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An elliptic system with bifurcation parameters on the boundary conditions

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A B S T R A C T

In this paper we consider the elliptic system $\Delta u = a(x)u^p v^q$, $\Delta v = b(x)u^r v^s$ in Ω, a smooth bounded domain, with boundary conditions $\frac{\partial u}{\partial \nu} = \lambda u$, $\frac{\partial v}{\partial \nu} = \mu v$ on $\partial\Omega$. Here λ and μ are regarded as parameters and $p, s > 1$, $q, r > 0$ verify $(p-1)(s-1) > qr$. We consider the case where $a(x) \geqslant 0$ in Ω and $a(x)$ is allowed to vanish in an interior subdomain Ω_0, while $b(x) > 0$ in $\overline{\Omega}$. Our main results include existence of nonnegative nontrivial solutions in the range $0 < \lambda < \lambda_1 \leqslant \infty$, $\mu > 0$, where λ_1 is characterized by means of an eigenvalue problem, and the uniqueness of such solutions. We also study their asymptotic behavior in all possible cases: as both $\lambda, \mu \to 0$, as $\lambda \to \lambda_1 < \infty$ for fixed μ (respectively $\mu \to \infty$ for fixed λ) and when both $\lambda, \mu \to \infty$ in case $\lambda_1 = \infty$.
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1. Introduction

Reaction–diffusion systems is a broad field most of whose main branches still remain open in multiple aspects. Namely, existence, uniqueness, bifurcation aspects together with limit profiles of solutions when parameters approach the boundary of existence regions, stability and dynamical behavior, maximum principles and many others (see [10,23,27] and [28] for comprehensive accounts on these subjects). Only some few classes of such equations are nowadays partially well understood. In view of their applications, specially in the realm of population dynamics, the so-called competitive

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systems constitute a main case of such systems (see, for example, [5–8,24] and the general texts cited above).

The aim of the present work is to provide a detailed study of positive solutions (in both components) of the following elliptic system of competitive type

$$\begin{cases} \Delta u = a(x)u^p v^q & \text{in } \Omega, \\ \Delta v = b(x)u^r v^s & \text{in } \Omega, \end{cases} \tag{1.1}$$

complemented with the flux boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$ (with ν the outward unit normal field), $a, b \in C(\overline{\Omega})$ are nonnegative functions, $p, s > 1$, $q, r > 0$. The real parameters λ, μ control the fluxes of u, v into the domain.

A main feature in our problem (1.1)–(1.2) is that the parameters appear in the boundary condition. In this sense this paper is a natural continuation of the two previous works [17] and [18] which dealt with a single equation. For the case of scalar equations, some few papers (see for instance [2] and [29]) have considered boundary conditions with parameters, although such conditions were nonlinear. This fact and the lack of suitable symmetries did not permit to perform a complete study of the bifurcation diagram as the one in our preceding jobs [17] or [18]. On the other hand, at the best of our knowledge, recent or past literature treating the dependence on parameters of boundary conditions does not practically exist.

Our intention in the present work is to fully describe the bifurcation diagram for problem (1.1)–(1.2). We will prove that under suitable conditions on a, b and the exponents p, q, r, s, there exists a unique positive weak solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ for $0 < \lambda < \lambda_1$ and $\mu > 0$, where $\lambda_1 \leqslant \infty$ is defined in terms of a suitable eigenvalue problem. Furthermore, $(u_{\lambda,\mu}, v_{\lambda,\mu})$ defines a global attractor for all nonnegative solutions to the corresponding parabolic system associated to (1.1) under the boundary conditions (1.2).

On the other hand, a significative part of the results will be oriented to determine the behavior of the solution when the parameters are varied, paying special attention to its asymptotic behavior when $\lambda \to \lambda_1 \leqslant \infty$ or $\mu \to +\infty$ (or both). We will find that in some situations there is a limit profile, which is a solution to (1.1) but with a singular boundary condition. Moreover, depending on the vanishing properties of coefficients a, b such finite profiles can only be sustained on certain subdomains of Ω. In some other cases the components of the solutions just go to zero or infinity uniformly. This means that asymptotically the system drives one of the species to extinction.

Next, we state the precise hypotheses that we impose on the weights a and b. They will be continuous functions in $\overline{\Omega}$ such that $b(x) > 0$, $a(x) \geqslant 0$ for all $x \in \overline{\Omega}$, being a nontrivial. In addition and to enlarge the scope of our analysis, we are allowing a to vanish in a whole subdomain Ω_0 of Ω (see [9,11,13,25] and [26] for a similar situation in the case of a single equation under Dirichlet or Robin boundary conditions which do not depend on parameters). More precisely, we are assuming that the set $\{x \in \Omega\colon\ a(x) = 0\}$ is the closure of a smooth (say C^2) subdomain $\Omega_0 \subset \Omega$ (the case $a > 0$ corresponding to $\Omega_0 = \emptyset$). For later use, we set $\Omega^+ = \Omega \setminus \overline{\Omega}_0$ together with $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega$, $\Gamma_2 = \partial\Omega_0 \cap \Omega$, $\Gamma^+ = \partial\Omega^+ \setminus \Gamma_2$. As in [17,18], we are making the simplifying additional hypothesis $\Gamma_2 = \overline{\Gamma}_2$ and hence

$$\Gamma_2 \subset \Omega. \tag{1.3}$$

This means that $\partial\Omega_0 \cap \Omega$ lies at a positive distance from $\partial\Omega_0 \cap \partial\Omega$. As studied in [19] (cf. also [20]), suppressing (1.3) only implies a certain loss of regularity in the solutions. On the other hand, observe that as a consequence of the smoothness of both Ω and Ω_0, all Γ_1, Γ_2 and Γ^+ consists of a finite union of smooth closed manifolds. Finally and as a simplifying assumption it will be also supposed that $a > 0$ on $\partial\Omega$ whenever $\Omega_0 \Subset \Omega$. All the preceding vanishing properties of a will be referred to in the current work as hypothesis (H).

Remark 1. The connectedness requirement on the null set Ω_0 of a is assumed in the present work by the sake of simplicity. However, the positivity region Ω^+ could exhibit several components (see below).

As for the exponents p, q, r, s, we are assuming that $p, s > 1$, $q, r > 0$ with

$$\delta := (p-1)(s-1) - qr > 0. \tag{1.4}$$

This assumption somehow measures the coupling between the two equations in (1.1), and it makes the system behave "essentially" as a single equation. More precisely (1.4) makes possible the construction of suitable sub- and supersolutions. Indeed, as was already mentioned, system (1.1) is of competitive type. This implies that comparison arguments can still be employed, although when defining sub- and supersolutions one of the inequalities has to be reversed (see [27]). On the other hand, it should be remarked that the particular prototype (1.1) was analyzed for instance in [16] and [12] but with boundary conditions of Dirichlet and blow-up type. See also [15] for a related system under the latter kind of boundary conditions.

Regarding the smoothness of solutions we are always dealing with weak nonnegative solutions (u, v) to (1.1)–(1.2), i.e. $u, v \in H^1(\Omega)$ such that

$$-\int_\Omega \nabla u \nabla \varphi + \lambda \int_{\partial\Omega} u\varphi = \int_\Omega a u^p v^q \varphi, \qquad -\int_\Omega \nabla v \nabla \psi + \mu \int_{\partial\Omega} v\psi = \int_\Omega a u^r v^s \psi,$$

for all $\varphi, \psi \in H^1(\Omega)$. However, such solutions are indeed more regular since it can be shown, via a standard iteration procedure, that actually $u, v \in L^\infty(\Omega)$ (see [19,20]). Hence $u, v \in W^{2,q}(\Omega) \cap C^{1,\eta}(\overline{\Omega})$ for every $q > 1$, $\eta \in (0, 1)$, and are indeed strong solutions (cf. [21,22]).

Now we arrive to the statements of our results. The first theorem clarifies the issues of existence and uniqueness of positive solutions to (1.1)–(1.2) and their dynamical rôle. It turns out that the principal eigenvalue (denoted by λ_1) of the problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_0, \\ \dfrac{\partial\phi}{\partial\nu} = \lambda\phi & \text{on } \Gamma_1, \\ \phi = 0 & \text{on } \Gamma_2, \end{cases} \tag{1.5}$$

will be determinant in the existence of positive solutions. Existence, uniqueness, variational characterization and further features concerning λ_1 were discussed in [17,18]. Under our assumptions it may perfectly be the case that $\overline{\Omega}_0 \subset \Omega$ (and so Γ_1 would be empty). If so, we are setting $\lambda_1 = \infty$.

Theorem 1. *Let Ω be a C^2 bounded domain of $\mathbb{R}^N$, and $a, b \in C(\overline{\Omega})$. Assume that $b(x) > 0$ in $\overline{\Omega}$ while $a(x)$ verifies hypothesis* (H). *If $p, s > 1$, $q, r > 0$ satisfy* (1.4), *then*:

(i) *Problem* (1.1)–(1.2) *can only have positive weak solutions if* $0 < \lambda < \lambda_1 \leqslant \infty$ *and* $\mu > 0$.
(ii) *For $\lambda \in (0, \lambda_1)$, $\lambda_1 \leqslant \infty$, and $\mu > 0$ there exists a unique positive weak solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$. Moreover, $(u_{\lambda,\mu}, v_{\lambda,\mu})$ defines an asymptotically stable equilibrium for the associated parabolic system which is a global attractor for all nonnegative solutions.*

After this important step is given, we are interested in the analysis of the dependence of the solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ with respect to the parameters λ and μ.

This analysis constitutes the main contribution of this paper. We are performing a rather complete study of this dependence, and the subsequent results will be stated in several different theorems to clarify the exposition.

In our first statement, we gather the monotonicity properties of the solution and the asymptotic behavior of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ for small λ and μ.

Theorem 2. *Under the assumptions of Theorem* 1, *let* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *be the unique positive weak solution to* (1.1)–(1.2) *for* $0 < \lambda < \lambda_1 \leqslant \infty$, $\mu > 0$. *Then*:

(i) $u_{\lambda,\mu}$ *is increasing in* λ *and decreasing in* μ, *while* $v_{\lambda,\mu}$ *is decreasing in* λ *and increasing in* μ.
(ii) *If* $\lambda \to 0$ *with fixed* μ, *then* $u_{\lambda,\mu} \to 0$ *and* $v_{\lambda,\mu} \to +\infty$ *uniformly in* $\overline{\Omega}$. *Similarly, if* $\mu \to 0$ *for fixed* λ, *then* $u_{\lambda,\mu} \to +\infty$ *and* $v_{\lambda,\mu} \to 0$ *uniformly in* $\overline{\Omega}$.
(iii) *For* $\lambda, \mu \to 0$ *it holds*:

$$
\begin{aligned}
u_{\lambda,\mu} &\sim \left\{ \left(\frac{\lambda}{a^*}\right)^{s-1} \left(\frac{b^*}{\mu}\right)^{q} \right\}^{\frac{1}{\delta}}, \\
v_{\lambda,\mu} &\sim \left\{ \left(\frac{\mu}{b^*}\right)^{p-1} \left(\frac{a^*}{\lambda}\right)^{r} \right\}^{\frac{1}{\delta}},
\end{aligned}
\tag{1.6}
$$

where

$$
a^* = \frac{1}{|\partial\Omega|} \int_{\Omega} a(x)\,dx, \qquad b^* = \frac{1}{|\partial\Omega|} \int_{\Omega} b(x)\,dx.
$$

Estimates (1.6) yield a complete picture of the asymptotic behavior of the solutions $u_{\lambda,\mu}, v_{\lambda,\mu}$ as both λ, μ approach zero. Our next result contains the full regime of behaviors. To provide the information in a concise way it is convenient to introduce the following notation. For $\mu_1 = \mu_1(\lambda)$, $\mu_2 = \mu_2(\lambda)$, positive functions of λ defined near zero and satisfying $\lim_{\lambda\to 0} \mu_i = 0$, $i = 1, 2$, we say $\mu_1 \ll \mu_2$ if $\lim_{\lambda\to 0+} \mu_1/\mu_2 = 0$, while $\mu_1 \approx \mu_2$ stands for $\lim_{\lambda\to 0+} \mu_1/\mu_2 = \kappa$, $0 < \kappa < \infty$ ($\mu_1 \sim \mu_2$ corresponds to the case $\kappa = 1$).

Theorem 3. *Under the assumptions of Theorem* 2 *let* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *be the positive solution to* (1.1)–(1.2) *and* $\theta_1 = \frac{r}{p-1}$, $\theta_2 = \frac{s-1}{q}$.

If $\mu = \mu(\lambda)$ *defines any positive function such that* $\mu \to 0$ *as* $\lambda \to 0$ *then* (*see Fig.* 1)

(1) $\lim_{\lambda\to 0}(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, \infty)$ *if* $\mu \gg \lambda^{\theta_1}$,
(2) $\lim_{\lambda\to 0}(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, c_1)$ *for a certain positive constant* c_1 *if* $\mu \approx \lambda^{\theta_1}$,
(3) $\lim_{\lambda\to 0}(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0, 0)$ *provided* $\lambda^{\theta_2} \ll \mu \ll \lambda^{\theta_1}$,
(4) $\lim_{\lambda\to 0}(u_{\lambda,\mu}, v_{\lambda,\mu}) = (c_2, 0)$, c_2 *certain positive constant if* $\mu \approx \lambda^{\theta_2}$,
(5) $\lim_{\lambda\to 0}(u_{\lambda,\mu}, v_{\lambda,\mu}) = (\infty, 0)$ *whenever* $\mu \ll \lambda^{\theta_2}$.

Next we describe the behavior of the unique positive weak solution to (1.1)–(1.2) when a parameter is kept fixed (say μ) and λ is moved to reach the limiting value λ_1, which can be finite or not. As a surprising fact, it turns out that when $\lambda_1 < \infty$ there could exist distinguished finite values of μ separating different "spatially located" limit behaviors of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ as $\lambda \to \lambda_1-$. Such values are associated to the connected pieces of Ω^+. In fact and while Ω_0 was assumed connected from the start (see Remark 1) this not need to be the case with Ω^+. Since Ω, Ω_0 are class C^2 domains then Ω^+ exhibits a finite number M of components Ω_i^+ all of them defining C^2 domains. To each compo-

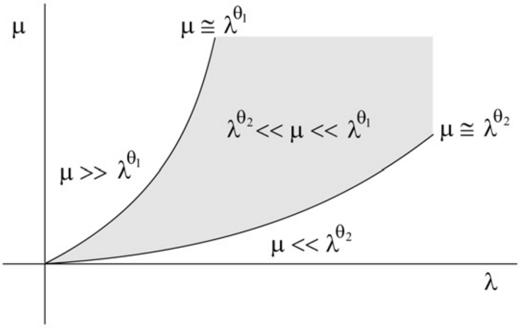

Fig. 1. A symbolic drawing of the five regions describing the change in the regime for the asymptotic behavior of the solutions $(u_{\lambda,\mu}, v_{\lambda,\mu})$ as $(\lambda, \mu) \to (0, 0)$. The shadowed area in the middle stands for the region for solutions that bifurcate from $u = 0$, $v = 0$.

