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Shift-invariant spaces on LCA groups

Carlos Cabrelli [a,b,*], Victoria Paternostro [a,b,1]

[a] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*
[b] *CONICET, Argentina*

Abstract

In this article we extend the theory of shift-invariant spaces to the context of LCA groups. We introduce the notion of H-invariant space for a countable discrete subgroup H of an LCA group G, and show that the concept of range function and the techniques of fiberization are valid in this context. As a consequence of this generalization we prove characterizations of frames and Riesz bases of these spaces extending previous results, that were known for $\mathbb{R}^d$ and the lattice $\mathbb{Z}^d$.
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1. Introduction

A *shift-invariant space* (SIS) is a closed subspace of $L^2(\mathbb{R})$ that is invariant under translations by integers. The Fourier transform of a shift-invariant space is a closed subspace that is invariant under integer modulations (multiplications by complex exponentials of integer frequency). Spaces that are invariant under integer modulations are called *doubly invariant spaces*. Every re-

sult on doubly invariant spaces can be translated to an equivalent result in shift-invariant spaces via the Fourier transform. Doubly invariant spaces have been studied in the sixties by Helson [7] and also by Srinivasan [16,10], in the context of operators related to harmonic analysis.

Shift-invariant spaces are very important in applications and the theory had a great development in the last twenty years, mainly in approximation theory, sampling, wavelets, and frames. In particular they serve as models in many problems in signal and image processing.

In order to understand the structure of doubly invariant spaces, Helson introduced the notion of *range function*. This became an essential tool in the modern development of the theory. See [3,4,14,1].

Range functions characterize completely shift-invariant spaces and provide a series of techniques known in the literature as *fiberization* that allow to have a different view and a deeper insight of these spaces.

Fiberization techniques are very important in the class of *finitely generated* shift-invariant spaces. A key feature of these spaces is that they can be generated by the integer translations of a finite number of functions. Using range functions allows us to translate problems on finitely generated shift-invariant spaces, into problems of linear algebra (i.e. finite dimensional problems).

Shift-invariant spaces generalize very well to several variables where the invariance is understood to be under the group $\mathbb{Z}^d$.

When looking carefully at the theory it becomes apparent that it is strongly based on the additive group operation of $\mathbb{R}^d$ and the action of the subgroup $\mathbb{Z}^d$.

It is therefore interesting to see if the theory can be set in a context of general *locally compact abelian* groups (LCA groups). The locally compact abelian group framework has several advantages. First because it is important to have a valid theory for the classical groups such as $\mathbb{Z}^d$, $\mathbb{T}^d$ and $\mathbb{Z}_n$. This will be crucial particularly in applications, as in the case of the generalization of the Fourier transform to LCA groups and also Kluvanek's theorem, where the Classical Sampling theorem is extended to this general context (see [13,5]).

On the other side, the LCA groups setting, unifies a number of different results into a general framework with a concise and elegant notation. This fact enables us to visualize hidden relationships between the different components of the theory, what, as a consequence, will translate in a deeper and better understanding of shift-invariant spaces, even in the case of the real line.

In this paper we develop the theory of shift invariant spaces in LCA groups. Our emphasis will be on range functions and fiberization techniques. The order of the subjects follows mainly the treatment of Bownik in $\mathbb{R}^d$, [1]. In [12] the authors study, in the context of LCA groups, *principal* shift-invariant spaces, that is, shift-invariant spaces generated by one single function. However they don't develop the general theory.

This article is organized in the following way. In Section 2 we give the necessary background on LCA groups and set the basic notation. In Section 3 we state our standing assumptions and prove the characterizations of H-invariant spaces using range functions. We apply these results in Section 4 to obtain a characterization of frames and Riesz bases of H-translations.

2. Background on LCA Groups

In this section we review some basic known results from the theory of LCA groups, that we need for the remainder of the article. In this way we set the notation that we will use in the following sections. Most proofs of the results are omitted unless it is considered necessary. For details and proofs see [15,8,9].

2.1. LCA Groups

Throughout this article, G will denote a locally compact abelian, Hausdorff group (LCA) and Γ (or $\widehat{G}$) its dual group. That is,

$$\Gamma = \{\gamma : G \to \mathbb{C} : \gamma \text{ is a continuous character of } G\},$$

where a character is a function such that:

(a) $|\gamma(x)| = 1$, $\forall x \in G$.
(b) $\gamma(x + y) = \gamma(x)\gamma(y)$, $\forall x, y \in G$.

Thus, characters generalize the exponential functions $\gamma_t(y) = e^{2\pi i t y}$, from the case $G = (\mathbb{R}, +)$.

Since in this context, both the algebraic and topological structures coexist, we will say that two groups G and G' are *topologically isomorphic* and we will write $G \approx G'$, if there exists a topological isomorphism from G onto G'. That is, an algebraic isomorphism which is a homeomorphism as well.

The following theorem states some important facts about LCA groups. Its proof can be found in [15].

Theorem 2.1. *Let G be an LCA group and Γ its dual. Then,*

(a) *The dual group Γ, with the operation $(\gamma + \gamma')(x) = \gamma(x)\gamma'(x)$, is an LCA group.*
(b) *The dual group of Γ is topologically isomorphic to G, with the identification $x \in G \leftrightarrow \phi_x \in \widehat{\Gamma}$, where $\phi_x(\gamma) := \gamma(x)$.*
(c) *G is discrete (compact) if and only if Γ is compact (discrete).*

As a consequence of item (b) of Theorem 2.1, it is convenient to use the notation (x, γ) for the complex number $\gamma(x)$, representing the character γ applied to x or the character x applied to γ.

Next we list the most basic examples that are relevant to Fourier analysis. As usual, we identify the interval $[0, 1)$ with the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Example 2.2.

(I) In case that $G = (\mathbb{R}^d, +)$, the dual group Γ is also $(\mathbb{R}^d, +)$, with the identification $x \in \mathbb{R}^d \leftrightarrow \gamma_x \in \Gamma$, where $\gamma_x(y) = e^{2\pi i \langle x, y \rangle}$.
(II) In case that $G = \mathbb{T}$, its dual group is topologically isomorphic to $\mathbb{Z}$, identifying each $k \in \mathbb{Z}$ with $\gamma_k \in \Gamma$, being $\gamma_k(\omega) = e^{2\pi i k \omega}$.
(III) Let $G = \mathbb{Z}$. If $\gamma \in \Gamma$, then $(1, \gamma) = e^{2\pi i \alpha}$ for same $\alpha \in \mathbb{R}$. Therefore, $(k, \gamma) = e^{2\pi i \alpha k}$. Thus, the complex number $e^{2\pi i \alpha}$ identifies the character γ. This proves that Γ is $\mathbb{T}$.
(IV) Finally, in case that $G = \mathbb{Z}_n$, the dual group is also $\mathbb{Z}_n$.

Let us now consider $H \subseteq G$, a closed subgroup of an LCA group G. Then, the quotient G/H is a regular (T3) topological group. Moreover, with the quotient topology, G/H is an LCA group and if G is second countable, the quotient G/H is also second countable.

For an LCA group G and $H \subseteq G$ a subgroup of G, we define the subgroup Δ of Γ as follows:

$$\Delta = \left\{ \gamma \in \Gamma \colon (h, \gamma) = 1, \ \forall h \in H \right\}.$$

This subgroup is called the *annihilator of H*. Since each character in Γ is a continuous function on G, Δ is a closed subgroup of Γ. Moreover, if $H \subseteq G$ is a closed subgroup and Δ is the annihilator of H, then H is the annihilator of Δ (see [15, Lemma 2.1.3.]).

The next result establishes duality relationships among the groups H, Δ, G/H and Γ/Δ.

Theorem 2.3. *If G is an LCA group and $H \subseteq G$ is a closed subgroup of G, then*:

(i) Δ *is topologically isomorphic to the dual group of G/H, i.e.: $\Delta \approx \widehat{(G/H)}$.*
(ii) Γ/Δ *is topologically isomorphic to the dual group of H, i.e.: $\Gamma/\Delta \approx \widehat{H}$.*

Remark 2.4. According to Theorem 2.1, each element of G induces one character in $\widehat{\Gamma}$. In particular, if H is a closed subgroup of G, each $h \in H$ induces a character that has the additional property of being Δ-periodic. That is, for every $\delta \in \Delta$, $(h, \gamma + \delta) = (h, \gamma)$ for all $\gamma \in \Gamma$.

The following definition will be useful throughout this paper. It agrees with the one given in [11].

Definition 2.5. Given G an LCA group, a *uniform lattice H in G* is a discrete subgroup of G such that the quotient group G/H is compact.

The next theorem points out a number of relationships which occur among G, H, Γ, Δ and their respective quotients.

Theorem 2.6. *Let G be a second countable LCA group. If $H \subseteq G$ is a countable (finite or countably infinite) uniform lattice, the following properties hold.*

(1) *G is separable.*
(2) *$H \subseteq G$ is closed.*
(3) *G/H is second countable and metrizable.*
(4) *$\Delta \subseteq \Gamma$, the annihilator of H, is closed, discrete and countable.*
(5) *$\widehat{H} \approx \Gamma/\Delta$ and $\widehat{(G/H)} \approx \Delta$.*
(6) *Γ/Δ is a compact group.*

Note that in particular, this theorem states that Δ is a countable uniform lattice in Γ.

2.2. Haar Measure on LCA groups

On every LCA group G, there exists a *Haar measure*. That is, a non-negative, regular borel measure m_G, which is not identically zero and *trans-lation-invariant*. This last property means that,

$$m_G(E + x) = m_G(E)$$

for every element $x \in G$ and every Borel set $E \subseteq G$. This measure is unique up to constants, in the following sense: if m_G and m'_G are two Haar measures on G, then there exists a positive constant λ such that $m_G = \lambda m'_G$.

Given a Haar measure m_G on an LCA group G, the integral over G is translation-invariant in the sense that,

$$\int_G f(x + y) \, dm_G(x) = \int_G f(x) \, dm_G(x)$$

for each element $y \in G$ and for each Borel-measurable function f on G.

As in the case of the Lebesgue measure, we can define the spaces $L^p(G, m_G)$, that we will denote as $L^p(G)$, in the following way

$$L^p(G) = \left\{ f : G \to \mathbb{C} \colon f \text{ is measurable and } \int_G |f(x)|^p \, dm_G(x) < \infty \right\}.$$

If G is a second countable LCA group, $L^p(G)$ is separable, for all $1 \leqslant p < \infty$. We will focus here on the cases $p = 1$ and $p = 2$

The next theorem is a generalization of the periodization argument usually applied in case $G = \mathbb{R}$ and $H = \mathbb{Z}$ (for details see [9, Theorem 28.54]).

