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Blow-up with logarithmic nonlinearities ☆

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Abstract

We study the asymptotic behaviour of nonnegative solutions of the nonlinear diffusion equation in the half-line with a nonlinear boundary condition,

$$\begin{cases} u_t = u_{xx} - \lambda(u+1)\log^p(u+1), & (x,t) \in \mathbb{R}_+ \times (0,T), \\ -u_x(0,t) = (u+1)\log^q(u+1)(0,t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+, \end{cases}$$

with $p, q, \lambda > 0$. We describe in terms of p, q and λ when the solution is global in time and when it blows up in finite time. For blow-up solutions we find the blow-up rate and the blow-up set and we describe the asymptotic behaviour close to the blow-up time, showing that a phenomenon of asymptotic simplification takes place. We finally study the appearance of extinction in finite time.

Keywords: Blow-up; Asymptotic behaviour; Nonlinear boundary conditions

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1. Introduction and main results

In this paper we study the asymptotic behaviour of the solutions to the semilinear problem

$$\begin{cases} u_t = u_{xx} - \lambda(u+1)\log^p(u+1), & (x,t) \in \mathbb{R}_+ \times (0,T), \\ -u_x(0,t) = (u+1)\log^q(u+1)(0,t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+, \end{cases} \tag{1}$$

with positive parameters p, q, λ. The initial condition, u_0, is a nonnegative continuous function, nonincreasing and compatible with the boundary condition.

The purpose of this work is twofold: first we characterize, for different values of the parameters in the problem, if there exist solutions that blow up in a finite time, and next, we describe the behaviour of these blow-up solutions when they exist.

The blow-up phenomenon has attracted an increasing interest in the last years, both from the point of view of the mathematics developed to understand parabolic equations and also for the possible applications, see for instance the book [17]. Recent works have shown a great number of situations in which blow-up of solutions in finite time occurs due to the presence of nonlinear source terms, either in the equation or in the boundary conditions, see [3,4,7,9,12]. In this paper, by finite time blow-up we will understand that a solution exists for $0 < t < T$ and becomes unbounded as t approaches T.

In problem (1) we have an absorption term in the equation together with a nonlinear boundary condition acting as a reaction. Therefore the resulting evolution should depend critically on the balance between both terms. At this respect we find that there exist three critical lines in the (p,q)-plane, namely

$$2q = 1, \quad 2q = p \quad \text{and} \quad 2q = p+1,$$

which separates different behaviours of the solutions. In order to explain these critical lines we perform as in [9] the change of variable $v = \log(1+u)$, obtaining the problem

$$\begin{cases} v_t = v_{xx} + |v_x|^2 - \lambda v^p, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ -v_x(0,t) = v^q(0,t), & t \in (0,T), \\ v(x,0) = v_0(x) \equiv \log\bigl(1+u_0(x)\bigr), & x \in \mathbb{R}_+. \end{cases} \tag{2}$$

Now let us look at the following formal argument: if the boundary condition in (2) holds also for some interval $0 < x < \varepsilon$, $-v_x(x,t) = v^q(x,t)$ then "differentiating" at $x = 0$ this condition we get $v_{xx}(0,t) = qv^{2q-1}(0,t)$ and substituting into the equation, we obtain

$$v_t = qv^{2q-1} + v^{2q} - \lambda v^p. \tag{3}$$

We immediately see in this expression the critical values $2q - 1 = p$ and $2q = p$. To get blow-up also the condition $2q > 1$ is needed. In the following sections we make rigorous this observation. On the other hand, the parameter λ cannot be scaled out, and we show that it is important for the evolution whenever $p \leqslant 2q \leqslant p+1$. In the borderlines we recognize the critical values of the coefficient from (3): $\lambda = q$ if $2q - 1 = p$ and $\lambda = 1$ if $2q = p$. As for blow-up solutions, it is only in the case $2q = p$ where the size of λ becomes relevant.

The presence of the absorption and the reaction, together with the logarithmic form of those terms, produces the appearance of the following interesting phenomena in (1): for some exponents p, q there is regional or global blow-up (note that the diffusion is linear); there is finite speed of propagation; concerning blow-up there is an asymptotic simplification and moreover, the blow-up rate is discontinuous with respect to the parameters of the problem. All of these features have already appeared in problems with some of the terms like the ones in problem (1), but not all at the same time. In fact, thanks to the competition between the reaction and the absorption terms, we find that for fixed $p < 1$, and depending on the initial data, there exist solutions with finite time extinction and also solutions with finite time blow-up.

At this point, let us comment briefly on related literature. Regional or global blow-up with linear diffusion have been observed for the problem

$$\begin{cases} u_t = \Delta u + (1+u)\log^q(1+u) & \text{in } \mathbb{R}^N, \\ u(x,0) = u_0(x) \end{cases} \tag{4}$$

when $1 < q \leqslant 2$, cf. [9,12]. Also an asymptotic simplification holds for blow-up solutions to problem (4), in the way described for our problem in Theorem 5: the term Δu simplifies to $|\nabla u|^2(1+u)^{-1}$, or which is the same, the term Δv, $v = \log(1+u)$, vanishes asymptotically, see [9]. On the other hand, finite speed of propagation and finite-time extinction holds for some solutions to the problem

$$\begin{cases} u_t = \Delta u - u^p & \text{in } \mathbb{R}^N, \\ u(x,0) = u_0(x) \end{cases}$$

when $p < 1$, cf. [14,16]. An example of a discontinuity in the blow-up rate with respect to the parameters of the problem is presented in the work [6].

As another precedent to our work, the balance between absorption and boundary reaction in the case of powers in a bounded domain, in one or several variables, has been studied in the papers [2,13]. More precisely, for the problem

$$\begin{cases} u_t = \Delta u - \lambda u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = u^q & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) \end{cases} \tag{5}$$

the results obtained in [2,13] are the following: if $2q < p+1$ then all solutions are global; if $2q > p+1$ then there exist blow-up solutions; if $2q = p+1$ then all solutions are global if $\lambda > q$ while there exist blow-up solutions whenever $\lambda < q$. The case $\lambda = q$ is settled only in dimension one, and it belongs to the blow-up case. We remark the importance in this case of dealing with a bounded domain.

