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J. Differential Equations 227 (2006) 80–101
Journal of Differential Equations
www.elsevier.com/locate/jde

An optimization problem with volume constraint for a degenerate quasilinear operator ☆

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Received 17 May 2005; revised 9 March 2006

Available online 17 April 2006

Abstract

We consider the optimization problem of minimizing $\int_\Omega |\nabla u|^p\,dx$ with a constraint on the volume of $\{u > 0\}$. We consider a penalization problem, and we prove that for small values of the penalization parameter, the constrained volume is attained. In this way we prove that every solution u is locally Lipschitz continuous and that the free boundary, $\partial\{u > 0\} \cap \Omega$, is smooth.

MSC: 35J20; 35P30; 49K20

Keywords: Optimal design problems; Free boundaries

1. Introduction

In the seminal paper [2], Aguilera, Alt and Caffarelli study an optimal design problem with a volume constraint by introducing a penalization term in the energy functional (the Dirichlet integral) and minimizing without the volume constraint. For fixed values of the penalization parameter, the penalized functional is very similar to the one considered in the paper [4]. So that, regularity results for minimizers of the penalized problem follow almost without change as in [4]. The main result in [2] that makes this method so useful is that the right volume is already

☆ Supported by ANPCyT PICT No. 03-13719 and 03-10608, UBA X052 and X066 and Fundación Antorchas 13900-5. J. Fernández Bonder and N. Wolanski are members of CONICET.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2006.03.006

attained for small values of the penalization parameter. In this way, all the regularity results apply to the solution of the optimal design problem.

This method has been applied to other problems with similar success. In all those cases, the differential equation satisfied by the minimizers is nondegenerate, uniformly elliptic. See, for instance, [3,9,13,16].

In this article we want to show that the same kind of results can be obtained for some nonlinear degenerate or singular elliptic equations. As an example, we study here the following problem which is a generalization of the one in [2] for $1 < p < \infty$:

We take Ω a smooth bounded domain in $\mathbb{R}^N$ and $\varphi_0 \in W^{1,p}(\Omega)$, a Dirichlet datum, with $\varphi_0 \geqslant c_0 > 0$ in $\bar{A}$, where A is a nonempty relatively open subset of $\partial\Omega$ such that $A \cap \partial\Omega$ is C^2. Let

$$\mathcal{K}_\alpha = \{u \in W^{1,p}(\Omega)\colon \ \big|\{u > 0\}\big| = \alpha,\ u = \varphi_0 \text{ on } \partial\Omega\}.$$

Our problem is to minimize $\mathcal{J}(u) = \int_\Omega |\nabla u|^p\,dx$ in $\mathcal{K}_\alpha$.

Problems similar to the one considered here appear in shape optimization. For instance, in optimization of torsional rigidity [13], insulation of pipelines for hot liquids [10], minimization of the current leakage from insulated wires and coaxial cables [1], minimization of the capacity of condensers and resistors, etc.

Although the existence of a minimizer is not difficult to establish by variational techniques, the regularity properties of such minimizers and their free boundaries $\partial\{u > 0\}$, are not easy to obtain since it is hard to make enough volume preserving perturbations without the previous knowledge of the regularity of $\partial\{u > 0\}$.

In order to solve our original problem in a way that allows us to perform non-volume preserving perturbations we consider instead the following penalized problem: we let

$$\mathcal{K} = \{u \in W^{1,p}(\Omega)\colon \ u = \varphi_0 \text{ on } \partial\Omega\}$$

and

$$\mathcal{J}_\varepsilon(u) = \int_\Omega |\nabla u|^p\,dx + F_\varepsilon\big(\big|\{u > 0\}\big|\big), \tag{1.1}$$

where

$$F_\varepsilon(s) = \begin{cases} \varepsilon(s - \alpha) & \text{if } s < \alpha,\\ \frac{1}{\varepsilon}(s - \alpha) & \text{if } s \geqslant \alpha. \end{cases}$$

Then, the penalized problem is

$$\text{find } u_\varepsilon \in \mathcal{K} \quad \text{such that} \quad \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{v \in \mathcal{K}} \mathcal{J}_\varepsilon(v). \tag{P$_\varepsilon$}$$

The existence of minimizers follows easily by direct minimization. Their regularity and the regularity of their free boundaries $\partial\{u_\varepsilon > 0\}$ follow as in [5] where a very similar problem was studied, namely, to minimize

$$\overline{\mathcal{J}}_\lambda(v) = \int_\Omega |\nabla v|^p\,dx + \lambda^p\big|\{v > 0\}\big|, \tag{1.2}$$

where $\lambda > 0$ is a constant. In particular, u_ε is a solution of the following free boundary problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda_\varepsilon & \text{on } \partial\{u > 0\} \cap \Omega, \end{cases}$$

where λ_ε is a positive constant and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian.

In [9] the authors study a problem closely related to [2]. The problem in [9] is to minimize the best Sobolev trace constant from $H^1(\Omega)$ into $L^q(\partial\Omega)$ for subcritical q, among functions that vanish in a set of fixed measure. We will sometimes refer to some of the proofs in [9] for the different treatment of the penalization term (which is piecewise linear in the measure of the positivity set) with respect to [4,5] where the function is linear in the measure.

As in [2], the reason why this penalization method is so useful is that there is no need to pass to the limit in the penalization parameter ε for which uniform, in ε, regularity estimates would be needed. In fact, we show that for small values of ε the right volume is already attained. This is, $|\{u_\varepsilon > 0\}| = \alpha$ for ε small. It is at this point where the main changes have to be made since the perturbations used in [2,9] make strong use of the linearity of the underlying equation.

In particular, the fact that, for small ε, any minimizer of $\mathcal{J}_\varepsilon$ satisfies $|\{u_\varepsilon > 0\}| = \alpha$ implies that any minimizer of our original optimization problem is also a minimizer of $\mathcal{J}_\varepsilon$ so that it is locally Lipschitz continuous with smooth free boundary.

We include at the end of the paper a couple of appendices where some properties of p-subharmonic functions are established. We use these results in Section 2. We believe that these results have independent interest.

The paper is organized as follows. In Section 2 we begin our analysis of problem (P_ε) for fixed ε. First we prove the existence of a minimizer, local Lipschitz regularity and nondegeneracy near the free boundary (Theorem 2.1) and with these results we have the regularity of the free boundary by adapting the results of [5].

The main results of this paper appear in Section 3 where we prove that for small values of ε we recover our original optimization problem.

The appendices are included at the end of the paper.

2. The penalized problem

In this section we look for minimizers of the functional $\mathcal{J}_\varepsilon$ and a representation theorem for solutions of $\mathcal{J}_\varepsilon$ as in [4, Theorem 4.5].

Observe that a solution to (P_ε) satisfies that

$$\Delta_p u = 0 \quad \text{in } \{u > 0\}^\circ.$$

In fact, let B be a ball such that $u > 0$ in B. Let v be the solution to

$$\Delta_p v = 0 \quad \text{in } B, \qquad v = u \quad \text{on } \partial B.$$

Let $\bar{v} \in W^{1,p}(\Omega)$, $\bar{v}(x) = v(x)$ for $x \in B$, $\bar{v}(x) = u(x)$ if $x \notin B$. Then, $\bar{v} \in \mathcal{K}$ so that

$$0 \leqslant \int_\Omega |\nabla \bar{v}|^p\,dx - \int_\Omega |\nabla u|^p\,dx + F_\varepsilon\big(\big|\{\bar{v} > 0\}\big|\big) - F_\varepsilon\big(\big|\{u > 0\}\big|\big) = \int_B |\nabla v|^p - |\nabla u|^p\,dx, \tag{2.1}$$

and (see [5, Section 3])

$$\int_B |\nabla v|^p - |\nabla u|^p \, dx \leqslant -c \int_B \big|\nabla(v-u)\big|^p \, dx \quad \text{if } p \geqslant 2, \tag{2.2}$$

$$\int_B |\nabla v|^p - |\nabla u|^p \, dx \leqslant -c \int_B \big|\nabla(v-u)\big|^2 \big(|\nabla v| + |\nabla u|\big)^{p-2} \, dx \quad \text{if } 1 < p \leqslant 2, \tag{2.3}$$

where c is a positive constant that depends on p. In any case, combining (2.1), (2.2) and (2.3) we get $|\nabla(v-u)| = 0$ in B. Thus, $u = v$ in B. So that, $\Delta_p u = 0$ in B.

