslant u_2(x, t) \qquad \textit{a.e.} \quad (x, t) \in (0, 1) \times (0, T).$$

Proof. Let $v_i(x, t) = \int_0^x u_i(s, t)\, ds$, then $D_t v_i \in L^1_{\mathrm{Loc}}(0, T; L^1(0, 1))$ and v_i is the solution of (3.1) with F_i the distribution function of the measure μ_i, and $v_i(x, t) \to F_i(x)$ in $L^1(0, 1)$ $(t \to 0)$.

As $\mu_1 \leqslant \mu_2$, $F_2 - F_1$ is nondecreasing and nonnegative because $F_1(0) = F_2(0) = 0$. Therefore $v_1(x, t) \leqslant v_2(x, t)$ a.e.

Let us remark that v_i is continuous in $(0, 1) \times (0, T)$. In fact, it is easy to see that $D_t v_i \in L^1_{\mathrm{Loc}}(0, T)$ $\forall x \in (0, 1)$. We will prove that $D_x v_i \in L^\infty_{\mathrm{Loc}}(0, T; L^2(0, 1))$ and deduce that $v \in C((0, 1) \times (0, T))$ as in Theorem 1.

In fact, let $t_0 > 0$, then $u_i(x, t_0) \in C([0, 1])$ and therefore it is a bounded function. As we know that $u_i(x, t) = S(t - t_0)\, u_i(x, t_0)$ for $t > t_0$,

$$|u_i(x, t)| \leqslant \max_{0 \leqslant x \leqslant 1} |u_i(x, t_0)|, \qquad t \geqslant t_0, \quad x \in (0, 1),$$

and therefore $D_x v_i = u_i \in L^\infty_{\mathrm{Loc}}(0, T; L^\infty(0, 1)) \subset L^\infty_{\mathrm{Loc}}(0, T; L^2(0, 1))$.

We will prove that $v_2 - v_1$ is a nondecreasing function of x for every $t > 0$. Suppose it does not happen. Let $t_0 > 0$, x_1, x_2 and c be such that

$$(v_2 - v_1)(x_1, t_0) > c > (v_2 - v_1)(x_2, t_0) \qquad \text{with} \quad x_1 < x_2.$$

We may suppose that $c = 0$; in fact we will prove the following, if v_2 is a solution of (3.1a, b, c), with $v_2(x, t) \to F_2(x)$ in $L^1(0, 1)$ and v_1 is a solution of (3.1a, c) with $v_1(x, t) \to F_1(x)$ in $L^1(0, 1)$ and satisfying

(b′) $v_1(0, t) = c > 0$ a.e. t,

and if $F_2 - F_1$ is nondecreasing, then it is impossible that

$$(v_2 - v_1)(x_1, t_0) > 0 > (v_2 - v_1)(x_2, t_0)$$

with $x_1 < x_2$.

This can be done because $w_1(x, t) = v_1(x, t) + c$ also satisfies (3.1a, c), and we know that $c > 0$ because $v_2 \geqslant v_1$ a.e.

We may also observe that if $F_2 - F_1$ is nondecreasing, then it is also true for $F_2 - (F_1 + c)$.

Let then G be the component of the open set,

$$\{(x, t) \in (0, 1) \times (0, t_0)/(v_2 - v_1)(x, t) > 0\}$$

such that $(x_1, t_0) \in \partial G$.

Let H be the component of the open set

$$\{(x, t) \in (0, 1) \times (0, t_0)/(v_2 - v_1)(x, t) < 0\}$$

such that $(x_2, t_0) \in \partial H$. Then,

$$\text{meas}\{x \in (0, 1)/(x, 0) \in \partial G\} > 0.$$

In fact, if not, v_2 would be a solution in G of the problem,

$$\begin{aligned}
&\text{(a)} \quad v_t = D_x(\varphi(v_x)) \text{ a.e.,}\\
&\text{(b)} \quad \text{if } (\tilde{x}, \tilde{t}) \in \partial_p G, \qquad 0 < \tilde{x} < 1,\ \tilde{t} > \bar{t}\\
&\qquad\qquad \text{(i)} \quad \lim_{x \to \tilde{x}} \chi_G(x, \tilde{t})\, v(x, \tilde{t}) = \lim_{x \to \tilde{x}} \chi_G(x, \tilde{t})\, v_1(x, \tilde{t})\\
&\qquad \text{or}\\
&\qquad\qquad \text{(ii)} \quad \lim_{t \to \tilde{t}} \chi_G(\tilde{x}, t) v(\tilde{x}, t) = \lim_{t \to \tilde{t}} \chi_G(\tilde{x}, t)\, v_1(\tilde{x}, t)\\
&\text{(b}') \quad \text{if } (0, t) \in \partial_p G, \\
&\qquad \lim_{x \to 0} \chi_G(x, t)\, v(x, t) = 0 < c = \lim_{x \to 0} \chi_G(x, t)\, v_1(x, t),\\
&\text{(c)} \quad \text{if } (1, t) \in \partial_p G,\\
&\qquad \lim_{x \to 1} \chi_G(x, t)\, v_x(x, t) = 0 = \lim_{x \to 1} \chi_G(x, t)(v_1)_x\, (x, t),\\
&\text{(d)} \quad \lim_{t \to \bar{t}} \int_0^1 \chi_G(x, t)\, |v(x, t) - v_1(x, t)|\, dx = 0,
\end{aligned} \tag{4.1}$$

where (d) holds because $v_i \in L^\infty((0, 1) \times (0, T)$ and

(i) if $\bar{t} > 0$, $\quad (v_2(x, t) - v_1(x, t))\, \chi_G(x, t) \to 0 \quad (t \to \bar{t})$

(ii) if $\bar{t} = 0$, $\quad \chi_G(x, t) \to 0 \quad (t \to 0)$ a.e. x.