Theorem 2.7. *Let G be an LCA group, $H \subseteq G$ a closed subgroup and $f \in L^1(G)$. Then, the Haar measures m_G, m_H and $m_{G/H}$ can be chosen such that*

$$\int_G f(x) \, dm_G(x) = \int_{G/H} \int_H f(x + h) \, dm_H(h) \, dm_{G/H}([x]),$$

where $[x]$ denotes the coset of x in the quotient G/H.

If G is a countable discrete group, the integral of $f \in L^1(G)$ over G, is determined by the formula

$$\int_G f(x) \, dm_G(x) = m_G(\{0\}) \sum_{x \in G} f(x),$$

since, due to the translations invariance, $m_G(\{x\}) = m_G(\{0\})$, for each element $x \in G$.

Definition 2.8. A *section* of G/H is a set of representatives of this quotient. That is, a subset C of G containing exactly one element of each coset. Thus, each element $x \in G$ has a unique expression of the form $x = c + h$ with $c \in C$ and $h \in H$.

We will need later in the paper to work with Borel sections. The existence of Borel sections is provided by the following lemma (see [11] and [6]).

Lemma 2.9. *Let G be an LCA group and H a uniform lattice in G. Then, there exists a section of the quotient G/H, which is Borel measurable.*

Moreover, there exists a section of G/H which is relatively compact.

A section $C \subseteq G$ of G/H is in one to one correspondence with G/H by the *cross-section map* $\tau : G/H \to C$, $[x] \mapsto [x] \cap C$. Therefore, we can carry over the topological and algebraic structure of G/H to C. Moreover, if C is a Borel section, $\tau : G/H \to C$ is measurable with respect to the Borel σ-algebra in G/H and the Borel σ-algebra in G (see [6, Theorem 1]). Therefore, the set value function defined by $m(E) = m_{G/H}(\tau^{-1}(E))$ is well defined on Borel subsets of C. In the next lemma, we will prove that this measure m is equal to m_G up to a constant.

Lemma 2.10. *Let G be an LCA group, H a countable uniform lattice in G and C a Borel section of G/H. Then, for every Borel set $E \subseteq C$*

$$m_G(E) = m_H(\{0\}) m_{G/H}(\tau^{-1}(E)),$$

where τ is the cross-section map.

In particular, $m_G(C) = m_H(\{0\}) m_{G/H}(G/H)$.

Proof. According to Lemma 2.9, there exists a relatively compact section of G/H. Let us call it C'. Therefore, if C is any other Borel section of G/H, it must satisfy $m_G(C) = m_G(C')$. Since C' has finite m_G measure, C must have finite measure as well.

Now, take $E \subseteq C$ a Borel set. Using Theorem 2.7,

$$m_G(E) = \int_G \chi_E(x) \, dm_G(x) = \int_{G/H} \int_H \chi_E(x+h) \, dm_H(h) \, dm_{G/H}([x])$$

$$= m_H(\{0\}) \int_{G/H} \sum_{h \in H} \chi_E(x+h) \, dm_{G/H}([x])$$

$$= m_H(\{0\}) \int_{G/H} \chi_{\tau^{-1}(E)}([x]) \, dm_{G/H}([x])$$

$$= m_H(\{0\}) m_{G/H}(\tau^{-1}(E)). \quad \square$$

Remark 2.11. Notice that C, together with the LCA group structure inherited by G/H through τ, has the Haar measure m. We proved that $m_G|_C$, the restriction of m_G to C, is a multiple of m. It follows that $m_G|_C$ is also a Haar measure on C.

In this paper we will consider C as an LCA group with the structure inherited by G/H and with the Haar measure m_G.

A *trigonometric polynomial* in an LCA group G is a function of the form $P(x) = \sum_{j=1}^n a_j(x, \gamma_j)$, where $\gamma_j \in \Gamma$ and $a_j \in \mathbb{C}$ for all $1 \leqslant j \leqslant n$.

As a consequence of Stone–Weierstrass Theorem, the following result holds, (see [15, p. 24]).

Lemma 2.12. *If G is a compact LCA group, then the trigonometric polynomials are dense in $\mathcal{C}(G)$, where $\mathcal{C}(G)$ is the set of all continuous complex-valued functions on G.*

Another important property of characters in compact groups is the following. For its proof see proof of [15, Theorem 1.2.5].

Lemma 2.13. *Let G be a compact LCA group and Γ its dual. Then, the characters of G verify the following orthogonality relationship*:

$$\int_G (x, \gamma)\overline{(x, \gamma')}\, dm_G(x) = m_G(G)\delta_{\gamma\gamma'},$$

for all $\gamma, \gamma' \in \Gamma$, where $\delta_{\gamma\gamma'} = 1$ if $\gamma = \gamma'$ and $\delta_{\gamma\gamma'} = 0$ if $\gamma \neq \gamma'$.

Let us now suppose that H is a uniform lattice in G. If Γ is the dual group of G and Δ is the annihilator of H, the following characterization of the characters of the group Γ/Δ will be useful to understand what follows.

For each $h \in H$, the function $\gamma \mapsto (h, \gamma)$ is constant on the cosets $[\gamma] = \gamma + \Delta$. Therefore, it defines a character on Γ/Δ. Moreover, each character on Γ/Δ is of this form. Thus, this correspondence between H and the characters of Γ/Δ, which is actually a topological isomorphism, shows the dual relationship established in Theorem 2.3.

Furthermore, since Γ/Δ is compact, we can apply Lemma 2.13 to Γ/Δ. Then, for $h \in H$, we have

$$\int_{\Gamma/\Delta} (h, [\gamma])\, dm_{\Gamma/\Delta}([\gamma]) = \begin{cases} m_{\Gamma/\Delta}(\Gamma/\Delta) & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases} \tag{1}$$

2.3. The Fourier transform on LCA groups

Given a function $f \in L^1(G)$ we define the *Fourier transform* of f, as

$$\widehat{f}(\gamma) = \int_G f(x)(x, -\gamma)\, dm_G(x), \quad \gamma \in \Gamma. \tag{2}$$

Theorem 2.14. *The Fourier transform is a linear operator from $L^1(G)$ into $\mathcal{C}_0(\Gamma)$, where $\mathcal{C}_0(\Gamma)$ is the subspace of $\mathcal{C}(\Gamma)$ of functions vanishing at infinite, that is, $f \in \mathcal{C}_0(\Gamma)$ if $f \in \mathcal{C}(\Gamma)$ and for all $\varepsilon > 0$ there exists a compact set $K \subseteq G$ with $|f(x)| < \varepsilon$ if $x \in K^c$.*
Furthermore, $\wedge : L^1(G) \to \mathcal{C}_0(\Gamma)$ satisfies

$$\widehat{f}(\gamma) = 0 \quad \forall \gamma \in \Gamma \quad \Rightarrow \quad f(x) = 0 \quad a.e. \ x \in G. \tag{3}$$

The Haar measure of the dual group Γ of G, can be normalized so that, for a specific class of functions, the following inversion formula holds (see [15, Section 1.5]),

$$f(x) = \int_\Gamma \widehat{f}(\gamma)(x, \gamma)\, dm_\Gamma(\gamma).$$

In the case that the Haar measures m_G and m_Γ are normalized such that the inversion formula holds, the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\Gamma)$, the so-called Plancherel transformation. We also denote this transformation by "$\wedge$".

Thus, the Parseval formula holds

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)} \, dm_G(x) = \int_\Gamma \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} \, dm_\Gamma(\gamma) = \langle \widehat{f}, \widehat{g} \rangle,$$

where $f, g \in L^2(G)$.

Let us now suppose that G is compact. Then Γ is discrete. Fix m_G and m_Γ in order that the inversion formula holds. Then, using the Fourier transform, we obtain that

$$1 = m_\Gamma(\{0\}) m_G(G). \tag{4}$$

The following lemma is a straightforward consequence of Lemma 2.13, Eq. (1) and statement (3).

Lemma 2.15. *If G is a compact LCA group and its dual Γ is countable, then the characters $\{\gamma : \gamma \in \Gamma\}$ form an orthogonal basis for $L^2(G)$.*

For an LCA group G and a countable uniform lattice H in G, we will denote by Ω a Borel section of Γ/Δ. In the remainder of this paper we will identify $L^2(\Omega)$ with the set $\{\varphi \in L^2(\Gamma) : \varphi = 0 \text{ a.e. } \Gamma \setminus \Omega\}$ and $L^1(\Omega)$ with the set $\{\phi \in L^1(\Gamma) : \phi = 0 \text{ a.e. } \Gamma \setminus \Omega\}$.

Let us now define the functions $\eta_h : \Gamma \mapsto \mathbb{C}$, as $\eta_h(\gamma) = (h, -\gamma)\chi_\Omega(\gamma)$. Using Lemma 2.15 we have:

Proposition 2.16. *Let G be an LCA group and H a countable uniform lattice in G. Then, $\{\eta_h\}_{h \in H}$ is an orthogonal basis for $L^2(\Omega)$.*

Remark 2.17. We can associate to each $\varphi \in L^2(\Omega)$, a function φ' defined on Γ/Δ as $\varphi'([\gamma]) = \sum_{\delta \in \Delta} \varphi(\gamma + \delta)$. The correspondence $\varphi \mapsto \varphi'$, is an isometric isomorphism up to a constant between $L^2(\Omega)$ and $L^2(\Gamma/\Delta)$, since

$$\|\varphi\|^2_{L^2(\Omega)} = m_\Delta(\{0\}) \|\varphi'\|^2_{L^2(\Gamma/\Delta)}.$$

Combining the above remark, Proposition 2.16, and the relationships established in Theorem 2.3, we obtain the following proposition, which will be very important on the remainder of the paper.

Proposition 2.18. *Let G be an LCA group, H countable uniform lattice on G, $\Gamma = \widehat{G}$ and Δ the annihilator of H. Fix Ω a Borel section of Γ/Δ and choose m_H and $m_{\Gamma/\Delta}$ such that the inversion formula holds. Then*

$$\|a\|_{\ell^2(H)} = \frac{m_H(\{0\})^{1/2}}{m_\Gamma(\Omega)^{1/2}} \left\| \sum_{h \in H} a_h \eta_h \right\|_{L^2(\Omega)},$$

for each $a = \{a_h\}_{h \in H} \in \ell^2(H)$.

Proof. Let $a \in \ell^2(H)$. So,

$$\|a\|_{\ell^2(H)} = \|\widehat{a}\|_{L^2(\Gamma/\Delta)}, \tag{5}$$

since $\widehat{H} \approx \Gamma/\Delta$ and therefore $\wedge : H \to \Gamma/\Delta$.