We begin by discussing the existence of extremals.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded and $1 < p < \infty$. Then there exists a solution to the problem* (P_ε)*. Moreover, any such solution u_ε has the following properties*:

(1) *u_ε is locally Lipschitz continuous in Ω.*
(2) *For every $D \Subset \Omega$, there exist constants $C, c > 0$ such that for every $x \in D \cap \{u_\varepsilon > 0\}$,*

$$c \operatorname{dist}\big(x, \partial\{u_\varepsilon > 0\}\big) \leqslant u_\varepsilon(x) \leqslant C \operatorname{dist}\big(x, \partial\{u_\varepsilon > 0\}\big).$$

(3) *For every $D \Subset \Omega$, there exists a constant $c > 0$ such that for $x \in \partial\{u > 0\}$ and $B_r(x) \subset D$,*

$$c \leqslant \frac{|B_r(x) \cap \{u_\varepsilon > 0\}|}{|B_r(x)|} \leqslant 1 - c.$$

The constants may depend on ε.

Proof. The proof of existence is standard. We state it here for the reader's convenience.

Take u_0 with $|\{u_0 > 0\}| \leqslant \alpha$, then $\mathcal{J}_\varepsilon(u_0) \leqslant C$ (uniformly in ε), also $\mathcal{J}_\varepsilon \geqslant -\alpha$. Therefore a minimizing sequence $(u_n) \subset \mathcal{K}$ exists. Then $\mathcal{J}_\varepsilon(u_n)$ is bounded, so $\|\nabla u_n\|_p \leqslant C$. As $u_n = \varphi_0$ in $\partial\Omega$, there exists a subsequence (that we still call u_n) and a function $u_\varepsilon \in W^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u_\varepsilon \quad \text{weakly in } W^{1,p}(\Omega), \qquad u_n \to u_\varepsilon \quad \text{a.e. in } \Omega.$$

Thus,

$$u_\varepsilon = \varphi_0 \quad \text{on } \partial\Omega, \qquad \big|\{u_\varepsilon > 0\}\big| \leqslant \liminf_{n\to\infty} \big|\{u_n > 0\}\big| \quad \text{and}$$

$$\int_\Omega |\nabla u_\varepsilon|^p \, dx \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx.$$

Hence $u_\varepsilon \in \mathcal{K}$ and

$$\mathcal{J}_\varepsilon(u_\varepsilon) \leqslant \liminf_{n\to\infty} \mathcal{J}_\varepsilon(u_n) = \inf_{v \in \mathcal{K}} \mathcal{J}_\varepsilon(v),$$

therefore u_ε is a minimizer of $\mathcal{J}_\varepsilon$ in $\mathcal{K}$.

The proof of (1), (2) and (3) follow as Theorem 3.3, Lemma 4.2 and Theorem 4.4 in [5]. The only difference being that the functional they analyze is linear in $|\{u_\varepsilon > 0\}|$ and ours is piecewise linear. The different treatment of this term is similar to the one in [9]. □

From now on we denote by u instead of u_ε a solution to (P_ε).

Lemma 2.1. *Let $u \in \mathcal{K}$ be a solution to (P_ε). Then u satisfies for every $\varphi \in C_0^\infty(\Omega)$ such that* $\operatorname{supp}(\varphi) \subset \{u > 0\}$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi = 0. \tag{2.4}$$

Moreover, the application

$$\lambda(\varphi) := -\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx$$

from $C_0^\infty(\Omega)$ into $\mathbb{R}$ defines a nonnegative Radon measure with support on $\Omega \cap \partial\{u > 0\}$.

Proof. See [5, Theorem 5.1]. □

Theorem 2.2 *(Representation theorem). Let $u \in \mathcal{K}$ be a solution to* (P_ε). *Then,*

(1) $\mathcal{H}^{N-1}(D \cap \partial\{u > 0\}) < \infty$ *for every $D \Subset \Omega$.*
(2) *There exists a Borel function q_u such that*

$$\Delta_p u = q_u \mathcal{H}^{N-1} \lfloor \partial\{u > 0\}.$$

(3) *For $D \Subset \Omega$ there are constants $0 < c \leqslant C < \infty$ depending on N, Ω, D and ε such that for $B_r(x) \subset D$ and $x \in \partial\{u > 0\}$,*

$$c \leqslant q_u(x) \leqslant C, \qquad cr^{N-1} \leqslant \mathcal{H}^{N-1}\big(B_r(x) \cap \partial\{u > 0\}\big) \leqslant Cr^{N-1}.$$

(4) *For $\mathcal{H}^{N-1}$-a.e. $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$,*

$$u(x_0 + x) = q_u(x_0)\big(x \cdot \nu(x_0)\big)^- + o\big(|x|\big) \quad \text{for } x \to 0$$

with $\nu(x_0)$ the outward unit normal to $\partial\{u > 0\}$ in the measure theoretic sense.
(5) $\mathcal{H}^{N-1}(\partial\{u > 0\} \setminus \partial_{\mathrm{red}}\{u > 0\}) = 0$.

Proof. The proof of (1)–(3) follow exactly as that of [4, Theorem 4.5].

Observe that $D \cap \partial\{u > 0\}$ has finite perimeter, thus, the reduce boundary $\partial_{\mathrm{red}}\{u > 0\}$ is defined as well as the measure theoretic normal $\nu(x)$ for $x \in \partial_{\mathrm{red}}\{u > 0\}$ (see [8]). For the proof of (4) see [5, Theorem 5.5].

Finally, (5) is a consequence of Theorem 2.1 and part (3) (see [8]). □

Theorem 2.3. *Let $u \in \mathcal{K}$ be a solution to* (P_ε) *and* q_u *the function as in Theorem* 2.2. *Then there exists a constant* λ_u *such that*

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \left|\nabla u(x)\right| = \lambda_u \quad \text{for every } x_0 \in \Omega \cap \partial\{u>0\}, \tag{2.5}$$

$$q_u(x_0) = \lambda_u, \qquad \mathcal{H}^{N-1}\text{-a.e. } x_0 \in \Omega \cap \partial\{u>0\}. \tag{2.6}$$

Moreover, if B is a ball contained in $\{u=0\}$ touching the boundary $\partial\{u>0\}$ at x_0. Then

$$\limsup_{\substack{x\to x_0\\ u(x)>0}} \frac{u(x)}{\operatorname{dist}(x,B)} = \lambda_u. \tag{2.7}$$

To prove this theorem, we have to prove first the following lemma.

Lemma 2.2. *Let $x_0, x_1 \in \partial\{u>0\}$ and $\rho_k \to 0^+$. For $i=0,1$ let $x_{i,k} \to x_i$ with $u(x_{i,k})=0$ such that $B_{\rho_k}(x_{i,k}) \subset \Omega$ and such that the blow-up sequence*

$$u_{i,k}(x) = \frac{1}{\rho_k} u(x_{i,k} + \rho_k x)$$

has a limit $u_i(x) = \lambda_i (x\cdot\nu_i)^-$, with $0<\lambda_i<\infty$ and ν_i a unit vector. Then $\lambda_0 = \lambda_1$.

Proof. Assume that $\lambda_1 < \lambda_0$ then we will perturb the minimizer u near x_0 and x_1 and get an admissible function with less energy. To this end, we take a nonnegative C_0^∞ function ϕ supported in the unit interval. For k large, define

$$\tau_k(x) = \begin{cases} x + \rho_k^2 \phi\big(\frac{|x-x_{0,k}|}{\rho_k}\big)\nu_0 & \text{for } x \in B_{\rho_k}(x_{0,k}),\\ x - \rho_k^2 \phi\big(\frac{|x-x_{1,k}|}{\rho_k}\big)\nu_1 & \text{for } x \in B_{\rho_k}(x_{1,k}),\\ x & \text{elsewhere,} \end{cases}$$

which is a diffeomorphism if k is big enough. Now let

$$v_k(x) = u\big(\tau_k^{-1}(x)\big),$$

that are admissible functions. Let us also define

$$\eta_i(y) = (-1)^i \phi\big(|y|\big)\nu_i. \tag{2.8}$$

We have

$$F_\varepsilon\big(\left|\{v_k>0\}\right|\big) - F_\varepsilon\big(\left|\{u>0\}\right|\big) = o\big(\rho_k^{N+1}\big). \tag{2.9}$$

To estimate the other term in $\mathcal{J}_\varepsilon$ we make a change of variables and then

$$\rho_k^{-N} \int\limits_{B_{\rho_k}(x_i)} \big(|\nabla v_k|^p - |\nabla u_\varepsilon|^p\big)\,dx$$
$$= \int\limits_{B_1(0)\cap\{u_{i,k}>0\}} \rho_k\big[|\nabla u_{i,k}|^p \operatorname{div}(\eta_i) - p|\nabla u_{i,k}|^{p-2}(\nabla u_{i,k})^t D\eta_i \nabla u_{i,k}\big] + o(\rho_k)\,dy.$$

