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J. Differential Equations 198 (2004) 91–128
Journal of Differential Equations
http://www.elsevier.com/locate/jde

A convex–concave problem with a nonlinear boundary condition

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Received February 11, 2003

Abstract

In this paper we study the existence of nontrivial solutions of the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

with $1<q<2(N-1)/(N-2)$ and $1<p\leqslant 2N/(N-2)$. In the concave–convex case, i.e., $1<q<2<p$, if λ is small there exist two positive solutions while for λ large there is no positive solution. When p is critical, and q subcritical we obtain existence results using the concentration compactness method. Finally, we apply the implicit function theorem to obtain solutions for λ small near $u_0 = 1$.

MSC: 35J65; 35J50; 35J55

Keywords: Nonlinear boundary conditions; Critical exponents

1. Introduction

In this paper we study the existence of nontrivial solutions for the following problem:

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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[1] Supported by project BFM2001-0183, M.C.Y.T, Spain.
[2] Supported by Fundacion Antorchas, CONICET and ANPCyT PICT 05009.

0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(03)00068-8

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, and $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

Problems with nonlinear boundary conditions of form (1.1) appear in a natural way when one considers the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$, see [15]. Also, one is led to nonlinear boundary conditions in the study of conformal deformations on Riemannian manifolds with boundary, see for example [7,13,14].

The study of existence when the nonlinear term is placed in the equation, that is if one considers a problem of the form $-\Delta u = f(u)$ with Dirichlet boundary conditions, has received considerable attention, see for example [3,6,16,24], etc. However, nonlinear boundary conditions have only been considered in recent years. For the Laplace operator with nonlinear boundary conditions see for example [8,10,15,18,29]. Also see [25] where a problem similar to (1.1) is studied.

We want to remark that we are facing two nonlinear terms in problem (1.1), one in the equation, $|u|^{p-2}u$, and one in the boundary condition, $|u|^{q-2}u$. Our interest here is to analyze the interplay between both.

In this work, by solutions to (1.1) we understand critical points of the associated energy functional (defined on $H^1(\Omega)$)

$$\mathscr{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q\,d\sigma, \tag{1.2}$$

where $d\sigma$ is the measure on the boundary.

This functional $\mathscr{F}$ is well defined and C^1 in $H^1(\Omega)$ if p and q verify

$$1 < q \leqslant q^* = \frac{2(N-1)}{N-2} \quad \text{and} \quad 1 < p \leqslant p^* = \frac{2N}{N-2}.$$

Along this paper we look for conditions that ensure the existence of nontrivial solutions of (1.1), focusing our attention on the existence of positive ones. We distinguish several cases.

1.1. *Convex–concave subcritical case*

We suppose that

$$1 < q < 2 < p. \tag{1.3}$$

We want to remark that the new feature of these problems is that we are facing a convex–concave problem where the convex nonlinearity appears in the equation and the concave one at the boundary condition. Notice that if we look at the positive solutions of these problems as the stationary states of the corresponding evolution equation, since the right-hand side of the equation represents a positive reaction term, and the boundary condition means a positive flux at the boundary, then some absorption is required to reach a nontrivial equilibrium. In our equation, this is the linear term $+u$.

First, we assume that the exponents involved are subcritical, that is,

$$1 < q < q^* = \frac{2(N-1)}{N-2} \quad \text{and} \quad 1 < p < p^* = \frac{2N}{N-2}, \tag{1.4}$$

and we prove the following theorems using standard variational arguments together with the Sobolev trace immersion that provides the necessary compactness.

Theorem 1.1. *Let p and q satisfy* (1.3) *and* (1.4). *Then there exists* $\lambda_0 > 0$ *such that if* $0 < \lambda < \lambda_0$ *then problem* (1.1) *has infinitely many nontrivial solutions.*

Now we concentrate on positive solutions for (1.1).

Theorem 1.2. *Let p and q satisfy* (1.3) *and* (1.4). *Then there exists* $\Lambda > 0$ *such that there exist at least two positive solutions of* (1.1) *for every* $\lambda < \Lambda$, *at least one positive solution for* $\lambda = \Lambda$ *and there is no positive solution of* (1.1) *for* $\lambda > \Lambda$. *Moreover there exists a constant C such that every positive solution verifies*

$$||u||_{L^\infty(\Omega)} \leqslant C.$$

1.2. Critical case

Next we analyze the existence of solution when we have a critical exponent $p = p^*$. Here we use the concentration compactness method introduced in [21,22] and follow some ideas from [16]. In these kind of problems the concentration is a priori possible on the boundary. This difficulty leads us to use technical estimates that are implicit in [23] and that we explicitly point out.

For $2 < q < q^*$ (notice that in this case q means a *convex* reaction term) we have,

Theorem 1.3. *Let* $p = p^*$ *with* $2 < q < q^*$, *then problem* (1.1) *has at least a positive nontrivial solution for every* $\lambda > 0$.

And for $1 < q < 2$,

Theorem 1.4. *If* $p = p^*$ *with* $1 < q < 2$, *then there exists* Λ *such that problem* (1.1) *has at least two positive solutions for* $\lambda < \Lambda$, *at least one positive solution for* $\lambda = \Lambda$ *and no positive solution for* $\lambda > \Lambda$.

1.3. Further results

Moreover, to obtain existence of solutions we can apply the implicit function theorem near $\lambda_0 = 0$, $u_0 = 1$ to get existence of solutions for any p and q, but imposing a restriction on the domain. We prove the following result.