Take $\varphi(\gamma) = \sum_{h \in H} a_h(h, -\gamma)\chi_\Omega(\gamma)$. Then, by Proposition 2.16, $\varphi \in L^2(\Omega)$. Furthermore, $\varphi'([\gamma]) = \varphi(\gamma)$, a.e. $\gamma \in \Omega$. So, as a consequence of Remark 2.17, we have

$$\left\|\varphi'\right\|^2_{L^2(\Gamma/\Delta)} = \frac{1}{m_\Delta(\{0\})}\|\varphi\|^2_{L^2(\Omega)}. \tag{6}$$

Now, $\widehat{a}([\gamma]) = m_H(\{0\}) \sum_{h \in H} a_h(h, -[\gamma])$. Therefore, substituting in Eqs. (5) and (6),

$$\|a\|_{\ell^2(H)} = \frac{m_H(\{0\})}{m_\Delta(\{0\})^{1/2}}\|\varphi\|_{L^2(\Omega)}.$$

Finally, since $m_\Gamma(\Omega) = m_\Delta(\{0\})m_{\Gamma/\Delta}(\Gamma/\Delta)$, using (4) we have that

$$\frac{m_H(\{0\})}{m_\Delta(\{0\})^{1/2}} = \frac{m_H(\{0\})^{1/2}}{m_\Gamma(\Omega)^{1/2}},$$

which completes the proof. $\square$

We finish this section with a result which is a consequence of statement (3) and Theorem 2.7.

Proposition 2.19. *Let G, H and Ω as in Proposition* 2.18. *If $\phi \in L^1(\Omega)$ and $\widehat{\phi}(h) = 0$ for all $h \in H$, then $\phi(\omega) = 0$ a.e. $\omega \in \Omega$.*

3. H-invariant spaces

In this section we extend the theory of shift-invariant spaces in $\mathbb{R}^d$ to general LCA groups. We will develop the concept of range function and the techniques of fiberization in this general context. The treatment will be for shift-invariant spaces following the lines of Bownik [1]. The conclusions for doubly invariant spaces will follow via the Plancherel theorem for the Fourier transform on LCA groups.

First we will fix some notation and set our standing assumptions that will be in force for the remainder of the manuscript.

Standing Assumptions 3.1. We will assume throughout the next sections that.

- G is a second countable LCA group.
- H is a countable uniform lattice on G.

We denote, as before, by Γ the dual group of G, by Δ the annihilator of H, and by Ω a fixed Borel section of Γ/Δ.

The choice of particular Haar measure in each of the groups considered in this paper does not affect the validity of the results. However, different constants will appear in the formulas.

Since we have the freedom to choose the Haar measures, we will fix the following normalization in order to avoid carrying over constants through the paper and to simplify the statements of the results.

We choose m_Δ and m_H such that $m_\Delta(\{0\}) = m_H(\{0\}) = 1$. We fix $m_{\Gamma/\Delta}$ such that $m_{\Gamma/\Delta}(\Gamma/\Delta) = 1$ and therefore the inversion formula holds between H and Γ/Δ. Then, we set m_Γ such that Theorem 2.7 holds for m_Γ, $m_{\Gamma/\Delta}$ and m_Δ. Finally, we normalize m_G such that the inversion formula holds for m_Γ and m_G.

As a consequence of the normalization given above and Lemma 2.10, it holds that $m_\Gamma(\Omega) = 1$. Note that under our Standing Assumptions 3.1, Theorem 2.6 applies. So we will use the properties of G, H, Γ and Δ stated in that theorem.

3.1. Preliminaries

The space $L^2(\Omega, \ell^2(\Delta))$ is the space of all measurable functions $\Phi : \Omega \to \ell^2(\Delta)$ such that

$$\int_\Omega \|\Phi(\omega)\|^2_{\ell^2(\Delta)} \, dm_\Gamma(\omega) < \infty,$$

where a function $\Phi : \Omega \to \ell^2(\Delta)$ is measurable, if and only if for each $a \in \ell^2(\Delta)$ the function $\omega \mapsto \langle \Phi(\omega), a \rangle$ is a measurable function from Ω into $\mathbb{C}$.

Remark 3.2. This is the usual notion of weak measurability for vector functions. If the values of the functions are in a separable space, as a consequence of Petti's Theorem, the notions of weak and strong measurability agree. As we have seen in Section 2 and according to our hypotheses, Δ is a countable uniform lattice on Γ. Therefore, $\ell^2(\Delta)$ is a separable Hilbert space. Then, in $L^2(\Omega, \ell^2(\Delta))$ we have only one measurability notion.

The space $L^2(\Omega, \ell^2(\Delta))$, with the inner product

$$\langle \Phi, \Psi \rangle := \int_\Omega \big\langle \Phi(\omega), \Psi(\omega) \big\rangle_{\ell^2(\Delta)} \, dm_\Gamma(\omega)$$

is a complex Hilbert space.

Note that for $\Phi \in L^2(\Omega, \ell^2(\Delta))$ and $\omega \in \Omega$

$$\|\Phi(\omega)\|_{\ell^2(\Delta)} = \bigg(\sum_{\delta \in \Delta} \big|(\Phi(\omega))_\delta\big|^2 \bigg)^{1/2},$$

where $(\Phi(\omega))_\delta$ denotes the value of the sequence $\Phi(\omega)$ in δ. If $\Phi \in L^2(\Omega, \ell^2(\Delta))$, the sequence $\Phi(\omega)$ is the *fiber of Φ at ω*.

The following proposition shows that the space $L^2(\Omega, \ell^2(\Delta))$ is isometric to $L^2(G)$.

Proposition 3.3. *The mapping $\mathcal{T} : L^2(G) \longrightarrow L^2(\Omega, \ell^2(\Delta))$ defined as*

$$\mathcal{T} f(\omega) = \big\{ \widehat{f}(\omega + \delta) \big\}_{\delta \in \Delta},$$

is an isomorphism that satisfies $\|\mathcal{T} f\|_2 = \|f\|_{L^2(G)}$.

The next periodization lemma will be necessary for the proof of Proposition 3.3.

Lemma 3.4. *Let $g \in L^2(\Gamma)$. Define the function $\mathcal{G}(\omega) = \sum_{\delta \in \Delta} |g(\omega + \delta)|^2$. Then, $\mathcal{G} \in L^1(\Omega)$ and moreover*

$$\|g\|_{L^2(\Gamma)} = \|\mathcal{G}\|_{L^1(\Omega)}.$$

Proof. Since Ω is a section of Γ/Δ, we have that $\Gamma = \bigcup_{\delta \in \Delta} \Omega - \delta$, where the union is disjoint. Therefore,

$$\int_\Gamma |g(\gamma)|^2 \, dm_\Gamma(\gamma) = \sum_{\delta \in \Delta} \int_{\Omega - \delta} |g(\omega)|^2 \, dm_\Gamma(\omega)$$

$$= \sum_{\delta \in \Delta} \int_\Omega |g(\omega + \delta)|^2 \, dm_\Gamma(\omega)$$

$$= \int_\Omega \sum_{\delta \in \Delta} |g(\omega + \delta)|^2 \, dm_\Gamma(\omega).$$

This proves that $\mathcal{G} \in L^1(\Omega)$ and $\|g\|_{L^2(\Gamma)} = \|\mathcal{G}\|_{L^1(\Omega)}$. $\square$

Proof of Proposition 3.3. First we prove that $\mathcal{T}$ is well defined. For this we must show that, $\forall f \in L^2(G)$, the vector function $\mathcal{T}f$ is measurable and $\|\mathcal{T}f\|_2 < \infty$.

According to Lemma 3.4, the sequence $\{\widehat{f}(\omega + \delta)\}_{\delta \in \Delta} \in \ell^2(\Delta)$, a.e. $\omega \in \Omega$, for all $f \in L^2(G)$. Then, given $a = \{a_\delta\}_{\delta \in \Delta} \in \ell^2(\Delta)$, the product $\langle \mathcal{T}f(\omega), a \rangle = \sum_{\delta \in \Delta} \widehat{f}(\omega + \delta) a_\delta$ is finite a.e. $\omega \in \Omega$. From here the measurability of f implies that $\omega \mapsto \langle \mathcal{T}f(\omega), a \rangle$ is a measurable function in the usual sense. This proves the measurability of $\mathcal{T}f$.

If $f \in L^2(G)$, as a consequence of Lemma 3.4, we have

$$\|\mathcal{T}f\|_2^2 = \int_\Omega \|\mathcal{T}f(\omega)\|_{\ell^2(\Delta)}^2 \, dm_\Gamma(\omega)$$

$$= \int_\Omega \sum_{\delta \in \Delta} |\widehat{f}(\omega + \delta)|^2 \, dm_\Gamma(\omega)$$

$$= \int_\Gamma |\widehat{f}(\gamma)|^2 \, dm_\Gamma(\gamma)$$

$$= \int_G |f(x)|^2 \, dm_G(x).$$

Thus, $\|\mathcal{T}f\|_2 < \infty$ and this also proves that $\|\mathcal{T}f\|_2 = \|f\|_{L^2(G)}$.

What is left is to show that $\mathcal{T}$ is onto. So, given $\Phi \in L^2(\Omega, \ell^2(\Delta))$ let us see that there exists a function $f \in L^2(G)$ such that $\mathcal{T}f = \Phi$. Using that the Fourier transform is an isometric isomorphism between $L^2(G)$ and $L^2(\Gamma)$, it will be sufficient to find $g \in L^2(\Gamma)$ such that $\{g(\omega + \delta)\}_{\delta \in \Delta} = \Phi(\omega)$ a.e. $\omega \in \Omega$ and then take $f \in L^2(G)$ such that $\widehat{f} = g$.

Given $\gamma \in \Gamma$, there exist unique $\omega \in \Omega$ and $\delta \in \Delta$ such that $\gamma = \omega + \delta$. So, we define $g(\gamma)$ as

$$g(\gamma) = \big(\Phi(\omega)\big)_\delta.$$

The measurability of g is straightforward.