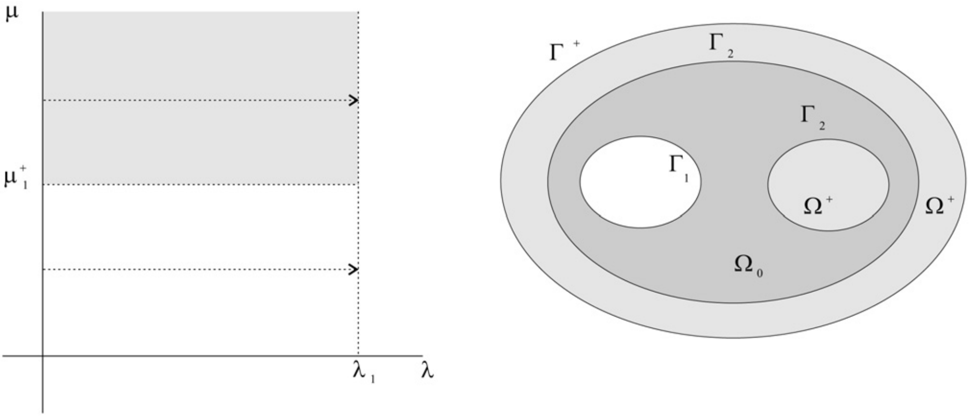

Fig. 2. A possible configuration for Ω: Ω^+ has two components, the outer one with $\mu_1^+ < \infty$, the inner one with $\mu_2^+ = \infty$, while $\lambda_1 < \infty$. For $\mu \leqslant \mu_1^+$ the solution $(u_{\lambda,\mu}, v_{\lambda,\mu}) \to (\infty, 0)$ as $\lambda \to \lambda_1$ while it keeps a finite profile in the outer component as $\lambda \to \lambda_1$ provided $\mu > \mu_1^+$.

nent Ω_i^+ such that $\Gamma_i^+ := \partial\Omega_i^+ \cap \partial\Omega = \emptyset$ (i.e. $\Omega_i^+ \Subset \Omega$) we associate the value $\mu_i^+ = \infty$ meanwhile $\mu = \mu_i^+$ is defined as the principal eigenvalue of the problem [17,18]

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega_i^+, \\ \dfrac{\partial\psi}{\partial\nu} = \mu\psi & \text{on } \Gamma_i^+, \\ \psi = 0 & \text{on } \Gamma_{2,i} := \Gamma_2 \cap \partial\Omega_i^+, \end{cases}$$

for all those components with $\Gamma_i^+ \neq \emptyset$. As it will be seen below the limit behavior of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ as $\lambda \to \lambda_1-$ in each Ω_i^+ will depend on the relative values of μ and μ_i^+.

On the contrary, particular values of μ have no relevance in the asymptotic behavior of the solutions (μ fixed) when $\lambda_1 = \infty$. The important information when $\lambda_1 = \infty$ is whether the exponent r is less than $(p-1)/2$ or not. See Fig. 2.

In the next statement we are denoting $d = \operatorname{dist}(x, \Gamma_2)$ and assuming that coefficient $a(x)$ under hypotheses (H) satisfies in addition the decay condition (observe that by continuity $a = 0$ on Γ_2):

$$a(x) = o\big(d(x)\big) \quad \text{as } d(x) \to 0.$$

Theorem 4. *Assume $a, b \in C(\overline{\Omega})$, a satisfies* (H) *while $\Omega_1^+, \dots, \Omega_M^+$ stand for the connected components of Ω^+. Let $(u_{\lambda,\mu}, v_{\lambda,\mu})$ be the unique positive solution to* (1.1)–(1.2) *for $0 < \lambda < \lambda_1 \leqslant \infty$, $\mu > 0$.*

(A) *Suppose $\lambda_1 < \infty$.*

(i) *If $0 < \mu \leqslant \mu_i^+$ for all $i \in \{1, \dots, M\}$, then $u_{\lambda,\mu} \to +\infty$, $v_{\lambda,\mu} \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to \lambda_1-$.*

(ii) *Assume that $\mu > \min \mu_i^+$. Then $u_{\lambda,\mu} \to +\infty$, $v_{\lambda,\mu} \to 0$ uniformly in*

$$\overline{\Omega}_0 \cup \left(\bigcup_j \overline{\Omega}_j^+\right),$$

as $\lambda \to \lambda_1-$, the union being extended to those Ω_j^+ with $\mu \leqslant \mu_j^+$. Furthermore, if $a(x)$ satisfies in addition ($d = \operatorname{dist}(x, \Gamma_2)$)

$$C_1 d(x)^\sigma \leqslant a(x) \leqslant C_2 d(x)^\sigma, \quad x \in \Omega_i^+, \tag{1.7}$$

near $\Gamma_{2,i}$ for some $\sigma > 0$ and positive constants C_1, C_2, then $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges uniformly on compacts of the remaining components $\Omega_i^+ \cup \Gamma_i^+$ where $\mu > \mu_i^+$ to a weak solution of the system

$$\begin{cases} \Delta u = a(x)u^p v^q & \text{in } \Omega_i^+, \qquad u = \infty \quad \text{on } \Gamma_{2,i}, \qquad \dfrac{\partial u}{\partial \nu} = \lambda_1 u \quad \text{on } \Gamma_i^+, \\ \Delta v = b(x)u^r v^s & \text{in } \Omega_i^+, \qquad v = 0 \quad \text{on } \Gamma_{2,i}, \qquad \dfrac{\partial v}{\partial \nu} = \mu v \quad \text{on } \Gamma_i^+, \end{cases} \tag{1.8}$$

where $\Gamma_{2,i} := \partial\Omega_i^+ \cap \Gamma_2$.

(B) *Assume $\lambda_1 = \infty$.*

(iii) *If $0 < r < (p-1)/2$, then $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges uniformly in compacts of Ω to the unique positive weak solution $(u_{\infty,\mu}, v_{\infty,\mu})$ of the system*

$$\begin{cases} \Delta u = a(x)u^p v^q & \text{in } \Omega, \qquad u = \infty \quad \text{on } \partial\Omega, \\ \Delta v = b(x)u^r v^s & \text{in } \Omega, \qquad \dfrac{\partial v}{\partial \nu} = \mu v \quad \text{on } \partial\Omega, \end{cases}$$

as $\lambda \to +\infty$.

(iv) *If $r \geqslant (p-1)/2$, then $u_{\lambda,\mu} \to +\infty$ and $v_{\lambda,\mu} \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to +\infty$.*

Remark 2. In the case $\Omega^+ \Subset \Omega$ (and so $\lambda_1 < \infty$) no special values of μ have influence on the limit behavior of the solutions and the conclusion of (i) holds true.

Statements symmetric to those in (iii) and (iv) hold when λ is kept fixed and $\mu \to +\infty$. Thus it only remains to study the behavior of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ when *both* λ and μ go to infinity. Accordingly, the existence of positive solutions is required for λ, μ free of upper limitations. Thus, as the weights a and b are not playing now a significative role, we are setting in the remaining statements $a(x) = b(x) = 1$ (as a minor remark, observe that solutions are now classical thanks to standard elliptic theory, see [1,21,22]). We show that, depending on the relative values of p, q, r, s and on the quotients λ/μ, μ/λ, the solutions converge to a finite profile or not. We remark that uniqueness of positive classical solutions to the system (1.9) below was proved in [16].

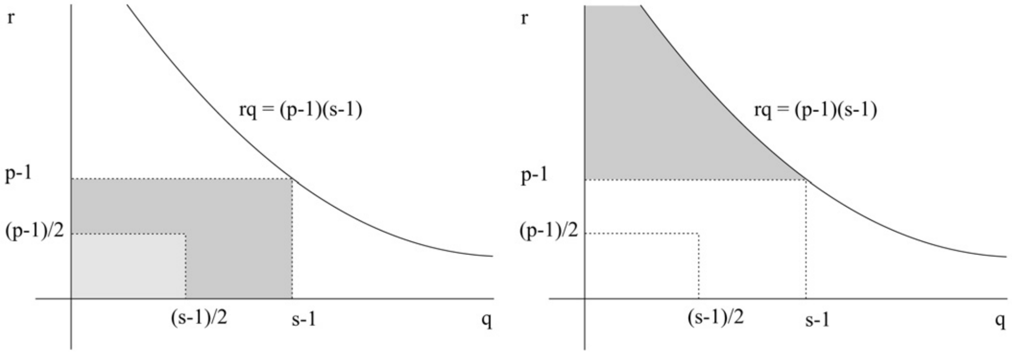

Fig. 3. On the left we highlight the regions of parameters corresponding to the asymptotic behaviors described in points (i) and (ii) of Theorem 5 and on the right the parametric regime leading to the behavior in (iii).

Theorem 5. *Assume $a(x) = b(x) = 1$, and let $(u_{\lambda,\mu}, v_{\lambda,\mu})$ be the unique positive weak solution to* (1.1)–(1.2)*.*

(i) *If $r < (p-1)/2$, $q < (s-1)/2$, then $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges uniformly on compacts of Ω to the unique positive weak solution (u_∞, v_∞) to*

$$\begin{cases} \Delta u = u^p v^q & \text{in } \Omega, \qquad u = \infty \quad \text{on } \partial\Omega, \\ \Delta v = u^r v^s & \text{in } \Omega, \qquad v = \infty \quad \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

as $\lambda, \mu \to \infty$.

(ii) *If $r < p-1$, $q < s-1$ and $\lambda, \mu \to \infty$ in such a way that μ/λ is bounded and bounded away from zero, then $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges uniformly on compacts of Ω to the unique positive weak solution (u_∞, v_∞) to* (1.9)*.*

(iii) *If $r > p-1$ (resp. $q > s-1$) and $\lambda, \mu \to \infty$ in such a way that μ/λ (resp. λ/μ) is bounded, then $u_{\lambda,\mu} \to +\infty$, $v_{\lambda,\mu} \to 0$ (resp. $u_{\lambda,\mu} \to 0$, $v_{\lambda,\mu} \to +\infty$) uniformly in $\overline{\Omega}$.*

See Fig. 3.

As a complement of the behavior observed in point (ii) of the precedent theorem, we show that even in the regime $r < p-1$, $s < q-1$, solutions do not converge to a finite profile as λ, $\mu \to \infty$ provided λ, μ vary along some curves of the form $\mu = C\lambda^\theta$ for certain values of $\theta \in (0,1)$. Such conclusion is attained under radial symmetry on x. However, we suspect that a similar assertion is true in any smooth bounded domain of $\mathbb{R}^N$.

Theorem 6. *Assume $(p-1)/2 < r \leqslant p-1$ and choose $\mu = C\lambda^\theta$ for any constant $C > 0$ and*

$$0 < \theta < \frac{2r - p + 1}{p - 1}. \tag{1.10}$$

If Ω is a ball or an annulus of $\mathbb{R}^N$ then the unique positive solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ to (1.1)–(1.2) *satisfies $u_{\lambda,\mu} \to +\infty$, $v_{\lambda,\mu} \to 0$ uniformly in $\overline{\Omega}$. Furthermore, the conclusion holds if Ω is an arbitrary simply connected domain of $\mathbb{R}^2$.*

The rest of the paper is organized as follows: Section 2 revises an already known auxiliary problem. In addition, several kind of singular eigenvalue problems—which are interesting by themselves—are considered in detail, and some new interesting results are obtained. In particular, some estimates near the boundary for some equations with Dirichlet boundary conditions and singular weights. The analysis in Section 2 will be mainly instrumental when elucidating the limit profiles of solutions to

(1.1)–(1.2). Section 3 is dedicated to prove Theorem 1, while the results on the asymptotic behavior of the solutions for varying λ and μ are all collected in Section 4.

2. Some scalar auxiliary problems

In this section, we consider some auxiliary problems which will turn out to be important in the rest of the paper. Some results are already known, but most of them are new and interesting in their own right.

We begin by analyzing the problem

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

$p > 1$, which was deeply studied in [17]. However, we would like to stress that our next result improves the knowledge of the asymptotic behavior of the solution both as $\lambda \to 0$ and $\lambda \to \infty$. In particular, the uniform estimates (2.4) and (2.5) below for large λ are not contained there. We denote by λ_1 the first eigenvalue of problem (1.5).

Theorem 7. *Assume $a \in C(\overline{\Omega})$ verifies* (H). *Then problem* (2.1) *admits a unique positive weak solution $U_\lambda \in H^1(\Omega) \cap L^\infty(\Omega)$ for every λ with $0 < \lambda < \lambda_1 \leqslant \infty$, while no positive solutions exist if $\lambda \leqslant 0$ or $\lambda \geqslant \lambda_1$ if $\lambda_1 < \infty$. In addition, U_λ is increasing and continuous in λ, and we have that*

$$U_\lambda \sim \left(\frac{\lambda}{a^*}\right)^{\frac{1}{p-1}} \tag{2.2}$$

as $\lambda \to 0+$, where $a^* = \frac{1}{|\partial\Omega|}\int_\Omega a$. When $\lambda_1 < \infty$, $U_\lambda \to \infty$ uniformly in $\overline{\Omega}_0$ as $\lambda \to \lambda_1-$, provided that $a(x) = o(d)$ as $d \to 0$, $d = \operatorname{dist}(x, \Gamma_2)$, while U_λ converges in $C^{1,\nu}(\Omega \cup \Gamma^+)$, $0 < \nu < 1$, to the minimal solution of the problem

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega^+, \\ u = \infty & \text{on } \Gamma_2, \\ \dfrac{\partial u}{\partial \nu} = \lambda_1 u & \text{on } \Gamma^+, \end{cases}$$

where the latter boundary condition is removed provided $\Gamma^+ = \emptyset$.

In case $\lambda_1 = \infty$, we have that U_λ converges to the minimal solution U_∞ to

$$\begin{cases} \Delta u = a(x)u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In addition, there exists a positive constant C which does not depend on λ such that

$$U_\lambda(x) \geqslant C\left(d(x) + \frac{\alpha}{\lambda}\right)^{-2/(p-1)} \tag{2.4}$$

in Ω for $\lambda \geqslant \lambda_0$, where $\alpha = 2/(p-1)$. If $a > 0$ on $\partial\Omega$ then we also have the complementary upper estimate

$$U_\lambda(x) \leqslant C'\left(d(x) + \frac{\alpha}{\lambda}\right)^{-2/(p-1)} \tag{2.5}$$

in Ω, for $\lambda \geqslant \lambda_0$, where C' is a positive constant which does not depend on λ.

Remark 3. To simplify somewhat the notation, we will denote by V_μ the unique solution to the corresponding problem for v where $b(x)$, s, μ replace $a(x)$, p, λ in (2.1). More precisely,

$$\begin{cases} \Delta v = b(x)v^s & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Proof of Theorem 7. Our analysis in [17] dealt with existence, uniqueness and limit profile properties of classical solutions to the more regular version of (2.1) where $a \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$. In addition, the existence of an H^1 weak solution to (2.1) was obtained there by a variational approach covering the more general framework $a \in L^\infty(\Omega)$. Furthermore, it was shown in [19] (see also [20]) that H^1 solutions are also in $L^\infty(\Omega)$ and so they are unique and define strong solutions (see above) to (2.1). Therefore, we are only proving (2.2), (2.4) and (2.5), the remaining assertions being essentially contained in Theorems 1, 2 and 3 of [17].