Theorem 1.5. *Given* $1 < p < \infty$, *let* Ω *be a domain such that* $(p-1) \notin \sigma_{\mathrm{Neu}}(-\Delta + I)$. *Then, for any* $q \in (1, \infty)$ *there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$ *there exists a* *positive solution* $u_\lambda \in C^\alpha$ *of* (1.1) *with* $u_\lambda \to 1$ *in* C^α *as* $\lambda \to 0$. *Here* $\sigma_{\mathrm{Neu}}(-\Delta + I)$ *stands for the spectrum of* $-\Delta + I$ *with homogeneous Neumann boundary conditions.*

Finally let us state a result for the remaining case, $q = 2$. In this case we have a bifurcation problem from the first eigenvalue of a related problem. Let λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases}$$

Notice that λ_1 is just the best constant in the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ in the sense that

$$\lambda_1 ||u||^2_{L^2(\partial\Omega)} \leqslant ||u||^2_{H^1(\Omega)}.$$

Then we have that

Theorem 1.6. *Let* $q = 2$ *with* $2 < p < p^*$. *Then there exists a positive solution of* (1.1) *if and only if* $0 < \lambda < \lambda_1$.

Our ideas can also be applied to

$$\begin{cases} -\Delta u + u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

For this problem we assume that

$$1 < q < 2 < p, \tag{1.6}$$

i.e., p stands for the convex term, and q for the concave one, and p is subcritical (notice that in this case this means $p < 2(N-1)/(N-2)$). The results presented here have analogous statements for (1.5). The proofs of the existence results are similar to the ones performed for problem (1.1) so we leave the details to the reader. The nonexistence result for positive solutions with λ large also holds here, but in this case the proof needs some major changes, therefore we include the details.

1.4. Organization of the paper

The rest of the paper is organized as follows. In Section 2 we deal with the subcritical case. In Section 3 we find L^∞ a priori bounds for positive solutions. In Section 4 we prove some regularity for the solutions. In Section 5 we prove nonexistence results for positive solutions with λ large. In Section 6 we find existence of at least two positive solutions for λ small. In Sections 7 and 8 we deal with critical exponents, and in Section 9 we use the implicit function theorem to obtain existence of solutions for λ small near $u_0 \equiv 1$. Finally in Section 10 we deal with the case $q = 2$.

2. The subcritical case. Proof of Theorem 1.1

In this section we study (1.1) using variational techniques.

Let us begin with the following lemma that will be helpful in order to prove the Palais–Smale condition.

Lemma 2.1. *Let $\phi \in H^1(\Omega)'$. Then there exists a unique $u \in H^1(\Omega)$ such that*

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega uv \, dx = \langle \phi, v \rangle, \quad \text{for all } v \in H^1(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $H^1(\Omega)$. Moreover, the operator $\mathscr{A} : \phi \mapsto u$ is continuous.

This lemma is just an application of the Lax–Milgram Theorem. Let us recall that $u \in H^1(\Omega)$ satisfying (2.1) is a critical point of the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + |u|^2 \, dx - \langle \phi, u \rangle.$$

Along this paper $\phi \in H^1(\Omega)'$ will be typically a pair $\phi = (g, f)$ where $g \in (L^{q^*}(\partial\Omega))'$ and $f \in (L^{p^*}(\Omega))'$, i.e.

$$\langle \phi, v \rangle = \int_{\partial\Omega} gv \, d\sigma + \int_\Omega fv \, dx.$$

Now we introduce a topological tool, the *genus*, that was introduced in [20]. We will use an equivalent definition due to [11]. Given a Banach Space X, we consider the class

$$\Sigma = \{A \subset X\colon\ A \text{ is closed},\ A = -A\}.$$

Over this class we define the genus, $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\{k \in \mathbb{N}\colon\ \text{there exists } \varphi \in C(A, \mathbb{R}^k - \{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

With this definition we have the following well known Lemma.

Lemma 2.2. *For every $n \in \mathbb{N}$ there exists a constant $\varepsilon > 0$ such that*

$$\gamma(\mathscr{F}^{-\varepsilon}) \geqslant n,$$

where $\mathscr{F}^c = \{u \in W^{1,p}(\Omega)\colon\ \mathscr{F}(u) \leqslant c\}$.

Proof. Let $E_n \subset H^1(\Omega)$ be a n-dimensional subspace such that $u|_{\partial\Omega} \not\equiv 0$ for all $u \in E_n$, $u \neq 0$. Hence we have, for $u \in E_n$, $||u||_{H^1(\Omega)} = 1$,

$$\mathscr{F}(tu) = \frac{t^2}{2} - \frac{\lambda t^q}{q} \int_{\partial\Omega} |u|^q - \frac{t^p}{p} \int_{\Omega} |u|^p \leqslant \frac{t^2}{2} - a_n \frac{\lambda t^q}{q}, \tag{2.2}$$

where

$$a_n = \inf \left\{ \int_{\partial\Omega} |u|^q \colon u \in E_n, \ ||u||_{H^1(\Omega)} = 1 \right\}.$$

Observe that $a_n > 0$ because E_n is finite dimensional. As $q < 2$ we obtain from (2.2) that there exists positive constants ρ and ε such that

$$\mathscr{F}(\rho u) < -\varepsilon \quad \text{for } u \in E_n, \qquad ||u||_{H^1(\Omega)} = \rho.$$

Therefore, if we set $S_{\rho,n} = \{u \in E_n \colon ||u||_{H^1(\Omega)} = \rho\}$, we have that $S_{\rho,n} \subset \mathscr{F}^{-\varepsilon}$. Hence by the monotonicity of the genus

$$\gamma(\mathscr{F}^{-\varepsilon}) \geqslant \gamma(S_{\rho,n}) = n,$$

as we wanted to show. □

Now we observe, using the Sobolev trace Theorem, that

$$\mathscr{F}(u) \geqslant \frac{1}{2} ||u||^2_{H^1(\Omega)} - \lambda c_1 ||u||^q_{H^1(\Omega)} - c_2 ||u||^p_{H^1(\Omega)} = j(||u||_{H^1(\Omega)}),$$

where $j(x) = \frac{1}{2}x^2 - \lambda c_1 x^q - c_2 x^p$. Let us emphasize that, if λ is small, j attains a local but not a global minimum (j is not bounded from below). Taking λ smaller if necessary, we can also assume that the maximum of j is positive. To localize this minimum, we have to perform some sort of truncation. To this end let x_0, x_1 be such that $m < x_0 < x_1 < M$ where m is the local minimum of j and M is the local maximum and $j(M) > j(x_1) > j(x_0) > 0 > j(m)$. For these values x_0 and x_1 we can choose a smooth cutoff function $\tau(x)$ such that $\tau(x) = 1$ if $x \leqslant x_0$, $\tau(x) = 0$ if $x \geqslant x_1$ and $0 \leqslant \tau(x) \leqslant 1$. Finally, let $\varphi(u) = \tau(||u||_{H^1(\Omega)})$ and define the truncated functional as follows:

$$\tilde{\mathscr{F}}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + |u|^2 \, dx - \frac{1}{p} \int_{\Omega} \varphi(u) |u|^p \, d\sigma - \frac{\lambda}{q} \int_{\partial\Omega} |u|^q \, d\sigma.$$