Once again, according to Lemma 3.4,

$$
\begin{aligned}
\int_\Gamma |g(\gamma)|^2 \, dm_\Gamma(\gamma) &= \int_\Omega \sum_{\delta \in \Delta} |g(\omega + \delta)|^2 \, dm_\Gamma(\omega) \\
&= \int_\Omega \sum_{\delta \in \Delta} |(\Phi(\omega)_\delta)|^2 \, dm_\Gamma(\omega) \\
&= \int_\Omega \|\Phi(\omega)\|_{\ell^2(\Delta)}^2 \, dm_\Gamma(\omega) \\
&= \|\Phi\|_2^2 < +\infty.
\end{aligned}
$$

Thus, $g \in L^2(\Gamma)$ and this completes the proof. $\quad\square$

The mapping $\mathcal{T}$ will be important to study the properties of functions of $L^2(G)$ in terms of their fibers, (i.e. in terms of the fibers $\mathcal{T}f(\omega)$).

3.2. H-invariant spaces and range functions

Definition 3.5. We say that a closed subspace $V \subseteq L^2(G)$ is *H-invariant* if

$$f \in V \Rightarrow t_h f \in V \quad \forall h \in H,$$

where $t_y f(x) = f(x - y)$ denotes the translation of f by an element y of G.

For a subset $\mathcal{A} \subseteq L^2(G)$, we define

$$E_H(\mathcal{A}) = \{t_h \varphi \colon \varphi \in \mathcal{A}, \ h \in H\} \quad \text{and} \quad S(\mathcal{A}) = \overline{\operatorname{span}} \, E_H(\mathcal{A}).$$

We call $S(\mathcal{A})$ the *H-invariant space generated by* $\mathcal{A}$. If $\mathcal{A} = \{\varphi\}$, we simply write $E_H(\varphi)$ and $S(\varphi)$, and we call $S(\varphi)$ a *principal H-invariant space*.

Our main goal is to give a characterization of *H*-invariant spaces. We first need to introduce the concept of range function.

Definition 3.6. A *range function* is a mapping

$$J : \Omega \longrightarrow \{\text{closed subspaces of } \ell^2(\Delta)\}.$$

The subspace $J(\omega)$ is called the *fiber space* associated to ω.

For a given range function J, we associate to each $\omega \in \Omega$ the orthogonal projection onto $J(\omega)$, $P_\omega : \ell^2(\Delta) \to J(\omega)$.

A range function J is *measurable* if for each $a \in \ell^2(\Delta)$ the function $\omega \mapsto P_\omega a$, from Ω into $\ell^2(\Delta)$, is measurable. That is, for each $a, b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega a, b \rangle$ is measurable in the usual sense.

Remark 3.7. Note that J is a measurable range function if and only if for all $\Phi \in L^2(\Omega, \ell^2(\Delta))$, the function $\omega \mapsto P_\omega(\Phi(\omega))$ is measurable. That is, $\forall b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega(\Phi(\omega)), b \rangle$ is measurable in the usual sense.

Given a range function J (not necessarily measurable) we define the subset M_J as

$$M_J = \left\{ \Phi \in L^2(\Omega, \ell^2(\Delta)): \ \Phi(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega \right\}.$$

Lemma 3.8. *The subset M_J is closed in $L^2(\Omega, \ell^2(\Delta))$.*

Proof. Let $\{\Phi_j\}_{j \in \mathbb{N}} \subseteq M_J$ such that $\Phi_j \to \Phi$ when $j \to \infty$ in $L^2(\Omega, \ell^2(\Delta))$. Let us consider the functions $g_j : \Omega \to \mathbb{R}_{\geqslant 0}$ defined as $g_j(\omega) := \|\Phi_j(\omega) - \Phi(\omega)\|^2_{\ell^2(\Delta)}$. Then, g_j is measurable for all $j \in \mathbb{N}$ and $\forall \alpha > 0$ it holds that

$$m_\Gamma(\{g_j > \alpha\}) \leqslant \frac{1}{\alpha} \int_\Omega g_j(\omega) \, dm_\Gamma(\omega) = \frac{1}{\alpha} \int_\Omega \|\Phi_j(\omega) - \Phi(\omega)\|^2_{\ell^2(\Delta)} \, dm_\Gamma(\omega) \to 0,$$

when $j \to \infty$. So, $g_j \to 0$ in measure and therefore, there exists a subsequence $\{g_{j_k}\}_{k \in \mathbb{N}}$ of $\{g_j\}_{j \in \mathbb{N}}$ which goes to zero a.e. $\omega \in \Omega$. Then, $\Phi_{j_k}(\omega) \to \Phi(\omega)$ in $\ell^2(\Delta)$ a.e. $\omega \in \Omega$ and hence, since $\Phi_{j_k}(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$ and $J(\omega)$ is closed, $\Phi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$. Therefore $\Phi \in M_J$. $\quad\square$

The following proposition is a generalization to the context of groups of a lemma of Helson, (see [7] and also [1]).

Proposition 3.9. *Let J be a measurable range function and P_ω the associated orthogonal projections. Denote by $\mathcal{P}$ the orthogonal projection onto M_J. Then,*

$$(\mathcal{P}\Phi)(\omega) = P_\omega(\Phi(\omega)), \quad a.e. \ \omega \in \Omega, \quad \forall \Phi \in L^2(\Omega, \ell^2(\Delta)).$$

Proof. Let $\mathcal{Q} : L^2(\Omega, \ell^2(\Delta)) \to L^2(\Omega, \ell^2(\Delta))$ be the linear mapping $\Phi \mapsto \mathcal{Q}\Phi$, where

$$(\mathcal{Q}\Phi)(\omega) := P_\omega(\Phi(\omega)).$$

We want to show that $\mathcal{Q} = \mathcal{P}$.

Since J is a measurable range function, due to Remark 3.7, $\mathcal{Q}\Phi$ is measurable for each $\Phi \in L^2(\Omega, \ell^2(\Delta))$. Furthermore, since P_ω is an orthogonal projection, it has norm one, and therefore

$$\|\mathcal{Q}\Phi\|^2_2 = \int_\Omega \|(\mathcal{Q}\Phi)(\omega)\|^2_{\ell^2(\Delta)} \, dm_\Gamma(\omega) = \int_\Omega \|P_\omega(\Phi(\omega))\|^2_{\ell^2(\Delta)} \, dm_\Gamma(\omega)$$

$$\leqslant \int_\Omega \|\Phi(\omega)\|^2_{\ell^2(\Delta)} \, dm_\Gamma(\omega) = \|\Phi\|^2_2 < \infty.$$

Then, $\mathcal{Q}$ is well defined and it has norm less or equal to 1.

From the fact that P_ω is an orthogonal projection, it follows that $\mathcal{Q}^2 = \mathcal{Q}$ and $\mathcal{Q}^* = \mathcal{Q}$. So, $\mathcal{Q}$ is also an orthogonal projection. To complete our proof let us see that $M = M_J$, where $M :=$ $Ran(\mathcal{Q})$.

By definition of $\mathcal{Q}$, $M \subseteq M_J$.

If we suppose that M is properly included in M_J, then there exists $\Psi \in M_J$ such that $\Psi \neq 0$ and $\Psi \perp M$. Then, $\forall \Phi \in L^2(\Omega, \ell^2(\Delta))$, $0 = \langle \mathcal{Q}\Phi, \Psi \rangle = \langle \Phi, \mathcal{Q}\Psi \rangle$.

Hence, $\mathcal{Q}\Psi = 0$ and therefore $P_\omega(\Psi(\omega)) = 0$ a.e. $\omega \in \Omega$. Since $\Psi \in M_J$, $\Psi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$, thus $P_\omega(\Psi(\omega)) = \Psi(\omega)$ a.e. $\omega \in \Omega$. Finally, $\Psi = 0$ a.e. $\omega \in \Omega$ and this is a contradiction. $\square$

We now give a characterization of H-invariant spaces using range functions.

Theorem 3.10. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}$ the map defined in Proposition 3.3. Then, V is H-invariant if and only if there exists a measurable range function J such that*

$$V = \big\{ f \in L^2(G) \colon \mathcal{T}f(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega \big\}.$$

Identifying range functions which are equal almost everywhere, the correspondence between H-invariant spaces and measurable range functions is one to one and onto.

Moreover, if $V = S(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$, the measurable range function J associated to V is given by

$$J(\omega) = \overline{\text{span}}\big\{ \mathcal{T}\varphi(\omega) \colon \varphi \in \mathcal{A} \big\}, \quad \text{a.e. } \omega \in \Omega.$$

For the proof, we need the following results.

Lemma 3.11. *If J and K are two measurable range functions such that $M_J = M_K$, then $J(\omega) = K(\omega)$ a.e. $\omega \in \Omega$. That is, J and K are equal almost everywhere.*

Proof. Let P_ω and Q_ω be the projections associate to J and K respectively. If $\mathcal{P}$ is the orthogonal projection onto $M_J = M_K$, by Proposition 3.9 we have that, for each $\Phi \in L^2(\Omega, \ell^2(\Delta))$

$$(\mathcal{P}\Phi)(\omega) = P_\omega\big(\Phi(\omega)\big) \quad \text{and} \quad (\mathcal{P}\Phi)(\omega) = Q_\omega\big(\Phi(\omega)\big) \quad \text{a.e. } \omega \in \Omega.$$

So, $P_\omega(\Phi(\omega)) = Q_\omega(\Phi(\omega))$ a.e. $\omega \in \Omega$, for all $\Phi \in L^2(\Omega, \ell^2(\Delta))$. In particular, if $e_\lambda \in \ell^2(\Delta)$ is defined by $(e_\lambda)_\delta = 1$ if $\delta = \lambda$ and $(e_\lambda)_\delta = 0$ otherwise, $P_\omega(e_\lambda) = Q_\omega(e_\lambda)$ a.e. $\omega \in \Omega$, for all $\lambda \in \Delta$. Hence, since $\{e_\lambda\}_{\lambda \in \Delta}$ is a basis for $\ell^2(\Delta)$, it follows that $P_\omega = Q_\omega$ a.e. $\omega \in \Omega$. Thus $J(\omega) = K(\omega)$ a.e. $\omega \in \Omega$. $\square$

Remark 3.12. Note that for $f \in L^2(G)$ and for $h \in H$,

$$\mathcal{T}t_h f(\omega) = (h, -\omega)\mathcal{T}f(\omega),$$

since $\forall y \in G$, $\widehat{t_y f}(\gamma) = (y, -\gamma)\widehat{f}(\gamma)$ and, as we showed in Remark 2.4, the character $(h, .)$ is Δ-periodic.

Proof of Theorem 3.10. Let us first suppose that V is H-invariant. Since $L^2(G)$ is separable, $V = S(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$.

We define the function J as $J(\omega) = \overline{\text{span}}\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$. Note that since $\mathcal{A}$ is a countable set, J is well defined a.e. $\omega \in \Omega$. We will prove that J satisfies:

(i) $V = \{f \in L^2(G): \mathcal{T}f(\omega) \in J(\omega) \text{ a.e. } \omega \in \Omega\}$,
(ii) J is measurable.