On the other hand, by Lemma B.1, we have

$$B_1(0)\cap\{u_{i,k}>0\} \to B_1(0)\cap\{y\cdot\nu_i<0\}, \quad \text{as } \rho\to 0, \quad \text{and}$$
$$\nabla u_{i,k} \to \nabla u_i = -\lambda_i \nu_i \chi_{\{y\cdot\nu_i<0\}} \quad \text{a.e. in } B_1(0).$$

Therefore

$$\rho_k^{-N-1} \int\limits_{B_{\rho_k}(x_i)} \big(|\nabla v_k|^p - |\nabla u_\varepsilon|^p\big)\,dx \to \int\limits_{B_1(0)\cap\{y\cdot\nu_i>0\}} \lambda_i^p\big(\operatorname{div}(\eta_i) - p\nu_i^t D\eta_i \nu_i\big)\,dy.$$

Using that

$$\operatorname{div}(\eta_i) - p\nu_i^t D\eta_i\nu_i = (-1)^i(1-p)\frac{\phi'(|y|)}{|y|}(y\cdot\nu_i) = (-1)^i(1-p)\operatorname{div}(\eta_i),$$

we obtain

$$\rho_k^{-N-1} \int\limits_{B_{\rho_k}(x_i)} \big(|\nabla v_k|^p - |\nabla u_\varepsilon|^p\big)\,dx \to (-1)^i(1-p)\lambda_i^p \int\limits_{B_1(0)\cap\{y\cdot\nu_i=0\}} \phi\big(|y|\big)\,d\mathcal{H}^{N-1}(y).$$

Hence

$$\int\limits_\Omega |\nabla v_k|^p\,dx - \int\limits_\Omega |\nabla u_\varepsilon|^p\,dx$$
$$= \rho_k^{N+1}\big(\lambda_1^p - \lambda_0^p\big) \int\limits_{B_1(0)\cap\{y_1=0\}} (p-1)\phi\big(|y|\big)\,d\mathcal{H}^{N-1}(y) + o\big(\rho_k^{N+1}\big). \tag{2.10}$$

If we take k large enough we get

$$\mathcal{J}_\varepsilon(v_k) < \mathcal{J}_\varepsilon(u),$$

a contradiction. □

Proof of Theorem 2.3. Now, the theorem follows as in steps 2 and 3 of Theorem 5.1 in [13], using [5, Lemma 5.4], Theorem A.1 and properties (1)–(8) of Lemma B.1. We sketch the proof here for the reader's convenience.

Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let

$$\lambda = \lambda(x_0) := \limsup_{\substack{x \to x_0 \\ u(x)>0}} \left|\nabla u(x)\right|.$$

Then there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0, \qquad \left|\nabla u(z_k)\right| \to \lambda.$$

Let y_k be the nearest point to z_k on $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider the blow-up sequence with respect to $B_{d_k}(y_k)$ with limit u_0, such that there exists

$$\nu := \lim_{k\to\infty} \frac{y_k - z_k}{d_k},$$

and suppose that $\nu = e_N$. Using the results of Appendix B, we can proceed as in [5, p. 13] to prove that $0 < \lambda < \infty$ and

$$u_0(x) = -\lambda x_N \quad \text{in } \{x_N \leqslant 0\}.$$

Finally, by Lemma B.1(8) we have that $0 \in \partial\{u_0 > 0\}$ and then, using Lemma B.1(6) we see that u_0 satisfies the hypotheses of Theorem A.1. Therefore $u_0 = 0$ in $\{x_N > 0\}$. Then $u_0 = \lambda \max(-x \cdot \nu, 0)$.

To complete the proof, we follow the lines in step 3 of Theorem 5.1 in [13]. That is, we apply Lemma 2.2 to this blow up sequence and to a blow up sequence centered at a regular point of the free boundary.

A similar argument proves (2.7). □

Summing up, we have the following theorem.

Theorem 2.4. *Let $u \in \mathcal{K}$ be a solution to* (P_ε). *Then u is a weak solution to the following free boundary problem*:

$$\Delta_p u = 0 \quad \text{in } \{u > 0\} \cap \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda_u \quad \text{on } \partial\{u > 0\} \cap \Omega,$$

where λ_u is the constant in Theorem 2.3. *More precisely, $\mathcal{H}^{N-1}$-a.e. point $x_0 \in \partial\{u > 0\}$ belongs to $\partial_{\mathrm{red}}\{u > 0\}$ and*

$$u(x_0 + x) = \lambda_u\big(x \cdot \nu(x_0)\big)^- + o\big(|x|\big) \quad \text{for } x \to 0.$$

Finally, we get an estimate of the gradient of u that will be needed in order to get the regularity of the free boundary.

Theorem 2.5. *Let $u \in \mathcal{K}$ be a solution to* (P_ε). *Given $D \Subset \Omega$, there exist constants $C = C(N, \varepsilon, D)$, $r_0 = r_0(N, D) > 0$ and $\gamma = \gamma(N, \varepsilon, D) > 0$ such that, if $x_0 \in D \cap \partial\{u > 0\}$ and $r < r_0$, then*

$$\sup_{B_r(x_0)} |\nabla u| \leqslant \lambda_u\big(1 + C r^\gamma\big).$$

Proof. The proof follows the lines of the proof of [5, Theorem 7.1]. □

As a corollary we have the following regularity result for the free boundary $\partial\{u > 0\}$.

Corollary 2.1. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to* (P_ε). *Then $\partial_{\mathrm{red}}\{u_\varepsilon > 0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has $\mathcal{H}^{N-1}$-measure zero.*

Proof. See [5, Corollary 9.2]. □

Remark 2.1. In dimension 2, Danielli and Petrosyan (see [6]) proved the full regularity of the free boundary of the minimizers of (1.2) if $2 - \delta \leqslant p < \infty$ for a small $\delta > 0$. Also, a similar result was proved by Petrosyan in dimension 3 for p close to 2 (see [15]).

3. Behavior of the minimizer for small ε

In this section, since we want to analyze the dependence of the problem with respect to ε we will again denote by u_ε a solution to problem (P_ε).

To complete the analysis of the problem, we will now show that if ε is small enough, then

$$\big|\{u_\varepsilon > 0\}\big| = \alpha.$$

To this end, we need to prove that the constant $\lambda_\varepsilon := \lambda_{u_\varepsilon}$ is bounded from above and below by positive constants independent of ε. We perform this task in a series of lemmas.

Lemma 3.1. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to* (P_ε). *Then, there exists a constant $C > 0$ independent of ε such that*

$$\lambda_\varepsilon := \lambda_{u_\varepsilon} \leqslant C.$$

Proof. The proof is similar to the one in [2, Theorem 3].

First we will prove that there exist $C, c > 0$, independent of ε, such that

$$c \leqslant \big|\{u_\varepsilon > 0\}\big| \leqslant C\varepsilon + \alpha.$$

In fact, as in Theorem 2.1 we have that $F_\varepsilon(|\{u_\varepsilon > 0\}|) \leqslant C$ thus obtaining the bound from above. On the other hand, taking $q < p$, using the Sobolev trace theorem, the Hölder inequality and the fact that $\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leqslant C$ (see Theorem 2.1) we have

$$\int\limits_{\partial\Omega} \varphi_0^q \, d\mathcal{H}^{N-1} \leqslant C\big|\{u_\varepsilon > 0\}\big|^{\frac{p-q}{p}} \|u\|_{W^{1,p}(\Omega)}^q \leqslant C\big|\{u_\varepsilon > 0\}\big|^{\frac{p-q}{p}},$$

and thus we obtain the bound from below.