As above, $\tilde{\mathscr{F}}(u) \geqslant \tilde{j}(||u||_{H^1(\Omega)})$ where $\tilde{j}(x) = \frac{1}{2}x^2 - c_1 x^p \tau(x) - c_2 \lambda x^q$. We observe that if $x \leqslant x_0$ then $\tilde{j}(x) = j(x)$ and if $x \geqslant x_1$ then $\tilde{j}(x) = \frac{1}{2}x^2 - \lambda c_2 x^q$.

In particular (and this is the main point of the truncation) if $\tilde{\mathscr{F}}(u) < 0$ then $\mathscr{F}$ and $\tilde{\mathscr{F}}$ coincide in a neighborhood of u.

Now we state a lemma that summarizes the main properties of $\tilde{\mathscr{F}}$.

Lemma 2.3. *The functional $\tilde{\mathscr{F}}$ is bounded from below and verifies the Palais–Smale condition.*

Proof. First, by the Sobolev-trace inequality and the performed truncation, there exists a constant C such that

$$\tilde{\mathscr{F}}(u) \geqslant \frac{1}{2}||u||^2_{H^1(\Omega)} - C\frac{\lambda}{q}||u||^q_{H^1(\Omega)} \equiv h(||u||_{H^1(\Omega)}),$$

where $h(t) = \frac{1}{2}t^2 - C\frac{\lambda}{q}t^q$. As $q<2$, $h(t)$ is bounded from below and we conclude that $\tilde{\mathscr{F}}$ is also bounded from below.

Now, to prove the Palais–Smale condition for the truncation, let $\{u_k\} \subset H^1(\Omega)$ be a Palais–Smale sequence. As $c = \lim_{k\to\infty} \tilde{\mathscr{F}}(u_k)$, using that $\tilde{\mathscr{F}}'(u_k) = \varepsilon_k \to 0$ in $H^1(\Omega)'$ we have that, for k large enough,

$$\begin{aligned} c+1 &\geqslant \tilde{\mathscr{F}}(u_k) - \frac{1}{p}\langle \tilde{\mathscr{F}}'(u_k)u_k\rangle + \frac{1}{p}\langle \tilde{\mathscr{F}}'(u_k)u_k\rangle \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)||u_k||^2_{H^1(\Omega)} + \frac{1}{p}\langle \tilde{\mathscr{F}}'(u_k)u_k\rangle - C||u_k||^q_{H^1(\Omega)} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)||u_k||^2_{H^1(\Omega)} - \frac{1}{p}||u_k||_{H^1(\Omega)}\varepsilon_k - C||u_k||^q_{H^1(\Omega)} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)||u_k||^2_{H^1(\Omega)} - \frac{1}{p}||u_k||_{H^1(\Omega)} - C||u_k||^q_{H^1(\Omega)}. \end{aligned}$$

Hence, as $q<2<p$, u_k is bounded in $H^1(\Omega)$.

We can assume that $u_k \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_k \to u$ strongly in $L^p(\Omega)$, $u_k \to u$ strongly in $L^q(\partial\Omega)$ and a.e. in $\partial\Omega$. Then, as the exponents are subcritical, it follows that,

$$|u_k|^{q-2}u_k \to |u|^{q-2}u \quad \text{in } (L^{\frac{2(N-1)}{N-2}}(\partial\Omega))'$$

and

$$|u_k|^{p-2}u_k \to |u|^{p-2}u \quad \text{in } (L^{\frac{2N}{N-2}}(\Omega))'$$

and hence in $H^1(\Omega)'$. Therefore, according to Lemma 2.1,

$$u_k \to \mathscr{A}(|u|^{q-2}u, |u|^{p-2}u), \quad \text{in } H^1(\Omega).$$

This completes the proof. □

Finally, the following Theorem gives the proof of Theorem 1.1, see [5] for an analogous result.

Theorem 2.1. *Let*

$$\Sigma = \{A \subset H^1(\Omega) - \{0\}\colon\ A \text{ is closed},\ A = -A\},$$

$$\Sigma_k = \{A \subset \Sigma\colon\ \gamma(A) \geqslant k\},$$

where γ stands for the genus. Then

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \mathscr{F}(u)$$

is a negative critical value of $\mathscr{F}$ and moreover, if $c = c_k = \cdots = c_{k+r}$, then $\gamma(K_c) \geqslant r + 1$, where $K_c = \{u \in W^{1,p}(\Omega)\colon\ \mathscr{F}(u) = c,\ \mathscr{F}'(u) = 0\}$.

Proof. We argue with $\tilde{\mathscr{F}}$. According to Lemma 2.2 for every $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\tilde{\mathscr{F}}^{-\varepsilon}) \geqslant k$. As $\tilde{\mathscr{F}}$ is even and continuous it follows that $\tilde{\mathscr{F}}^{-\varepsilon} \in \Sigma_k$, and therefore $c_k \leqslant -\varepsilon < 0$. Moreover by Lemma 2.3, $\tilde{\mathscr{F}}$ is bounded from below so $c_k > -\infty$. Let us now see that c_k is in fact a critical value for $\tilde{\mathscr{F}}$. To this end let us suppose that $c = c_k = \cdots = c_{k+r}$. As $\tilde{\mathscr{F}}$ is even it follows that K_c is symmetric, and the Palais–Smale condition implies that K_c is compact. Now, assume by contradiction that $\gamma(K_c) \leqslant r$. Then, by the continuity property of the genus (see [26]) there exists a neighborhood of K_c, $N_\delta(K_c) = \{v \in H^1(\Omega)\colon\ d(v, K_c) \leqslant \delta\}$, such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \leqslant r$.