On the other hand, some particular cases of logarithmic nonlinearities are included in the study performed in [15], again in a bounded domain:

$$\begin{cases} u_t = \Delta u - f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega, \\ u(x,0) = u_0(x). \end{cases} \tag{6}$$

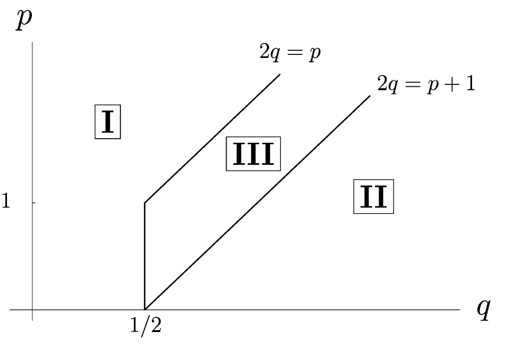

Fig. 1. I: All solutions are global. II: There exist blow-up and also global solutions. III: There exists blow-up; existence of global solutions depends on λ.

If f and g have the form of problem (1) with $2q = p \leqslant 1$, the authors prove global existence provided λ is large and existence of blow-up if $\lambda < 1$.

Now we present our main results. We begin in Section 2 with a description of the stationary solutions to problem (2) and we prove

Theorem 1. *There exist stationary solutions to problem* (2) *if and only if either* $2q > p + 1$*, or* $2q < p$*, or* $p \leqslant 2q \leqslant p + 1$ *and* λ *is large.*

Our next step is to characterize when there exist solutions with finite time blow-up (see Fig. 1), depending on whether $2q \neq p$ or $2q = p$:

Theorem 2. *Let* v *be a solution to problem* (2).

(i) *If* $2q \leqslant 1$ *or* $2q < p$*, then* v *is global.*
(ii) *If* $2q > \max(1, p)$*, then* v *blows up provided the initial value is large.*

Theorem 3. *Assume* $2q = p > 1$ *and let* v *be a solution to problem* (2).

(i) *If* $\lambda > 1$*, then* v *is global.*
(ii) *If* $\lambda < 1$*, then there exist blow-up solutions.*
(iii) *If* $\lambda = 1$*, then* v *can blow up if and only if* $q > 1$.

These theorems are proved in Section 3. We devote the next sections to study asymptotics for solutions that blow up. This includes the blow-up rate, Theorem 4, the blow-up profile, Theorem 5, and the blow-up set, Theorem 6. These three results are proved, respectively, in Sections 4, 5 and 6. Assume then, from now on, that v is a solution to problem (2) that blows up at a finite time T. As to the blow-up rate we have

Theorem 4. *In the above hypotheses it holds*

(i) *if* $2q > p$ *or* $2q = p$ *with* $\lambda < 1$*, then* $v(0, t) \sim (T - t)^{-1/(2q-1)}$;
(ii) *if* $2q = p$ *and* $\lambda = 1$*, then* $v(0, t) \sim (T - t)^{-1/(2q-2)}$.

Here by $f \sim g$ we mean that there are two constants c_1, c_2 such that $0 < c_1 \leqslant f/g \leqslant c_2 < \infty$. Note that the blow-up rate is discontinuous in terms of the exponents p, q or the coefficient λ. The blow-up rate with the same logarithmic boundary flux in a bounded domain, but without absorption, has been obtained in [1], and the exponent turns to be in that case $-1/(2q-1)$.

Now we rescale the solution according to the blow-up rate, with α the exponent given in (i) or (ii) of the precedent theorem, and take $\beta = (q-1)\alpha$:

$$f(\xi,\tau) = (T-t)^{\alpha} v(x,t), \quad \xi = x(T-t)^{-\beta},\ \tau = -\log(T-t). \tag{7}$$

In case there is nonuniqueness of solution of the limit problem, see Section 5, we state the result in terms of the ω-limit set. If f_0 is the corresponding initial value of the rescaled solution f, then it is defined as

$$\omega(f_0) = \big\{h \in C(\mathbb{R}_+)\colon\ \exists \tau_j \text{ such that } f(\cdot,\tau_j) \to h(\cdot) \text{ as } \tau_j \to \infty \\ \text{uniformly on compact subsets of } \mathbb{R}_+\big\}. \tag{8}$$

Theorem 5.

(i) *If* $2q > p$ *then*

$$\lim_{\tau\to\infty} f(\xi,\tau) = F(\xi) \tag{9}$$

uniformly on compact sets, where F *is the unique solution to the problem*

$$\begin{cases} \alpha F + \beta \xi F' = (F')^2, & \xi > 0, \\ -F'(0) = F^q(0). \end{cases}$$

(ii) *If* $2q = p$ *with* $\lambda < 1$ *then*

$$\lim_{\tau\to\infty} f(\xi,\tau) = G(\xi) \tag{10}$$

uniformly on compact sets, where G *is the unique solution to the problem*

$$\begin{cases} \alpha G + \beta \xi G' = (G')^2 - \lambda G^{2q}, & \xi > 0, \\ -G'(0) = G^q(0). \end{cases}$$

(iii) *If* $2q = p$ *with* $\lambda = 1$ *then the* ω*-limit set is a subset of*

$$\big\{H(\xi) = \big[A + (q-1)\xi\big]^{-1/(q-1)},\ A > 0\big\}.$$

Since the solutions are nonincreasing in x, there exist only three possibilities for the set where the solution blows up: a single point (the origin), a bounded interval or the whole $\mathbb{R}_+$, and all three actually occur. We have

Theorem 6.

(i) *If* $q < 1$ *then blow-up is global, i.e.,* $B(v) = \mathbb{R}_+$.

(ii) *If* $q > 1$ *then the blow-up set reduces to the origin,* $B(v) = \{0\}$.
(iii) *If* $q = 1$ *then blow-up is regional*; *more precisely, the blow-up set is* $B(v) = [0, L]$, *where* $L = 2$ *if* $p < 2$ *and for* $p = 2$ *we have* $\frac{1}{\sqrt{\lambda}} \log(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}) \leqslant L \leqslant \frac{2}{\sqrt{1-\lambda}}$.

A final section, Section 7, is dedicated to the case $p < 1$. We prove finite speed of propagation, Theorem 7, and finite time extinction, Theorem 8.

Theorem 7. *Assume* v_0 *has compact support. If* $p < 1$ *then the solution to problem* (2) *has compact support in* x *for all* $t \in [0, T)$, T *being finite or infinite. Moreover, for* $q \geqslant 1$, *if* T *is finite then the support is compact even for* $t = T$ (*localization property*).

Theorem 8. *Let* $p < 1$. *There exist solutions to problem* (2) *which vanishes identically in finite time if and only if* $2q > p + 1$ *or* $2q = p + 1$ *and* $\lambda > q$.