Take $D \Subset \Omega$ smooth, such that $\theta = |D| > \alpha$ and $|\Omega \setminus D| < c$ then,

$$\big|D \cap \{u_\varepsilon > 0\}\big| \leqslant \alpha + C\varepsilon < \theta$$

for ε small enough. On the other hand,

$$\big|D \cap \{u_\varepsilon > 0\}\big| \geqslant \big|\{u_\varepsilon > 0\}\big| - |\Omega \setminus D| \geqslant c - |\Omega \setminus D| > 0.$$

Therefore by the relative isoperimetric inequality we have

$$\mathcal{H}^{N-1}\big(D \cap \partial\{u_\varepsilon > 0\}\big) \geqslant c \min\big\{\big|D \cap \{u_\varepsilon > 0\}\big|, \big|D \cap \{u_\varepsilon = 0\}\big|\big\}^{\frac{N-1}{N}} \geqslant c > 0.$$

Now let w be the p-harmonic function in Ω with boundary data equal to φ_0. Using Theorems 2.2 and 2.3 we have,

$$C \geqslant \int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(u_\varepsilon - w)\, dx = \int_{\Omega \cap \partial\{u_\varepsilon > 0\}} w\lambda_\varepsilon \, d\mathcal{H}^{N-1} \geqslant \int_{D \cap \partial\{u_\varepsilon > 0\}} w\lambda_\varepsilon \, d\mathcal{H}^{N-1}$$
$$\geqslant \lambda_\varepsilon \Big(\inf_D w\Big)\mathcal{H}^{N-1}\big(D \cap \partial\{u_\varepsilon > 0\}\big) \geqslant c\lambda_\varepsilon.$$

Now the result follows. □

Lemma 3.2. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to (P_ε), $B_r \Subset \Omega$ and v a solution to*

$$\Delta_p v = 0 \quad \text{in } B_r, \qquad v = u_\varepsilon \quad \text{on } \partial B_r.$$

Then

$$\int_{B_r} \big|\nabla(u_\varepsilon - v)\big|^q \, dx \geqslant C\big|B_r \cap \{u_\varepsilon = 0\}\big| \left(\frac{1}{r}\left(\fint_{B_r} u_\varepsilon^\gamma \, dx\right)^{1/\gamma}\right)^q$$

for all $q \geqslant 1$ and for any $\gamma < \frac{N(p-1)}{N-p}$ if $p \leqslant N$, $\gamma < \infty$ if $p > N$, and C is a constant independent of ε.

Proof. The idea of the proof is similar to [4, Lemma 3.2]. We include the details since there are differences due to the fact that we are dealing with the p-Laplacian instead of the Laplacian.

First let us assume that $B_r = B_1(0)$. For $|z| \leqslant \frac{1}{2}$ we consider the change of variables from B_1 into itself such that z becomes the new origin. We call $u_z(x) = u((1 - |x|)z + x)$, $v_z(x) = v((1 - |x|)z + x)$ and define

$$r_\xi = \inf\left\{r\colon\ \frac{1}{8} \leqslant r \leqslant 1 \text{ and } u_z(r\xi) = 0\right\},$$

if this set is nonempty. Observe that this change of variables leaves the boundary fixed.

Now, for almost every $\xi \in \partial B_1$ we have

$$v_z(r_\xi \xi) = \int_{r_\xi}^1 \frac{d}{dr}(u_z - v_z)(r\xi)\, dr \leqslant (1 - r_\xi)^{1/q'}\left(\int_{r_\xi}^1 \big|\nabla(u_z - v_z)(r\xi)\big|^q \, dr\right)^{1/q}. \tag{3.1}$$

Let us assume that the following inequality holds

$$v_z(r_\xi \xi) \geqslant C(N,\Omega)(1-r_\xi)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}. \tag{3.2}$$

Then, using (3.1) and (3.2), integrating first over ∂B_1 and then over $|z| \leqslant 1/2$ we obtain as in [4],

$$\int_{B_1} \left|\nabla(u-v)\right|^q\,dx \geqslant C\left|B_1 \cap \{u=0\}\right|\left(\fint_{B_1} u^\gamma\,dx\right)^{q/\gamma}.$$

If we take $u_r(x) = \frac{1}{r}u(x_0+rx)$ (where x_0 is the center of the ball B_r) then

$$\int_{B_1} \left|\nabla(u_r-v_r)\right|^q\,dx = r^{-N}\int_{B_r} \left|\nabla(u-v)\right|^q\,dy,$$

$$\left|B_1 \cap \{u_r=0\}\right| = r^{-N}\left|B_r \cap \{u=0\}\right| \quad \text{and} \quad \left(\fint_{B_1} u_r^\gamma\,dx\right)^{1/\gamma} = \frac{1}{r}\left(\fint_{B_r} u^\gamma\,dy\right)^{1/\gamma},$$

so we have the desired result.

Therefore we only have to prove (3.2). If $|(1-r_\xi)z + r_\xi\xi| \leqslant \frac{3}{4}$, by Harnack inequality,

$$v_z(r_\xi\xi) \geqslant C_N v(0).$$

By [17, Theorem 1.2] we have

$$v(0) \geqslant \alpha(N,\Omega)\left(\fint_{B_1} v^\gamma\,dx\right)^{1/\gamma} \geqslant \alpha(N,\Omega)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma}. \tag{3.3}$$

If $|(1-r_\xi)z + r_\xi\xi| \geqslant \frac{3}{4}$ we prove by a comparison argument that inequality (3.2) also holds. In fact, again by [17, Theorem 1.2],

$$v \geqslant C_N\alpha\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma} \quad \text{in } B_{3/4}.$$

Let $w(x) = e^{-\lambda|x|^2} - e^{-\lambda}$. There exists $\lambda = \lambda(N,\alpha)$ such that

$$\begin{cases} \Delta_p w \geqslant 0 & \text{in } B_1 \setminus B_{3/4}, \\ w \leqslant C_N\alpha & \text{on } \partial B_{3/4}, \\ w = 0 & \text{on } \partial B_1, \end{cases}$$

so that,

$$v \geqslant w\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma} \geqslant C\big(1-|x|\big)\left(\fint_{B_1} u^\gamma\,dx\right)^{1/\gamma} \quad \text{in } B_1 \setminus B_{3/4}.$$

Therefore

$$v_z(r_\xi \xi) \geqslant C\big(1 - \big|(1-r_\xi)z + r_\xi \xi\big|\big)\bigg(\fint_{B_1} u^\gamma\,dx\bigg)^{1/\gamma} \geqslant C(1-r_\xi)\bigg(\fint_{B_1} u^\gamma\,dx\bigg)^{1/\gamma}$$

since $|z| \leqslant \frac{1}{2}$. So that (3.2) holds for every $r_\xi \geqslant \frac{1}{8}$.

This completes the proof. □

Lemma 3.3. *Let $u_\varepsilon \in \mathcal{K}$ be a solution to* (P_ε)*, then*

$$\lambda_\varepsilon \geqslant c > 0,$$

where c is independent of ε.

Proof. We proceed as in [2, Lemma 6]. We will use the following fact that we prove in Lemma 3.4 bellow: for every $\varepsilon > 0$ there is a neighborhood of A in Ω where $u_\varepsilon > 0$.

Let $y_0 \in A$ and let D_t with $0 \leqslant t \leqslant 1$ be a family of open sets with smooth boundary and uniformly (in ε and t) bounded curvatures such that D_0 is an exterior tangent ball at y_0, D_1 contains a free boundary point, $D_0 \Subset D_t$ for $t > 0$ and $D_t \cap \partial\Omega \subset A$.

Let $t \in (0,1]$ be the first time such that D_t touches the free boundary and let $x_0 \in \partial D_t \cap \partial\{u_\varepsilon > 0\} \cap \Omega$. Now, take w such that $\Delta_p w = 0$ in $D_t \setminus \overline{D}_0$ with $w = c_0$ on ∂D_0 and $w = 0$ on ∂D_t. Thus $w \leqslant u_\varepsilon$ in $D_t \cap \Omega$ and $\partial_{-\nu} w(x_0) \geqslant c c_0$ with $c > 0$ independent of ε. Therefore, for r small enough,

$$\bigg(\fint_{B_r(x_0)} u_\varepsilon^\gamma\,dx\bigg)^{1/\gamma} \geqslant \bigg(\fint_{B_r(x_0)} w^\gamma\,dx\bigg)^{1/\gamma} \geqslant r\bar{c}c_0, \tag{3.4}$$

with $\bar{c}$ is independent of ε.

If v_0 is the solution to

$$\begin{cases} \Delta_p v_0 = 0 & \text{in } B_r(x_0), \\ v_0 = u_\varepsilon & \text{on } \partial B_r(x_0), \end{cases}$$

then, by Lemma 3.2, we have

$$\int_{B_r} \big|\nabla(u_\varepsilon - v_0)\big|^p\,dx \geqslant C\big|B_r \cap \{u_\varepsilon = 0\}\big|\bigg(\frac{1}{r}\bigg(\fint_{B_r} u_\varepsilon^\gamma\,dx\bigg)^{1/\gamma}\bigg)^p \quad \text{for } p \geqslant 2,$$

$$\int_{B_r} \big|\nabla(u_\varepsilon - v_0)\big|^2\,dx \geqslant C\big|B_r \cap \{u_\varepsilon = 0\}\big|\bigg(\frac{1}{r}\bigg(\fint_{B_r} u_\varepsilon^\gamma\,dx\bigg)^{1/\gamma}\bigg)^2 \quad \text{for } 1 < p \leqslant 2.$$

Then using (2.2) we obtain,

$$\int_{B_r} \big(|\nabla u_\varepsilon|^p - |\nabla v_0|^p\big)\,dx \geqslant C\big|B_r \cap \{u_\varepsilon = 0\}\big|\bigg(\frac{1}{r}\bigg(\fint_{B_r} u_\varepsilon^\gamma\,dx\bigg)^{1/\gamma}\bigg)^p \tag{3.5}$$

for $p \geqslant 2$.