By the usual deformation lemma, we get

$$\eta(1, \tilde{\mathscr{F}}^{c+\varepsilon/2} - N_\delta(K_c)) \subset \tilde{\mathscr{F}}^{c-\varepsilon/2}.$$

On the other hand, by the definition of c_{k+r} there exists $A \subset \Sigma_{k+r}$ such that $A \subset \tilde{\mathscr{F}}^{c+\varepsilon/2}$ hence

$$\eta(1, A - N_\delta(K_c)) \subset \tilde{\mathscr{F}}^{c-\varepsilon/2}. \tag{2.3}$$

Now by the monotonicity of the genus (see [26]), we have

$$\gamma(\overline{A - N_\delta(K_c)}) \geqslant \gamma(A) - \gamma(N_\delta(K_c)) \geqslant k.$$

As $\eta(1, \cdot)$ is an odd homeomorphism it follows that (see [26])

$$\gamma(\eta(1, \overline{A - N_\delta(K_c)})) \geqslant \gamma(\overline{A - N_\delta(K_c)}) \geqslant k.$$

But as $\eta(1, \overline{A - N_\delta(K_c)}) \in \Sigma_k$ then

$$\sup_{u \in \eta(1, \overline{A - N_\delta(K_c)})} \tilde{\mathscr{F}}(u) \geqslant c = c_k,$$

a contradiction with (2.3) that proves that $\gamma(K_c) \geqslant r + 1$, and, in particular, c_k is a critical value for $\tilde{\mathscr{F}}$. Finally, as $c_k < 0$ it is also a critical value for $\mathscr{F}$. $\square$

3. A priori bounds in $L^\infty(\Omega)$

In this section we consider positive solutions of (1.1) and prove a priori bounds in $L^\infty(\Omega)$ using the blow-up technique.

Let us fix $0<\lambda<\Lambda$ and, arguing by contradiction, let us assume that there exists a sequence u_n of positive solutions of (1.1) such that

$$a_n = ||u_n||_{L^\infty(\Omega)} \to +\infty.$$

We denote by x_n a point where the maximum of u_n is located. By the compactness of $\bar{\Omega}$, we can assume that $x_n \to x_0 \in \bar{\Omega}$. Let us consider

$$v_n(y) = \frac{u_n(a_n^\alpha y - x_n)}{a_n}.$$

This function v_n is defined in $\Omega_n = \{y \in \mathbb{R}^N;\ a_n^\alpha y + x_n \in \Omega\}$. We observe that $0 \leqslant v_n \leqslant 1$ and $v_n(0) = 1$. By a simple calculation we get that v_n satisfies

$$\begin{cases} -\Delta v_n = -a_n^{2\alpha} v_n + a_n^{2\alpha+p-2} v_n^{p-1} & \text{in } \Omega_n, \\ \frac{\partial v_n}{\partial \nu} = a_n^{\alpha+q-2} \lambda v_n^{q-1} & \text{on } \partial\Omega_n. \end{cases}$$

Recall that we are dealing with $p>2>q>1$ and then $p>2(q-1)$. Let us choose $2\alpha + p - 2 = 0$. With this choice, $2\alpha<0$ and also $\alpha + q - 2 < 0$. Therefore, as we are assuming that $a_n \to +\infty$, we get that $a_n^{2\alpha} \to 0$ and $a_n^{\alpha+q-2} \to 0$. Passing to the limit, a subsequence of v_n converges to v that is a solution of

$$\begin{cases} -\Delta v = v^{p-1} & \text{in } D, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial D, \end{cases}$$

with $v(0) = 1$. Here D is the whole $\mathbb{R}^N$ or the half-space $\mathbb{R}^N_+$ according to $x_0 \in \Omega$ or $x_0 \in \partial\Omega$, respectively. In case we have $D = \mathbb{R}^N_+$ we can perform a reflection using that v has null normal derivative. Hence in any case we get

$$\begin{cases} -\Delta v = v^{p-1} & \text{in } \mathbb{R}^N, \\ v(0) = 1 \quad v \geqslant 0 & . \end{cases}$$

This is a contradiction with the fact that p is subcritical, $1<p<2N/(N-2)$. We have proved that

Theorem 3.1. *If $2<p<2N/(N-2)$ with $p>2(q-1)$ then, for every $\Lambda>0$ there exists a constant $C = C(\Lambda)$ such that every nonnegative solution u of* (1.1) *with $0<\lambda<\Lambda$ verifies*

$$||u||_{L^\infty(\Omega)} \leqslant C.$$

Let us remark that if $1<q<2<p<2N/(N-2)$ we are in this case. Therefore all positive solutions (if there exist any) are uniformly bounded in $L^\infty(\Omega)$ when one considers $\lambda<\Lambda$.

Now we want to make a remark regarding an analogous a priori bound for nonnegative solutions of problem (1.5) (i.e., concave reaction in Ω, and convex nonlinearity at $\partial\Omega$).

We have $1<q<2<p$, hence $q<2(p-1)$. In this case we take $\alpha=2-p<0$. With this choice we have $2\alpha<0$ and $2\alpha+q-2<0$.

First we observe that if we have a sequence u_n with

$$a_n = ||u_n||_{L^\infty(\Omega)} \to +\infty,$$

then the maximum of u_n must be located at the boundary. In fact if x_n is an interior maximum of u_n we have $-\Delta u_n(x_n)<0$ for n large, a contradiction. Therefore we can assume that x_n lies on the boundary of Ω. As before, we can pass to the limit as $n\to\infty$ and obtain a solution of

$$\begin{cases} -\Delta v = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial v}{\partial \nu} = v^{p-1} & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{3.1}$$

with $v(0)=1$. The corresponding Liouville Theorem for (3.1) was proved in [18]: if p is subcritical the only bounded nonnegative solution is $v\equiv 0$. We want to remark that in the critical case, $p=2(N-1)/(N-2)$, there exists nontrivial nonnegative solutions, see [10].