To show (i) it is sufficient to prove that $M = M_J$, where $M := \mathcal{T}V$. Let $\Phi \in M$. Then, $\mathcal{T}^{-1}\Phi \in V = \overline{\text{span}}\{t_h\varphi: h \in H, \varphi \in \mathcal{A}\}$. Therefore, there exists a sequence $\{g_j\}_{j\in\mathbb{N}} \subseteq \text{span}\{t_h\varphi: h \in H, \varphi \in \mathcal{A}\}$ such that $\mathcal{T}g_j := \Phi_j$ converges in $L^2(\Omega, \ell^2(\Delta))$ to Φ, when $j \to \infty$.

Due to the definition of J and Remark 3.12, $\Phi_j(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$. Thus, in the same way that in Lemma 3.8, we can prove that $\Phi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$ and therefore $\Phi \in M_J$. So, $M \subseteq M_J$.

Let us suppose that there exists $\Psi \in L^2(\Omega, \ell^2(\Delta))$, such that $\Psi \neq 0$ and Ψ is orthogonal to M. Then, for each $\Phi \in M$, $\langle \Phi, \Psi \rangle = 0$. In particular, if $\Phi \in \mathcal{T}\mathcal{A} \subseteq \mathcal{T}V = M$ and $h \in H$, we have that $(h, .)\Phi(.) \in \mathcal{T}V = M$ since $(h, .)\Phi(.) = \mathcal{T}(t_{-h}\mathcal{T}^{-1}\Phi)(.)$ and $t_{-h}\mathcal{T}^{-1}\Phi \in V$.

So, as $(h, .)$ is Δ-periodic,

$$0 = \langle (h, .)\Phi(.), \Psi \rangle = \int_\Omega (h, \omega) \langle \Phi(\omega), \Psi(\omega) \rangle_{\ell^2(\Delta)} \, dm_\Gamma(\omega).$$

Hence, by Proposition 2.19, $\langle \Phi(\omega), \Psi(\omega) \rangle_{\ell^2(\Delta)} = 0$ a.e. $\omega \in \Omega$, and this holds $\forall \Phi \in \mathcal{T}(\mathcal{A})$. Therefore $\Psi(\omega) \in J(\omega)^\perp$ a.e. $\omega \in \Omega$.

Now, if M is properly included in M_J, there exists $\Psi \in M_J$, with $\Psi \neq 0$ and orthogonal to M. Hence, $\Psi(\omega) \in J(\omega)^\perp$ a.e. $\omega \in \Omega$. On the other hand since $\Psi \in M_J$, $\Psi(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$. Thus, $\Psi(\omega) = 0$ a.e. $\omega \in \Omega$ and this is a contradiction. Therefore $M = M_J$.

It remains to prove that the range function J is measurable. For this we must show that, for all $a, b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega a, b \rangle$ is measurable, where $P_\omega : \ell^2(\Delta) \to J(\omega)$ are the orthogonal projections associated to $J(\omega)$,

Let $\mathcal{I}$ be the identity mapping in $L^2(\Omega, \ell^2(\Delta))$ and $\mathcal{P} : L^2(\Omega, \ell^2(\Delta)) \to M$ the orthogonal projection associated to M. If $\Psi \in L^2(\Omega, \ell^2(\Delta))$, the function $(\mathcal{I} - \mathcal{P})\Psi$ is orthogonal to M and, by the above reasoning, $(\mathcal{I} - \mathcal{P})\Psi(\omega) \in J(\omega)^\perp$, a.e. $\omega \in \Omega$. Then,

$$P_\omega\big((\mathcal{I} - \mathcal{P})\Psi(\omega)\big) = P_\omega\big(\Psi(\omega) - \mathcal{P}\Psi(\omega)\big) = 0$$

a.e. $\omega \in \Omega$ and therefore $P_\omega(\Psi(\omega)) = P_\omega(\mathcal{P}\Psi(\omega)) = \mathcal{P}\Psi(\omega)$ a.e $\omega \in \Omega$. In particular, $P_\omega a = \mathcal{P}a(\omega)$ a.e. $\omega \in \Omega$, $\forall a \in \ell^2(\Delta)$. Thus, since $\omega \mapsto \langle \mathcal{P}a(\omega)a, b \rangle$ is measurable $\forall b \in \ell^2(\Delta)$, $\omega \mapsto \langle P_\omega a, b \rangle$ is measurable as well.

Conversely, if J is a measurable range function, let us see that the closed subspace in $L^2(G)$, defined by $V := \mathcal{T}^{-1}(M_J)$ is H-invariant. For this, let us consider $f \in V$ and $h \in H$ and let us prove that $t_h f \in V$.

Since $\mathcal{T}(t_h f)(\omega) = (h, -\omega)\mathcal{T}f(\omega)$ a.e. $\omega \in \Omega$ and $\mathcal{T}f \in M_J$, we have that $(h, -\omega)\mathcal{T}f(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$. Then, $\mathcal{T}(t_h f) \in M_J$ and therefore $t_h f \in V$.

Furthermore, $V = S(\mathcal{A})$ for some countable set $\mathcal{A}$ of $L^2(G)$. Then,

$$K(\omega) = \overline{\text{span}}\big\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\big\} \quad \text{a.e. } \omega \in \Omega,$$

defines a measurable range function which satisfies $V = \mathcal{T}^{-1}(M_K)$. Thus, $M_K = \mathcal{T}V = M_J$. Since J and K are both measurable range functions, Lemma 3.11 implies that $J = K$ a.e. $\omega \in \Omega$.

This also shows that the correspondence between V and J is onto and one to one. $\quad\square$

4. Frames and Riesz basis for H-invariant spaces

Let $\mathcal{H}$ be a Hilbert space and $\{u_i\}_{i \in I}$ a sequence of $\mathcal{H}$.

The sequence $\{u_i\}_{i \in I}$ is a *Bessel sequence* in $\mathcal{H}$ with constant B if

$$\sum_{i \in I} |\langle f, u_i \rangle|^2 \leqslant B \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

The sequence $\{u_i\}_{i \in I}$ is a *frame* for $\mathcal{H}$ with constants A and B if

$$A \|f\|^2 \leqslant \sum_{i \in I} |\langle f, u_i \rangle|^2 \leqslant B \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

The frame $\{u_i\}_{i \in I}$ is a *tight frame* if $A = B$, and the frame $\{u_i\}_{i \in I}$ is a *Parseval frame* if $A = B = 1$.

The sequence $\{u_i\}_{i \in I}$ is a *Riesz sequence* for $\mathcal{H}$ if there exist positive constants A and B such that

$$A \sum_{i \in I} |a_i|^2 \leqslant \left\| \sum_{i \in I} a_i u_i \right\|_{\mathcal{H}}^2 \leqslant B \sum_{i \in I} |a_i|^2$$

for all $\{a_i\}_{i \in I}$ with finite support. Moreover, a Riesz sequence that in addition, is a complete family in $\mathcal{H}$, is a *Riesz basis* for $\mathcal{H}$.

We are now ready to prove a result which characterizes when $E_H(\mathcal{A})$ is a frame of $L^2(G)$ in terms of the fibers $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$. It generalizes Theorem 2.3 of [1] to the context of groups.

Theorem 4.1. *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated to $S(\mathcal{A})$ and $A \leqslant B$ positive constants. Then, the following propositions are equivalent*:

(i) *The set $E_H(\mathcal{A})$ is a frame for $S(\mathcal{A})$ with constants A and B.*
(ii) *For almost every $\omega \in \Omega$, the set $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a frame for $J(\omega)$ with constants A and B.*

Proof. Since $\langle f, g \rangle_{L^2(G)} = \langle \mathcal{T}f, \mathcal{T}g \rangle_{L^2(\Omega, \ell^2(\Delta))}$, by Remark 3.12 we have that

$$\sum_{h \in H} \sum_{\varphi \in \mathcal{A}} |\langle t_h \varphi, f \rangle_{L^2(G)}|^2 = \sum_{h \in H} \sum_{\varphi \in \mathcal{A}} |\langle \mathcal{T}(t_h \varphi), \mathcal{T}f \rangle_{L^2(\Omega, \ell^2(\Delta))}|^2$$

$$= \sum_{\varphi \in \mathcal{A}} \sum_{h \in H} \left| \int_\Omega (h, -\omega) \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)} \, dm_\Gamma(\omega) \right|^2.$$

Let us define for each $\varphi \in \mathcal{A}$, the following,

$$R(\varphi) = \sum_{h \in H} \left| \int_{\Omega} (h, -\omega) \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)} \, dm_\Gamma(\omega) \right|^2$$

and

$$T(\varphi) = \int_{\Omega} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)} \right|^2 dm_\Gamma(\omega).$$

(i) $\Rightarrow$ (ii) If $E_H(\mathcal{A})$ is a frame for $S(\mathcal{A})$, in particular it holds that $\forall f \in S(\mathcal{A})$, $\sum_{h \in H} \sum_{\varphi \in \mathcal{A}} |\langle t_h \varphi, f \rangle|^2 < \infty$.

Then, for each $\varphi \in \mathcal{A}$, we have that $R(\varphi) < \infty$. Therefore, the sequence $\{c_h\}_{h \in H}$, with

$$c_h := \int_{\Omega} (h, \omega) \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)} \, dm_\Gamma(\omega),$$

belongs to $\ell^2(H)$.

Let us consider the function $F(\omega) := \sum_{h \in H} c_h \eta_h(\omega)$, where η_h are the functions defined in Lemma 2.16. Then, since $\{c_h\}_{h \in H} \in \ell^2(H)$ and $\{\eta_h\}_{h \in H}$ is an orthogonal basis of $L^2(\Omega)$, we have that $F \in L^2(\Omega) \subseteq L^1(\Omega)$ (recall that $m_\Gamma(\Omega) < \infty$).

On the other hand, the function $\psi(\omega) := \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)}$ belongs to $L^1(\Omega)$. So, $\psi - F \in L^1(\Omega)$ and moreover

$$\int_{\Omega} (h, -\omega) \big(\psi(\omega) - F(\omega) \big) \, dm_\Gamma(\omega) = c_{-h} - c_{-h} = 0$$

for all $h \in H$. Thus, Proposition 2.19 yields that $F = \psi$ a.e. $\omega \in \Omega$. Therefore $\psi \in L^2(\Omega)$ and

$$\psi(\omega) = \sum_{h \in H} c_h \eta_h(\omega),$$

a.e. $\omega \in \Omega$.