We may once more apply the comparison theorem in [3] and deduce

$$v_2(x, t) \leqslant v_1(x, t) \qquad \text{a.e.} \quad \text{in } G,$$

which is an absurd.

As in Theorem 1, we deduce that $(0, t) \notin \partial H$ for every $t > 0$ and we deduce that

$$\text{meas}(\{x \in (0, 1)/(x, 0) \in \partial H\}) > 0.$$

Let us prove that

$$\text{meas}(\{x \in (0, 1)/(x, 0) \in \partial G \text{ and } F_2(x) > F_1(x)\} > 0,$$

$$(\text{meas}(\{x \in (0, 1)/(x, 0) \in \partial H \text{ and } F_2(x) < F_1(x)\}) > 0).$$

In fact, in G (in H) v_i is a solution of (4.1a, b, b′) (this condition does not appear in the case of H), (4.1c) and

(d′) $$\lim_{t \to 0} \int_0^1 \chi_G(x, t)\, |v(x, t) - F_i(x)|\, dx = 0$$

$$\left(\lim_{t \to 0} \int_0^1 \chi_H(x, t)\, |v(x, t) - F_i(x)|\, dx = 0\right).$$

Therefore if we have $F_2(x) \leqslant F_1(x)$ a.e. in $\{x \in (0, 1)/(x, 0) \in \partial G\}$ $(F_2(x) \geqslant F_1(x)$ a.e. in $\{x \in (0, 1)/(x, 0) \in \partial H\})$, we deduce

$$v_2(x, t) \leqslant v_1(x, t) \text{ a.e. in } G,$$

$$(v_2(x, t) \geqslant v_1(x, t) \text{ a.e. in } H),$$

which is absurd.

Let then x_1^0, x_2^0 be such that $F_2(x_1^0) > F_1(x_1^0)$, $(x_1^0, 0) \in \partial G$ and $F_2(x_2^0) < F_1(x_2^0)$, $(x_2^0, 0) \in \partial H$. As x_1^0 must be less than x_2^0 we have a contradiction.

Therefore $v_2 - v_1$ is nondecreasing as a function of x for every $t > 0$ and then

$$u_2(x, t) - u_1(x, t) = D_x(v_2(x, t) - v_1(x, t)) \geqslant 0 \qquad \text{a.e.}$$

The proof is finished.

Remark. With an argument similar to those used in Theorems 1 and 4, it can be proved the following estimate (which was obtained by Pierre (see [9]) when μ_i are nonnegative measures),

$$\int_0^1 |u_1(x,t)-u_2(x,t)|\,dx \leqslant \int_0^1 d\,|\mu_1-\mu_2|$$

if u_i is a solution of (2.1) with

$$u_i(x,t) \rightharpoonup \mu_i, \qquad i=1,2.$$

References

1. D. G. Aronson, Nonnegative solutions of linear parabolic equations, *Ann. Sc. Norm. Sup. Pisa Classe di Scienze* **22** (1968), 607–694.
2. Ph. Bénilan, H. Brézis, and M. G. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$, *Ann. Sc. Norm. Sup. Pisa, Ser.* IV, **2** (1975), 523–555.
3. J. E. Bouillet and C. Atkinson, A generalized diffusion equation: Radial symmetries and comparison theorems, *J. Math. Anal. Appl.*, in press.
4. H. Brézis and M. G. Crandall, Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$, *J. Math. Pures Appl* **58** (1979), 153–163.
5. M. G. Crandall, Semigroups of nonlinear transformations in Banach spaces, *in* "Contributions to Nonlinear Functional Analysis" (E. Zarantonello, Ed.), Academic Press, New York, 1971.
6. M. G. Crandall and M. Pierre, "Regularizing Effects for $u_t - \Delta\varphi(u) = 0$," MRC Technical Summary Report 2166, January 1981.
7. L. C. Evans, Differentiability of a nonlinear semigroup in L^1, *J. Math. Anal. Appl.* **60**, No. 3 (1977), 703–715.
8. S. Kamin, Source-type solutions for equations of nonstationary filtration, *J. Math. Anal. Appl.* **64** (1978), 263–276.
9. M. Pierre, "Uniqueness of the Solutions of $u_t - \Delta\varphi(u) = 0$ with Initial Datum a Measure," MRC Technical Summary Report 2171, January 1981.
10. R. M. Redheffer and W. Walter, The total variation of solutions of parabolic differential equations and a Maximum Principle in unbounded domains, *Math. Ann.* **209** (1974), 57–67.
11. J. L. Vázquez, Large-time behaviour of the solutions of the one-dimensional porous media equation, *in* "Proceedings of the Symposium on Free Boundary Problems: Theory and Applications, Montecatini, Italy, 1981," Lecture Notes in Mathematics, Springer-Verlag, Berlin/New York/Heidelberg, in press.
12. D. V. Widder, Positive temperature on the infinite rod, *Trans. Amer. Math. Soc.* **55** (1944), 85–95.
13. N. I. Wolanski, Degenerate nonlinear parabolic equations. Comparison and existence theorems, submitted.
14. Ya. B. Zel'dovich and Yu. P. Raizer, "Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena," Vol. II, Academic Press, New York, 1969.
15. O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, R. I., 1968.
16. H. Brezis, "Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam, 1973.