Using this result we get a contradiction which proves the a priori bound. In summary, we have proved the following result.

Theorem 3.2. *If* $2<p<2(N-1)/(N-2)(=q^*)$ *and* $q<2(p-1)$, *then there exists a constant* C *such that every nonnegative solution to* (1.5) u *verifies*

$$||u||_{L^\infty(\Omega)} \leqslant C.$$

4. Some remarks on regularity

From [7] we have that weak positive solutions of (1.1) are $C^\infty(\bar{\Omega})$. In this section we will prove $C^\alpha(\bar{\Omega})$ estimates for the solutions of (1.1). We include some details for the sake of completeness.

First, we deal with the subcritical case. Namely, $1<q<2(N-1)/(N-2)$, $1<p<2N/(N-2)$. The idea is to adapt the classical bootstrapping argument, taking into account the nonlinear boundary condition.

We start by recalling some linear results.

Proposition 4.1. (I) *Assume that* $g \in L^r(\Omega)$ *with* $r > \frac{2N}{N+2}$ *and let* $\phi \in H^1(\Omega)$ *be the weak solution to*

$$\begin{cases} -\Delta\phi + \phi = g & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

then $||\phi||_{W^{1,\beta}(\Omega)} \leqslant C||g||_{L^r(\Omega)}$ *with* $\beta = \frac{Nr}{(N-r)} > 2$.

(II) *Assume that* $h \in L^s(\partial\Omega)$ *with* $s > \frac{2(N-1)}{N}$, *and let* ψ *be the weak solution to problem*

$$\begin{cases} -\Delta\psi + \psi = 0 & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\nu} = h & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Then $||\psi||_{W^{1,\gamma}(\Omega)} \leqslant C||h||_{L^s(\partial\Omega)}$ *with* $\gamma = \frac{Ns}{(N-1)} > 2$.

Proof. Part (I) can be considered as the simplest case of the results in [28]. In this case the proof is easier: just integrating by parts we find

$$\left| \int_\Omega \nabla\phi\nabla\rho + \phi\rho \right| \leqslant ||g||_{L^r(\Omega)} ||\rho||_{L^{r'}(\Omega)},$$

with $\frac{1}{r} + \frac{1}{r'} = 1$, and by Sobolev embedding we can take a test function $\rho \in W^{1,\beta'}(\Omega)$ with $\beta' = \frac{Nr'}{N+r'}$.

As a consequence, using Proposition 1 of [7], we get

$$\phi \in W^{1,\beta}(\Omega) \quad \text{and} \quad ||\phi||_{W^{1,\beta}(\Omega)} \leqslant C||g||_{L^r(\Omega)}, \qquad \text{where } \beta = \frac{Nr}{N-r},$$

and since $r > \frac{2N}{N+2}$ it follows $\beta > 2$.

As for part (II), if ψ is the weak solution, multiplying by a regular test function $\eta \in C^1(\Omega)$ we get

$$\left| \int_\Omega \nabla\psi\nabla\eta + \int_\Omega \psi\eta \right| \leqslant ||h||_{L^s(\partial\Omega)} ||\eta||_{L^{s'}(\partial\Omega)},$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Then by density we can take $\eta \in W^{1,\gamma}(\Omega)$ and therefore, by the *trace theorem*,

$$\eta|_{\partial\Omega} \in W^{1-(\frac{1}{\gamma'}),\gamma'}(\partial\Omega) \subset L^{\frac{\gamma'(N-1)}{N-\gamma'}}(\partial\Omega),$$

where $s' = \gamma'(N-1)/(N-\gamma')$, which implies that $\gamma = \frac{Ns}{N-1}$. Hence, by Proposition 1 of [7], we get that

$$\psi \in W^{1,\gamma}(\Omega) \quad \text{and} \quad ||\psi||_{W^{1,\gamma}(\Omega)} \leqslant C||h||_{L^s(\partial\Omega)}.$$

This finishes the proof. $\square$

Next, we decompose our original problem, taking $g = |u|^{p-2}u$ and $h = \lambda|u|^{q-2}u$, in such a way that $u = \phi + \psi$, where ϕ and ψ are the corresponding solutions to linear problems (4.1) and (4.2).

The idea to prove regularity for solutions of (1.1) is that we can iterate the estimates in Proposition 4.1, improving from step to step the regularity of u. The argument is as follows:

We start assuming $g \in L^{r_0}(\Omega)$, and $h \in L^{s_0}(\partial\Omega)$, where

$$r_0 = \frac{2N}{(N-2)(p-1)}, \quad \text{and} \quad s_0 = \frac{2(N-1)}{(N-2)(q-1)}.$$

In particular, if $r_0 > N/2$ (that is, $p < (N+2)/(N-2)$) we get an exponent $\beta_0 > N$ such that $\phi \in W^{1,\beta_0}(\Omega) \subset C^\alpha(\bar{\Omega})$.

On the other hand, if $q < N/(N-2)$, we get that $\psi \in W^{1,\gamma_0}(\Omega)$ with $\gamma_0 > N$ and in this case $\psi \in C^\alpha(\bar{\Omega})$.

As a consequence, the C^α regularity for u is proved in the case $q < N/(N-2)$, $p < (N+2)/(N-2)$.