To show (2.2), let $\lambda_n \to 0$, and denote for simplicity $u_n = U_{\lambda_n}$. Proceeding as in the proof of Theorem 1 in [17] it follows that $|u_n|_\infty \to 0$. Thus $v_n := u_n/|u_n|_\infty$ solves

$$\begin{cases} \Delta v = a(x)|u_n|_\infty^{p-1} v^p & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda_n v & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

The right-hand side of the equation in (2.7) is bounded and so, also proceeding as in [17], one obtains a subsequence, still named v_n, such that $v_n \to v$ in $C^{1,\eta}(\overline{\Omega})$ for every $\eta \in (0,1)$, being v a strong solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $|v|_\infty = 1$. Hence $v = 1$. On the other hand, integrating the equation in (2.7) we get

$$\lambda_n \int_{\partial\Omega} v_n = |u_n|_\infty^{p-1} \int_\Omega a(x) v_n^{p-1},$$

and we arrive at

$$u_n \sim |u_n|_\infty \sim \left(\frac{1}{\partial\Omega} \int_\Omega a(x) \right)^{-\frac{1}{p-1}} \lambda_n^{\frac{1}{p-1}}.$$

Since the sequence λ_n is arbitrary, this proves (2.2).

To prove (2.4), we construct a suitable subsolution in a neighborhood of the boundary. Since Ω is C^2, there exists δ_0 such that $d(x)$ is C^2 in $0 < d < \delta_0$ and $|\nabla d| = 1$ there (cf. [21]). We search for our subsolution in the form

$$\underline{u} = \varepsilon\big(d(x) + \theta\big)^{-\alpha}, \tag{2.8}$$

where ε is small, $\alpha = 2/(p-1)$ and $\theta > 0$ is to be chosen. On the boundary we have

$$\left(\frac{\partial \underline{u}}{\partial \nu} - \lambda \underline{u} \right)\bigg|_{\partial\Omega} = \left(-\frac{\partial \underline{u}}{\partial d} - \lambda \underline{u} \right)\bigg|_{d=0} = \varepsilon\big(\alpha\theta^{-\alpha-1} - \lambda\theta^{-\alpha}\big),$$

so it suffices with setting $\theta = \alpha/\lambda$. On the other hand:

$$\Delta \underline{u} - a(x)\underline{u}^p = \varepsilon(d+\theta)^{-\alpha-2}\big(\alpha(\alpha+1) - \alpha(d+\theta)\Delta d - a(x)\varepsilon^{p-1}\big).$$

Thus $\underline{u}$ will be a subsolution provided

$$\alpha\left((\alpha+1) - \left(d + \frac{\alpha}{\lambda}\right)\Delta d\right) \geqslant \varepsilon^{p-1} \sup_{0<\delta<\delta_0} a.$$

We can choose λ_0 and diminish δ_0 if necessary to have

$$\inf_{0<d<\delta_0}\left\{(\alpha+1) - \left(d + \frac{\alpha}{\lambda}\right)\Delta d\right\} > 0$$

for $\lambda \geqslant \lambda_0$. This allows us to take a conveniently small ε (independent of λ) so that $\underline{u}$ is a subsolution in $0 < d < \delta_0$. Notice that δ_0 is also independent of λ. Next consider the problem

$$\begin{cases} \Delta u = a(x)u^p & \text{in } 0 < d < \delta_0, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } d = 0, \\ u = U_\lambda & \text{on } d = \delta_0, \end{cases}$$

which has a unique solution $u = U_\lambda$. If we choose a sufficiently small ε, we have $\underline{u} \leqslant U_\lambda$ on $d = \delta_0$. Since U_λ is increasing in λ, this choice can still be made independent of λ. Moreover, MU_λ is a supersolution for $M > 1$ large enough, and $MU_\lambda > \underline{u}$ in $0 < d < \delta_0$. It follows that $\underline{u} \leqslant U_\lambda$ in $0 < d < \delta_0$, that is

$$U_\lambda(x) \geqslant C\left(d(x) + \frac{\alpha}{\lambda}\right)^{-2/(p-1)}$$

if $\lambda \geqslant \lambda_0$, $0 < d(x) < \delta_0$, where C does not depend on λ. Finally, since U_λ converges to a finite profile as $\lambda \to \infty$, this estimate is valid throughout Ω for $\lambda \geqslant \lambda_0$, taking a smaller C if necessary. This proves (2.4).

When $a > 0$ in $\partial\Omega$, a supersolution similar to the subsolution in (2.8) can be constructed near $\partial\Omega$, so the proof of (2.5) is entirely similar. We leave the details to the reader. □

We are now concerned with a more general version of problem (2.1). We are allowing the weight $a(x)$ to be discontinuous but keeping its boundedness. We also assume that it depends on a parameter ε and becomes singular—in two different possible ways—as $\varepsilon \to 0$. More precisely, we consider

$$\begin{cases} \Delta u = A_\varepsilon(x)u^s & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \mu u & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

$s > 1$, $\mu > 0$, where we are assuming that $A_\varepsilon \in L^\infty(\Omega)$, $\varepsilon > 0$, is a family of bounded functions which verify either of the two following conditions. Namely,

$$A_\varepsilon(x) \to \infty \quad \text{uniformly in } \Omega' \tag{2.10}$$

as $\varepsilon \to 0$ in a smooth subdomain Ω' of Ω satisfying the structure conditions of Ω_0 in hypothesis (H) (cf. Section 1). In this scenario we define $\Omega'' = \Omega \setminus \overline{\Omega}'$ and we are supposing in addition that A_ε remains uniformly bounded in Ω''.

An alternative condition that we are studying is

$$A_\varepsilon(x) \geqslant C\big(d(x)+\varepsilon\big)^{-\theta} \quad \text{in } \Omega \tag{2.11}$$

for a certain $\theta \geqslant 1$ and a positive constant C.

We are interested in analyzing the behavior of the unique positive solution $U_{\mu,\varepsilon}$ to (2.9) as $\varepsilon \to 0$, for fixed $\mu > 0$. We remark that the results in Theorem 7 still hold for bounded weights with no essential modifications. The main features of problem (2.9) when the coefficient A_ε behaves in the singular way that are described below.

Theorem 8. *Suppose $A_\varepsilon \in L^\infty(\Omega)$, $\varepsilon > 0$, is a family of functions such that $A_\varepsilon(x)$ decreases in $\varepsilon > 0$, verifies* (2.10) *and remains uniformly bounded in $\Omega \setminus \overline{\Omega}'$. Then the unique solution $u = U_{\mu,\varepsilon}$ to* (2.9) *converges uniformly to zero in $\overline{\Omega}'$ as $\varepsilon \to 0$. Furthermore, $U_{\mu,\varepsilon}$ also converges uniformly to zero in every connected piece Ω''_i of Ω'' such that $\mu \leqslant \mu_{1,i}$ where $\mu_{1,i} = \infty$ if $\Omega''_i \Subset \Omega$ or $\mu = \mu_{1,i}$ stands for the principal eigenvalue to the problem* [17,18]

$$\begin{cases} \Delta\psi = 0, & x \in \Omega''_i, \\ \psi = 0, & x \in \partial\Omega''_i \cap \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = \mu\psi, & x \in \partial\Omega''_i \cap \partial\Omega. \end{cases}$$

Proof. We are using the notation u_ε instead $U_{\mu,\varepsilon}$ for simplicity. In addition we put $\Gamma'_1 = \partial\Omega' \cap \Omega$, $\Gamma'' = \partial\Omega'' \cap \partial\Omega$, $\Gamma' = \partial\Omega' \cap \Omega = \partial\Omega'' \cap \Omega$. Remark that, according to (H), $\Gamma' \subset \Omega$ is a closed manifold which is always nonempty while either Γ'_1 or Γ'' could be possibly empty, but not simultaneously. We are next dealing with the more elaborate case where both Γ'_1 and Γ'' are nonempty (the remaining possibilities are handled in the same way). We also denote $A_0(x) = \sup_{\varepsilon>0} A_\varepsilon(x) = \lim_{\varepsilon\to 0} A_\varepsilon(x)$ for $x \in \Omega''$. Observe that $A_0 \in L^\infty(\Omega'')$.

The auxiliary problem:

$$\begin{cases} \Delta u = A_0 u^s, & x \in \Omega'', \\ u = 0, & x \in \Gamma', \\ \dfrac{\partial u}{\partial\nu} = \mu u, & x \in \partial\Omega'', \end{cases} \tag{2.12}$$

has a unique positive strong solution $\hat{u}_0 \in W^{2,p}(\Omega'') \cap C^{1,\eta_0}(\overline{\Omega}'')$ for every $p > 1$, $0 < \eta_0 < 1$.

On the other hand, the positive strong solution u_ε to (2.9) belongs to $W^{2,p}(\Omega) \cap C^{1,\eta_0}(\overline{\Omega})$, $p > 1$, $0 < \eta_0 < 1$, and is increasing in ε. Therefore the function u_0 given as

$$u_0(x) = \lim u_\varepsilon(x) = \inf_{\varepsilon>0} u_\varepsilon(x), \quad x \in \Omega,$$

is well defined, lies in $L^\infty(\Omega)$ while the limit holds in $L^p(\Omega)$ for all $p \geqslant 1$. We are next showing that $u_0 = 0$ a.e. in Ω' together with $u_0(x) = \hat{u}_0(x)$ for *all* $x \in \Omega''$.

First, observe that $\hat{u}_0 \in H^1_{\Gamma'}(\Omega'') = \{u \in H^1(\Omega''):\ u_{|\Gamma'} = 0\}$ defines the minimum of the variational problem

$$\inf_{u \in H^1_{\Gamma'}(\Omega'')} J_0(u),$$

where

$$J_0(u) = \frac{1}{2}\int_{\Omega''} |\nabla u|^2 + \frac{1}{s+1}\int_{\Omega''} A_0 u^{s+1} - \frac{\mu}{2}\int_{\Gamma''} u^2.$$

Similarly, $J_\varepsilon(u_\varepsilon) = \inf_{H^1(\Omega)} J_\varepsilon(u)$ where

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{s+1}\int_\Omega A_\varepsilon u^{s+1} - \frac{\mu}{2}\int_{\partial\Omega} u^2.$$

Thus, by letting $\bar{u}_0 \in H^1(\Omega)$ be the extension by zero of $\hat{u}_0$ to Ω, we achieve

$$J_\varepsilon(u_\varepsilon) \leqslant J_\varepsilon(\bar{u}_0) \leqslant J_0(\hat{u}_0), \quad \varepsilon > 0. \tag{2.13}$$

This implies the boundedness of u_ε in $H^1(\Omega)$, hence $u_\varepsilon \rightharpoonup u_0$ and so $u_0 \in H^1(\Omega)$. It follows in addition from (2.13) that

$$m_\varepsilon \int_{\Omega'} u_\varepsilon^{s+1} = O(1) \quad \text{as } \varepsilon \to 0,$$

with $m_\varepsilon \to \infty$. From this $u_0 = 0$ a.e. in Ω' which implies that the restriction of u_0 to Ω'' belongs to $H^1_{\Gamma'}(\Omega'')$. By taking now limits in (2.13) we obtain

$$J_0(u_0) \leqslant J(\hat{u}_0).$$

From the uniqueness of the weak solution to (2.12) we conclude $u_0 = \hat{u}_0$ a.e. in Ω''. However, the interior version of the $W^{2,p}$ estimates in [1] can be used to show that the convergence $u_\varepsilon \to u_0$ actually occurs in $C^{1,\eta_0}(\Omega'' \cup \Gamma'')$, $0 < \eta_0 < 1$. Thus $u_0 \in C^{1,\eta_0}(\Omega'' \cup \Gamma'')$ which in turn ensures that $u_0(x) = \hat{u}_0(x)$ for *every* $x \in \Omega''$.

Let us finish by showing the uniform convergence of u_ε to zero in $\overline{\Omega}'$. For $\delta > 0$ small define $Q_\delta = \{x \in \Omega:\ \operatorname{dist}(x, \Omega') < \delta\}$ the δ neighborhood of Ω' in Ω, $\Gamma'_\delta = \{x \in \partial Q_\delta:\ \operatorname{dist}(x, \Omega') = \delta\}$. Observe that

$$-\Delta u_\varepsilon + m u_\varepsilon \leqslant m u_\varepsilon,$$

for $m > 0$ conveniently large, so that we achieve

$$u_\varepsilon(x) \leqslant \tilde{u}_{\varepsilon,\delta}(x), \quad x \in Q_\delta,$$

where $u = \tilde{u}_{\varepsilon,\delta} \in C^{1,\eta_0}(\overline{Q}_\delta)$ stands for the strong positive solution to the problem

$$\begin{cases} -\Delta u + mu = mu_\varepsilon, & x \in Q_\delta, \\ u = u_\varepsilon, & x \in \Gamma'_\delta, \\ \dfrac{\partial u}{\partial \nu} = \mu u, & x \in \Gamma'_1. \end{cases} \tag{2.14}$$

Observe now that $\Gamma'_\delta \subset \Omega''$ and thus we get uniform estimates of the $W^{2-1/p,p}(\Gamma'_\delta)$ norm of u_ε. By employing the $W^{2,p}$ estimates in [1] we conclude that $\tilde{u}_{\varepsilon,\delta} \to \tilde{u}_{0,\delta}$ in $C^{1,\eta_0}(\overline{Q}_\delta)$, being $\tilde{u}_{0,\delta}$ the strong solution to the problem

$$\begin{cases} -\Delta u + mu = m\bar{u}_0, & x \in Q_\delta, \\ u = \hat{u}_0, & x \in \Gamma'_\delta, \\ \dfrac{\partial u}{\partial \nu} = \mu u, & x \in \Gamma'_1. \end{cases} \tag{2.15}$$

Therefore,

$$u_0(x) \leqslant \tilde{u}_{0,\delta}(x) \quad \text{for all } x \in \overline{Q}_\delta. \tag{2.16}$$

We are finally proving that $\tilde{u}_{0,\delta}$ converges uniformly to zero in $\overline{\Omega}'$ as $\delta \to 0$. In fact, a smooth family of diffeomorphisms $x = T_\delta(y)$, $T_\delta : \overline{\Omega}' \to \overline{Q}_\delta$ exists which leave invariant $\overline{\Omega}' \setminus U_\delta$, U_δ a small δ-varying neighborhood of Γ_δ' in $\overline{Q}_\delta$ and such that $T_\delta(\Gamma') = \Gamma_\delta'$ for every $\delta > 0$ (see [25]). Setting $y = H_\delta(x) := T_\delta^{-1}(x)$ the inverse diffeomorphism, the "mayorant" problem (2.15) is transformed into

$$\begin{cases} -\displaystyle\sum_{k,l=1}^{N} \big\langle \nabla(H_\delta)_k, \nabla(H_\delta)_l \big\rangle \frac{\partial^2 v}{\partial y_k \partial y_l} - \sum_{k=1}^{N} \Delta(H_\delta)_k \frac{\partial v}{\partial y_k} + mv = m\bar{u}_0\big(T_\delta(y)\big), & y \in \Omega', \\ v = \hat{u}_0\big(T_\delta(y)\big), & y \in \Gamma', \\ \dfrac{\partial v}{\partial \nu} = \mu v, & x \in \Gamma_1'. \end{cases} \tag{2.17}$$

The unique strong solution to (2.17) is provided by $v_\delta = \tilde{u}_{0,\delta} \circ T_\delta$. Since $\hat{u}_0 \in W^{2,p}(\Omega'') \cap C^{1,\eta_0}(\overline{\Omega}'')$ we are in possession of uniform bounds in $W^{2-1/p,p}(\Gamma')$ of $\hat{u}_0 \circ T_\delta$. Therefore, the $W^{2,p}$ estimates in [1] once again imply the convergence $v_\delta \to v_0$ in $W^{2,p}(\Omega') \cap C^{1,\eta_0}(\overline{\Omega}')$ where v_0 is the unique solution of the limit problem obtained from (2.17) as $\delta \to 0$. Taking into account that $\langle \nabla(H_\delta)_k, \nabla(H_\delta)_l \rangle = \delta_{kl}$ and $\Delta(H_\delta)_k = 0$ at $\delta = 0$ or $1 \leqslant k, l \leqslant N$ (see [25]) the limit problem becomes

$$\begin{cases} -\Delta v + mv = 0, & x \in \Omega', \\ v = 0, & x \in \Gamma', \\ \dfrac{\partial v}{\partial \nu} = \mu v, & x \in \Gamma_1'. \end{cases} \tag{2.18}$$

However, if m is large enough the unique solution of (2.18) is $v_0 = 0$. Therefore and taking limits in (2.16) as $\delta \to 0$ it is obtained that $u_0 = 0$ at *every* $x \in \Omega'$. The uniform character of the convergence $u_\varepsilon \to 0$ in Ω' is implicit in the preceding argument. Alternatively, Dini's theorem can be employed.