By Theorem 2.3 and Lemma 3.1 we have that, near x_0, $|\nabla u_\varepsilon|$ is bounded from above by a constant independent of ε. Then by (2.3) we obtain that (3.5) also holds for $1 < p \leqslant 2$ if r is small enough (depending on ε). Then by (3.4)

$$\int\limits_{B_r(x_0)} \left(|\nabla u_\varepsilon|^p - |\nabla v_0|^p\right) dx \geqslant c\delta_r, \tag{3.6}$$

where $\delta_r = |B_r(x_0) \cap \{u_\varepsilon = 0\}|$ and c is a constant independent of ε.

Consider now a free boundary point x_1 away from x_0. We can choose x_1 to be regular.

Let us take

$$\tau_\rho(x) = \begin{cases} x - \rho^2 \phi\left(\frac{|x-x_1|}{\rho}\right) \nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1), \\ x & \text{elsewhere,} \end{cases}$$

where $\phi \in C_0^\infty(-1,1)$ with $\phi'(0) = 0$.

Now choose ρ such that

$$\delta_r = \rho^2 \int\limits_{B_\rho(x_1) \cap \partial\{u_\varepsilon > 0\}} \phi\left(\frac{|x - x_1|}{\rho}\right) d\mathcal{H}^{N-1}.$$

Take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ and

$$v = \begin{cases} v_0 & \text{in } B_r(x_0), \\ v_\rho & \text{in } B_\rho(x_1), \\ u_\varepsilon & \text{elsewhere.} \end{cases}$$

Thus, we have that

$$\left|\{v > 0\}\right| = \left|\{u_\varepsilon > 0\}\right|. \tag{3.7}$$

On the other hand as in Lemma 2.2, we have

$$\begin{aligned} &\int\limits_{B_\rho(x_1)} \left(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\right) dy \\ &\quad = \int\limits_{\tau_\rho(B_\rho(x_1)) \cap \{v_\rho > 0\}} |\nabla v_\rho|^p \, dy - \int\limits_{B_\rho(x_1)} |\nabla u|^p \, dx \\ &\quad = \int\limits_{B_\rho(x_1) \cap \{u > 0\}} \rho\left(|\nabla u_\varepsilon|^p \operatorname{div} \eta - p\,|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon D\eta \nabla u_\varepsilon\right) + o(\rho)\, dx, \end{aligned}$$

where $\eta(y) = -\phi(|y|)\nu(x_1)$. Using the fact that η is bounded from above by a constant k independent of ρ and ε, and that $|\nabla u_\varepsilon| = \lambda_\varepsilon + O(\rho^2)$ in $B_\rho(x_1)$ we have

$$\int\limits_{B_\rho(x_1)} \left(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\right) dy \leqslant k\lambda_\varepsilon^p \rho^{N+1} + o(\rho)\rho^N,$$

but δ_r has the same order of ρ^{N+1} then

$$\int_{B_\rho(x_1)} \left(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\right) dy \leqslant k\lambda_\varepsilon^p \delta_r + o(\delta_r). \tag{3.8}$$

Therefore by (3.6), (3.8) and (3.7) we have

$$0 \leqslant \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) \leqslant -c\delta_r + k\lambda_\varepsilon^p \delta_r + o(\delta_r)$$

and then $\lambda_\varepsilon \geqslant c > 0$. □

Now we prove the positivity result that was used in the previous lemma.

Lemma 3.4. *For every $\varepsilon > 0$ there exists a neighborhood of A in Ω such that $u_\varepsilon > 0$ in this neighborhood.*

Proof. Let $y_0 \in A$ and let $B_\delta(z_0)$ be an exterior tangent ball to $\partial\Omega$ at y_0 such that $\overline{\Omega} \cap \overline{B} = \{y_0\}$. Let us take δ small enough so that $B_{2\delta}(z_0) \cap \partial\Omega \Subset A$. Let w_ε be a minimizer of

$$\widetilde{J}_\varepsilon(w) := \int_{\mathcal{R}} |\nabla w|^p \, dx + \frac{1}{\varepsilon}\left|\{w > 0\} \cap \mathcal{R}\right| \tag{3.9}$$

in $\{w \in W^{1,p}(\mathcal{R}),\ w = 0 \text{ on } \partial B_{2\delta}(z_0),\ w = c_0 \text{ on } \partial B_\delta(z_0)\}$. Here $\mathcal{R} = B_{2\delta}(z_0) \setminus \overline{B}_\delta(z_0)$.

Every minimizer of (3.9) is radially symmetric and radially decreasing with respect to z_0. This is seen by using Schwartz symmetrization after extending w_ε to $B_\delta(z_0)$ as the constant function c_0 (see [12]). This symmetrization preserves the distribution function and strictly decreases the L^p norm of the gradient unless the function is already radially symmetric and radially decreasing. Moreover, these minimizers are ordered and their supports are nested. Let us take as w_ε the smallest minimizer.

By the properties of w_ε there holds that w_ε is strictly positive in a ring around $B_\delta(z_0)$. Also w_ε is continuous in $\mathcal{R}$. Recall that u_ε is continuous in Ω. Let us see that $u_\varepsilon \geqslant w_\varepsilon$ in $\mathcal{R} \cap \Omega$. This will prove the statement.

Assume instead that $\{w_\varepsilon > u_\varepsilon\} \neq \emptyset$.

Let us first consider the function $v = \min\{u_\varepsilon, w_\varepsilon\}$ in $\mathcal{R} \cap \Omega$. Since $u_\varepsilon \geqslant c_0 \geqslant w_\varepsilon$ on $\partial\Omega \cap \mathcal{R}$ and $u_\varepsilon \geqslant 0 = w_\varepsilon$ on $\Omega \cap \partial\mathcal{R}$ there holds that $v = w_\varepsilon$ on $\partial(\mathcal{R} \cap \Omega)$. Therefore, the function $\underline{v} = v$ in $\mathcal{R} \cap \Omega$, $\underline{v} = w_\varepsilon$ in $\mathcal{R} \setminus \Omega$ is an admissible function for the minimization problem (3.9). Since w_ε is the smallest minimizer and, by assumption $\underline{v} \leqslant w_\varepsilon$ and $\underline{v} \neq w_\varepsilon$, there holds that $\widetilde{J}_\varepsilon(\underline{v}) > \widetilde{J}_\varepsilon(w_\varepsilon)$. Since $\underline{v} = w_\varepsilon$ in $\mathcal{R} \setminus \Omega$ and in $\mathcal{R} \cap \Omega \cap \{w_\varepsilon \leqslant u_\varepsilon\}$ and equal to u_ε outside those sets there holds that (with $\mathcal{D} = \mathcal{R} \cap \Omega \cap \{w_\varepsilon > u_\varepsilon\}$)

$$\int_{\mathcal{D}} |\nabla u_\varepsilon|^p \, dx + \frac{1}{\varepsilon}\left|\{u_\varepsilon > 0\} \cap \mathcal{D}\right| > \int_{\mathcal{D}} |\nabla w_\varepsilon|^p \, dx + \frac{1}{\varepsilon}\left|\{w_\varepsilon > 0\} \cap \mathcal{D}\right|. \tag{3.10}$$

Let now $\bar{v} = \max\{u_\varepsilon, w_\varepsilon\}$ in $\mathcal{R} \cap \Omega$, $\bar{v} = u_\varepsilon$ in $\Omega \setminus \mathcal{R}$. This function is admissible for (P_ε) so that

$$\int_\Omega |\nabla \bar{v}|^p\,dx + F_\varepsilon\big(\big|\{\bar{v} > 0\}\big|\big) \geqslant \int_\Omega |\nabla u_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big).$$

Since $\bar{v} = w_\varepsilon$ in $\mathcal{D}$ and $\bar{v} = u_\varepsilon$ in $\Omega \setminus \mathcal{D}$,

$$\begin{aligned}&\int_{\mathcal{D}} |\nabla w_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big| + \big|\{w_\varepsilon > 0\} \cap \mathcal{D}\big| - \big|\{u_\varepsilon > 0\} \cap \mathcal{D}\big|\big)\\ &\quad \geqslant \int_{\mathcal{D}} |\nabla u_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big).\end{aligned} \tag{3.11}$$

By (3.10) and (3.11) (with $C_w = |\{w_\varepsilon > 0\} \cap \mathcal{D}|$ and $C_u = |\{u_\varepsilon > 0\} \cap \mathcal{D}|$) we have,

$$\begin{aligned}\int_{\mathcal{D}} |\nabla u_\varepsilon|^p\,dx &> \int_{\mathcal{D}} |\nabla w_\varepsilon|^p\,dx + \frac{1}{\varepsilon}(C_w - C_u)\\ &\geqslant \int_{\mathcal{D}} |\nabla u_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big) - F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big| + C_w - C_u\big) + \frac{1}{\varepsilon}(C_w - C_u).\end{aligned}$$

Thus,

$$F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big| + C_w - C_u\big) - F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big) > \frac{1}{\varepsilon}(C_w - C_u)$$

which is a contradiction since $F_\varepsilon(A) - F_\varepsilon(B) \leqslant \frac{1}{\varepsilon}(A - B)$ if $A \geqslant B$ and $C_w \geqslant C_u$ by assumption.