As a consequence of Proposition 2.18, we obtain that $R(\varphi) = T(\varphi)$ holds for all $\varphi \in \mathcal{A}$.

We will now prove that, for almost every $\omega \in \Omega$, $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$ is a frame with constants A and B for $J(\omega)$.

Let us suppose that

$$A \| P_\omega d \|^2_{\ell^2(\Delta)} \leqslant \sum_{\varphi \in \mathcal{A}} \left| \langle \mathcal{T}\varphi(\omega), P_\omega d \rangle \right|^2 \leqslant B \| P_\omega d \|^2_{\ell^2(\Delta)} \tag{7}$$

a.e. $\omega \in \Omega$, for each $d \in \mathcal{D}$, where $\mathcal{D}$ is a dense countable subset of $\ell^2(\Delta)$ and P_ω are the orthogonal projections associated to J. Then, for each $d \in \mathcal{D}$, let $Z_d \subseteq \Omega$ be a measurable set

with $m_\Gamma(Z_d) = 0$ such that (7) holds for all $\omega \in \Omega \setminus Z_d$. So the set $Z = \bigcup_{d \in \mathcal{D}} Z_d$ has null m_Γ-measure. Therefore for $\omega \in \Omega \setminus Z$ and $a \in J(\omega)$, using a density argument it follows from (7) that

$$A\|a\|^2 \leqslant \sum_{\varphi \in \mathcal{A}} \left|\langle \mathcal{T}\varphi(\omega), a \rangle\right|^2 \leqslant B\|a\|^2.$$

Thus, it is sufficient to show that (7) holds. For this, we will suppose that this is not so and we will prove that there exist $d_0 \in \mathcal{D}$, a measurable set $W \subseteq \Omega$ with $m_\Gamma(W) > 0$, and $\varepsilon > 0$ such that

$$\sum_{\varphi \in \mathcal{A}} \left|\langle \mathcal{T}\varphi(\omega), P_\omega d_0 \rangle\right|^2 > (B + \varepsilon)\|P_\omega d_0\|^2, \quad \forall \omega \in W$$

or

$$\sum_{\varphi \in \mathcal{A}} \left|\langle \mathcal{T}\varphi(\omega), P_\omega d_0 \rangle\right|^2 < (A - \varepsilon)\|P_\omega d_0\|^2, \quad \forall \omega \in W.$$

So, let us take $d_0 \in \mathcal{D}$ for which (7) fails. Then at least one of this sets

$$\left\{\omega \in \Omega\colon K(\omega) - B\|P_\omega d_0\|^2 > 0\right\}, \qquad \left\{\omega \in \Omega\colon K(\omega) - A\|P_\omega d_0\|^2 < 0\right\}$$

has positive measure, where $K(\omega) := \sum_{\varphi \in \mathcal{A}} |\langle \mathcal{T}\varphi(\omega), P_\omega d_0 \rangle|^2$. Let us suppose, without loss of generality, that

$$m_\Gamma\left(\left\{\omega \in \Omega\colon K(\omega) - B\|P_\omega d_0\|^2 > 0\right\}\right) > 0.$$

Since

$$\left\{\omega \in \Omega\colon K(\omega) - B\|P_\omega d_0\|^2 > 0\right\} = \bigcup_{j \in \mathbb{N}}\left\{\omega \in \Omega\colon K(\omega) - \left(B + \frac{1}{j}\right)\|P_\omega d_0\|^2 > 0\right\},$$

there exists at least one set in the union, in the right-hand side of this equality, with positive measure and this proves our claim.

Then, we can suppose that

$$\sum_{\varphi \in \mathcal{A}} \left|\langle \mathcal{T}\varphi(\omega), P_\omega d_0 \rangle\right|^2 > (B + \varepsilon)\|P_\omega d_0\|^2, \quad \forall \omega \in W \tag{8}$$

holds. Now take $f \in S(\mathcal{A})$ such that $\mathcal{T}f(\omega) = \chi_W(\omega)P_\omega d_0$. Note that this is possible since, by Theorem 3.10, $\chi_E(\omega)P_\omega d_0$ is a measurable function.

As $E_H(\mathcal{A})$ is a frame for $S(\mathcal{A})$ and

$$\sum_{h \in H}\sum_{\varphi \in \mathcal{A}}\left|\langle t_h\varphi, f \rangle_{L^2(G)}\right|^2 = \sum_{\varphi \in \mathcal{A}}\int_\Omega \left|\langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)}\right|^2 dm_\Gamma(\omega),$$

we have that

$$A\|f\|^2 \leqslant \sum_{\varphi \in \mathcal{A}} \int_{\Omega} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle \right|^2 dm_\Gamma(\omega) \leqslant B\|f\|^2. \tag{9}$$

Using Proposition 3.3, we can rewrite (9) as

$$A\|\mathcal{T}f\|^2 \leqslant \sum_{\varphi \in \mathcal{A}} \int_{\Omega} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle \right|^2 dm_\Gamma(\omega) \leqslant B\|\mathcal{T}f\|^2. \tag{10}$$

Now,

$$\|\mathcal{T}f\|^2 = \int_{\Omega} \chi_W(\omega) \|P_\omega d_0\|^2 dm_\Gamma(\omega)$$

and if we integrate in (8) over W, we obtain

$$\sum_{\varphi \in \mathcal{A}} \int_{\Omega} \left| \langle \mathcal{T}\varphi(\omega), \chi_W(\omega) P_\omega d_0 \rangle \right|^2 dm_\Gamma(\omega) \geqslant (B + \varepsilon)\|\mathcal{T}f\|^2.$$

This is a contradiction with inequality (10). Therefore, we proved inequality (7).

(ii) $\Rightarrow$ (i) If now $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$ is a frame for $J(\omega)$ a.e. $\omega \in \Omega$ with constants A and B, we have that

$$A\|a\|^2 \leqslant \sum_{\varphi \in \mathcal{A}} \left| \langle \mathcal{T}\varphi(\omega), a \rangle \right|^2 \leqslant B\|a\|^2$$

for all $a \in J(\omega)$. In particular, if $f \in S(\mathcal{A})$, by Theorem 3.10, $\mathcal{T}f(\omega) \in J(\omega)$ a.e. $\omega \in \Omega$ and then,

$$A\|\mathcal{T}f(\omega)\|^2 \leqslant \sum_{\varphi \in \mathcal{A}} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle \right|^2 \leqslant B\|\mathcal{T}f(\omega)\|^2 \tag{11}$$

a.e. $\omega \in \Omega$.

Thus, integrating (11) over Ω, we obtain

$$A\|\mathcal{T}f\|^2 \leqslant \int_{\Omega} \sum_{\varphi \in \mathcal{A}} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle \right|^2 dm_\Gamma(\omega) \leqslant B\|\mathcal{T}f\|^2. \tag{12}$$

So, $\langle \mathcal{T}\varphi(.), \mathcal{T}f(.) \rangle$ belongs to $L^2(\Omega)$ for each $\varphi \in \mathcal{A}$ and the equality $R(\varphi) = T(\varphi)$, can be obtained in a similar way as we did before.

Finally, since $\|\mathcal{T}f\|_2^2 = \|f\|_2^2$ and

$$\sum_{h \in H} \sum_{\varphi \in \mathcal{A}} \left| \langle t_h \varphi, f \rangle_{L^2(G)} \right|^2 = \sum_{\varphi \in \mathcal{A}} \int_{\Omega} \left| \langle \mathcal{T}\varphi(\omega), \mathcal{T}f(\omega) \rangle_{\ell^2(\Delta)} \right|^2 dm_\Gamma(\omega),$$

inequality (12) implies that $E_H(\mathcal{A})$ is a frame for $S(\mathcal{A})$ with constants A and B. $\square$

Theorem 4.1 reduces the problem of when $E_H(\mathcal{A})$ is a frame for $S(\mathcal{A})$ to when the fibers $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$ form a frame for $J(\omega)$. The advantage of this reduction is that, for example, when $\mathcal{A}$ is a finite set, the fiber spaces $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$ are finite dimensional while $S(\mathcal{A})$ has infinite dimension.

If $\mathcal{A} = \{\varphi\}$, Theorem 4.1 generalizes a known result for the case $G = \mathbb{R}^d$ to the context of groups. This is stated in the next corollary, which was proved in [12]. We give here a different proof.

Corollary 4.2. *Let $\varphi \in L^2(\Omega)$ and $\Omega_\varphi = \{\omega \in \Omega: \sum_{\delta \in \Delta} |\widehat{\varphi}(\omega + \delta)|^2 \neq 0\}$. Then, the following are equivalent*:

 (i) *The set $E_H(\varphi)$ is a frame for $S(\varphi)$ with constants A and B.*
(ii) $A \leqslant \sum_{\delta \in \Delta} |\widehat{\varphi}(\omega + \delta)|^2 \leqslant B$, *a.e. $\omega \in \Omega_\varphi$.*

Proof. Let J be the measurable range function associated to $S(\varphi)$. Then, by Theorem 3.10, $J(\omega) = \text{span}\{\mathcal{T}\varphi(\omega)\}$ a.e $\omega \in \Omega$. Thus, each $a \in J(\omega)$ can be written as $a = \lambda \mathcal{T}\varphi(\omega)$ for some $\lambda \in \mathbb{C}$.

Therefore, by Theorem 4.1, (i) holds if and only if, for almost every $\omega \in \Omega$ and for all $\lambda \in \mathbb{C}$,

$$A\|\lambda \mathcal{T}\varphi(\omega)\|^2 \leqslant |\lambda|^2 \|\mathcal{T}\varphi(\omega)\|^4 \leqslant B\|\lambda \mathcal{T}\varphi(\omega)\|^2. \tag{13}$$

Then, since $\|\mathcal{T}\varphi(\omega)\|^2 = \sum_{\delta \in \Delta} |\widehat{\varphi}(\omega + \delta)|^2$, (13) holds if and only if

$$A \leqslant \sum_{\delta \in \Delta} |\widehat{\varphi}(\omega + \delta)|^2 \leqslant B, \quad \text{a.e. } \omega \in \Omega_\varphi. \qquad \square$$

For the case of Riesz basis, we have an analogue result to Theorem 4.1.

Theorem 4.3. *Let $\mathcal{A}$ be a countable subset of $L^2(G)$, J the measurable range function associated to $S(\mathcal{A})$ and $A \leqslant B$ positive constants. Then, they are equivalent*:

 (i) *The set $E_H(\mathcal{A})$ is a Riesz basis for $S(\mathcal{A})$ with constants A and B.*
(ii) *For almost every $\omega \in \Omega$, the set $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a Riesz basis for $J(\omega)$ with constants A and B.*

For the proof we will need the next lemma.