At the present moment Ω'' has been regarded as a "whole". The proof of the theorem is completed with the additional remark that $\hat{u}_0 = 0$ at every connected piece Ω_i'' of Ω'' such that either $\Omega_i'' \Subset \Omega'$ or $\mu \leqslant \mu_{1,i}$ (cf. [17]). $\square$

A second result describing the behavior of positive solutions to problem (2.9) when the weight A_ε develops a singularity on the boundary is the following.

Theorem 9. *Consider a family $A_\varepsilon \in L^\infty(\Omega)$, $\varepsilon > 0$, which is decreasing in ε and verifies the condition* (2.11) *for a certain $\theta \geqslant 1$ while $u = U_{\mu,\varepsilon}$ stands for the unique positive solution to* (2.9)*. Then $U_{\mu,\varepsilon} \to 0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$.*

Proof. To simplify, let us define as before $u_\varepsilon = U_{\mu,\varepsilon}$ the unique positive solution to (2.9). Since A_ε is decreasing in ε, u_ε is increasing in ε, and then $u_\varepsilon \to u_0$ as $\varepsilon \to 0$, where u_0 is a nonnegative function. In addition, such a convergence holds in $L^p(\Omega)$ for all $p > 1$ while proceeding as in the proof of Theorem 8 it follows that both $u_\varepsilon \to u_0$ weakly in $H^1(\Omega)$ and in $C^{1,\eta_0}(\Omega)$, $0 < \eta_0 < 1$. We deduce from (2.9) and (2.11) that

$$C \int_\Omega \big(d(x) + \varepsilon\big)^{-\theta} u_\varepsilon^{s+1} \leqslant \frac{2}{s+1} \int_\Omega A_\varepsilon(x) u_\varepsilon^{s+1} = \mu \int_{\partial\Omega} u_\varepsilon^2 - \int_\Omega |\nabla u_\varepsilon|^2 \leqslant \mu \int_{\partial\Omega} u_\varepsilon^2.$$

We can now pass to the limit as $\varepsilon \to 0$, use Fatou's theorem and obtain

$$C\int_{\Omega} d(x)^{-\theta} u_0^{p+1} \leqslant \mu \int_{\partial\Omega} u_0^2. \tag{2.19}$$

We claim that the convergence of the integral in the right-hand side of (2.19) implies, in view of $\theta \geqslant 1$, that $u_0 = 0$ on $\partial\Omega$. Thus (2.19) and the continuity of u_0 readily give $u_0(x) = 0$ for every $x \in \Omega$. We are next showing the uniform convergence to zero in $\overline{\Omega}$. First, $u = u_\varepsilon$ satisfies for $0 < \varepsilon < \varepsilon_0$,

$$-\Delta u + mu \leqslant mu,$$

for a conveniently large m. That is why

$$u_\varepsilon(x) \leqslant \hat{u}_\varepsilon(x) \quad \text{for all } x \in \Omega,$$

where $u = \hat{u}_\varepsilon$ is the unique strong (even classical!) solution to the majorant problem

$$\begin{cases} -\Delta u + mu = mu_\varepsilon, & x \in \Omega', \\ \dfrac{\partial u}{\partial \nu} = \mu u, & x \in \partial\Omega. \end{cases} \tag{2.20}$$

By arguing as in the proof of Theorem 8 it follows that $\hat{u}_\varepsilon$ converges in $C^{1,\eta_0}(\overline{\Omega})$ to the unique solution $\hat{u}_0$ to the limit problem of (2.20), namely

$$\begin{cases} -\Delta u + mu = 0, & x \in \Omega', \\ \dfrac{\partial u}{\partial \nu} = \mu u, & x \in \partial\Omega. \end{cases}$$

Choosing a large m guarantees that $\hat{u}_0(x) = 0$ for every $x \in \Omega$ and so $u_\varepsilon \to 0$ uniformly in $\overline{\Omega}$.

We finally outline the proof of the claim: observe first that if a nonnegative u lies in say $H^1(I)$, $I = (0,1)$ the unit interval, and makes finite the integral $\int_0^\rho x^{-\theta} u(x)\,dx$ for a certain $\rho > 0$ then, since $u \in C[0,1]$, $u(0)$ must be necessarily zero provided $\theta \geqslant 1$. In the N-dimensional case above, after a change of variables near the boundary, u_0 belongs to $H^1(I_\delta)$ where I_δ stands for a uniform one-dimensional interval of length $\delta > 0$ on the normal inner semiline to $\partial\Omega$, for "almost all normal lines", the "almost all" being considered with respect to the $(N-1)$-dimensional measure on $\partial\Omega$. Moreover, an integral exactly as the considered above must be finite for almost all those normals. Therefore, $u_0(x) = 0$ for almost all $x \in \partial\Omega$ and we are done. □

Our next step is to consider problem (2.1), when the weight is allowed to be singular on $\partial\Omega$, that is, we study

$$\begin{cases} \Delta v = B(x)v^s & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega, \end{cases} \tag{2.21}$$

where the weight $B(x)$ is a continuous, positive function in Ω, and we require an upper bound for the singularity of the form

$$B(x) \leqslant Cd(x)^{-\tau} \tag{2.22}$$

for some $\tau < 1$ and $C > 0$ (for its use in Section 4, we have replaced $a(x), p, \lambda$ by $B(x), s, \mu$). Then we have the following result.

Theorem 10. *Let B be a positive continuous function in Ω which verifies* (2.22)*. Then problem* (2.21) *admits a unique positive weak solution $\tilde{V}_\mu \in H^1(\Omega) \cap L^\infty(\Omega)$ for every $\mu > 0$. Moreover, $\tilde{V}_\mu$ is increasing in μ and converges as $\mu \to \infty$ to the minimal positive solution $\tilde{V}_\infty$ to*

$$\begin{cases} \Delta V = B(x)V^s & \text{in } \Omega, \\ V = \infty & \text{on } \partial\Omega. \end{cases} \tag{2.23}$$

Proof. Let us first show existence. We truncate the weight B multiplying by a smooth cut-off function. To this aim, let $\psi \in C^\infty(\mathbb{R})$ such that $0 \leqslant \psi \leqslant 1$, $\psi(t) = 0$ if $t \leqslant 1$ while $\psi(t) = 1$ for $t \geqslant 2$, and ψ is increasing. If we denote $B_k(x) = \psi(kd(x))B(x)$, we obtain a family of increasing, bounded weights such that $B_k \to B$ uniformly on compacts of Ω as $k \to \infty$. We consider the truncated problem

$$\begin{cases} \Delta v = B_k(x)v^s & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

which has a unique positive weak solution v_k for every $\mu > 0$, thanks to Theorem 7. Moreover, v_k is decreasing in k, since v_{k+1} is a subsolution to problem (2.24) while Mv_k is a supersolution with a large enough M. To be able to pass to the limit, we need a uniform subsolution, to guarantee that v_k is bounded away from zero. Recall that $B_k(x) \leqslant B(x) \leqslant C\,d(x)^{-\tau}$, and let ϕ be the unique (positive) solution to the equation

$$\begin{cases} -\Delta\phi = Cd(x)^{-\tau} & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

We remark that (2.25) has a solution $\phi \in C^{1,1-\tau}(\overline{\Omega})$ since $\tau < 1$, thanks to Theorem 8.34 in [21]. We are taking the subsolution as $\underline{v} = \varepsilon - \varepsilon^s\phi$, for small positive ε. We have

$$\Delta\underline{v} = \varepsilon^s Cd(x)^{-\tau} \geqslant \varepsilon^s B_k(x) \geqslant B_k(x)\varepsilon^s\left(1 - \varepsilon^{s-1}\phi\right)^s = B_k(x)\underline{v}^s$$

in Ω, while

$$\frac{\partial\underline{v}}{\partial\nu} = -\varepsilon^s\frac{\partial\phi}{\partial\nu} \leqslant \mu\varepsilon = \mu\underline{v}$$

on $\partial\Omega$, for small ε. Since there are large supersolutions, we deduce $v_k \geqslant \underline{v}$ in Ω. Moreover:

$$\int_\Omega |\nabla v_k|^2 = \mu\int_{\partial\Omega} v_k^2 - \int_\Omega B_k(x)v_k^{s+1} \leqslant \mu\int_{\partial\Omega} v_k^2,$$

so that $v_k \to v$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, and $v \geqslant \underline{v}$. In particular, since for every $\psi \in H^1(\Omega)$ we have

$$\int_\Omega \nabla v_k\nabla\psi - \mu\int_{\partial\Omega} v_k\psi = \int_\Omega B_k(x)v_k^s\psi \tag{2.26}$$

and $0 \leqslant B_k \leqslant B \in L^1(\Omega)$ (due to $\tau < 1$), the dominated convergence theorem allows us to pass to the limit in (2.26) and obtain that v is a weak positive solution to (2.21).

To show uniqueness we first observe that every nonnegative weak solution $w \in H^1(\Omega)$ to (2.21) lies necessarily in $L^\infty(\Omega)$ (see [20]). This in particular implies, in virtue of the uniqueness of solutions to (2.24) that $w \leqslant v$. If, however, w is nontrivial (and so positive) since $B \in L^1(\Omega)$, we can argue as in [17] to obtain that $w = v$. Thus problem (2.21) admits a unique positive solution.

Finally, the asymptotic behavior of $\tilde{V}_\mu$ is obtained as in [17] (we refer also to [4] for existence and uniqueness results on problem (2.23) and related ones). $\square$

Another problem that will be necessary in Section 4 is obtained when the weight function is supported in a subdomain Ω^+ of Ω, and different boundary conditions are imposed on two parts of $\partial\Omega^+$. More precisely, we are interested in the case $\Omega^+ = \Omega \setminus \overline{\Omega}_0$, where Ω_0 is the same as in hypothesis (H) on $a(x)$ and Ω^+ might exhibit multiple connected pieces Ω_i^+. Recalling the notation $\Gamma^+ = \partial\Omega^+ \setminus \Gamma_2$ (with $\Gamma_2 = \partial\Omega_0 \cap \Omega$) we are dealing with the following problem, related to (2.1) but with a singular boundary condition

$$\begin{cases} \Delta w = A(x)w^p & \text{in } \Omega^+, \\ w = \infty & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial \nu} = \lambda w & \text{on } \Gamma^+. \end{cases} \tag{2.27}$$

The function $A(x)$ essentially behaves as a power of the distance $d(x) = \operatorname{dist}(x, \Gamma_2)$. Problem (2.27) for bounded weights was considered in [17] (although no estimates were provided there). We are also including here for completeness the case of singular weights.

Our result for problem (2.27) is as follows.

Theorem 11. *Let A be a continuous positive function in $\Omega^+ \cup \Gamma^+$ such that*

$$C_1 d(x)^\tau \leqslant A(x) \leqslant C_2 d(x)^\tau,$$

$d(x) = \operatorname{dist}(x, \Gamma_2)$, for some positive constants C_1, C_2 and $\tau > -2$. Then the problem (2.27) *admits a unique positive weak solution w_λ. Moreover, there exist positive constants D_1, D_2 such that*

$$D_1 d(x)^{-\frac{2+\tau}{p-1}} \leqslant w_\lambda(x) \leqslant D_2 d(x)^{-\frac{2+\tau}{p-1}}, \quad x \in \Omega. \tag{2.28}$$

Proof. The proof is an adaptation of that of Theorem 1 in [4]. We may assume $\tau < 0$, since when $\tau \geqslant 0$ the existence result is contained in [17]. We first fix $n \in \mathbb{N}$ and truncate the weight $A(x)$ as in the proof of Theorem 10 to obtain a bounded weight $A_k(x)$ and deal with the family of problems,

$$\begin{cases} \Delta w = A_k(x)w^p & \text{in } \Omega^+, \\ w = n & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial \nu} = \lambda w & \text{on } \Gamma^+. \end{cases} \tag{2.29}$$

Problem (2.29) admits for every $k, n \in \mathbb{N}$ a unique strong solution $w_{k,n}$, which is in addition unique thanks to Lemma 8 in [17]. In fact, $\underline{w} = 0$ is a subsolution. To construct large supersolutions we distinguish two cases. For $\lambda > \lambda_1$, λ_1 the principal eigenvalue to (1.5) regarded in Ω^+, and a small enough δ, the problem

$$\begin{cases} \Delta u = A_k(x)u^p & \text{in } \Omega_\delta^+, \\ u = 0 & \text{on } \Gamma_{2,\delta}, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma^+, \end{cases}$$

with $\Omega_\delta^+ = \Omega \cup \Gamma_2 \cup \{x\colon\ \operatorname{dist}(x, \Gamma_2) < \delta\}$, $\Gamma_{2,\delta} = \{x \in \partial\Omega_\delta^+\colon\ \operatorname{dist}(x, \Gamma_2) = \delta\}$ admits a unique positive strong solution $u_{\lambda,\delta}$. To see this it suffices with proceeding as in [17] where an entirely similar problem is considered. Thus, since $u_{\lambda,\delta}$ is positive on Γ_2, $\overline{w} = Mu_{\lambda,\delta}$ defines, for large $M > 0$, a supersolution as large as desired. In the second case, where $\lambda \leqslant \lambda_1$ and a small enough $\delta > 0$ again, the eigenvalue problem [17]

$$\begin{cases} \Delta\phi = \sigma\phi & \text{in } \Omega_\delta^+, \\ \phi = 0 & \text{on } \Gamma_{2,\delta}, \\ \dfrac{\partial\phi}{\partial\nu} = \lambda\phi & \text{on } \Gamma^+, \end{cases}$$

admits a unique principal eigenvalue $\sigma = \sigma_1 < 0$ with a positive associated eigenfunction $\phi_{1,\delta}$. Being $\phi_{1,\delta}$ positive on Γ_2 it is then clear that $\overline{w} = M\phi_{1,\delta}$ defines a large supersolution to (2.29) modulated by $M > 0$. Notice in addition that this choice of $\overline{w}$ also works in the present case for $\lambda > \lambda_1$ since A is positive in $\overline{\Omega}$ (our previous construction covers A nonnegative).