2. Stationary solutions

We study in this section the existence of stationary solutions to problem (2), in terms of the parameters p, q and λ. We then look for solutions to the problem

$$\begin{cases} V'' + (V')^2 - \lambda V^p = 0, & x > 0, \\ -V'(0) = V^q(0). \end{cases} \tag{11}$$

We prove Theorem 1, which we formulate here in a much more precise way. In particular, the sentence "λ large" in that theorem, when $p \leqslant 2q \leqslant p + 1$ means $\lambda > \lambda^*$ (or $\lambda \geqslant \lambda^*$). We find the values $\lambda^* = 1$ if $2q = p$ and $\lambda^* = q$ if $2q = p + 1$.

Theorem 9.

(i) *If* $2q < p$ *or* $2q > p + 1$ *then problem* (11) *has a unique bounded solution.*
(ii) *If* $2q = p$ *then there are no solutions if* $\lambda \leqslant 1$ *and exactly one bounded solution if* $\lambda > 1$.
(iii) *If* $p < 2q < p + 1$ *then there are no solutions if* $\lambda < \lambda^*$, *one bounded solution if* $\lambda = \lambda^*$ *and two bounded solutions if* $\lambda > \lambda^*$, *where* λ^* *depends on* p *and* q.
(iv) *If* $2q = p + 1$ *then there are no solutions if* $\lambda \leqslant q$ *and exactly one bounded solution if* $\lambda > q$.

In these cases the bounded solutions are nondecreasing and have compact support if and only if $p < 1$. *Also, whenever there exist bounded solutions, and only in this case, there also exist* (*infinite*) *unbounded solutions.*

Proof. Local existence of a solution to (11) is standard. This local solution can be continued and, since there cannot exist points of maxima (from the equation if $V' = 0$ we have $V'' \geqslant 0$), we have three possibilities: the solution is always decreasing and strictly positive, or the solution is decreasing until it meets the horizontal axis, or it has a positive minimum and the solution is increasing from this point. If $V(x_0) = 0$, then the solution cannot be continued by zero beyond x_0 unless $V'(x_0) = 0$. In this case we have that V has compact support. On the other hand, if V is strictly positive and decreasing for every $x > 0$, we must have $\lim_{x\to\infty} V(x) = \lim_{x\to\infty} V'(x) = 0$.

By direct integration we get the following expression for V

$$(V')^2 e^{2V} = B^{2q} e^{2B} - 2\lambda \int_V^B z^p e^{2z}\, dz, \tag{12}$$

where we fix the value at the origin $V(0) = B$. We want to study the values of B for which the corresponding solution satisfies $V'(x_0) = 0$, $V(x_0) \geqslant 0$, x_0 being finite or infinite. Putting then $V = V' = 0$ in (12) we are thus lead to characterize the positive roots of the function

$$H(B) = B^{2q} e^{2B} - 2\lambda \int_0^B z^p e^{2z}\, dz. \tag{13}$$

These are the values that give bounded (nonincreasing) solutions. We observe that, $H(0) = 0$ and

$$H'(B) = 2B^{2q-1} e^{2B} \big(B + q - \lambda B^{p+1-2q}\big).$$

We study the function $G(B) = B + q - \lambda B^\gamma$ for $B > 0$, where $\gamma = p + 1 - 2q$. We have:

- If $\gamma < 0$ then G is monotone from $-\infty$ to $+\infty$, with a unique root. Thus H is first decreasing and then increasing. Since

$$H(B) \geqslant B^{2q} e^{2B} - 2\lambda B^p \int_0^B e^{2z}\, dz = B^p\big[e^{2B}\big(B^{1-\gamma} - \lambda\big) + \lambda\big],$$

  we get $H(B) > 0$ for $B \geqslant \lambda^{1/(1-\gamma)}$. We remark that this property holds whenever $\gamma < 1$. In summary, the function $H(B)$ has a unique positive root, and thus problem (11) has a unique solution.
- If $\gamma = 0$ the function $G(B)$ is an increasing straight line, $G(B) = B + q - \lambda$, positive if $\lambda \leqslant q$, with a root if $\lambda > q$. Then as in the previous case we get a unique root if $\lambda > q$, though if $\lambda \leqslant q$ we have that $H(B)$ is monotone increasing, thus positive, and no solution exists in this case.
- If $0 < \gamma < 1$ we have that $G(B)$ has a minimum at a point $B_0 = (\lambda\gamma)^{1/(1-\gamma)}$. The value of $G(B_0)$ is nonnegative if $\lambda \leqslant \lambda_0 = (q/(1-\gamma))^{1-\gamma}\gamma^{-\gamma}$. Thus in this case H is nondecreasing and no solution can exist. When $\lambda > \lambda_0$ we have that G has two roots, which means that $H(B)$ has a maximum and a minimum. Recall that $H(B)$ is positive for small B as well as for large B. On the other hand, we can estimate H from above

$$H(B) \leqslant B^2 q\left(e^{2B} - \frac{2\lambda}{p+1} B^\gamma\right).$$

Observing that the function

$$J(B) = e^{2B} - \frac{2\lambda}{p+1} B^\gamma$$

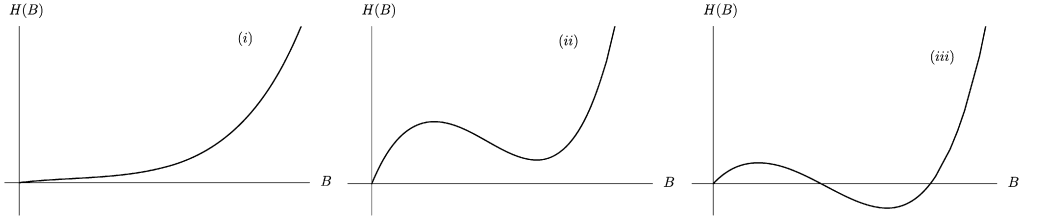

Fig. 2. The intermediate case $0 < \gamma < 1$: (i) $\lambda < \lambda_0$; (ii) $\lambda_0 < \lambda < \lambda^*$; (iii) $\lambda > \lambda^*$.

is negative for some interval provided $\lambda > \lambda_1 = \frac{p+1}{2} e^{\gamma} (\frac{\gamma}{2})^{-\gamma}$, we have that $H(B)$ has exactly two roots if $\lambda > \lambda_1$. By continuity there exists some $\lambda^* \in (\lambda_0, \lambda_1)$ for which $H(B)$ has exactly one root. We represent in Fig. 2 the curve $H(B)$ in this case $0 < \gamma < 1$ for different values of λ.