Lemma 4.4. *For each $m \in L^\infty(\Omega)$ there exists a sequence of trigonometric polynomials $\{P_k\}_{k \in \mathbb{N}}$ such that*:

 (i) $P_k(\omega) \to m(\omega)$, *a.e. $\omega \in \Omega$,*
(ii) *There exists $C > 0$, such that $\|P_k\|_\infty \leqslant C$, for all $k \in \mathbb{N}$.*

Proof. By Lemma 2.12, taking into account Remark 2.11, we have that the trigonometric polynomials are dense in $\mathcal{C}(\Omega)$.

By Lusin's Theorem, for each $k \in \mathbb{N}$, there exists a closed set $E_k \subseteq \Omega$ such that $m_\Gamma(\Omega \setminus E_k) < 2^{-k}$ and $m|_{E_k}$ is a continuous function where $m|_{E_k}$ denotes the function m restricted to E_k.

Since Ω is compact, E_k is compact as well. Therefore, $m|_{E_k}$ is bounded.

Let $m_1, m_2 : E_k \to \mathbb{R}$ be continuous function such that $m|_{E_k} = m_1 + im_2$. As a consequence of Tietze's Extension Theorem, it is possible to extend m_1 and m_2, continuously to all Ω keeping

their norms in $L^\infty(E_k)$. Let us call the extensions $\overline{m_1}$ and $\overline{m_2}$ and let $\overline{m_k} = \overline{m_1} + i\overline{m_2}$. Then, we have:

(1) $\overline{m_k}|_{E_k} = m|_{E_k}$,
(2) $\|\overline{m_k}\|_\infty \leqslant \|\overline{m_1}\|_\infty + \|\overline{m_2}\|_\infty \leqslant \|m_1\|_\infty + \|m_2\|_\infty \leqslant 2\|m\|_\infty$.

Now, by Lemma 2.12, there exists a trigonometric polynomial P_k such that $\|P_k - \overline{m_k}\|_\infty < 2^{-k}$. So,

(a) $|P_k(\omega) - m(\omega)| < 2^{-k}$, for all $\omega \in E_k$,
(b) $\|P_k\|_\infty \leqslant \|P_k - \overline{m_k}\|_\infty + \|\overline{m_k}\|_\infty \leqslant 2^{-k} + 2\|m\|_\infty \leqslant 1 + 2\|m\|_\infty$.

Repeating this argument for each $k \in \mathbb{N}$, we obtain a sequence $\{P_k\}_{k \in \mathbb{N}}$ of trigonometric polynomials and a sequence $\{E_k\}_{k \in \mathbb{N}}$ of sets, which satisfy conditions (a) and (b).

Let $E = \bigcup_{j=1}^\infty \bigcap_{k=j}^\infty E_k$. It is a straightforward to see that $m_\Gamma(\Omega \setminus E) = 0$. Let us prove that if $\omega \in E$, $P_k(\omega) \to m(\omega)$, for $k \to \infty$. Since $\omega \in E$, there exists $k_0 \in \mathbb{N}$ for which $\omega \in E_k$, $\forall k \geqslant k_0$. Then, for all $k \geqslant k_0$, we obtain that $|P_k(\omega) - m(\omega)| = |P_k(\omega) - m_k(\omega)| < 2^{-k} \to 0$, when $k \to \infty$. This proves part (i) of this lemma and taking $C := 1 + 2\|m\|_\infty$ we have that (ii) holds. $\quad\square$

Proof of Theorem 4.3. Since $S(\mathcal{A}) = \overline{\text{span}}\, E_H(\mathcal{A})$ and, by Theorem 3.10, $J(\omega) = \overline{\text{span}}\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\}$, we only need to show that $E_H(\mathcal{A})$ is a Riesz sequence for $S(\mathcal{A})$ with constants A and B if and only if for almost every $\omega \in \Omega$, the set $\{\mathcal{T}\varphi(\omega): \varphi \in \mathcal{A}\} \subseteq \ell^2(\Delta)$ is a Riesz sequence for $J(\omega)$ with constants A and B.

For the proof of the equivalence in the theorem, we will use the following reasoning.

Let $\{a_{\varphi,h}\}_{(\varphi,h) \in \mathcal{A} \times H}$ be a sequence of finite support and let P_φ be the trigonometric polynomials defined by

$$P_\varphi(\omega) = \sum_{h \in H} a_{\varphi,h} \eta_h(\omega),$$

with $\omega \in \Omega$ and η_h as in Proposition 2.16.

Note that, since $\{a_{\varphi,h}\}_{(\varphi,h) \in \mathcal{A} \times H}$ has finite support, only a finite number of the polynomials P_φ are not zero.

Now, as a consequence of Proposition 3.3 we have

$$\left\| \sum_{(\varphi,h) \in \mathcal{A} \times H} a_{\varphi,h} t_h \varphi \right\|_{L^2(G)}^2 = \left\| \sum_{(\varphi,h) \in \mathcal{A} \times H} a_{\varphi,h} \mathcal{T} t_h \varphi \right\|_{L^2(\Omega, \ell^2(\Delta))}^2$$

$$= \int_\Omega \left\| \sum_{(\varphi,h) \in \mathcal{A} \times H} a_{\varphi,h}(-h, \omega) \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega)$$

$$= \int_\Omega \left\| \sum_{\varphi \in \mathcal{A}} P_\varphi(\omega) \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega). \tag{14}$$

Furthermore, by Lemma 2.18,

$$\sum_{h\in H}|a_{\varphi,h}|^2 = \big\|\{a_{\varphi,h}\}_{h\in H}\big\|_{\ell^2(H)}^2 = \|P_\varphi\|_{L^2(\Omega)}^2,$$

and adding over $\mathcal{A}$, we obtain

$$\sum_{(\varphi,h)\in\mathcal{A}\times H}|a_{\varphi,h}|^2 = \sum_{\varphi\in\mathcal{A}}\|P_\varphi\|_{L^2(\Omega)}^2. \tag{15}$$

(ii) $\Rightarrow$ (i) If we suppose that for almost every $\omega\in\Omega$, $\{\mathcal{T}\varphi(\omega)\colon \varphi\in\mathcal{A}\}\subseteq\ell^2(\Delta)$ is a Riesz sequence for $J(\omega)$ with constants A and B,

$$A\sum_{\varphi\in\mathcal{A}}|a_\varphi|^2 \leqslant \bigg\|\sum_{\varphi\in\mathcal{A}}a_\varphi\mathcal{T}\varphi(\omega)\bigg\|_{\ell^2(\Delta)}^2 \leqslant B\sum_{\varphi\in\mathcal{A}}|a_\varphi|^2 \tag{16}$$

for all $\{a_\varphi\}_{\varphi\in\mathcal{A}}$ with finite support.

In particular, the above inequality holds for $\{a_\varphi\}_{\varphi\in\mathcal{A}}=\{P_\varphi(\omega)\}_{\varphi\in\mathcal{A}}$. Now, in (16), we can integrate over Ω with $\{a_\varphi\}_{\varphi\in\mathcal{A}}=\{P_\varphi(\omega)\}_{\varphi\in\mathcal{A}}$, in order to obtain

$$A\sum_{\varphi\in\mathcal{A}}\|P_\varphi\|_{L^2(\Omega)}^2 \leqslant \int_\Omega \bigg\|\sum_{\varphi\in\mathcal{A}}P_\varphi(\omega)\mathcal{T}\varphi(\omega)\bigg\|_{\ell^2(\Delta)}^2\,dm_\Gamma(\omega)$$

$$\leqslant B\sum_{\varphi\in\mathcal{A}}\|P_\varphi\|_{L^2(\Omega)}^2. \tag{17}$$

Using Eqs. (14) and (15) we can rewrite (17) as

$$A\sum_{(\varphi,h)\in\mathcal{A}\times H}|a_{\varphi,h}|^2 \leqslant \bigg\|\sum_{(\varphi,h)\in\mathcal{A}\times H}a_{\varphi,h}t_h\varphi\bigg\|_{L^2(G)}^2 \leqslant B\sum_{(\varphi,h)\in\mathcal{A}\times H}|a_{\varphi,h}|^2.$$

Therefore $E_H(\mathcal{A})$ is a Riesz sequence of $S(\mathcal{A})$ with constants A and B.

(i) $\Rightarrow$ (ii) We want to prove that, for every $a=\{a_\varphi\}_{\varphi\in\mathcal{A}}\in\ell^2(\mathcal{A})$ with finite support, we have a.e. $\omega\in\Omega$

$$A\sum_{\varphi\in\mathcal{A}}|a_\varphi|^2 \leqslant \bigg\|\sum_{\varphi\in\mathcal{A}}a_\varphi\mathcal{T}\varphi(\omega)\bigg\|_{\ell^2(\Delta)}^2 \leqslant B\sum_{\varphi\in\mathcal{A}}|a_\varphi|^2. \tag{18}$$

Let us suppose that (18) fails. Then, using a similar argument as in Theorem 4.1, we can see that there exist $a=\{a_\varphi\}_{\varphi\in\mathcal{A}}\in\ell^2(\mathcal{A})$ with finite support, a measurable set $W\subseteq\Omega$ with $m_\Gamma(W)>0$ and $\varepsilon>0$ such that

$$\bigg\|\sum_{\varphi\in\mathcal{A}}a_\varphi\mathcal{T}\varphi(\omega)\bigg\|_{\ell^2(\Delta)}^2 > (B+\varepsilon)\sum_{\varphi\in\mathcal{A}}|a_\varphi|^2, \quad \forall\omega\in W \tag{19}$$

or

$$\left\| \sum_{\varphi \in \mathcal{A}} a_\varphi \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 < (A - \varepsilon) \sum_{\varphi \in \mathcal{A}} |a_\varphi|^2, \quad \forall \omega \in W. \tag{20}$$

With $a = \{a_\varphi\}_{\varphi \in \mathcal{A}}$ and W, we define for each $\varphi \in \mathcal{A}$, $m_\varphi := a_\varphi \chi_W$. Thus, $m_\varphi \in L^\infty(\Omega)$ and only finitely many of these functions are not null.

By Lemma 4.4, for each $\varphi \in \mathcal{A}$ there exists a polynomial sequence $\{P_k^\varphi\}_{k \in \mathbb{N}}$ such that

(i) $P_k^\varphi \to m_\varphi$,
(ii) $\|P_k^\varphi\|_\infty \leqslant 1 + 2\|m_\varphi\|_\infty, \forall k \in \mathbb{N}$.