Moreover, since A_k is increasing in k, $w_{k,n}$ is decreasing in k, and it is increasing in n. By fixing n it follows that $w_{k,n}$ converges in $C^{1,\eta_0}(\Omega^+ \cup \Gamma^+) \cup W^{2,q}(\Omega^+ \cap \{d > \delta\})$ to a strong solution w_n of the equation satisfying the flux condition. To achieve the continuity up to Γ_2 we can now argue as in [4] to construct a local barrier near Γ_2. Thus we obtain that w_n defines a strong solution to

$$\begin{cases} \Delta w = A(x)w^p & \text{in } \Omega^+, \\ w = n & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial\nu} = \lambda w & \text{on } \Gamma^+. \end{cases}$$

In addition, w_n is increasing in n. Since $A(x) \geqslant A_0 > 0$ in Ω, it follows that w_n is locally bounded in Ω. Indeed, the upper bound is provided by the minimal solution to the previous problem with $A(x), n$ replaced by A_0, ∞, respectively. Thus we can pass to the limit to obtain that $w_n \to w$ locally uniformly, where w is a weak solution to (2.27).

Estimates (2.28) are proved exactly as in Theorem 3.1 in [4] (we remark that the estimates are local in nature). Finally, the uniqueness is a consequence of the estimates (2.28) by proceeding as in Theorem 3.4 of [4]. $\square$

We finally turn to consider the perhaps most interesting of our auxiliary problems. In this case, the weight is singular on Γ_2 (behaving essentially as a power of the distance $d(x) = \operatorname{dist}(x, \Gamma_2)$) and a homogeneous Dirichlet condition is imposed there. Such boundary condition makes that the problem can always be solved independently of the singularity of the weight, in contrast for example with Theorem 11. Imitating our framework described in hypothesis (H) we are considering a bounded smooth domain Ω^+ (in future applications such domain will be a connected piece of $\{a > 0\}$) whose boundary splits off in two separate groups Γ_2, Γ^+, of closed $(N-1)$-dimensional manifolds. Our next problem will be

$$\begin{cases} \Delta z = B(x)z^s & \text{in } \Omega^+, \\ z = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial z}{\partial\nu} = \mu z & \text{on } \Gamma^+, \end{cases} \tag{2.30}$$

with B positive and continuous in $\Omega^+ \cup \Gamma^+$ but singular on Γ_2. As mentioned above, the case where B is continuous up to Γ_2 can be treated as in [17], to show that there exists a unique weak solution provided $\mu > \mu^+$, where $\mu = \mu^+$ is the principal eigenvalue of the problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega^+, \\ \phi = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\phi}{\partial\nu} = \mu\phi & \text{on } \Gamma^+. \end{cases}$$

When B is singular, the existence of solutions is not at all straightforward. We remark that the hard task in this case is to obtain estimate near Γ_2 for the (unique) solution. These estimates will be important later on.

Theorem 12. *Let B be continuous and positive in $\Omega^+ \cup \Gamma^+$, and assume there exist positive constants C_1, C_2 and τ such that*

$$C_1 d(x)^{-\tau} \leqslant B(x) \leqslant C_2 d(x)^{-\tau}, \quad x \in \Omega^+,$$

where $d(x) = \operatorname{dist}(x, \Gamma_2)$. Then problem (2.30) *can only have positive solutions if $\mu > \mu^+$ and in fact such solutions exist for each $\mu > \mu^+$. Furthermore, provided $\tau \neq s+1$, positive weak solutions are* unique *in that range. More importantly, if $z = z_\mu$ stands for the solution to* (2.30)*, then there exist positive constants D_1, D_2 such that*

$$D_1 d(x)^\theta \leqslant z_\mu(x) \leqslant D_2 d(x)^\theta \tag{2.31}$$

in Ω, where $\theta = \max\{1, (\tau-2)/(s-1)\}$.

Remark 4. A close analysis of symmetric cases shows that estimates (2.31) fail when $\tau = s+1$ and in fact z_μ decays near Γ_2 as $h(d)d$ with h involving a negative power of $\log d^{-1}$. However, since this precise information is not to be used in this paper we are not sharpening the estimates in this case.

Proof of Theorem 12. Let us show that no positive solutions exist when $\mu \leqslant \mu^+$. Assume there exists a positive weak solution z to (2.30). Let $\Omega_n^+ = \{x \in \Omega^+ \colon d(x) > 1/n\}$, $d(x) = \operatorname{dist}(x, \Gamma_2)$, and μ_n^+, ϕ_n be the principal eigenvalue and corresponding eigenfunction in Ω_n of

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_n^+, \\ \phi = 0 & \text{on } \Gamma_{2,n}, \\ \dfrac{\partial\phi}{\partial\nu} = \mu\phi & \text{on } \Gamma^+, \end{cases}$$

where $\Gamma_{2,n} = \partial\Omega_n^+ \setminus \Gamma^+$. It is not hard to show that $\mu_n^+ \to \mu^+$, while $\phi_n \to \phi$ uniformly on compacts of $\Omega^+ \cup \Gamma^+$ (notice that only the Dirichlet boundary condition is perturbed). If we multiply (2.30) by ϕ_n and integrate in Ω_n^+ we get

$$\int_{\Omega_n^+} B(x) z^s \phi_n = \left(\mu - \mu_n^+\right) \int_{\Gamma^+} z\phi_n - \int_{\Gamma_{2,n}} \frac{\partial\phi_n}{\partial\nu} z. \tag{2.32}$$

The last term goes to zero as $n \to \infty$. Indeed, notice that estimates (2.31)—which will be proved later on—imply that $z \in C(\overline{\Omega}^+)$ and $z = 0$ on Γ_2 in the usual pointwise sense. Thus, given a small $\varepsilon > 0$ and taking a large enough n we can assume that $0 < z \leqslant \varepsilon$ on $\Gamma_{2,n}$. Thus,

$$\left| \int_{\Gamma_{2,n}} \frac{\partial\phi_n}{\partial\nu} z \right| \leqslant -\varepsilon \int_{\Gamma_{2,n}} \frac{\partial\phi_n}{\partial\nu} = \varepsilon \int_{\Gamma^+} \frac{\partial\phi_n}{\partial\nu} = \varepsilon\mu_n^+ \int_{\Gamma^+} \phi_n = O(\varepsilon),$$

as $\varepsilon \to 0+$. Since ε is arbitrary, we can pass to the limit in (2.32) by means of the dominated convergence theorem to arrive at

$$\int_{\Omega^+} B(x) z^s \phi = \left(\mu - \mu^+\right) \int_{\Gamma^+} z\phi,$$

and we deduce $\mu > \mu^+$, since z and ϕ are strictly positive on Γ^+.

Now assume $\mu > \mu^+$, and let us show that there exists a positive solution to (2.30). Since $B(x)$ is bounded in Ω_n^+ and $\mu > \mu_n^+$ for a sufficiently large n, it follows that (2.30) has a solution in Ω_n^+ (by replacing Ω^+ by Ω_n^+ and Γ_2 by $\Gamma_{2,n}$). This solution is in addition unique, thanks to Lemma 8 in [17]. Let us denote it by z_n. We have $z_n \leqslant z_{n+1}$, since z_{n+1} is a supersolution to the problem in Ω_n^+, while εz_n is a subsolution for small positive ε. On the other hand, it is possible to obtain a uniform bound by taking MZ_μ, where Z_μ is the solution to (2.30) with $B \equiv 1$ (notice that $Z_\mu > 0$ on $\Gamma_{2,n}$) where M is large and independent of n. We deduce then that $z_n \leqslant MZ_\mu$. It is now standard to conclude that $z_n \to z$ in $C^1(\Omega^+ \cup \Gamma^+)$, where z is a positive weak solution to (2.30). Notice that $z = 0$ on Γ_2, since $z \leqslant MZ_\mu$ and $Z_\mu = 0$ on Γ_2.

Let us now prove that every positive solution to (2.30) satisfies the estimates (2.31). Notice first that, thanks to Hopf's maximum principle, $Z_\mu(x) \leqslant Cd(x)$. Thus, every positive solution z verifies $z \leqslant Cd$. Now we use an argument from [4]. Take x near Γ_2, and introduce the function

$$w(y) = d(x)^{-\sigma} z\big(x + d(x)y\big)$$

with $\sigma = (\tau - 2)/(s - 1)$ and $y \in B_{1/2}(0)$. We have $\Delta w \geqslant Cw^s$ in $B_{1/2}(0)$, and hence $w \leqslant W$, the unique solution to $\Delta W = CW^s$ in $B_{1/2}$ with $W_{|\partial B_{1/2}} = \infty$. Setting $y = 0$, we arrive at $z(x) = w(0)d(x)^\sigma \leqslant W(0)d(x)^\sigma$. Thus we have shown

$$z(x) \leqslant Cd(x)^\theta,$$

where $\theta = \max\{1, (\tau - 2)/(s - 1)\}$.

The lower estimate is more delicate. If $\sigma > 1$, it is easily seen that $\underline{u} = \varepsilon d(x)^\sigma$ is a subsolution in a neighborhood of Γ_2 of the form $0 < d < \delta$ provided ε and δ are small enough. Indeed,

$$\begin{aligned} \Delta\underline{u} - B(x)\underline{u}^s &\geqslant \varepsilon\sigma(\sigma - 1)d^{\sigma-2} + \varepsilon\sigma d^{\sigma-1}\Delta d - C\varepsilon^s d^{-\tau+s\sigma} \\ &= \varepsilon d^{\sigma-2}\big(\sigma(\sigma - 1) + \sigma d\Delta d - C\varepsilon^{s-1}\big), \end{aligned}$$

and this quantity can be made positive when $\sigma > 1$, by taking ε and δ adequately small.

Now let z be a positive solution to (2.30). Then $w = z$ clearly satisfies ($d = \operatorname{dist}(x, \Gamma_2)$)

$$\begin{cases} \Delta w = B(x)w^s & \text{in } 0 < d < \delta, \\ w = 0 & \text{on } d = 0, \\ w = z & \text{on } d = \delta. \end{cases}$$

By diminishing ε if necessary, we can achieve $\underline{u} < z$ on $d = \delta$. This implies $\underline{u} \leqslant z$ in $0 < d < \delta$. In fact, let $D = \{\underline{u} > z\} \cap \{0 < d < \delta\}$, and assume $D \neq \emptyset$. In D we have $\Delta\underline{u} > \Delta z$, and by the maximum principle, since $\underline{u} = z$ on ∂D, we arrive at $\underline{u} \leqslant z$ in D, which is impossible. Hence, $D = \emptyset$, that is, $z \geqslant \underline{u}$, so that

$$z(x) \geqslant Cd(x)^\sigma,$$

provided $\sigma > 1$.

We are now considering the case $\sigma < 1$. For $x_0 \in \Gamma_2$, take an annulus $A = \{x\colon\ R_1 < |x - \tilde{x}| < R_2\}$, tangent to Γ_2 at x_0, and such that $A \subset \Omega^+$. With no loss of generality, we can assume $\tilde{x} = 0$. Consider the problem

$$\begin{cases} \Delta w = C\big(R_2 - |x|\big)^{-\tau} w^s & \text{in } A, \\ w = \varepsilon & \text{on } |x| = R_1, \\ w = 0 & \text{on } |x| = R_2, \end{cases} \tag{2.33}$$

where $C > 0$ and ε is sufficiently small. Problem (2.33) has at least a radial solution w, which can be constructed as before, by approximating A by sub-annulus which avoid the boundary $|x| = R_2$. Moreover, it follows again that $z \geqslant w$. Notice in addition that $w \leqslant C(R_2 - r)$. Let us obtain a lower estimate for w near $|x| = R_2$. To this aim, we perform in the radial version of (2.33) the change of variables:

$$y = \begin{cases} \frac{1}{N-2}\left(\frac{1}{r^{N-2}} - \frac{1}{R_2^{N-2}}\right), & N \geqslant 3, \\ \log\left(\frac{R_2}{r}\right), & N = 2, \end{cases}$$

where $r = |x|$, and obtain, in the new variable y:

$$\begin{cases} w'' = b(y)w^s, & y > 0, \\ w(0) = 0, \end{cases}$$

where $b(y)$ is continuous in $y > 0$ and verifies $C_1 y^{-\tau} \leqslant b(y) \leqslant C_2 y^{-\tau}$ near $y = 0$. Also, $w(y) \leqslant Cy$, and we have to prove that

$$\liminf_{y \to 0} \frac{w(y)}{y} > 0. \tag{2.34}$$

Notice first of all that w is convex. Thus, w' is increasing and we deduce that necessarily $w' \geqslant 0$ for $y > 0$ small, since $w(0) = 0$ and $w > 0$. Moreover, w' has a limit at $y = 0$. Assume for a contradiction that (2.34) does not hold, so that $\lim_{y \to 0} w'(y) = 0$.

Choose $y_0 > 0$ and integrate the equation between y_0 and y; we obtain

$$w(y) = w(y_0) + w'(y_0)(y - y_0) + \int_{y_0}^{y} \int_{y_0}^{t} b(r) w(r)^s \, dr \, dt.$$

Let $w_\delta = \sup_{[0,\delta]} w(y)/y$. We already know that $w_\delta \leqslant C$ for sufficiently small δ. Hence,

$$w(y) \leqslant w(y_0) + w'(y_0)(y - y_0) + C w_\delta^s \left(\left(y^{-\tau+s+2} - y_0^{-\tau+s+2} \right) - y_0^{-\tau+s+1}(y - y_0) \right),$$

where C is a positive constant, whose exact value is irrelevant. Now observe that $-\tau + s + 1 > 0$—since $\sigma < 1$—so that, letting $y_0 \to 0$ and dividing by y we obtain

$$\frac{w(y)}{y} \leqslant C w_\delta^s y^{-\tau+s+1}.$$

Taking supremums and dividing by w_δ, we arrive at $1 \leqslant C w_\delta^{s-1} \delta^{-\tau+s+1}$, which is a clear contradiction when $\delta \to 0$. Thus (2.34) holds.

Going back to the original variables, we have shown $w(r) \geqslant C(R_2 - r)$, so that

$$z(x) \geqslant C d(x),$$

when $\sigma < 1$, which concludes the proof of (2.31).

Finally we prove uniqueness. Let z, w be positive solutions to (2.30). Thanks to (2.31), it follows that z/w, w/z are bounded functions. Moreover, $B(x)z^{s+1}$ and $B(x)w^{s+1}$ are integrable. Hence, we can proceed as in [17] (see also [3]) to obtain uniqueness. $\square$

3. Existence and uniqueness

This section is devoted to the proof of Theorem 1. We begin by showing that positive weak solutions exist only when $0 < \lambda < \lambda_1 \leqslant \infty$ (see Section 1 for the definition of λ_1) and $\mu > 0$. Let us remark that, since $p, s > 1$, the strong maximum principle implies that nonnegative solutions (u, v) to (1.1)–(1.2) verify $u > 0$, $v > 0$ in $\overline{\Omega}$ unless $u \equiv 0$ or $v \equiv 0$ in Ω. We also mention in passing that when $\lambda = 0$ (respectively $\mu = 0$) there exist semitrivial solutions $(u, v) = (k, 0)$, $k \in \mathbb{R}$ (resp. $(u, v) = (0, k')$, $k' \in \mathbb{R}$).