- If $\gamma = 1$ the function $G(B)$ is again a straight line, $G(B) = (1 - \lambda)B + q$. It is clear then than there exist solutions (exactly one) if and only if $\lambda > 1$.
- If $\gamma > 1$ the function $G(B)$ is first increasing and then decreasing, with a unique root. This implies that $H(B)$ has also a unique root, no matter the value of λ.

When B^* is a root of $H(B)$ we obtain a nonincreasing solution to the stationary problem (11). This solution has compact support if and only if $p < 1$. In fact

$$\begin{aligned} V^*(x) &\sim C_1 x^{-2/(p-1)}, && \text{for } x \to \infty, \text{ if } p > 1, \\ V^*(x) &\sim C_2 e^{-\sqrt{\lambda} x}, && \text{for } x \to \infty, \text{ if } p = 1, \\ V^*(x) &\sim C_3 (x^* - x)_+^{2/(1-p)}, && \text{for } x \sim x^*, \text{ if } p < 1, \end{aligned}$$

with $C_1 = C_3 = (\frac{2(p+1)}{\lambda(p-1)^2})^{1/(p-1)}$ and some $C_2 > 0$, $x^* = x^*(B^*)$. Moreover, since it holds

$$(V')^2 e^{2V} = \int_0^V z^p e^{2z}\,dz,$$

the solution can be written implicitly as

$$\int_{V^*(x)}^{B^*} \frac{dz}{R(z)} = x, \quad \text{with } R(z) = \sqrt{2}\, e^{-z} \left(\int_0^z s^p e^{2s}\,ds \right)^{1/2}.$$

From this we obtain, in the case $p < 1$, $x^* = \int_0^{B^*} \frac{dz}{R(z)} < \infty$.

As we have said, other values of B give unbounded solutions if $H(B) < 0$, with a minimum of height C determined by the relation

$$H(B) = -2\lambda \int_C^B z^p e^{2z}\,dz,$$

or local solutions vanishing at some finite point x_0 with $V'(x_0) < 0$, thus not being global solutions when we extend them by zero, if $H(B) > 0$. □

3. Blow-up results

In this section we prove Theorems 2 and 3. The basic idea is to compare with subsolutions that blow up or with supersolutions that are global in time.

By a subsolution (supersolution) we mean a function which satisfies the problem with $\leqslant$ ($\geqslant$) instead of $=$ in the equation, the boundary condition and the initial data. We formulate here the comparison principle, its proof is standard and we omit it.

Lemma 10. *Let $\overline{v}$ be a supersolution, v be a solution, and $\underline{v}$ be a subsolution to problem* (2). *If $\overline{v}(x,0) \geqslant v(x,0) \geqslant \underline{v}(x,0)$ then $\overline{u}(x,t) \geqslant u(x,t) \geqslant \underline{u}(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,T)$.*

First we deal with the case $2q \neq p$.

Proof of Theorem 2. (i) We are going to prove that all solutions are global finding large global supersolutions. If $2q \leqslant 1$ solutions to the heat equation with flux $-u_x(0,t) = (u+1) \times \log^q(u+1)(0,t)$ are supersolutions to our problem and they are globally defined whenever $2q \leqslant 1$ (a supersolution of the form $\varphi(a(x)+b(t))$ can be constructed, see [18]).

In the case $2q < p$ we use a supersolution in self-similar form for the problem in the v variable, i.e., problem (2). Take

$$z(x,t) = e^t\left(1+e^{-\xi}\right), \quad \xi = xe^{ct}. \tag{14}$$

In order to have a supersolution we must have

$$\begin{cases} 1+e^{-\xi} - c\xi e^{-\xi} \geqslant e^{-\xi} e^{(2c+1)t}\left(1+e^{-t}e^{-\xi}\right) - \lambda\left(1+e^{-\xi}\right)e^{(p-1)t}, & \xi > 0, \\ e^{(c+1)t} \geqslant 2^q e^{qt}. \end{cases}$$

To get the boundary condition fulfilled we need $c > q-1$ and $t \geqslant t_0$ large. Concerning the equation, it is satisfied if we impose

$$\lambda e^{(p-1)t} \geqslant e^{(2c+1)t}\left(1+e^{-t}\right) + k$$

for some $k > 0$. Hence it suffices to have $2c < p-2$. Therefore the condition required is

$$2(q-1) < c < p-2,$$

that is,

$$2q < p.$$

If we now consider $\tilde{z}(x,t) = z(x,t+t_0)$, the comparison of the initial conditions means

$$e^{t_0} \geqslant u_0(x),$$

which holds if t_0 is large.

(ii) We construct a blow-up subsolution of the form

$$w(x,t) = (T-t)^{-\gamma} f(\xi), \quad \xi = x(T_0 - t)^{-\delta},$$

with $\gamma = 1/(2q-1)$, $\delta = (q-1)\gamma$ and $f(\xi) = (A - A^q \xi)_+$. The boundary condition holds with equality. As for the equation, the condition to be a subsolution is

$$\gamma A + \lambda T_0^{(2q-p)\gamma} A^p \leqslant A^{2q}.$$

This holds if A is chosen large enough, since $2q > p$ and $2q > 1$.

An alternative way to obtain a blow-up solution consists in imposing the condition $\max_{x\geqslant 0} |v_0'(x)| = |v_0'(0)|$ on the initial datum. This implies $\max_{x\geqslant 0} |v_x(x,t)| = |v_x(0,t)|$, and a fortiori $v_{xx}(0,t) \geqslant 0$. Thus, $2q > p$ implies

$$v_t(0,t) \geqslant v^{2q}(0,t) - \lambda v^p(0,t) \geqslant c v^{2q}(0,t), \tag{15}$$

provided $v_0(0) > \Lambda = \lambda^{-1/(2q-p)}$. This gives blow up if $2q > 1$. □

Remark 11. We observe that both methods presented to prove existence of blow-up also work when $2q = p$ and $\lambda < 1$. On the other hand, we remark that none of the above arguments gives that all the solutions blow up, since these subsolutions are not small. In fact, if $q > 2$ we will see that it is not the case and there exist small global solutions.

Now we prove a Fujita type result. That is, there exists a region of parameters where every nontrivial solution blows up.