Since $E_H(\mathcal{A})$ is a Riesz sequence for $S(\mathcal{A})$ with constants A and B,

$$A \sum_{(\varphi,h) \in \mathcal{A} \times H} |a_{\varphi,h}|^2 \leqslant \left\| \sum_{(\varphi,h) \in \mathcal{A} \times H} a_{\varphi,h} t_h \varphi \right\|_{L^2(G)}^2$$

$$\leqslant B \sum_{(\varphi,h) \in \mathcal{A} \times H} |a_{\varphi,h}|^2,$$

for each sequence $\{a_{\varphi,h}\}_{(\varphi,h) \in \mathcal{A} \times H}$ with finite support.

Now, for each $k \in \mathbb{N}$ take $\{a_{\varphi,h}\}_{(\varphi,h) \in \mathcal{A} \times H}$ to be the sequence formed with the coefficients of the polynomials $\{P_k^\varphi\}_{\varphi \in \mathcal{A}}$.

Then, using (14) and (15), we have for each $k \in \mathbb{N}$

$$A \sum_{\varphi \in \mathcal{A}} \|P_k^\varphi\|_{L^2(\Omega)}^2 \leqslant \int_\Omega \left\| \sum_{\varphi \in \mathcal{A}} P_k^\varphi(\omega) \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega) \leqslant B \sum_{\varphi \in \mathcal{A}} \|P_k^\varphi\|_{L^2(\Omega)}^2. \tag{21}$$

Therefore, since $m_\Gamma(\Omega) < \infty$ and by the Dominated Convergence Theorem, inequality (21) can be extended to m_φ as

$$A \sum_{\varphi \in \mathcal{A}} \|m_\varphi\|_{L^2(\Omega)}^2 \leqslant \int_\Omega \left\| \sum_{\varphi \in \mathcal{A}} m_\varphi(\omega) \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega) \leqslant B \sum_{\varphi \in \mathcal{A}} \|m_\varphi\|_{L^2(\Omega)}^2. \tag{22}$$

So, if (19) occurs, integrating over Ω we obtain

$$\int_\Omega \left\| \sum_{\varphi \in \mathcal{A}} m_\varphi(\omega) \mathcal{T}\varphi(\omega) \right\|_{\ell^2(\Delta)}^2 dm_\Gamma(\omega) > (B + \varepsilon) \int_\Omega \sum_{\varphi \in \mathcal{A}} |m_\varphi(\omega)|^2 dm_\Gamma,$$

which contradicts inequality (22). We can proceed analogously if (20) occurs. Hence, (18) holds. $\square$

For the case of principal H-invariant spaces we have the following corollary.

Corollary 4.5. *Let* $\varphi \in L^2(\Omega)$. *Then, the following are equivalent*:

  (i) *The set* $E_H(\varphi)$ *is a Riesz basis for* $S(\varphi)$ *with constants A and B*.
 (ii) $A \leqslant \sum_{\delta \in \Delta} |\widehat{\varphi}(\omega + \delta)|^2 \leqslant B$, *a.e.* $\omega \in \Omega$.

Proof. The proof is a straightforward consequence of Theorem 4.3 and Theorem 3.10. □

We now want to give another characterization of when the set $E_H(\mathcal{A})$ is a frame (Riesz sequence) for $S(\mathcal{A})$ with constants A and B. For this we will introduce what in the classical case are the synthesis and analysis operators.

For an LCA group G and for a subgroup H as in (3.1) let us consider a subset $\mathcal{A} = \{\varphi_i : i \in I\}$ of $L^2(G)$ where I is a countable set.

Let Ω be a Borel section of Γ/Δ. Fix $\omega \in \Omega$ and let $\mathcal{D}$ be the set of sequences in $\ell^2(I)$ with finite support. Define the operator $K_\omega : \mathcal{D} \to \ell^2(\Delta)$ as

$$K_\omega(c) = \sum_{i \in I} c_i \mathcal{T} \varphi_i(\omega). \tag{23}$$

The proof of the following proposition can be found in [2, Theorem 3.2.3].

Proposition 4.6. *The operator* K_ω *defined above is bounded if and only the set* $\{\mathcal{T}\varphi_i(\omega) : i \in I\}$ *is a Bessel sequence in* $\ell^2(\Delta)$.

In that case the adjoint operator of K_ω, $K_\omega^* : \ell^2(\Delta) \to \ell^2(I)$, *is given by*

$$K_\omega^*(a) = \left(\langle \mathcal{T}\varphi_i(\omega), a \rangle_{\ell^2(\Delta)} \right)_{i \in I}.$$

The operator K_ω is called the *synthesis operator* and K_ω^* the *analysis operator*.

Definition 4.7. Let $\{\varphi_i : i \in I\} \subseteq L^2(G)$ be a countable subset and K_ω and K_ω^* the synthesis and analysis operators. We define the *Gramian* of $\{\mathcal{T}\varphi_i(\omega) : i \in I\}$ as the operator $\mathcal{G}_\omega : \ell^2(I) \to \ell^2(I)$ given by $\mathcal{G}_\omega = K_\omega^* K_\omega$, and we also define the *dual Gramian* of $\{\mathcal{T}\varphi_i(\omega) : i \in I\}$ as the operator $\tilde{\mathcal{G}}_\omega : \ell^2(\Delta) \to \ell^2(\Delta)$ given by $\tilde{\mathcal{G}}_\omega = K_\omega K_\omega^*$.

The Gramian $\mathcal{G}_\omega$ can be associated with the (possible) infinite matrix

$$\mathcal{G}_\omega = \left(\sum_{\delta \in \Delta} \widehat{\varphi_i}(\omega + \delta) \overline{\widehat{\varphi_j}(\omega + \delta)} \right)_{i,j \in I}$$

since $\langle \mathcal{G}_\omega e_i, e_j \rangle = \langle \mathcal{T}\varphi_i(\omega), \mathcal{T}\varphi_j(\omega) \rangle$, where $\{e_i\}_{i \in I}$ be the standard basis of $\ell^2(I)$. In a similar way, considering the basis $\{e_\delta\}_{\delta \in \Delta}$ of $\ell^2(\Delta)$, we can associate the dual Gramian $\tilde{\mathcal{G}}_\omega$ with the matrix

$$\tilde{\mathcal{G}}_\omega = \left(\sum_{i \in I} \widehat{\varphi_i}(\omega + \delta) \overline{\widehat{\varphi_i}(\omega + \delta')} \right)_{\delta, \delta' \in \Delta}.$$

Remark 4.8. The operator K_ω (K_ω^*) is bounded if and only if $\mathcal{G}_\omega$ ($\tilde{\mathcal{G}}_\omega$) is bounded. In that case we have $\|K_\omega\|^2 = \|K_\omega^*\|^2 = \|\mathcal{G}_\omega\| = \|\tilde{\mathcal{G}}_\omega\|$.

Now we will give a characterization of when $E_H(\mathcal{A})$ is a frame (Riesz sequence) for $S(\mathcal{A})$ in terms of the Gramian $\mathcal{G}_\omega$ and the dual Gramian $\tilde{\mathcal{G}}_\omega$.

Proposition 4.9. *Let $\mathcal{A} = \{\varphi_i \colon i \in I\} \subseteq L^2(G)$ be a countable set. Then,*

(1) *The following are equivalent*:
   (a_1) $E_H(\mathcal{A})$ *is a Bessel sequence with constant B.*
   (b_1) $\text{supess}_{\omega \in \Omega} \|\mathcal{G}_\omega\| \leqslant B$.
   (c_1) $\text{supess}_{\omega \in \Omega} \|\tilde{\mathcal{G}}_\omega\| \leqslant B$.
(2) *The following are equivalent*:
   (a_2) $E_H(\mathcal{A})$ *is a frame for $S(\mathcal{A})$ with constants A and B.*
   (b_2) *For almost every $\omega \in \Omega$,*

$$A\|a\|^2 \leqslant \langle \tilde{\mathcal{G}}_\omega a, a \rangle \leqslant B\|a\|^2,$$

   for all $a \in \text{span}\{\mathcal{T}\varphi_i(\omega) \colon i \in I\}$.
   (c_2) *For almost every $\omega \in \Omega$,*

$$\sigma(\tilde{\mathcal{G}}_\omega) \subseteq \{0\} \cup [A, B].$$

(3) *The following are equivalent*:
   (a_3) $E_H(\mathcal{A})$ *is a Riesz sequence for $S(\mathcal{A})$ with constants A and B.*
   (b_3) *For almost every $\omega \in \Omega$,*

$$A\|c\|^2 \leqslant \langle \mathcal{G}_\omega c, c \rangle \leqslant B\|c\|^2,$$

   for all $c \in \ell^2(I)$.
   (c_3) *For almost every $\omega \in \Omega$*

$$\sigma(\mathcal{G}_\omega) \subseteq [A, B].$$

Proof. It follows easily from Theorem 4.1, Theorem 4.3, Proposition 4.6 and Remark 4.8. □

Note that Corollary 4.2 and Corollary 4.5 can also be obtained from the previous proposition.

Definition 4.10. For an H-invariant space $V \subseteq L^2(G)$ we define the *dimension function of V* as the map $\dim_V \colon \Omega \to \mathbb{N}_0$ given by $\dim_V(\omega) = \dim J(\omega)$, where J is the range function associated to V. We also define the *spectrum of V* as $s(V) = \{\omega \in \Omega \colon J(\omega) \neq 0\}$.

As in the $\mathbb{R}^d$ case, every H-invariant space can be decomposed into an orthogonal sum of principal H-invariant spaces. This can be easily obtained as a consequence of Zorn's Lemma as in [12]. The next theorem establishes a decomposition of H-invariant space with additional properties as in [1]. We do not include its proof since it follows readily from the $\mathbb{R}^d$ case (see [1, Theorem 3.3]).

Theorem 4.11. *Let us suppose that V is an H-invariant space of $L^2(G)$. Then V can be decomposed as an orthogonal sum*

$$V = \bigoplus_{n \in \mathbb{N}} S(\varphi_n),$$

where $E_H(\varphi_n)$ is a Parseval frame for $S(\varphi_n)$ and $s(S(\varphi_{n+1})) \subseteq s(S(\varphi_n))$ for all $n \in \mathbb{N}$.
Moreover, $\dim_{S(\varphi_n)}(\omega) = \|\mathcal{T}\varphi_n(\omega)\|$ for all $n \in \mathbb{N}$, and

$$\dim_V(\omega) = \sum_{n \in \mathbb{N}} \|\mathcal{T}\varphi_n(\omega)\|, \quad a.e.\ \omega \in \Omega.$$

Acknowledgments

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