Lemma 13. *Problem* (1.1)–(1.2) *can only have positive solutions when* $0 < \lambda < \lambda_1 \leqslant \infty$, $\mu > 0$.

Proof. Assume there exists a positive solution (u, v) to (1.1)–(1.2). Integrating the first equation in (1.1) in Ω we get

$$\int_{\Omega} a(x) u^p v^q = \lambda \int_{\partial\Omega} u,$$

so that if $\lambda \leqslant 0$ then $u \equiv 0$ or $v \equiv 0$ by the strong maximum principle. This contradicts the positiveness of both u and v. When $\mu \leqslant 0$ we proceed similarly. Hence both λ, μ must be positive.

Let us see now that $0 < \lambda < \lambda_1$ is also necessary. Denote by ϕ the positive normalized eigenfunction associated to λ_1. Multiplying the first equation in (1.1) by ϕ, and then integrating by parts in Ω_0 we obtain

$$0 = \int_{\Omega_0} \phi \Delta u = (\lambda - \lambda_1) \int_{\Gamma_1} u\phi - \int_{\Gamma_2} u \frac{\partial \phi}{\partial \nu}.$$

Since $u > 0$ in $\overline{\Omega}$, $\phi > 0$ on Γ_1 and $\partial\phi/\partial\nu < 0$ on Γ_2, we obtain $\lambda < \lambda_1$. $\square$

Now we show that when $0 < \lambda < \lambda_1$, $\mu > 0$, problem (1.1)–(1.2) has at least a positive solution (u, v). We use the notations introduced in Section 2.

Lemma 14. *Assume* $0 < \lambda < \lambda_1 \leqslant \infty$, $\mu > 0$. *Then problem* (1.1)–(1.2) *admits a positive weak solution* (u, v).

Proof. We are obtaining sub- and supersolutions by means of the solutions U_λ, V_μ of the auxiliary problems (2.1) and (2.6), respectively. By choosing a small ε and a large M, the pair $(\varepsilon U_\lambda, M V_\mu)$ defines a subsolution. Notice indeed that the boundary conditions are automatic, while

$$\begin{cases} \Delta(\varepsilon U_\lambda) = \varepsilon a(x) U_\lambda^p \geqslant a(x) \varepsilon^p U_\lambda^p M^q V_\mu^q, \\ \Delta(M V_\mu) = M b(x) V_\mu^s \leqslant b(x) \varepsilon^r U_\lambda^r M^s V_\mu^s \end{cases}$$

holds provided

$$\varepsilon^{p-1} M^q \sup_{\Omega} V_\mu^q \leqslant 1, \qquad \varepsilon^r M^{s-1} \inf_{\Omega} U_\lambda^r \geqslant 1. \tag{3.1}$$

By setting $M = \varepsilon^{-\gamma}$, (3.1) can be achieved for small ε if γ is chosen to satisfy $p - 1 - \gamma q > 0 > r - \gamma(s - 1)$, that is,

$$\frac{r}{s-1} < \gamma < \frac{p-1}{q}, \tag{3.2}$$

which is of course possible since $(p-1)(s-1)-qr>0$. A large supersolution is constructed similarly. Hence, for $0<\lambda<\lambda_1$, $\mu>0$, problem (1.1)–(1.2) has a positive solution. $\square$

We now turn to consider the question of uniqueness of positive solutions. Although an argument similar to the one employed later on in Section 4 could be used, we prefer to obtain it by means of a sweeping argument.

Lemma 15. *Assume* $0<\lambda<\lambda_1\leqslant\infty$ *and* $\mu>0$ *then problem* (1.1)–(1.2) *admits a unique positive solution* $(u_{\lambda,\mu},v_{\lambda,\mu})$. *Moreover,* $(u_{\lambda,\mu},v_{\lambda,\mu})$ *is an asymptotically stable equilibrium for the parabolic system associated to* (1.1)–(1.2) *which is globally attractive among nonnegative solutions.*

Proof. Let (u_1,v_1), (u_2,v_2) be positive solutions to (1.1)–(1.2). If $t\geqslant1$ and exponent $\gamma>0$ is selected as in (3.2), $(tu_1,t^{-\gamma}v_1)$ is a supersolution. Indeed,

$$\begin{cases}\Delta(tu_1)=ta(x)u_1^pv_1^q\leqslant a(x)t^pu_1^pt^{-\gamma q}v_1^q,\\ \Delta\big(t^{-\gamma}v_1\big)=t^{-\gamma}b(x)u_1^rv_1^s\geqslant b(x)t^ru_1^rt^{-\gamma s}v_1^s\end{cases}$$

holds provided $t^{p-1-\gamma q}\geqslant1$, $t^{r-\gamma(s-1)}\leqslant1$, that is, when $t\geqslant1$, while the boundary conditions remain unchanged. We now use a sweeping argument. If t is large enough, we have $tu_1>u_2$, $t^{-\gamma}v_1<v_2$. Set $t_0=\inf\{t>1:\ tu_1>u_2,\ t^{-\gamma}v_1<v_2\}$. We claim that $t_0=1$. To prove the claim, choose $M>0$ such that the function $f(\tau)=a(x)\tau^pv_2^q-M\tau$ is decreasing (with fixed x) in the interval $[\inf u_2,\sup(t_0u_1)]$. Then

$$\begin{aligned}\Delta(t_0u_1)-M(t_0u_1)&\leqslant a(x)(t_0u_1)^p\big(t_0^{-\gamma}v_1\big)^q-M(t_0u_1)\leqslant a(x)(t_0u_1)^pv_2^q-M(t_0u_1)\\&\leqslant a(x)u_2^pv_2^q-Mu_2=\Delta u_2-Mu_2.\end{aligned}$$

By the strong maximum principle and Hopf's principle, we deduce that either $t_0u_1>u_2$ in $\overline{\Omega}$ or $t_0u_1\equiv u_2$. Assume $t_0u_1>u_2$. An argument like the one we have just used implies $t_0^{-\gamma}v_1<v_2$ or $t_0^{-\gamma}v_1\equiv v_2$. Let us see that $t_0^{-\gamma}v_1<v_2$. As a matter of fact, if $t_0^{-\gamma}v_1\equiv v_2$, then we would obtain $t_0^{-\gamma}\Delta v_1=\Delta v_2$, that is $t_0^{-\gamma}u_1^rv_1^s=u_2^rv_2^s$, and hence

$$u_1=t_0^{-\frac{\gamma(s-1)}{r}}u_2\leqslant t_0^{-1}u_2,$$

which is not possible. Thus $t_0u_1>u_2$, $t_0^{-\gamma}v_1<v_2$ in $\overline{\Omega}$, contradicting the minimality of t_0.

The unique possible option is $t_0u_1\equiv u_2$, which leads to $t_0^{-\gamma}v_1\equiv v_2$. Substituting in the equation we arrive at $t_0=1$. Hence $u_1\geqslant u_2$, $v_1\leqslant v_2$, and the symmetric argument proves $u_1=u_2$, $v_1=v_2$. Uniqueness is proved.

The asymptotic stability of $(u_{\lambda,\mu},v_{\lambda,\mu})$ comes from the fact that it is the unique solution to (1.1)–(1.2) located between a sub- and a supersolution (cf. [27]). Regarding the global attractiveness of $(u_{\lambda,\mu},v_{\lambda,\mu})$, it can be shown that every nontrivial and nonnegative solution to the parabolic problem becomes, immediately after the initial time, positive and sufficiently smooth. Thus, it enters an interval bounded by a sub- and a supersolution and, by the preceding assertion, asymptotically converges to $(u_{\lambda,\mu},v_{\lambda,\mu})$ (cf. [5,24] and Theorem 1 in [17]). This concludes the proof. $\square$

4. Dependence on λ and μ

In this final section we prove Theorems 2–6 that describe the dependence of the solution $(u_{\lambda,\mu},v_{\lambda,\mu})$ to (1.1)–(1.2) on the parameters λ and μ.

Proof of Theorem 2(i). Let $\mu > 0$ be fixed. Then if $\lambda < \lambda'$, $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a subsolution to (1.1)–(1.2) with λ', μ. Since we have arbitrarily large supersolutions, we arrive at $u_{\lambda,\mu} < u_{\lambda',\mu}$, $v_{\lambda,\mu} > v_{\lambda',\mu}$. Hence, $u_{\lambda,\mu}$ is increasing in λ and $v_{\lambda,\mu}$ decreasing in λ for fixed μ. Similarly, for fixed λ, $u_{\lambda,\mu}$ is decreasing in μ and $v_{\lambda,\mu}$ increasing in μ. $\square$

Proof of Theorem 2(ii) and (iii). Let us obtain some estimates for the solutions which will turn to be useful for small λ or μ. To this aim, we are selecting "optimal" sub- and supersolutions, and take advantage of the uniqueness. The best subsolutions can be achieved by imposing the equality in (3.1). This gives for ε and M:

$$\varepsilon = \left(\frac{\inf_\Omega U_\lambda^r}{\sup_\Omega V_\mu^{s-1}}\right)^{\frac{q}{\delta}}, \qquad M = \left(\frac{\sup_\Omega V_\mu^q}{\inf_\Omega U_\lambda^{p-1}}\right)^{\frac{r}{\delta}}.$$

Since there exist arbitrarily large supersolutions, we arrive at

$$u_{\lambda,\mu} \geqslant \left(\frac{\inf_\Omega U_\lambda^r}{\sup_\Omega V_\mu^{s-1}}\right)^{\frac{q}{\delta}} U_\lambda, \qquad v_{\lambda,\mu} \leqslant \left(\frac{\sup_\Omega V_\mu^q}{\inf_\Omega U_\lambda^{p-1}}\right)^{\frac{r}{\delta}} V_\mu.$$

With a similar argument, we arrive at a lower bound for $v_{\lambda,\mu}$ and an upper one for $u_{\lambda,\mu}$. Thus,

$$\begin{gathered}\left(\frac{\inf_\Omega U_\lambda^r}{\sup_\Omega V_\mu^{s-1}}\right)^{\frac{q}{\delta}} U_\lambda \leqslant u_{\lambda,\mu} \leqslant \left(\frac{\sup_\Omega U_\lambda^r}{\inf_\Omega V_\mu^{s-1}}\right)^{\frac{q}{\delta}} U_\lambda, \\ \left(\frac{\inf_\Omega V_\mu^q}{\sup_\Omega U_\lambda^{p-1}}\right)^{\frac{r}{\delta}} V_\mu \leqslant v_{\lambda,\mu} \leqslant \left(\frac{\sup_\Omega V_\mu^q}{\inf_\Omega U_\lambda^{p-1}}\right)^{\frac{r}{\delta}} V_\mu. \end{gathered} \tag{4.1}$$

Several conclusions can be drawn at once from (4.1). Let $\mu > 0$ be fixed. Then

$$u_{\lambda,\mu} \to 0, \qquad v_{\lambda,\mu} \to +\infty$$

uniformly in $\overline{\Omega}$ as $\lambda \to 0+$, thanks to Theorem 7, since $U_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to 0+$. In the same way, if $\lambda > 0$ is kept fixed while $\mu \to 0$:

$$u_{\lambda,\mu} \to +\infty, \qquad v_{\lambda,\mu} \to 0$$

uniformly in $\overline{\Omega}$.

On the other hand, if $\lambda, \mu \to 0$, we obtain, thanks to estimates (2.2) in Theorem 7:

$$u_{\lambda,\mu} \sim \left(a^*\right)^{-\frac{s-1}{\delta}} \left(b^*\right)^{\frac{q}{\delta}} \left(\frac{\lambda^{s-1}}{\mu^q}\right)^{\frac{1}{\delta}}, \qquad v_{\lambda,\mu} \sim \left(b^*\right)^{-\frac{p-1}{\delta}} \left(a^*\right)^{\frac{r}{\delta}} \left(\frac{\mu^{p-1}}{\lambda^r}\right)^{\frac{1}{\delta}},$$

with $a^* = \frac{1}{|\partial\Omega|}\int_\Omega a$, $b^* = \frac{1}{|\partial\Omega|}\int_\Omega b$. This finishes the proof. $\square$

Proof of Theorem 3. It follows easily from estimates (1.6) and the proof of Theorem 2. $\square$

We are next elucidating the asymptotic behavior of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ as $\lambda \to \lambda_1$, for fixed μ. We recall that $\lambda_1 = \infty$ implies $\Omega_0 \Subset \Omega$, and thus $a > 0$ on $\partial\Omega$.

Proofs of Theorem 4(i) and first assertion in (ii). As μ is going to be kept fixed here we use the shorter (u_λ, v_λ) instead of $(u_{\lambda,\mu}, v_{\lambda,\mu})$.

In case (i), $\lambda_1 < \infty$, $0 < \mu \leqslant \mu_i^+$ for all $1 \leqslant i \leqslant M$. We know from Theorem 7 that $U_\lambda \to +\infty$ uniformly in $\overline{\Omega}_0$. Thanks to (4.1), we obtain that $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}_0$.

Let us see next that $v_\lambda \to 0$ uniformly on $\overline{\Omega}$. Notice that $u_\lambda(x)^r \geqslant (\inf_{\Omega_0} u_\lambda)^r \chi_{\overline{\Omega}_0} + c_0 \chi_{\Omega^+}$, for some $c_0 > 0$, and denote

$$A_\lambda(x) = b(x)\Big(\Big(\inf_{\Omega_0} u_\lambda\Big)^r \chi_{\overline{\Omega}_0} + c_0 \chi_{\Omega^+}\Big),$$

so that $A_\lambda(x) \to \infty$ uniformly in $\overline{\Omega}_0$ while it keeps uniformly bounded in Ω^+ as $\lambda \to \lambda_1-$. It follows that $v_\lambda \leqslant V_\lambda$, the unique solution to

$$\begin{cases} \Delta V = A_\lambda(x) V^s & \text{in } \Omega, \\ \dfrac{\partial V}{\partial \nu} = \mu V & \text{on } \partial\Omega. \end{cases}$$

According to Theorem 8 we obtain that $V_\lambda \to 0$ uniformly in $\overline{\Omega}_0$ as $\lambda \to \lambda_1-$, and hence $v_\lambda \to 0$ uniformly in $\overline{\Omega}_0$. Furthermore, it also follows from that result that $v_\lambda \to 0$ uniformly in Ω_i^+ for each component Ω_i^+ of Ω^+ such that $\lambda \leqslant \mu_i^+$. In case (i) this means that $v_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to \lambda_1-$.

Let us show now that $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$. Take $\varepsilon > 0$ as small as desired. There exists λ_ε, $0 < \lambda^* < \lambda_\varepsilon < \lambda_1$ (λ^* not depending on ε) such that for $\lambda \in (\lambda_\varepsilon, \lambda_1)$, we have $v_\lambda \leqslant \varepsilon$ in $\overline{\Omega}$. Thus

$$\Delta u_\lambda \leqslant \varepsilon^q a(x) u_\lambda^p \quad \text{in } \Omega,$$

and we deduce that

$$u_\lambda \geqslant \varepsilon^{-\frac{q}{p-1}} U_\lambda \geqslant \varepsilon^{-\frac{q}{p-1}} U_{\lambda^*},$$

which implies that $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$.