If not, in any case we have proved that $u \in W^{1,\tau_0}(\Omega)$ with $\tau_0 = \min\{\beta_0, \gamma_0\} > 2$. Then we can iterate exactly the same calculation as before, starting with $g \in L^{r_1}(\Omega)$ and $h \in L^{s_1}(\partial\Omega)$ where

$$r_1 = \frac{N\tau_0}{(N-\tau_0)(p-1)} \quad \text{and} \quad s_1 = \frac{(N-1)\tau_0}{(N-\tau_0)(q-1)}.$$

If r_1 and s_1 were both large enough (namely, $r_1 > N/2$ and $s_1 > N-1$), then we have finished. If not, we get that $u \in W^{1,\tau_1}(\Omega)$, where

$$\tau_1 = \begin{cases} \frac{Ns_1}{N-1}, & \text{if } r_1 > N/2 \text{ and } s_1 \leqslant N-1, \\ \min\left\{\frac{Ns_1}{N-1}, \frac{Nr_1}{N-r_1}\right\}, & \text{if } r_1 \leqslant N/2. \end{cases}$$

Let us estimate these quantities in terms of the starting exponent τ_0. Since $\tau_0 > 2$, we have,

$$\frac{Ns_1}{N-1} = \frac{N}{(N-\tau_0)(q-1)}\tau_0 \geqslant \frac{N}{(N-2)(q-1)}\tau_0.$$

And, on the other hand, it is easy to see that

$$\frac{Nr_1}{N-r_1} = \frac{N}{p(N-\tau_0)-N}\tau_0 > \frac{N}{p(N-2)-N}\tau_0.$$

Therefore (taking into account that p and q are subcritical) we have proved that there exists a constant $C = C(N,p,q) > 1$ such that

$$\tau_1 \geqslant C\tau_0,$$

and, in general, $\tau_k \geqslant C^k \tau_0$. This implies that in a finite number of steps we reach that $u \in W^{1,\tau^*}(\Omega)$ with $\tau^* > N$, and hence $u \in C^\alpha(\bar{\Omega})$.

Next, we will sketch briefly the arguments in the critical case, $p = p^*$. In this case, the problem comes from the first iteration, since there is no margin to improve directly the initial exponent, getting $W^{1,\beta_0}(\Omega)$ regularity for some $\beta_0 > 2$. To overcome this difficulty we can use a truncation argument by Trudinger (see [30]) which proves that $||u||_{L^\tau(\Omega)} \leqslant C(|\Omega|, ||u||_{L^{p^*}(\Omega)})$, where $\tau > p^*$. The sketch of the argument is as follows: consider the problem

$$\begin{cases} -\Delta u + u = \lambda |u|^{p^*-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = |u|^{q-2} u & \text{on } \partial\Omega, \end{cases}$$

where q is subcritical. Assume that $u \in W^{1,2}(\Omega)$, $u > 0$ is a solution and let us prove that $u \in L^\tau(\Omega)$ for some $\tau > p^*$. The main idea is to choose a suitable truncation of u^β as test function with β greater but close to one. After some manipulations, that in our case involve the Sobolev trace inequality to handle the integrals over the boundary that appear, we arrive to $u \in L^{\beta p^*}(\Omega)$. As β is greater than one this estimate gives the required starting point, after which the argument follows as in the previous case, getting finally $u \in C^\alpha(\bar{\Omega})$.

The case $q = q^*$ with p subcritical can be handled in a similar way. With the argument given by Trudinger, [30], we can begin the iterative procedure and also in this case we get $u \in C^\alpha(\bar{\Omega})$.

5. Nonexistence results

In this section we prove nonexistence results of positive regular solutions for (1.1) and (1.5) when λ is large.

The idea of the proof of nonexistence of positive solutions for λ large is that, in this case, the sublinear term (which dominates near $u = 0$) is very strong, and forces the solution to problem (1.1) to become large. But when the solution is large, the main term is the superlinear one. Then, this superlinear reaction term allows us to prove blow-up in an associated parabolic problem whose solution should be bounded by u, getting a contradiction.

Theorem 5.1. *There exists $\Lambda > 0$ such that problem* (1.1) *with* $1 < q < 2 < p$ *has no positive solution for* $\lambda > \Lambda$.

Proof. Let $v(x,t)$ be a positive solution of the associated evolution problem

$$\begin{cases} v_t = \Delta v - v + v^{p-1} & \text{in } \Omega \times (0,T), \\ \frac{\partial v}{\partial \nu} = \lambda v^{q-1} & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{5.1}$$

Let us remark a first blow-up result for this problem: for any positive λ, by a comparison argument with solutions that do not depend on x, we get that any solution v with initial data $v(x,0)>1$ blows up in finite time, that is, there exists T such that

$$\lim_{t\nearrow T} ||v(\cdot,t)||_{L^\infty(\Omega)} = \infty. \tag{5.2}$$

As mentioned, the idea of the proof is to see that, for every λ large enough, any positive solution of (5.1) blows up in finite time. This proves that no positive stationary solution exists, giving the desired conclusion.

To this end, let us see that there is no positive global (defined for every t) solution. The argument follows by contradiction: we will see that if such a positive global solution exists for λ large, then it becomes greater than one at some positive time, and this is impossible, by the first blow-up result.

We begin by recalling that there exists $\underline{u}$ a nontrivial solution of

$$\begin{cases} \underline{u}_t = \Delta \underline{u} & \text{in } \Omega\times(0,T),\\ \frac{\partial \underline{u}}{\partial \nu} = \underline{u}^{q-1} & \text{on } \partial\Omega\times(0,T),\\ \underline{u}(x,0)=0 \end{cases}$$

with $\underline{u}(x,t)>0$ for all $t>0$. This solution exists thanks to the fact that $q-1<1$, see [12], which implies that this problem does not have uniqueness. As $\underline{u}$ is positive for positive times, we have that there exists $\delta>0$ such that $\underline{u}(x,1)\geqslant\delta$ for all $x\in\Omega$. By taking $w(x,t)=\lambda^{2-q}\underline{u}(x,t)$ we obtain a solution of

$$\begin{cases} w_t = \Delta w & \text{in } \Omega\times(0,T),\\ \frac{\partial w}{\partial \nu} = \lambda w^{q-1} & \text{on } \partial\Omega\times(0,T),\\ w(x,0)=0, \end{cases} \tag{5.3}$$

that verifies

$$w(x,1)\geqslant \lambda^{2-q}\delta > e, \tag{5.4}$$

for λ large enough. Now, we observe that if v is a solution of problem (5.1), then $z(x,t)=e^t v(x,t)$ is a supersolution to problem (5.3) and hence, using a comparison argument based on Hopf Lemma, we get $z(x,t)\geqslant w(x,t)$. By (5.4) we have $ev(x,1)=z(x,1)\geqslant w(x,1)>e$. Hence $v(x,t)>1$ for some time as we wanted to show. □

Remark 5.1. A direct elliptic proof could be done by using the same type of arguments as in Theorem 2.3 of [1].