Theorem 12. *Assume* $2q > \max\{p, 1\}$. *Then every nontrivial solution blows up in finite time if* $2q < p + 1$, $q \leqslant 1$ *and* $\lambda < \lambda^*$.

Proof. The proof is an immediate consequence of Theorems 2(ii), 11 and 13 below. Assume that $v(0,t) \leqslant \Lambda$ for every $t > 0$, otherwise the solution must blow up. Then through a subsequence if necessary we have that $v(\cdot, t_n)$ tends to a stationary solution as $t_n \to \infty$. This is a contradiction since no stationary solution exist in this range of parameters and the solution cannot go to zero due to the following result. □

Theorem 13. *Let* $q < 1$. *If* $2q < p + 1$ *or* $2q = p + 1$ *and* $\lambda \leqslant q$ *then problem* (11) *admits small subsolutions.*

Proof. The subsolution mentioned takes the form

$$w(x) = \big(a - (1-q)x\big)_+^{1/(1-q)}. \tag{16}$$

The boundary condition holds with equality. As for the equation, the condition to be a subsolution is

$$\lambda w^{(p+1-2q)/(1-q)} \leqslant q + w^{1/(1-q)},$$

and it is clear that this holds taking $a > 0$ small. □

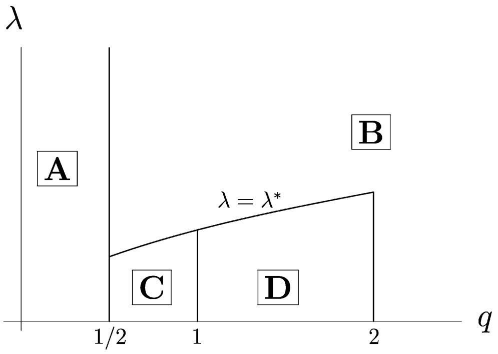

Fig. 3. Case $p < 2q \leqslant p + 1$. A: All solutions are global. B: There exists blow-up and also global solutions. C: All solutions blow up. D: There exists blow-up.

In Fig. 3 we represent the existence or not of global solutions in the strip $p < 2q \leqslant p+1$, i.e., region III from Fig. 1. Thus we put $2q = p + \varepsilon$, $0 < \varepsilon \leqslant 1$, and study the existence in terms of q and λ.

Remark 14.

(i) The fact that there exists small global solutions when $q > 2$ follows by comparison with a solution z to the heat equation with boundary flux $-z_x(0,t) = Cz^q(0,t)$ that stays less than one and goes to zero as t goes to infinity (the existence of such a solution is proved in [8]). Since $Cz^q(0,t) \geqslant (1+z)\log^q(1+z)(0,t)$, we have that z is a supersolution to our problem.

(ii) It is left as an open problem if there exist global solutions when $p < 2q \leqslant p+1$, $\lambda < \lambda^*$, $1 < q \leqslant 2$.

Next we deal with the case $2q = p$. Here the result depends critically on whether the absorption coefficient λ is greater or less than one.

Proof of Theorem 3. (i) A modification of the function defined in (14) is a supersolution in this case. Take

$$w(x,t) = e^t\left(A + e^{-b\xi}\right), \quad \xi = xe^{(q-1)t}. \tag{17}$$

We choose $b = (A+1)^q$ to fix the boundary condition. Now, the restriction imposed by the equation becomes

$$\lambda A^p \geqslant b^2\left(1 + e^{-t_0}\right) = (A+1)^p\left(1 + e^{-t_0}\right).$$

Since $\lambda > 1$, this condition holds if A and t_0 are large.

(ii) See Remark 11.

(iii) The same function as in case (i) works here as a supersolution when $q \leqslant 1$. Conversely, if $q > 1$ we can construct a blow-up subsolution of the form

$$V(x,t) = (T-t)^{-\gamma}\varphi(\xi), \quad \xi = x(T-t)^{-1/2},$$

with $\varphi(\xi) = (A - B\xi)_+^2$, $\gamma = 1/(2(q-1))$, $B = A^{2q-1}/2$, $A > 0$ small. □

4. Blow-up rates

We study in this section the speed at which a blow-up solution approaches infinity near the blow-up time. This is the blow-up rate. Let v be a solution to problem (2) that blows up at time T. We prove Theorem 4, which can be stated as

$$v(0,t) \sim (T-t)^{-\alpha} \tag{18}$$

for $t \nearrow T$, where

$$\alpha = \begin{cases} \frac{1}{2q-1}, & \text{if } 2q \neq p \text{ or } 2q = p \text{ with } \lambda < 1, \\ \frac{1}{2(q-1)}, & \text{if } 2q = p \text{ with } \lambda = 1. \end{cases} \tag{19}$$

The main point is that the blow-up rate is discontinuous with respect to the parameters of the problem.

To prove estimate (18) we need a preliminary result that asserts that eventually any blow-up solution is convex near the origin.

Lemma 15. *If v is a blow-up solution, then for any time close enough to the blow-up time the maximum of $|v_x|$ is achieved at the origin.*

Proof. It suffices to look at the equation satisfied by $z = -v_x$,

$$\begin{cases} z_t = z_{xx} - 2zz_x - \lambda p v^{p-1} z, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ z(0,t) = v^q(0,t). \end{cases}$$

By hypotheses we know that $z(x,t) \geqslant 0$ and $z(0,t) > 0$. Let $x_0(t)$ be the point of maximum of z at each time t, and put $M(t) = z(x_0(t),t)$. Whenever $x_0(t) > 0$ we have

$$z_x\big(x_0(t),t\big) = 0, \qquad z_{xx}\big(x_0(t),t\big) \leqslant 0,$$

and therefore $M'(t) < -C$. This results in a contradiction if v blows up in finite time. Therefore there exists a time $t_0 < T$ for which z attains its maximum at $x = 0$. It is easy to see that this property holds also for every $t_0 < t < T$. □

Proof of Theorem 4. (i) The previous lemma implies $v_{xx}(0,t) \geqslant 0$ for t close to T. Then we get the estimate

$$v_t(0,t) \geqslant C v^{2q}(0,t)$$

whenever $2q > p$ or $2q = p$ and $\lambda < 1$, see (15). Then we can integrate to get the upper bound

$$v(0,t) \leqslant C(T-t)^{-1/(2q-1)}.$$

In order to obtain the lower bound, we use a rescaling technique inspired in the work [10], see also [11]. The difference lies in the final step: we do not pass to the limit, but instead we estimate the blow-up time of the rescaled function. This is translated into a blow-up rate for the original solution, see [5].