As for the first part of (ii) let Ω_i^+ be a connected piece with $\mu \leqslant \mu_i^+ < \infty$. We already know that $u_\lambda \to \infty$ on $\Gamma_{2,i}$ while $v_\lambda \to 0$ uniformly in $\overline{\Omega}_i^+$ as $\lambda \to \lambda_1-$. For $\varepsilon > 0$ given and $\lambda \geqslant \lambda_\varepsilon$, $u = u_\lambda$ defines a supersolution to

$$\begin{cases} \Delta u = a\varepsilon^q u^p, & x \in \Omega_i^+, \\ u = u_\lambda, & x \in \Gamma_{2,i}, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, & x \in \Gamma_i^+. \end{cases}$$

Taking limits as $\lambda \to \lambda_1-$ we obtain that

$$\lim_{\lambda \to \lambda_1} u_\lambda \geqslant \varepsilon^{-q/(p-1)} v_\infty,$$

$u = v_\infty$ being the minimal solution to (see [17] for a discussion of this and other related problems)

$$\begin{cases} \Delta u = au^p, & x \in \Omega_i^+, \\ u = \infty, & x \in \Gamma_{2,i}, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, & x \in \Gamma_i^+. \end{cases}$$

The desired conclusion follows from the precedent estimate by letting $\varepsilon \to 0$. The proof in the case $\Omega_i^+ \Subset \Omega$ is identical. $\square$

Proof of Theorem 4(ii) completed. It only remains to show that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges to a finite profile in Ω_i^+ as $\lambda \to \lambda_1$ in every connected piece Ω_i^+ of Ω^+ with associated μ_i^+ smaller than μ. To abbreviate we write (u_λ, v_λ) instead $(u_{\lambda,\mu}, v_{\lambda,\mu})$.

It is enough to find a convenient supersolution in Ω_i^+. This can be done with the aid of the auxiliary problems (2.27) and (2.30) which were analyzed in Section 2. Indeed, once the weights $A(x)$, $B(x)$ are properly chosen, there exists a supersolution of the form $(tw_{\lambda_1}, t^{-\gamma} z_\mu)$, where γ verifies (3.2), $t > 0$ is large enough and w_{λ_1}, z_μ stand for the solutions to (2.27) with $\lambda = \lambda_1$ and (2.30), respectively, now regarded in Ω_i^+ and boundary conditions in $\Gamma_{2,i}$ and Γ_i^+. Such solutions are provided by Theorems 11 and 12 while condition $\mu > \mu_i^+$ is required for the existence of z_μ.

To find $A(x)$ and $B(x)$ first notice that the pair $(tw_{\lambda_1}, t^{-\gamma} z_\mu)$ is a supersolution to (1.1) in Ω_i^+ provided

$$A(x) \leqslant a(x) t^{p-1-\gamma q} z_\mu^q, \qquad B(x) \geqslant b(x) t^{r-\gamma(s-1)} w_{\lambda_1}^r$$

in Ω_i^+. Thanks to the choice of γ and property (1.7) on $a(x)$, it is enough to have for some positive constant C

$$A(x) \leqslant C d(x)^\sigma z_\mu^q, \qquad C B(x) \geqslant w_{\lambda_1}^r, \tag{4.2}$$

where $d(x) = \operatorname{dist}(x, \Gamma_2)$. Let us now choose $B(x) = d(x)^{-\tau}$, for some $\tau > (s+1) > 2$ to be found. Then, according to Theorem 12, the solution z_μ verifies

$$C_1 d(x)^\theta \leqslant z_\mu(x) \leqslant C_2 d(x)^\theta,$$

where $\theta = (\tau - 2)/(s-1) > 1$. We now set $A(x) = d(x)^\sigma z_\mu^q$, so that the first inequality in (4.2) holds for $C > 1$. To verify the second inequality it suffices with seeing that,

$$w_{\lambda_1}(x) \leqslant C d(x)^{-\frac{\tau}{r}}.$$

Since $w_{\lambda_1}(x) \leqslant C d(x)^{-\beta}$, where $\beta = (\sigma + 2 + q\theta)/(p-1)$, this reduces to have $\beta \leqslant \tau/r$, that is,

$$\frac{\delta}{s-1}\tau \geqslant (\sigma+2)r - 2\frac{qr}{s-1},$$

which can always be achieved by taking τ large enough. Thus $(tw_{\lambda_1}, t^{-\gamma} z_\mu)$ is a supersolution to (1.1) in Ω_i^+, provided that $t > 0$ is large enough.

Now notice that for our solution $u_\lambda < \infty$ while $v_\lambda > 0$ on $\Gamma_{2,i}$, so the boundary conditions are coherent with the chosen supersolution, while the corresponding ones in Γ_i^+ are exactly verified. Thus $u_\lambda < tw_{\lambda_1}$, $v_\lambda > t^{-\gamma} z_\mu$. This gives local interior bounds for u_λ while v_λ is uniformly bounded and remains bounded away from zero in Ω_i^+. It is then standard to conclude that $(u_\lambda, v_\lambda) \to (u_\infty, v_\infty)$ in $C^{1,\eta_0}(\Omega_i^+)$, where (u_∞, v_∞) stands here for a positive weak solution to (1.1) in Ω^+. Finally, from the analysis in the proof of part (i) and Theorem 8 it follows that $(u_\lambda, v_\lambda) \to (\infty, 0)$ uniformly in Γ_2, in particular in $\Gamma_{2,i}$. This means that (u_∞, v_∞) defines a positive weak solution to the boundary value problem (1.8). $\square$

Proof of Theorem 4(iii). In the present situation, $\lambda_1 = \infty$ and $0 < r < (p-1)/2$. To show that the solution $(u_\lambda, v_\lambda) = (u_{\lambda,\mu}, v_{\lambda,\mu})$ converges to a finite profile, it suffices again with finding a large supersolution. We are looking for it in the form $(tU_\infty, t^{-\gamma} \tilde{V}_\mu)$, $t > 1$, where U_∞ is the unique solution to

$$\begin{cases} \Delta u = a(x) u^p & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

and $\tilde{V}_\mu$ is to be chosen. Notice that $a > 0$ on $\partial\Omega$, and hence U_∞ is unique and verifies in particular $U_\infty(x) \leqslant Cd(x)^{-\frac{2}{p-1}}$, $d(x) = \mathrm{dist}(x, \partial\Omega)$, for some positive constant C (see for instance [4]). This implies that $B(x) = b(x)U_\infty^r$ verifies (2.22), with $\tau = 2r/(p-1) < 1$. Thus, thanks to Theorem 10, there exists a unique positive solution $\tilde{V}_\mu$ to (2.21). To have that $(tU_\infty, t^{-\gamma}\tilde{V}_\mu)$ is a supersolution, we need

$$1 \leqslant t^{p-1-\gamma q}\tilde{V}_\mu^q, \qquad 1 \geqslant t^{r-\gamma(s-1)},$$

which is true for large t if γ is chosen to satisfy (3.2). Hence, thanks to uniqueness, we have for $\lambda \geqslant \lambda^*$:

$$u_{\lambda^*} \leqslant u_\lambda \leqslant tU_\infty, \qquad t^{-\gamma}\tilde{V}_\mu \leqslant v_\lambda \leqslant v_{\lambda^*}.$$

It is then standard to conclude that $u_\lambda \to u_\infty$, $v_\lambda \to v_\infty$ in $C^{1,\eta_0}(\Omega)$, where (u_∞, v_∞) verifies (1.1) in the strong sense.

On the other hand, we deduce from (4.1) and Theorem 7 that $u_\infty = \infty$ on $\partial\Omega$. We now need to analyze the boundary condition for v_∞. Indeed, we have from (1.1):

$$\int_\Omega |\nabla v_\lambda|^2 = \mu \int_{\partial\Omega} v_\lambda^2 - \int_\Omega b(x)u_\lambda^r v_\lambda^{s+1} \leqslant \mu \int_{\partial\Omega} v_\lambda^2,$$

so that $v_\lambda \to v_\infty$ weakly in $H^1(\Omega)$ and strongly in $L^2(\partial\Omega)$. Thanks to the weak formulation of (1.1):

$$\int_\Omega \nabla v_\lambda \nabla\psi - \mu \int_{\partial\Omega} v_\lambda\psi = -\int_\Omega b(x)u_\lambda^r v_\lambda^s\psi, \tag{4.4}$$

for every $\psi \in H^1(\Omega)$. Since $u_\lambda^r \leqslant t^r U_\infty^r \leqslant Cd(x)^{-\tau} \in L^1(\Omega)$, we can pass to the limit in (4.4) to deduce that v_∞ satisfies the boundary condition

$$\frac{\partial v_\infty}{\partial \nu} = \mu v_\infty$$

in the weak sense.

To summarize, we have proved that $(u_\lambda, v_\lambda) = (u_{\lambda,\mu}, v_{\lambda,\mu})$ converges to (u_∞, v_∞), which is a weak solution to

$$\begin{cases} \Delta u = a(x)u^p v^q \quad \text{in } \Omega, & u = \infty \quad \text{on } \partial\Omega, \\ \Delta v = b(x)u^r v^s \quad \text{in } \Omega, & \dfrac{\partial v}{\partial \nu} = \mu v \quad \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

We now claim to finish the proof that (4.5) has a unique positive solution. To this aim, we adapt the argument of Lemma 10 in [16]. Let (u_1, v_1), (u_2, v_2) be positive solutions to (4.5). Since $v_i > 0$ in $\overline{\Omega}$, it follows that at every $x_0 \in \partial\Omega$

$$u_i(x) \sim \left(a(x_0)v_i(x_0)^q\right)^{-\frac{1}{p-1}} d(x)^{-\alpha} \tag{4.6}$$

as $x \to x_0 \in \partial\Omega$, where $\alpha = 2/(p-1)$ (cf. for instance [14]). Now set

$$w = \frac{u_1}{u_2}, \qquad z = \frac{v_1}{v_2}.$$

It is not hard to see that z verifies:

$$\begin{cases} \Delta z + 2\dfrac{\nabla v_2}{v_2}\nabla z + b\big(1 - w^r z^{s-1}\big)u_2^r v_2^{s-1} z = 0 & \text{in } \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

and, thanks to (4.6) we have $w = z^{-\frac{q}{p-1}}$ on $\partial\Omega$.

Assume $k = \sup z > 1$, and let $x_0 \in \overline{\Omega}$ be a point where the maximum of z is achieved. We claim that we can always assume that $x_0 \in \Omega$. For if we assume $x_0 \in \partial\Omega$, since $1 - w^r z^{s-1} = 1 - k^{s-1-\frac{qr}{p-1}} < 0$ in x_0, we deduce that the coefficient of z in (4.7) is negative in a neighborhood of x_0, and from Hopf's principle, z is constant in a neighborhood of x_0.

Thus we will assume $x_0 \in \Omega$. Then $\nabla z(x_0) = 0$, $\Delta z(x_0) \leqslant 0$. From Eq. (4.7) we obtain $w(x_0) \leqslant k^{-\frac{s-1}{r}}$.

We now claim that $w \geqslant k^{-\frac{q}{p-1}}$ in Ω. To show this, we consider the set $\Omega' = \{w < k^{-\frac{q}{p-1}}\}$, and assume $\Omega' \neq \emptyset$. Since $w = z^{-\frac{q}{p-1}} \geqslant k^{-\frac{q}{p-1}}$ on $\partial\Omega$, we conclude $w = k^{-\frac{q}{p-1}}$ on $\partial\Omega'$. In addition, w satisfies

$$\Delta w + 2\frac{\nabla u_2}{u_2}\nabla w = a\big(z^q w^{p-1} - 1\big)u_2^{p-1} v_2^q w \leqslant 0$$

in Ω'. From the maximum principle, $w > k^{-\frac{q}{p-1}}$ in Ω', which is a clear contradiction. Hence $\Omega' = \emptyset$, that is, $w \geqslant k^{-\frac{q}{p-1}}$ in Ω. Particularizing at x_0 we arrive at $k^{-\frac{q}{p-1}} \leqslant k^{-\frac{s-1}{r}}$, which is not possible since $k > 1$ and $\delta = (p-1)(s-1) - qr > 0$.

In conclusion, $k \leqslant 1$, that is $v_1 \leqslant v_2$. The symmetric argument shows $v_1 = v_2$ and hence $u_1 = u_2$. This proves uniqueness. □

Remark 5. Recovering our original notation and putting $u_\infty = u_{\infty,\mu}$, $v_\infty = v_{\infty,\mu}$ it follows that

$$v_{\infty,\mu} \geqslant C\tilde{V}_\mu \tag{4.8}$$

in Ω, for a constant that does not depend on μ for large μ, where $\tilde{V}_\mu$ is the unique solution to (2.21) with $B(x) = d(x)^{-\alpha r}$ and $\alpha = 2/(p-1)$. Indeed, notice that $v_{\infty,\mu}$ is increasing in μ, and thus $v_{\infty,\mu} \geqslant v_{\infty,\mu_0}$ when $\mu \geqslant \mu_0$. We deduce then that $\Delta u_{\infty,\mu} \geqslant C u_{\infty,\mu}^p$ for some positive constant C, and hence $u_{\infty,\mu} \leqslant CU_\infty$, where U_∞ is the unique solution to (2.3) with $a(x) = 1$. Thus $\Delta v_{\infty,\mu} \leqslant CU_\infty^r v_{\infty,\mu}^s \leqslant Cd(x)^{-\alpha r} v_{\infty,\mu}^s$, and this implies (4.8).

Proof of Theorem 4(iv). We have again $\lambda_1 = \infty$ but in this occasion $r \geqslant (p-1)/2$. We will show that $v_\lambda = v_{\lambda,\mu} \to 0$ uniformly in $\overline{\Omega}$, and then it will follow as in the proof of Theorem 4(i) that $u_\lambda = u_{\lambda,\mu} \to +\infty$ uniformly in $\overline{\Omega}$. From the first equation in (1.1) we have $\Delta u_\lambda \leqslant (\sup_\Omega v_\lambda)^q a(x) u_\lambda^p$ in Ω, which implies

$$u_\lambda \geqslant \Big(\sup_\Omega v_\lambda\Big)^{-\frac{q}{p-1}} U_\lambda \geqslant C\Big(\sup_\Omega v_\lambda\Big)^{-\frac{q}{p-1}} \left(d(x) + \frac{\alpha}{\lambda}\right)^{-\alpha}$$

in Ω, thanks to Theorem 7, where $\alpha = 2/(p-1)$. It follows from the second equation in (1.1) that

$$\Delta v_\lambda \geqslant C\Big(\sup_\Omega v_\lambda\Big)^{-\frac{qr}{p-1}} \left(d(x) + \frac{\alpha}{\lambda}\right)^{-\alpha r} b(x) v_\lambda^s.$$

Thus $v_\lambda \leqslant C(\sup_\Omega v_\lambda)^{\frac{qr}{(p-1)(s-1)}} \tilde{v}_\lambda$, where $v = \tilde{v}_\lambda$ is the unique positive solution to

$$\begin{cases} \Delta v = b(x)\left(d(x) + \dfrac{\alpha}{\lambda}\right)^{-\alpha r} v^s & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

given by Theorem 7. We conclude

$$\left(\sup_\Omega v_\lambda\right)^{1-\frac{qr}{(p-1)(s-1)}} \leqslant C \sup_\Omega \tilde{v}_\lambda, \tag{4.10}$$

and thanks to Theorem 9 we obtain $\tilde{v}_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to +\infty$. By (4.10), we have $v_\lambda = v_{\lambda,\mu} \to 0$ uniformly in $\overline{\Omega}$, as we wanted to show. $\square$

We finally prove Theorem 5, that is, the asymptotic behavior of $(u_{\lambda,\mu}, v_{\lambda,\mu})$ when both λ and μ go to infinity.