Therefore, $u_\varepsilon \geqslant w_\varepsilon$ in $\mathcal{R} \cap \Omega$ and the lemma is proved. □

With these uniform bounds on λ_ε, we can prove the desired result.

Theorem 3.1. *There exists $\varepsilon_0 > 0$ such that if $u_\varepsilon \in \mathcal{K}$ is a solution to (P_ε) and $\varepsilon < \varepsilon_0$ there holds that $|\{u_\varepsilon > 0\}| = \alpha$. Therefore, u_ε is a minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$.*

Proof. Arguing by contradiction, assume first that $|\{u_\varepsilon > 0\}| > \alpha$. Let $x_1 \in \partial\{u_\varepsilon > 0\} \cap \Omega$ be a regular point. We will proceed as in the proof of Lemma 3.3. Given $\delta > 0$, we perturb the domain $\{u_\varepsilon > 0\}$ in a neighborhood of x_1, decreasing its measure by δ. We choose δ small so that the measure of the perturbed set is still larger than α. Take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$, and let

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1),\\ u_\varepsilon & \text{elsewhere,}\end{cases}$$

where τ_ρ is the function that we have considered in the previous lemma.

We have

$$0 \leqslant \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) = \int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla u_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{v > 0\}\big|\big) - F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big)$$

$$\leqslant k\lambda_\varepsilon^p\delta + o_\varepsilon(\delta) - \frac{1}{\varepsilon}\delta \leqslant \left(kC^p - \frac{1}{\varepsilon}\right)\delta + o_\varepsilon(\delta) < 0,$$

if $\varepsilon < \varepsilon_0$ and then $\delta < \delta_0(\varepsilon)$. A contradiction.

Now assume that $|\{u_\varepsilon > 0\}| < \alpha$. This case, is a little bit different from the other. First, we proceed as in the previous case but this time we perturb in a neighborhood of x_1 the set $\{u_\varepsilon > 0\}$ increasing its measure by δ. That is, take

$$\tau_\rho(x) = \begin{cases} x + \rho^2\phi\big(\frac{|x-x_1|}{\rho}\big)\nu_{u_\varepsilon}(x_1) & \text{for } x \in B_\rho(x_1),\\ x & \text{elsewhere,}\end{cases}$$

where $\phi \in C_0^\infty$ supported in the unit interval, take $v_\rho(\tau_\rho(x)) = u_\varepsilon(x)$ and

$$v = \begin{cases} v_\rho & \text{in } B_\rho(x_1),\\ u_\varepsilon & \text{elsewhere.}\end{cases}$$

For ρ small enough we have $|\{v > 0\}| < \alpha$ and

$$\big|\{v > 0\}\big| - \big|\{u_\varepsilon > 0\}\big| = C\rho^{N+1} + o\big(\rho^{N+1}\big),$$

therefore

$$F_\varepsilon\big(\big|\{v > 0\}\big|\big) - F_\varepsilon\big(\big|\{u_\varepsilon > 0\}\big|\big) \leqslant C\varepsilon\rho^{N+1} + o_\varepsilon\big(\rho^{N+1}\big). \tag{3.12}$$

In order to estimate the other term, we will make use of a blow up argument as in Lemma 2.2. In fact, we take $u_\rho(y) = \frac{1}{\rho}u(x_1 + \rho y)$ and we change variables to obtain

$$\rho^{-N}\int_{B_\rho(x_1)} \big(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\big)\,dx$$
$$= \int_{B_1(0)\cap\{u_\rho>0\}} \rho\big[|\nabla u_\rho|^p \operatorname{div}(\eta) - p|\nabla u_\rho|^{p-2}(\nabla u_\rho)^t D\eta\nabla u_\rho\big] + o(\rho)\,dy,$$

where $\eta(y) = \phi(|y|)\nu(x_1)$. Now, as in Lemma 2.2, we get

$$\rho^{-N-1}\int_{B_\rho(x_1)} \big(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\big)\,dx \to (1-p)\lambda_\varepsilon^p \int_{B_1(0)\cap\{y\cdot\nu=0\}} \phi\big(|y|\big)\,d\mathcal{H}^{N-1}(y).$$

Therefore

$$\int_{B_\rho(x_1)} \big(|\nabla v_\rho|^p - |\nabla u_\varepsilon|^p\big)\,dx = C\rho^{N+1}(1-p)\lambda_\varepsilon^p + o\big(\rho^{N+1}\big). \tag{3.13}$$

Finally, combining (3.12) and (3.13) we have

$$
\begin{aligned}
0 \leqslant \mathcal{J}_\varepsilon(v) - \mathcal{J}_\varepsilon(u_\varepsilon) &= \int_\Omega |\nabla v|^p\,dx - \int_\Omega |\nabla u_\varepsilon|^p\,dx + F_\varepsilon\big(\big|\{v>0\}\big|\big) - F_\varepsilon\big(\big|\{u_\varepsilon>0\}\big|\big) \\
&\leqslant C(1-p)\lambda_\varepsilon^p\delta + o_\varepsilon(\delta) + C\varepsilon\delta \leqslant C\big(-c^p+\varepsilon\big)\delta + o_\varepsilon(\delta) < 0,
\end{aligned}
$$

if $\varepsilon < \varepsilon_1$ and then $\delta < \delta_0(\varepsilon)$. Again a contradiction that ends the proof. □

As a corollary, we have the desired result for our problem.

Corollary 3.1. *For ε small any minimizer u of $\mathcal{J}$ in $\mathcal{K}_\alpha$ is a locally Lipschitz continuous function and $\partial_{\mathrm{red}}\{u>0\}$ is a $C^{1,\beta}$ surface locally in Ω and the remainder of the free boundary has $\mathcal{H}^{N-1}$-measure zero.*

Proof. If u is minimizer of $\mathcal{J}$ in $\mathcal{K}_\alpha$, by Theorem 3.1 we have that for small ε there exists a solution u_ε to (P_ε) such that $|\{u_\varepsilon>0\}| = \alpha$, then u is a solution to (P_ε), therefore the result follows. □

Acknowledgment

The authors thank Professor Arshak Petrosyan for providing the proof of Lemma 3.4.
While this paper was under review we found out that similar results where obtained in [14].

Appendix A. A result on p-harmonic functions with linear growth

In this section we will prove some properties of p-subharmonic functions. From now on, we note $B_r^+ = B_r(0)\cap\{x_N>0\}$.

Theorem A.1. *Let u be a Lipschitz function in $\mathbb{R}^N$ such that*

(1) $u \geqslant 0$ *in* $\mathbb{R}^N$, $\Delta_p u = 0$ *in* $\{u>0\}$.
(2) $\{x_N<0\}\subset\{u>0\}$, $u=0$ *in* $\{x_N=0\}$.
(3) *There exists* $0<\lambda_0<1$ *such that* $\frac{|\{u=0\}\cap B_R(0)|}{|B_R(0)|} > \lambda_0$, $\forall R>0$.

Then $u=0$ in $\{x_N>0\}$.

In order to prove this theorem we follow ideas from [13]. To this end, we need to prove a couple of lemmas.

Lemma A.1. *Let u be a p-subharmonic function in B_r^+ such that, $0\leqslant u\leqslant \alpha x_N$ in B_r^+, $u\leqslant \delta_0\alpha x_N$ on $\partial B_r^+\cap B_{r_0}(\bar{x})$ with $\bar{x}\in\partial B_r^+$, $\bar{x}_N>0$ and $0<\delta_0<1$.*

Then there exist $0<\gamma<1$ and $0<\varepsilon\leqslant 1$, depending only on r and N, such that $u(x)\leqslant\gamma\alpha x_N$ in B_ε^+.

Proof. By homogeneity of the p-Laplacian we can suppose that $r=1$.