Fix $t \in (0, T)$, define $M = u(0, t)$, and consider the function

$$\phi_M(y, s) = \frac{1}{M} u\big(M^{1-q} y, M^{1-2q} s + t\big).$$

This function is defined for $y \geqslant 0$ and $s \in (0, S)$, where $S = M^{2q-1}(T - t)$. In particular it blows up at $s = S$. On the other hand, it solves the problem

$$\begin{cases} (\phi_M)_s = \dfrac{1}{M}(\phi_M)_{yy} + \big((\phi_M)_y\big)^2 - \lambda M^{p-2q}(\phi_M)^p & \text{in } \mathbb{R}_+ \times (0, S), \\ -(\phi_M)_y(0, s) = (\phi_M)^q(0, s), \\ \phi_M(y, 0) = \dfrac{1}{M} u\big(M^{1-q} y, t\big). \end{cases}$$

Notice that $\phi_M(0, 0) = 1$ and $(\phi_M)_y(0, 0) = -1$. Therefore, $\phi_M(y, 0) \leqslant 1$ for $y > 0$. We now construct a supersolution for this problem. Let

$$w(y, s) = (S_1 - s)^{-\alpha} \left(A + \frac{\alpha}{4}(L - \xi)_+^2 \right), \quad \xi = y(S_1 - s)^{-\beta},$$

where $\alpha = 1/(2q - 1)$ and $\beta = (q - 1)\alpha$. In order to obtain a supersolution of the equation, the parameters must satisfy

$$2A - \beta\xi(L - \xi)_+ \geqslant \frac{S_1^\alpha}{M_0}, \tag{20}$$

for all $M > M_0$. Recall that the condition on M means $t_0 < t < T$ for some t_0. The boundary condition imposes the restriction

$$\frac{\alpha}{2} L \geqslant \left(A + \frac{\alpha}{4} L^2 \right)^q. \tag{21}$$

Finally, comparison between w and our rescaled solution ϕ_M at time $s = 0$ requires

$$S_1^{-\alpha} A \geqslant 1. \tag{22}$$

In the case $\beta \leqslant 0$, i.e., $q \leqslant 1$, it is enough to consider A small, then we obtain L which verifies the boundary inequality, S_1 small to fix the condition at $s = 0$ and finally M_0 large to verify the equation.

For the case $\beta > 0$, i.e., $q > 1$, we take $A = \beta L^2/2$, and L and S_1 small to satisfy the boundary and initial conditions. As to restriction (20), we note that from the choice of A we have

$$\left(2 - \frac{1}{M_0} \right) A \geqslant \beta\xi(L - \xi).$$

Thus taking M_0 large we are done.

Summing up, we have a supersolution independent of M, for all M large enough. Therefore the blow-up time of ϕ_M is greater than S_1, that is $M^{2q-1}(T-t) \geqslant S_1$. This implies

$$u(0,t) \geqslant C(T-t)^{-1/(2q-1)},$$

and the lower bound is obtained.

(ii) Assume now $2q = p$ and $\lambda = 1$. The lower estimate just obtained, is also valid here but is not sharp. Instead we perform the change of variables

$$\phi_M(y,s) = \frac{1}{M} u\big(M^{1-q} y, M^{2-2q} s + t\big).$$

Then ϕ_M satisfies

$$\begin{cases} (\phi_M)_s = (\phi_M)_{yy} + M\big[\big((\phi_M)_y\big)^2 - (\phi_M)^{2q}\big] & \text{in } \mathbb{R}_+ \times (0, \widetilde{S}), \\ -(\phi_M)_y(0,s) = (\phi_M)^q(0,s), \\ \phi_M(y,0) = \dfrac{1}{M} u\big(M^{1-q} y, t\big). \end{cases}$$

In this case $\widetilde{S} = M^{2q-2}(T-t)$. The supersolution takes the self-similar form

$$w(y,s) = (S_2 - s)^{-\alpha}\big(A + B(L-\xi)_+^2\big), \quad \xi = y(S_2 - s)^{-1/2},$$

where $\alpha = 1/(2q-2)$, $M > M_0$ large, B and S_2 are small, $L = C(q)B^{q-1}$ and $A = (2BL)^{1/q} - BL^2$. Therefore, for M large we obtain that the blow-up time of ϕ_M is greater than S_2, which implies

$$u(0,t) \geqslant C(T-t)^{-1/(2q-2)}.$$

To get the upper bound we construct a subsolution in the form

$$z(y,s) = \big(\alpha^2 - s\big)^{-\alpha} \frac{\alpha^{2\alpha+2}}{4}\left(\frac{2}{\alpha} - \xi\right)_+^2, \quad \xi = y\big(\alpha^2 - s\big)^{-1/2}.$$

Now, in order to compare the initial values, we consider the function

$$P(y,s) = \left(1 - \frac{1}{2} y s^{-1/2}\right)_+^2,$$

which is a subsolution of the equation for $0 \leqslant s \leqslant 1$. Moreover, $P(0,s) = 1$ and $P(y,0) = 0$ for all $y > 0$. On the other hand, $\phi_M(0,0) = 1$ and $(\phi_M)_s(0,s) \geqslant 0$. Therefore, by comparison we obtain that for all $M > 0$

$$\phi_M(y,s) \geqslant P(y,s) \quad \text{for } 0 \leqslant s \leqslant 1.$$

But, since $P(y,1) = z(y,0)$ we have that

$$\phi_M(y, s+1) \geqslant z(y,s).$$

Therefore, the blow-up time of ϕ_M satisfies $\widetilde{S} \leqslant 1 + \alpha^2$. This implies

$$u(0,t) \leqslant C(T-t)^{-1/(2q-2)},$$

and the theorem is proved. □

5. Asymptotic behaviour

In this section we consider the rescaled function f given by (7) and study its behaviour for $\tau \to \infty$. This proves Theorem 5.

The problem satisfied by f is the following

$$\begin{cases} e^{-\delta\tau}(f_\tau + \alpha f + \beta\xi f_\xi) = e^{-\alpha\tau} f_{\xi\xi} + (f_\xi)^2 - \lambda e^{-\varepsilon\tau} f^p, & \xi > 0, \tau > 0, \\ -f_\xi(0,\tau) = f^q(0,\tau), & \tau > 0, \end{cases} \tag{23}$$

where α is given by (19), we put $\beta = (q-1)\alpha$, and the exponents ε and δ vary. We have three possible situations:

Case 1: if $2q > p$ then $\delta = 0$, $\varepsilon = (2q-p)\alpha > 0$;
Case 2: if $2q = p$ and $\lambda < 1$ then $\delta = \varepsilon = 0$;
Case 3: if $2q = p$ and $\lambda = 1$ then $\delta = \alpha > 0$, $\varepsilon = 0$.