Proof of Theorem 5(i). Fix $\mu_0 > 0$. For $\mu \geqslant \mu_0$, we have $v_{\lambda,\mu} \geqslant v_{\lambda,\mu_0}$, since $v_{\lambda,\mu}$ is increasing in μ. Thanks to Theorem 4(iii), v_{λ,μ_0} converges to a finite profile v_{∞,μ_0} as $\lambda \to \infty$ (recall that $r < (p-1)/2$). This shows that $v_{\lambda,\mu}$ is bounded from below, and hence $u_{\lambda,\mu}$ is bounded from above in compacts of Ω. A similar reasoning using $q < (s-1)/2$ shows that $u_{\lambda,\mu}$ is bounded from below and $v_{\lambda,\mu}$ bounded from above in compacts of Ω. Thus it is standard to conclude that for every pair of sequences $\lambda_n, \mu_n \to \infty$, the corresponding solutions, denoted (u_n, v_n) for the sake of brevity, converge uniformly on compacts of Ω to a pair (u_∞, v_∞), which will be a weak solution of (1.1). We claim that $u_\infty = v_\infty = \infty$ on $\partial\Omega$.

Let $(u_{\infty,\mu_0}, v_{\infty,\mu_0})$ be the unique solution to (4.5) with $\mu = \mu_0$. Since $u_n \leqslant u_{\lambda_n,\mu_0}$, $v_n \geqslant v_{\lambda_n,\mu_0}$, we obtain, letting $n \to \infty$,

$$u_\infty \leqslant u_{\infty,\mu_0}, \qquad v_\infty \geqslant v_{\infty,\mu_0}.$$

We now use the inequality (4.8) in Remark 5 to deduce that $v_\infty \geqslant C\tilde{V}_{\mu_0}$, where C does not depend on μ_0. Letting $\mu_0 \to \infty$ and using Theorem 10, we conclude that $v_\infty \geqslant \tilde{V}_\infty$, where $\tilde{V}_\infty$ is the unique solution to (2.23) with $B(x) = d(x)^{-\alpha r}$, $d(x) = \operatorname{dist}(x, \partial\Omega)$. This shows that $v_\infty = \infty$ on $\partial\Omega$, and $u_\infty = \infty$ on $\partial\Omega$ is proved similarly. Thus (u_∞, v_∞) is the unique solution to the system

$$\begin{cases} \Delta u = u^p v^q & \text{in } \Omega, \qquad u = \infty \quad \text{on } \partial\Omega, \\ \Delta v = u^r v^s & \text{in } \Omega, \qquad v = \infty \quad \text{on } \partial\Omega \end{cases} \tag{1.9}$$

(cf. [16] for the proof of uniqueness). Since the limit is the same for every pair of sequences λ_n, $\mu_n \to \infty$, we have shown that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges to the unique solution to (1.9). $\square$

Proof of Theorem 5(ii). We are next showing that in the range $r < p-1$, $q < s-1$ (of course assuming $r \geqslant (p-1)/2$ or $q \geqslant (s-1)/2$, to be out of part (i)) the solutions also converge to a finite profile *provided* that λ/μ, μ/λ are both bounded.

The key is to introduce the numbers

$$\alpha_1 = \frac{2(s-1-q)}{\delta}, \qquad \beta_1 = \frac{2(p-1-r)}{\delta}$$

and set $p_1 = 1 + 2/\alpha_1$, $s_1 = 1 + 2/\beta_1$. Observe that $\alpha_1, \beta_1 > 0$ and hence $p_1, s_1 > 1$. Let $z = \tilde{z}_\lambda$, $w = \tilde{w}_\mu$ be the unique solutions to the problems

$$\begin{cases} \Delta z = z^{p_1} & \text{in } \Omega, \\ \dfrac{\partial z}{\partial \nu} = \lambda z & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \Delta w = w^{s_1} & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = \mu w & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

respectively. We look for a supersolution of the form $(tw, t^{-\gamma} z)$, with a large enough t and γ verifying (3.2). Thus we need to have

$$t^{p-1-\gamma q} z^{p-p_1} w^q \geqslant 1, \qquad t^{r-\gamma(s-1)} z^r w^{s-s_1} \leqslant 1,$$

which in turn will hold for large t if the functions $(\tilde{z}_\lambda)^{p-p_1} (\tilde{w}_\mu)^q$, $(\tilde{z}_\lambda)^r (\tilde{w}_\mu)^{s-s_1}$ are bounded from below and from above, respectively, independently of λ and μ. If we now use the estimates (2.4) and (2.5) in Theorem 7 for the solutions of problems (4.11) and (4.12), this is equivalent to show

$$\begin{aligned} \left(d(x) + \frac{\alpha_1}{\lambda}\right)^{(p-p_1)\alpha_1} \left(d(x) + \frac{\beta_1}{\mu}\right)^{q\beta_1} &\geqslant C, \\ \left(d(x) + \frac{\alpha_1}{\lambda}\right)^{r\alpha_1} \left(d(x) + \frac{\beta_1}{\mu}\right)^{(s-s_1)\beta_1} &\leqslant C. \end{aligned} \tag{4.13}$$

We remark that the exponents in (4.13) verify $(p - p_1)\alpha_1 + q\beta_1 = 0$, $r\alpha_1 + (s - s_1)\beta_1 = 0$, and since λ and μ are of the same order then (4.13) holds. This provides a supersolution, and in a similar way a subsolution can be constructed. Hence we can argue as before and obtain that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ converges to a pair (u_∞, v_∞) which is a solution to (1.1) (of course this convergence is in principle through a subsequence). Since the sub- and supersolution we have constructed imply in this case

$$u_{\lambda,\mu} \geqslant C\left(d(x) + \frac{\alpha_1}{\lambda}\right)^{-\alpha_1}, \qquad v_{\lambda,\mu} \geqslant C\left(d(x) + \frac{\beta_1}{\mu}\right)^{-\beta_1},$$

it follows immediately that $u_\infty = v_\infty = \infty$ on $\partial\Omega$, and thus (u_∞, v_∞) is the unique solution to (1.9). This concludes the proof. $\square$

Proof of Theorem 5(iii). As we have shown in the proof of Theorem 4(iv) (cf. (4.10)):

$$\left(\sup_\Omega v_{\lambda,\mu}\right)^{1-\frac{qr}{(p-1)(s-1)}} \leqslant C \sup_\Omega \tilde{v}_{\lambda,\mu}, \tag{4.14}$$

where $\tilde{v}_{\lambda,\mu}$ (a subindex μ has now been added) stands for the unique solution to

$$\begin{cases} \Delta v = \left(d(x) + \dfrac{\alpha}{\lambda}\right)^{-\alpha r} v^s & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

cf. Theorem 7. We now need to obtain good estimates of the solution $\tilde{v}_{\lambda,\mu}$ to (4.15) when both λ and μ go to infinity. Fix $\delta > 0$. Then, since $(d + 1/\lambda)^{-\alpha r} \geqslant (\delta + 1/\lambda)^{-\alpha r}$ in $0 < d < \delta$, we arrive at

$$\tilde{v}_{\lambda,\mu} \leqslant \left(\delta + \frac{\alpha}{\lambda}\right)^{\frac{\alpha r}{s-1}} v$$

in $0 < d < \delta$, where v is the unique solution to

$$\begin{cases} \Delta v = v^s & \text{in } 0 < d < \delta, \\ \dfrac{\partial v}{\partial \nu} = \mu v & \text{on } d = 0, \\ v = \infty & \text{on } d = \delta. \end{cases} \tag{4.16}$$

Moreover, if $x \in \Omega$ verifies $d(x) = \delta/2$, we have the universal estimate

$$v(x) \leqslant C\left(\frac{\delta}{2}\right)^{-\frac{2}{s-1}},$$

where C does not depend on δ. Next, we construct a supersolution to (4.16) in the set $0 < d < \delta/2$ of the form

$$z = A\left(d(x) + \frac{\beta}{\mu}\right)^{-\beta},$$

where $\beta = 2/(s-1)$. It is easily seen that z will be indeed a supersolution to (4.16) provided $A \geqslant A_0 = A_0(\delta_0, \mu_0)$, when $\delta \leqslant \delta_0$, $\mu \geqslant \mu_0$. It suffices to take $A \geqslant 2\beta(\beta + 1)$ for small δ and large μ. We now choose A so that

$$A\left(\frac{\delta}{2} + \frac{\beta}{\mu}\right)^{-\beta} = C\left(\frac{\delta}{2}\right)^{-\beta},$$

that is, $A = C(1 + 2\beta/\delta\mu)^{\beta}$, and it will follow that $v \leqslant z$ on $d = \delta/2$. Hence $v \leqslant z$ in $0 \leqslant d \leqslant \delta/2$. In particular, for $x \in \partial\Omega$, we have

$$\tilde{v}_{\lambda,\mu}(x) \leqslant C\left(\frac{1}{2} + \frac{\beta}{\delta\mu}\right)^{\frac{2}{s-1}} \left(\delta + \frac{\alpha}{\lambda}\right)^{\frac{\alpha r}{s-1}} \mu^{\frac{2}{s-1}},$$

and since $\tilde{v}_{\lambda,\mu}$ is subharmonic:

$$\sup_{\Omega} \tilde{v}_{\lambda,\mu} \leqslant C\left(1 + \frac{1}{\delta\mu}\right)^{\frac{2}{s-1}} \left(\delta\mu + \alpha\frac{\mu}{\lambda}\right)^{\frac{\alpha r}{s-1}} \mu^{\frac{2-\alpha r}{s-1}}.$$

Now choose $\delta = 1/\mu$:

$$\sup_{\Omega} \tilde{v}_{\lambda,\mu} \leqslant C\left(1 + \alpha\frac{\mu}{\lambda}\right)^{\frac{\alpha r}{s-1}} \mu^{\frac{2-\alpha r}{s-1}}.$$

If μ/λ is bounded, and since $\alpha r > 2$ we arrive at $\sup \tilde{v}_{\lambda,\mu} \to 0$, hence $v_{\lambda,\mu} \to 0$ and $u_{\lambda,\mu} \to +\infty$ uniformly in $\overline{\Omega}$. $\square$

Proof of Theorem 6. As in the proof of Theorem 5(iii), and thanks to (4.14), it suffices to show that the unique positive solution $\tilde{v}_{\lambda,\mu}$ to (4.15) (with Ω replaced by B) converges to zero uniformly in $\overline{B}$ when λ, $\mu \to \infty$. If we multiply the equation in (4.15) by $\tilde{v}_{\lambda,\mu}$ and integrate in B—taken here as the unit ball for simplicity—dropping the term in the gradient, we arrive at

$$\frac{1}{\mu}\int_{B_\delta}\left(d+\frac{1}{\lambda}\right)^{-\alpha r}\tilde{v}_{\lambda,\mu}^{s+1} \leqslant \int_{\partial B}\tilde{v}_{\lambda,\mu}^{2}, \tag{4.17}$$

where $B_\delta = \{x \in B\colon\ 1-\delta < |x| < 1\}$. Taking into account that the solution $\tilde{v}_{\lambda,\mu}$ is radial in this case (by uniqueness), and $d(t) = 1-t$, (4.17) gets transformed into

$$\frac{1}{\mu}\int_{1-\delta}^{1}\left(1-t+\frac{1}{\lambda}\right)^{-\alpha r}\tilde{v}_{\lambda,\mu}(t)^{s+1}dt \leqslant C\tilde{v}_{\lambda,\mu}(1)^2, \tag{4.18}$$

where C is a positive constant not depending on λ or μ. Now, thanks to the radial version of (4.15), we obtain that the function $r^{N-1}\tilde{v}'_{\lambda,\mu}$ is increasing (where $r = |x|$ and $' = d/dr$). Thus, using the mean value theorem and the boundary condition, we have, for every $r \in (1-\delta, 1)$:

$$\begin{aligned}\tilde{v}_{\lambda,\mu}(r) &= \tilde{v}_{\lambda,\mu}(1) - \tilde{v}'_{\lambda,\mu}(\xi)(1-r)\\ &\geqslant \tilde{v}_{\lambda,\mu}(1) - \frac{(1-r)}{\xi^{N-1}}\tilde{v}'_{\lambda,\mu}(1) \geqslant \tilde{v}_{\lambda,\mu}(1)\left(1-\frac{\mu\delta}{(1-\delta)^{N-1}}\right).\end{aligned}$$

Hence from (4.18):

$$\tilde{v}_{\lambda,\mu}(1)^{-(s-1)} \geqslant \frac{C}{\mu}\left(1-\frac{\mu\delta}{(1-\delta)^{N-1}}\right)^{s+1}\int_{1-\delta}^{1}\left(1-t+\frac{1}{\lambda}\right)^{-\alpha r}dt. \tag{4.19}$$

We now choose $\delta = 1/(2\mu)$, and obtain from (4.19):

$$\tilde{v}_{\lambda,\mu}(1)^{-(s-1)} \geqslant \frac{C}{\mu}\left(\left(\frac{1}{\lambda}\right)^{1-\alpha r} - \left(\frac{1}{\lambda}+\frac{1}{2\mu}\right)^{1-\alpha r}\right),$$

for some positive constant C which is independent of λ and μ when they are large enough. If we now set $\mu = \lambda^\theta$ with $0 < \theta < 1$, we have that

$$\begin{aligned}\liminf_{\lambda\to+\infty}\tilde{v}_{\lambda,\mu}(1)^{-(s-1)} &\geqslant C\liminf_{\lambda\to+\infty}\lambda^{\alpha r-\theta-1}\left(1-\left(1+\frac{\lambda^{1-\theta}}{2}\right)^{1-\alpha r}\right)\\ &\geqslant C\liminf_{\lambda\to+\infty}\lambda^{\alpha r-\theta-1} = +\infty,\end{aligned}$$

since $\theta < \alpha r - 1$ thanks to (1.10). Thus $\tilde{v}_{\lambda,\mu}(1) \to 0$, and since $\tilde{v}_{\lambda,\mu}$ is subharmonic, $\tilde{v}_{\lambda,\mu} \to 0$ uniformly in $\overline{B}$. This implies that $v_{\lambda,\mu} \to 0$, $u_{\lambda,\mu} \to +\infty$ uniformly in $\overline{B}$ when $\lambda, \mu \to \infty$.

The proof when Ω is an annulus is identical while if $\Omega \subset \mathbb{R}^2$ is any smooth enough simply connected domain then $\overline{\Omega}$ can be mapped one to one onto the closed unit ball $\overline{B}$ by means of a holomorphic mapping $\zeta = g(z)$, $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$. In the ζ variables (1.1) becomes $\Delta_\zeta u = |g'|^{-2}u^p v^q$, $\Delta_\zeta v = |g'|^{-2}u^r v^s$ and the boundary conditions are transformed in $\partial u/\partial\nu = \lambda|g'|^{-1}u$, $\partial v/\partial\nu = \mu|g'|^{-1}v$. Thus, the boundedness of $|g'|$, comparison and the previous analysis lead to the conclusion. □

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