Let ψ be a p-harmonic function in B_1^+, with smooth boundary data, such that

$$\begin{cases} \psi = x_N & \text{on } \partial B_1^+ \setminus B_{r_0}(\bar{x}), \\ \delta_0 x_N \leqslant \psi \leqslant x_N & \text{on } \partial B_1^+ \cap B_{r_0}(\bar{x}), \\ \psi = \delta_0 x_N & \text{on } \partial B_1^+ \cap B_{r_0/2}(\bar{x}). \end{cases}$$

Therefore, by comparison $u \leqslant \alpha\psi$ in B_1^+. Let us see that there exist $0 < \gamma < 1$ and $\varepsilon > 0$, independent of α, such that $\psi \leqslant \gamma x_N$ in B_ε^+.

First, $\psi \in C^{1,\beta}(\overline{B_1^+})$ for some $\beta > 0$. Then (cf. [11]), ψ is a viscosity solution of

$$|\nabla\psi|^{p-4}\left\{|\nabla\psi|^2\Delta\psi + (p-2)\sum_{i,j=1}^{N}\psi_{x_i}\psi_{x_j}\psi_{x_i x_j}\right\} = 0.$$

If $|\nabla\psi| \geqslant \mu > 0$ in some open set U, we have that ψ is a solution of the linear uniformly elliptic equation

$$\sum_{i,j=1}^{N} a_{ij}\psi_{x_i x_j} = 0 \quad \text{in } U, \tag{A.1}$$

where

$$\min\{1, p-1\}|\nabla\psi|^2|\xi|^2 \leqslant \sum_{i,j}^{N} a_{ij}\xi_i\xi_j \leqslant \max\{1, p-1\}|\nabla\psi|^2|\xi|^2.$$

Therefore, $\psi \in C^{2,\beta}(U)$ and is a classic solution of (A.1).

Let $w = x_N - \psi$ then $w \in C^{1,\beta}(\overline{B_1^+})$ and is a solution of

$$\mathcal{L}w = \sum_{i,j=1}^{N} a_{ij} w_{x_i x_j} = 0$$

in any open set U, where $|\nabla\psi| \geqslant \mu > 0$.

On the other hand, as $\psi \leqslant x_N$ in ∂B_1^+ and both functions are solutions of the p-Laplacian we have, by comparison, that $\psi \leqslant x_N$ in B_1^+. Therefore $w \geqslant 0$ in B_1^+.

Moreover, we have $w > 0$ in B_1^+. In fact, suppose that there exists x_0 such that $\psi(x_0) = x_{0,N}$. As $\psi \leqslant x_N$, we have that $\nabla\psi(x_0) = e_N$. Then $|\nabla\psi(x_0)| = 1$ and by continuity, $|\nabla\psi| \geqslant \frac{1}{2} > 0$ in a neighborhood U of x_0. Therefore $w \geqslant 0$, $w(x_0) = 0$ and $\mathcal{L}w = 0$ in U with $\mathcal{L}$ uniformly elliptic in U. Then by the strong maximum principle, $w \equiv 0$ in U.

So, we have that the set

$$\mathcal{A} = \left\{x \in B_1^+ \colon \psi(x) = x_N\right\},$$

is a relative open and close subset of B_1^+. Then if there exists x_0 such that $\psi(x_0) = x_{0,N}$, we have that $\psi \equiv x_N$. Since this is not the case in some part of ∂B_1^+, we arrive at a contradiction. Therefore $\psi < x_N$, and this implies that $w > 0$.

On the other hand, since $\psi \leqslant x_N$ and $\psi = 0$ on $B_1^+ \cap \{x_N = 0\}$, we have that $\psi_{x_N} \leqslant 1$ on $B_1^+ \cap \{x_N = 0\}$. Let us see that $\psi_{x_N} < 1$ in $B_1^+ \cap \{x_N = 0\}$.

Assume that there exists $x_0 \in B_1^+ \cap \{x_N = 0\}$ such that $\psi_{x_N}(x_0) = 1$ (so that $w_{x_N}(x_0) = 0$). Then, $|\nabla \psi| \geqslant 1/2$ in a neighborhood of x_0. But w is a positive solution of $\mathcal{L}w = 0$ in $B_1^+ \cap B_{r_0}(x_0)$ for some $r_0 > 0$, with $\mathcal{L}$ uniformly elliptic and $w = 0$ on $\{x_N = 0\}$. Thus, by Hopf's lemma, $w_{x_N}(x_0) > 0$, a contradiction.

Therefore $\psi_{x_N} < 1$ in $B_1^+ \cap \{x_N = 0\}$. This implies that there exist $0 < \gamma < 1$ and $\varepsilon > 0$ such that $\psi_{x_N} < \gamma$ in B_ε^+. From this, $\psi \leqslant \gamma x_N$ in B_ε^+, and then we have $u \leqslant \gamma \alpha x_N$ in B_ε^+, where ε and γ only depend on ψ. □

Lemma A.2. *Let w be a function that satisfies,*

(1) *w is a Lipschitz function in $\mathbb{R}^N$ with constant L.*
(2) *$w \geqslant 0$ in $\mathbb{R}^N$, $\Delta_p w = 0$ in $\{w > 0\}$.*
(3) *$\{x_N < 0\} \subset \{w > 0\}$, $w = 0$ in $\{x_N = 0\}$.*
(4) *There exists $0 < \lambda_0 < 1$ such that $\frac{|\{w=0\} \cap B_1(0)|}{|B_1(0)|} > \lambda_0$.*
(5) *There exists $0 \leqslant \alpha \leqslant L$ such that $w(x) \leqslant \alpha x_N$ in $B_1(0) \cap \{x_N > 0\}$.*

Then there exist $0 < \gamma < 1$ and $0 < \varepsilon \leqslant 1$ depending only on λ_0 and N, such that $w(x) \leqslant \gamma \alpha x_N$ in $B_\varepsilon(0) \cap \{x_N > 0\}$.

Proof. Let $\beta = \frac{\lambda_0}{2^{N-1}} < 1$, then by (3) and (4) there exists $x_0 \in B_1(0)$, with $x_{0,N} > \beta$ such that $w(x_0) = 0$. By (1), $w(x) \leqslant L|x - x_0|$, then if we take $r_0 = \frac{\alpha\beta}{4L}$, we have $w(x) \leqslant \frac{\alpha\beta}{4}$ for $|x - x_0| < r_0$. As $\alpha/L \leqslant 1$, in that set there holds that $x_N \geqslant \frac{3\beta}{4}$. Then we have that

$$w(x) \leqslant \frac{\alpha x_N}{3} \quad \text{in } \partial B_r^+ \cap B_{r_0}(x_0),$$

where $r = |x_0| > \beta$. Taking in Lemma A.1 $\delta_0 = 1/3$ and $\bar{x} = x_0$ we have that there exist $0 < \gamma < 1$ and $0 < \varepsilon \leqslant 1$, depending on r and N, such that $w(x) \leqslant \gamma \alpha x_N$ in B_ε^+.

As $r > \beta$ what we obtain is that γ and ε only depend on λ_0. Therefore the result follows. □

Now we are ready to proceed with the proof of the theorem.

Proof of Theorem A.1. Once we have proved Lemma A.2 we consider the same iteration as in [13, Theorem A.1, step 2] and the result follows. □

As a remark we mention that with Lemma A.1 we can also prove the asymptotic development of p-harmonic functions. This result was originally proved in [7].

Lemma A.3. *Let u be Lipschitz continuous in $\overline{B_1^+}$, $u \geqslant 0$ in B_1^+, p-harmonic in $\{u > 0\}$ and vanishing on $\partial B_1^+ \cap \{x_N = 0\}$. Then, in B_1^+, u has the asymptotic development*

$$u(x) = \alpha x_N + o(|x|),$$

with $\alpha \geqslant 0$.

Proof. Let

$$\alpha_j = \inf\big\{l\colon\ u \leqslant l x_n \text{ in } B^+_{2^{-j}}\big\}.$$

Let $\alpha = \lim_{j\to\infty} \alpha_j$.

Given $\varepsilon_0 > 0$ there exists j_0 such that for $j \geqslant j_0$ we have $\alpha_j \leqslant \alpha + \varepsilon_0$. From here, we have $u(x) \leqslant (\alpha + \varepsilon_0)x_N$ in $B^+_{2^{-k}}$ so that

$$u(x) \leqslant \alpha x_N + o\big(|x|\big) \quad \text{in } B_1^+.$$

If $\alpha = 0$ the result follows. Assume that $\alpha > 0$ and let us suppose that $u(x) \neq \alpha x_N + o(|x|)$. Then there exist $x_k \to 0$ and $\bar\delta > 0$ such that

$$u(x_k) \leqslant \alpha x_{k,N} - \bar\delta |x_k|.$$

Let $r_k = |x_k|$ and $u_k(x) = r_k^{-1} u(r_k x)$. Then, there exists u_0 such that, for a subsequence that we still call u_k, $u_k \to u_0$ uniformly in $\overline{B_1^+}$ and

$$u_k(\bar x_k) \leqslant \alpha \bar x_{k,N} - \bar\delta, \qquad u_k(x) \leqslant (\alpha + \varepsilon_0)x_N \quad \text{in } B_1^+,$$

where $\bar x_k = \frac{x_k}{r_k}$, and we can assume that $\bar x_k \to x_0$.