Therefore, in each case it is easy to get intuition of the limit problem to be considered. Thanks to Theorem 4 we have that there exist two positive constants such that

$$C_1 \leqslant f(0,\tau), \qquad \left|f_\tau(0,\tau)\right| \leqslant C_2, \tag{24}$$

for every $\tau > 0$. Also, since the maximum of f and $|f_\xi|$ are located at the origin, see Lemma 15, we conclude that both functions are bounded. This implies that if we define the orbits $h_j(\cdot,\tau) = f(\cdot,\tau + s_j)$ we have, by the usual compactness arguments, the convergence

$$\lim_{s_j \to \infty} h_j(\cdot,\tau) = h(\cdot,\tau).$$

Now let us consider Case 1. Passing to the limit in problem (23), in its weak formulation, we get that h satisfies the simplified problem

$$\begin{cases} h_\tau = (h_\xi)^2 - \beta\xi h_\xi - \alpha h, & \xi > 0, \tau > 0, \\ -h_\xi(0,\tau) = h^q(0,\tau), & \tau > 0. \end{cases} \tag{25}$$

Also, the bounds (24) are true for h. On the other hand, notice that at $\xi = 0$ the above equation reads

$$h_\tau(0,\tau) = h^{2q}(0,\tau) - \alpha h(0,\tau).$$

Therefore, if for some $\tau > 0$ we have $h(0,\tau) > \mu = \alpha^\alpha$, we deduce that h blows up at some finite $\tau = \tau_0$. This is a contradiction. Analogous contradiction is obtained if $h(0,\tau) < \mu$ for some

$\tau > 0$. We conclude that $h(0,\tau) = \mu$ for every $\tau > 0$. This also implies $h_\xi(0,\tau) = -\mu^q$. We end the proof by observing that the unique solution to the overdetermined problem

$$\begin{cases} h_\tau = (h_\xi)^2 - \beta\xi h_\xi - \alpha h, & \xi > 0, \tau > 0, \\ h(0,\tau) = \mu, & \tau > 0, \\ h_\xi(0,\tau) = -\mu^q, & \tau > 0, \end{cases} \tag{26}$$

is the stationary solution constructed in Theorem 16 below.

In Case 2, the same argument gives convergence to a solution of the reduced problem

$$\begin{cases} h_\tau = (h_\xi)^2 - \beta\xi h_\xi - \alpha h - \lambda h^{2q}, & \xi > 0, \tau > 0, \\ h(0,\tau) = \mu', & \tau > 0, \\ h_\xi(0,\tau) = -(\mu')^q, & \tau > 0, \end{cases} \tag{27}$$

with $\mu' = (\alpha/(1-\lambda))^\alpha$. The unique solution to this problem is given by the stationary solution obtained in Theorem 17.

Finally, in Case 3 we obtain the convergence of the orbits to the reduced (stationary) equation

$$(h_\xi)^2 - h^{2q} = 0, \quad \xi \geqslant 0, \tag{28}$$

where the boundary condition implies the minus sign in h_ξ. The solutions to this last problem are $H(\xi) = [A + (q-1)\xi]^{-1/(q-1)}$, $A > 0$. Thus the limit is not unique in this case and we have to consider the ω-limit set, see (8).

In order to set up the convergence result in all cases, we now study the stationary reduced problems previously mentioned.

Theorem 16. *The problem*

$$\begin{cases} \alpha F + \beta\xi F' = (F')^2, & \xi > 0, \\ -F'(0) = F^q(0) \end{cases} \tag{29}$$

with $\alpha = 1/(2q-1)$, $2q > 1$, $\beta = (q-1)\alpha$, has a unique solution. The solution has compact support if and only if $q \leqslant 1$. Moreover, if $q = 1$ this solution is explicit $F(\xi) = (1-\xi/2)_+^2$, and if $q > 1$ it behaves like $\xi^{-1/(q-1)}$ as $\xi \to \infty$.

Proof. A solution F_1 to the equation in problem (29) is obtained in [17, p. 171] with the boundary condition $F_1(0) = 1$. The unique solution to our problem can now be obtained just by rescaling, $F(\xi) = \mu F_1(\xi/\sqrt{\mu})$ with $\mu = \alpha^\alpha$. □

Theorem 17. *The problem*

$$\begin{cases} \alpha G + \beta\xi G' = (G')^2 - \lambda G^{2q}, & \xi > 0, \\ -G'(0) = G^q(0) \end{cases} \tag{30}$$

with $\alpha = 1/(2q-1)$, $2q > 1$, $\beta = (q-1)\alpha$, and $\lambda < 1$, has a unique solution. It has compact support if and only if $q > 1$.

Proof. Local existence is standard. The value at the origin is determined by the equation, and it is $G(0)=\mu'$. Moreover, it is also immediate to see that the solution can be continued and is decreasing until it reaches the horizontal axis, if it does. If this happens we extend the solution by zero.

Assume first $q\leqslant 1$. We have

$$(G')^2-\alpha G-\lambda G^{2q}=\beta\xi G'\geqslant 0.$$

Thus

$$G'\leqslant -\sqrt{\alpha G+\lambda G^{2q}}\leqslant -\sqrt{\alpha G},$$

which means that there exists $\xi_0>0$ such that $G(\xi_0)=0$. This proves that the support is compact.

Now consider $q=1$. Since $\beta=0$ in this case, integrating the equation we get

$$\int_{G(\xi)}^{\mu'}\frac{ds}{\sqrt{s+\lambda s^2}}=\xi,$$

where $\mu'=1/(1-\lambda)$ in this case. From here we obtain the support of G,

$$L(\lambda)=\int_0^{1/(1-\lambda)}\frac{ds}{\sqrt{s+\lambda s^2}}=\frac{1}{\sqrt{\lambda}}\log\left(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}\right).$$

Observe that $L(\lambda)>2$ for every $0<\lambda<1$. In fact, $L(\lambda)$ is an increasing function in this interval with $\lim_{\lambda\to 0}L(\lambda)=2$, $\lim_{\lambda\to 1}L(\lambda)=\infty$.