Now we state and prove an analogous theorem for solutions to problem (1.5).

Theorem 5.2. *There exists $\Lambda>0$ such that problem* (1.5) *with* $1<q<2<p$ *has no positive solution for* $\lambda>\Lambda$.

Proof. First, we observe that if u is a positive solution, by the maximum principle u attains its minimum at some point, x_0, that must lie in Ω and therefore

$$u(x_0) \geqslant -\Delta u(x_0) + u(x_0) = \lambda(u(x_0))^{q-1}$$

and we conclude that

$$\lambda \leqslant (u(x_0))^{2-q}.$$

Hence all positive solutions must be large uniformly in Ω for large values of λ.

Now we consider the auxiliary evolution problem

$$\begin{cases} w_t = \Delta w - w & \text{in } \Omega \times (0, T), \\ \frac{\partial w}{\partial \nu} = w^{p-1} & \text{on } \partial\Omega \times (0, T), \\ w(x, 0) = w_0(x) > 0. \end{cases} \tag{5.5}$$

It is known that for any large initial data w_0, solutions of (5.5) blow-up in finite time in $L^\infty(\Omega)$, see [9].

The proof of this blow-up result can be sketched as follows: let us consider

$$z(x, t) = e^t w(x, t).$$

This function z verifies for any $T \leqslant 1$,

$$\begin{cases} z_t = \Delta z & \text{in } \Omega \times (0, T), \\ \frac{\partial z}{\partial \nu} = e^{-t(p-2)} z^{p-1} \geqslant c_1 z^{p-1} & \text{on } \partial\Omega \times (0, T), \\ z(x, 0) = w_0(x) > 0. \end{cases} \tag{5.6}$$

Hence if we show that there exists a solution of

$$\begin{cases} \phi_t = \Delta \phi & \text{in } \Omega \times (0, T), \\ \frac{\partial \phi}{\partial \nu} = c_1 \phi^{p-1} & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{5.7}$$

with finite time blow-up $T_\phi < 1$, we get that there exists a solution z of (5.6) with finite time blow-up (just use a comparison argument, based on Hopf Lemma, to obtain $\phi \leqslant z$). As $w = e^{-t} z$, we get that w blows-up, as we wanted to see.

Assume that there exists a (positive) solution ϕ of (5.7) with finite time blow-up. Considering $\phi(x, T - 1/2)$ as initial data if necessary (if $T \geqslant 1/2$), we get a solution which blows-up at time $T = 1/2 < 1$. Hence we only have to see that there exists a solution of (5.7) with finite time blow-up.

This is a well-known fact that can be found in [19,31]. An alternative argument is as follows, see [27]. Let

$$m(t) = \int_\Omega \left(\int_{\phi(x,t)}^{+\infty} \frac{1}{(s)^{p-1}} \, ds \right) dx.$$

We observe that $m(t)$ is well defined for every $t \in (0,T)$ where T is the maximal time of existence for $\phi(x,t)$ (T finite or not). Also $m(t)$ is positive.

We claim that there exists a constant $k>0$ such that

$$m'(t) \leqslant -k < 0.$$

In order to prove this claim we compute $m'(t)$,

$$m'(t) = -\int_\Omega \frac{\phi_t}{\phi^{p-1}}\,dx.$$

Using that $\phi(x,t)$ is a solution of the heat equation we get,

$$m'(t) = -\int_\Omega \frac{\Delta\phi}{\phi^{p-1}}\,dx,$$

which can be written as

$$m'(t) = -\int_\Omega \operatorname{div}\left(\frac{\nabla\phi}{\phi^{p-1}}\right)dx - \int_\Omega \phi^{-p}||\nabla\phi||^2\,dx.$$

The second integral is nonnegative and by Gauss Theorem we get,

$$m'(t) \leqslant -\int_{\partial\Omega} \frac{1}{\phi^{p-1}}\frac{\partial\phi}{\partial\nu}\,d\sigma = -c_1|\partial\Omega|.$$

This proves the claim.

We finish the argument by making the following remark. As $m(t)$ is decreasing, and positive, t must be less than or equal to $m(0)/(c_1|\partial\Omega|)$ and therefore,

$$T \leqslant \frac{1}{c_1|\partial\Omega|}\int_\Omega \left(\int_{\phi_0(x)}^{+\infty} \frac{1}{(s)^{p-1}}\,ds\right)dx.$$

This completes the proof of the existence of blow-up solutions for (5.5). Next, we use this result in a comparison argument:

Take λ large enough in order to make $u(x) > w_0(x)$. As u is a supersolution of (5.5) we can use a comparison principle, based mainly on Hopf Lemma, to obtain that

$$u(x) \geqslant w(x,t) \quad \forall\ t \in (0,T),$$

a contradiction. $\square$

6. Positive solutions

In this section we prove that there exists at least two positive solutions for λ small, in the concave–convex subcritical case; namely $1<q<2<p<p^*$, and the concave reaction term acting on the boundary of the domain, i.e., problem (1.1).

Consider the functional,

$$\mathscr{F}_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2\,dx - \frac{1}{p}\int_\Omega (u_+)^p\,dx - \frac{\lambda}{q}\int_{\partial\Omega}(u_+)^q\,d\sigma. \tag{6.1}$$

For this functional we will find two nontrivial critical points, the first one by minimization and the second one by a mountain pass argument. This can be done only if λ is small.

These nontrivial critical points are weak solutions of

$$\begin{cases} -\Delta u + u = (u_+)^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda (u_+)^{q-1} & \text{on } \partial\Omega. \end{cases}$$

The maximum principle and the Hopf Lemma say that the minimum of u must be positive, therefore these two weak solutions are in fact weak solutions of (1.1).