In fact, $u(x) \leqslant (\alpha + \varepsilon_0)x_N$ in $B^+_{2^{-j_0}}$, therefore $u_k(x) \leqslant (\alpha + \varepsilon_0)x_N$ in $B^+_{r_k/2^{-j_0}}$, and if k is big enough $r_k / 2^{-j_0} \geqslant 1$.

If we take $\bar\alpha = \alpha + \varepsilon_0$ we have

$$\begin{cases} \Delta_p u_k \geqslant 0 & \text{in } B_1^+,\\ u_k = 0 & \text{on } \{x_N = 0\},\\ 0 \leqslant u_k \leqslant \bar\alpha x_N & \text{on } \partial B_1^+,\\ u_k \leqslant \delta_0 \bar\alpha x_N & \text{on } \partial B_1^+ \cap B_{\bar r}(\bar x) \end{cases}$$

for some $\bar x \in \partial B_1^+$, $\bar x_N > 0$ and some small $\bar r > 0$.

In fact, as u_k are continuous with uniform modulus of continuity, we have

$$u_k(x_0) \leqslant \alpha x_{0,N} - \frac{\bar\delta}{2} \quad \text{if } k \geqslant \bar k.$$

Moreover there exists $r_0 > 0$ such that $u_k(x) \leqslant \alpha x_N - \frac{\bar\delta}{4}$ in $B_{2r_0}(x_0)$. If $x_{0,N} > 0$ we take $\bar x = x_0$, if not, we take $\bar x \in B_{2r_0}(x_0)$ with $\bar x_N > 0$ and

$$u_k(x) \leqslant \alpha x_N - \frac{\bar\delta}{4} \quad \text{in } B_{r_0}(\bar x) \Subset \{x_N > 0\}.$$

As $B_{r_0}(\bar x) \Subset \{x_N > 0\}$ there exists δ_0 such that $\alpha x_N - \frac{\bar\delta}{4} \leqslant \delta_0 \alpha x_N \leqslant \delta_0 \bar\alpha x_N$ in $B_{\bar r}(\bar x)$ for some small $\bar r$, and the claim follows.

Now, by Lemma A.1, there exist $0 < \gamma < 1$, $\varepsilon > 0$ independent of ε_0 and k, such that $u_k(x) \leqslant \gamma(\alpha + \varepsilon_0)x_N$ in B_ε^+. As γ and ε are independent of k and ε_0, taking $\varepsilon_0 \to 0$, we have

$$u_k(x) \leqslant \gamma\alpha x_N \quad \text{in } B_\varepsilon^+.$$

So that,

$$u(x) \leqslant \gamma\alpha x_N \quad \text{in } B_{r_k\varepsilon}^+.$$

Now if j is big enough we have $\gamma\alpha < \alpha_j$ and $2^{-j} \leqslant r_k\varepsilon$. But this contradicts the definition of α_j. Therefore,

$$u(x) = \alpha x_N + o(|x|),$$

as we wanted to prove. □

Appendix B. Blow-up limits

Now we give the definition of blow-up sequence, and we collect some properties of the limits of these blow-up sequences for certain classes of functions that are used throughout the paper.

Let u be a function with the following properties:

(B1) u is Lipschitz in Ω with constant $L > 0$, $u \geqslant 0$ in Ω and $\Delta_p u = 0$ in $\Omega \cap \{u > 0\}$.

(B2) Given $0 < \kappa < 1$, there exist two positive constants C_κ and r_κ such that for every ball $B_r(x_0) \subset \Omega$ and $0 < r < r_\kappa$,

$$\frac{1}{r}\left(\fint_{B_r(x_0)} u^\gamma\,dx\right)^{1/\gamma} \leqslant C_\kappa \quad \text{implies that } u \equiv 0 \quad \text{in } B_{\kappa r}(x_0).$$

(B3) There exist constants $r_0 > 0$ and $0 < \lambda_1 \leqslant \lambda_2 < 1$ such that, for every ball $B_r(x_0) \subset \Omega$ x_0 on $\partial\{u > 0\}$ and $0 < r < r_0$

$$\lambda_1 \leqslant \frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leqslant \lambda_2.$$

Definition B.1. Let $B_{\rho_k}(x_k) \subset \Omega$ be a sequence of balls with $\rho_k \to 0$, $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$. Let

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x).$$

We call u_k a blow-up sequence with respect to $B_{\rho_k}(x_k)$.

Since u is locally Lipschitz continuous, there exists a blow-up limit $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that for a subsequence,

$$u_k \to u_0 \quad \text{in } C_{\text{loc}}^\alpha(\mathbb{R}^N) \text{ for every } 0 < \alpha < 1,$$
$$\nabla u_k \to \nabla u_0 \quad *\text{-weakly in } L_{\text{loc}}^\infty(\mathbb{R}^N),$$

and u_0 is Lipschitz in $\mathbb{R}^N$ with constant L.

Lemma B.1. *If u satisfies properties* (B1)–(B3) *then*:

(1) $u_0 \geqslant 0$ *in* Ω *and* $\Delta_p u_0 = 0$ *in* $\{u_0 > 0\}$,
(2) $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
(3) $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$,
(4) *if* $K \Subset \{u_0 = 0\}$, *then* $u_k = 0$ *in* K *for big enough* k,
(5) *if* $K \Subset \{u_0 > 0\} \cup \{u_0 = 0\}^\circ$, *then* $\nabla u_k \to \nabla u_0$ *uniformly in* K,
(6) *there exists a constant* $0 < \lambda < 1$ *such that*

$$\frac{|B_R(y_0) \cap \{u_0 = 0\}|}{|B_R(y_0)|} \geqslant \lambda, \quad \forall R > 0,\ \forall y_0 \in \partial\{u_0 > 0\},$$

(7) $\nabla u_k \to \nabla u_0$ *a.e. in* Ω,
(8) *if* $x_k \in \partial\{u > 0\}$, *then* $0 \in \partial\{u_0 > 0\}$.

Proof. As u_k are p-harmonic and $u_k \to u_0$ uniformly in compacts subsets of $\mathbb{R}^N$ then (1) holds. For the proof of (2)–(8) see [13]. □

References

[1] A. Acker, An extremal problem involving distributed resistance, SIAM J. Math. Anal. 12 (1981) 169–172.
[2] N. Aguilera, H.W. Alt, L.A. Caffarelli, An optimization problem with volume constraint, SIAM J. Control Optim. 24 (2) (1986) 191–198.
[3] N. Aguilera, L.A. Caffarelli, J. Spruck, An optimization problem in heat conduction, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 14 (3) (1988) 355–387.
[4] H.W. Alt, L.A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981) 105–144.
[5] D. Danielli, A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, Calc. Var. Partial Differential Equations 23 (1) (2005) 97–124.
[6] D. Danielli, A. Petrosyan, Full regularity of the free boundary in a Bernoulli-type problem in two dimensions, Math. Res. Lett., submitted for publication.
[7] D. Danielli, A. Petrosyan, H. Shahgholian, A singular perturbation problem for the p-Laplace operator, Indiana Univ. Math. J. 52 (2) (2003) 457–476.
[8] H. Federer, Geometric Measure Theory, Springer, Berlin, 1969.
[9] J. Fernández Bonder, J.D. Rossi, N. Wolanski, Regularity of the free boundary in an optimization problem related to the best Sobolev trace constant, SIAM J. Control Optim. 44 (5) (2005) 1612–1635.
[10] M. Flucher, An asymptotic formula for the minimal capacity among sets of equal area, Calc. Var. Partial Differential Equations 1 (1993) 71–86.
[11] P. Juutinen, P. Lindqvist, J.J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasilinear equation, SIAM J. Math. Anal. 33 (3) (2001) 699–717.
[12] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Math., vol. 1150, Springer, New York, 1985.
[13] C. Lederman, A free boundary problem with a volume penalization, Ann. Sc. Norm. Super. Pisa Cl. Sci. 23 (2) (1996) 249–300.
[14] K. Oliveira, E. Teixeira, An optimization problem with free boundary governed by a degenerate quasilinear operator, preprint.
[15] A. Petrosyan, On the full regularity of the free boundary in a class of variational problems, preprint.
[16] E. Teixeira, The nonlinear optimization problem in heat conduction, Calc. Var. Partial Differential Equations 24 (1) (2005) 21–46.
[17] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. Pure Appl. Math. 20 (1967) 721–747.