If now $q>1$, we fix some $\varepsilon>0$ small and take $\xi_1>0$ such that $G(\xi_1)=(\varepsilon/\lambda)^{\alpha}$. Then, for $\xi\geqslant\xi_1$ we have

$$\beta\xi G'+(\alpha+\varepsilon)G=(G')^2+\varepsilon G-\lambda G^{2q}\geqslant 0,$$

which implies that G is positive with $G(\xi)\geqslant c\xi^{-(\alpha+\varepsilon)/\beta}$ for $\xi\geqslant\xi_1$. □

6. Blow-up sets

Here we study at which spatial points the solution goes to infinity. This is called the blow-up set and can be defined as

$$B(v)=\left\{x\geqslant 0\colon\ \exists x_n\to x,\ t_n\nearrow T\ \text{with}\ v(x_n,t_n)\to\infty\right\}.$$

Since v is nonincreasing in x, then $B(v)$ is a connected interval containing the origin. We say that we have single point blow-up if $B(v)$ reduces to the origin, that blow-up is regional if $B(v)$ is a nontrivial bounded interval, and that it is global if $B(v)$ is the whole half-line. We are going to show that the three possibilities occur in our problem, in spite of the diffusion being linear. We prove Theorem 6.

Proof of Theorem 6. (i) By the Mean Value Theorem, and the fact that $|v_x|$ attains its maximum at $x=0$, see Lemma 15, we obtain

$$v(x,t)-v(0,t)=xv_x(\eta,t)\geqslant -xv^q(0,t).$$

Therefore,

$$v(x,t)\geqslant v(0,t)\bigl(1-xv^{q-1}(0,t)\bigr).$$

Since $q<1$, we have that all points are in the blow-up set.

(ii) We use comparison in u variables (see (1)), with the problem

$$\begin{cases} z_t=z_{xx}, & x>0,\ 0<t<T,\\ z(0,t)=e^{C(T-t)^{-\alpha}}, & 0<t<T,\\ z(x,0)=u_0(x). \end{cases} \tag{31}$$

Note that z blows up at the same time as u does. Thanks to the blow-up rates (Theorem 4) we have that u is a subsolution to this problem, and thus $u\leqslant z$ for every $0<t<T$. We end by observing, using the explicit representation of z in terms of the Green's function, that z is bounded for every $x>0$. Actually

$$z(x,t)=\int_0^\infty G(x,y,t,0)u_0(y)\,dy+\int_0^t G(x,0,t,\tau)e^{C(T-t)^{-\alpha}}\,d\tau, \tag{32}$$

where

$$G(x,y,t,\tau)=G_\infty(x-y,t-\tau)+G_\infty(x+y,t-\tau),$$
$$G_\infty(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}.$$

We have

$$z(x,t)\leqslant M+C\int_0^t (t-\tau)^{-1/2}e^{-\frac{x^2}{4(t-\tau)}}e^{\frac{C}{(T-t)^{\alpha}}}\,d\tau,$$

which is finite for every $x>0$ due to the fact that $\alpha<1$.

(iii) For $q=1$ we have $\alpha=1$, and the above representation gives z bounded for every $x>2\sqrt{C}$. We have thus the estimate $B(u)\subset[0,2\sqrt{C}]$.

Moreover, by the convergence to the limit profile we can take the value of the self-similar profile at the origin as the constant in the blow-up rates. This means $C=1$ if $p<2$, and $C=1/(1-\lambda)$ if $p=2$. Therefore,

$$B(u)\subseteq\begin{cases}[0,2], & p<2,\\ [0,\frac{2}{\sqrt{1-\lambda}}], & p=2.\end{cases}$$

On the other hand, the blow-up set must contain the support of the limit profile. From where it follows that

$$B(u) \supseteq \begin{cases} [0,2], & p<2, \\ [0, \frac{1}{\sqrt{\lambda}} \log(\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}})], & p=2. \end{cases}$$

This ends the proof. □

7. The case $p < 1$

In this section we study some properties of the solutions to our problem (2) in the case $p < 1$. In particular, we prove finite speed of propagation, localization of the support and existence of a "small" solution with finite time extinction.

Proof of Theorem 7. Consider the following problem

$$\begin{cases} V'' + (V')^2 - \lambda V^p = 0, & x > 0, \\ V(0) = M, \\ V'(0) = -N. \end{cases}$$

It is clear (see the analogous study performed in Section 2), that for each $M > 0$ there exists $N > 0$ such that the corresponding solution is nonincreasing with compact support $[0, \ell(M)]$. Now let $[0, L]$ be the support of v_0. For every $0 < t_0 < T$ fixed, take $M = \max_{0 \leqslant t \leqslant t_0} v(L,t)$. We then have

$$v(x,t) \leqslant V(x-L), \quad x \geqslant L,\ 0 \leqslant t \leqslant t_0.$$

This gives $\operatorname{supp}(v(\cdot, t_0)) \subset [0, L + \ell(M)]$.

Finally, observe that for $q \geqslant 1$ the blow-up set of v is bounded. We can therefore take $t_0 = T$ in the above argument. The localization property holds. □

Remark 18. Notice that for $p \geqslant 1$ we have infinite speed of propagation. Indeed, we use as a subsolution of our problem a solution of $w_t = w_{xx} - \lambda w^p$ which is positive for all positive times.

We end with a characterization of the property of extinction in finite time.

Proof of Theorem 8. The requirements needed to have extinction follow from Theorem 13. Conversely, in order to construct a supersolution with the property of finite time extinction in the range $2q > p + 1$ we consider the self-similar function

$$\overline{V}(x,t) = (T-t)^{\gamma} F\big(x(T-t)^{\sigma}\big),$$

with $\gamma = 1/(1-p)$ and $\sigma = (q-1)\gamma$. To choose the profile F we impose the condition $-F' = F^q$ to hold not only on the boundary. Therefore, the condition to be a supersolution becomes

$$\lambda F^p - \gamma F + \sigma \xi F^q \geqslant q(T-t)^{(2q-p-1)\gamma} F^{2q-1} + (T-t)^{2(q-p)\gamma} F^{2q}.$$

We observe that, if $p < \min\{1, 2q - 1\}$, the smallest power of F in the expression above is p and the powers of $(T - t)$ are both positive. Then we can take T and $F(0) = A$ small enough such that the required inequality holds. In the case $p = 2q - 1 < 1$ we arrive at the same conclusion provided $\lambda > q$. □

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