Moreover, let

$$\Lambda = \sup\{\lambda \text{ such that (1.1) has a nontrivial positive solution}\}. \tag{6.2}$$

In the previous section we have seen that Λ is bounded.

We will see that for every $\lambda<\Lambda$ there exists at least two positive solutions. To prove this fact let us first see that for $\lambda<\Lambda$ there exists a minimal positive solution.

As mentioned in the previous section, there exists $\underline{u}$ a nontrivial solution of

$$\begin{cases} \underline{u}_t = \Delta \underline{u} & \text{in } \Omega\times(0,T), \\ \frac{\partial \underline{u}}{\partial \nu} = \underline{u}^{q-1} & \text{on } \partial\Omega\times(0,T), \\ \underline{u}(x,0) = 0. \end{cases} \tag{6.3}$$

with $\underline{u}(x,t)>0$ for all $t>0$, see [12].

Let us take $\mu\leqslant\lambda$ and consider

$$\underline{v}(x,t) = \mu e^{-t}\underline{u}(x,t).$$

This function $\underline{v}$ is a subsolution of

$$\begin{cases} v_t = \Delta v - v + v^{p-1} & \text{in } \Omega\times(0,T), \\ \frac{\partial v}{\partial \nu} = \mu v^{q-1} & \text{on } \partial\Omega\times(0,T), \\ v(x,0) = 0, \end{cases} \tag{6.4}$$

that satisfies $\underline{v}(x,t)>0$, for every $t>0$.

Let us point out that problem (6.4) has a comparison principle, which says that a nonnegative subsolution and a nonnegative supersolution, with ordered initial data remain ordered for any time. The proof of this fact follows by using the strong maximum principle inside Ω, and Hopf Lemma at the boundary.

Assuming the existence of a positive solution (and therefore, by maximum principle, strictly positive) to problem (1.1) u, using the comparison principle and the usual iterative argument starting from the subsolution $\underline{v}$ we find a solution of (6.4), v, which satisfies $u(x) \geqslant v(x,t) \geqslant \underline{v}(x,t) > 0$, for every $t > 0$.

Let U be a function in the ω-limit set of $v(x,t)$. Taking into account that the problem has a Lyapunov functional, we see that this function U must be a stationary solution of (6.4). Therefore U is a solution to (1.1) and has to be positive and minimal since $u(x) \geqslant v(x,t)$ for any positive solution to (1.1).

We have proved the following result:

Lemma 6.1. *Let* $1 < q < 2 < p$ *and* $\lambda < \Lambda$ (Λ *defined by* (6.2). *Then there exists a positive minimal solution of* (1.1).

Next, let us prove that minimal solutions are strictly ordered.

Lemma 6.2. *Given* $0 < \lambda < \Lambda$, *consider* λ_1, λ_2 *such that* $0 < \lambda_1 < \lambda < \lambda_2 < \Lambda$ *and minimal positive solutions* u_1 *and* u_2 *corresponding to* λ_1 *and* λ_2, *respectively. It holds*

$$u_1(x) < u_2(x) \quad \text{in } \bar{\Omega}.$$

Proof. Since u_2 is a supersolution of (1.1) with λ_1, it is immediate that $u_1 \leqslant u_2$ and $u_1 \not\equiv u_2$. We can conclude the strict inequality using the maximum principle together with the Hopf Lemma. $\square$

Let

$$\tilde{f}(x,s) = \begin{cases} u_1^{p-1}(x) & s \leqslant u_1(x), \\ s^{p-1} & u_1(x) < s < u_2(x), \\ u_2^{p-1}(x) & s \geqslant u_2(x), \end{cases}$$

and

$$\tilde{g}(x,s) = \begin{cases} u_1^{q-1}(x) & s \leqslant u_1(x), \\ s^{q-1} & u_1(x) < s < u_2(x), \\ u_2^{q-1}(x) & s \geqslant u_2(x), \end{cases}$$

and $\tilde{F}$, $\tilde{G}$ two primitives (with respect to s) of $\tilde{f}$ and $\tilde{g}$ respectively. We consider the following auxiliary functional,

$$\tilde{\mathscr{F}}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + |u|^2 \, dx - \int_\Omega \tilde{F}(x,u) \, dx - \lambda \int_{\partial\Omega} \tilde{G}(x,u) \, d\sigma.$$

In particular, notice that if u is a critical point of $\tilde{F}$ such that $u_1 < u < u_2$, then u is also a critical point of F . We have,

Lemma 6.3. *For every $\lambda < \Lambda$ the functional $\mathcal{F}$ has a solution, $\tilde{u}$, that is a local minimum in the C^0 topology.*

Proof. Let $\lambda_1 < \lambda < \lambda_2 < \Lambda$ and u_1, u_2 two minimal solutions corresponding to λ_1 and λ_2, respectively. From Lemma 6.2 we have that $u_1 < u_2$ in $\bar{\Omega}$. We set $\tilde{\mathcal{F}}$ as before.

One can check that this functional $\tilde{\mathcal{F}}$ has a global minimum at some u_0 that belongs to $C(\bar{\Omega})$ (by our regularity results, see Section 4). Using again the maximum principle and Hopf's Lemma we get that $u_1 < u_0 < u_2$ in $\bar{\Omega}$. Hence u_0 is a minimum for $\mathcal{F}$ in C^0 topology. $\square$

Remark 6.1. We observe that u_0 has negative energy $\mathcal{F}(u_0) < 0$.

Next, we fix $\lambda \in (0, \Lambda)$. Let us look for a second positive solution of the form

$$u = u_0 + v,$$

with $v > 0$. We will follow closely the ideas in [3].

The function v satisfies

$$\begin{cases} -\Delta v + v = (u_0 + v)^{p-1} - u_0^{p-1} & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda (u_0 + v)^{q-1} - \lambda u_0^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Let

$$\bar{f}(x, s) = \begin{cases} (u_0 + s)^{p-1} - u_0^{p-1} & s \geqslant 0, \\ 0 & s < 0, \end{cases}$$

and

$$\bar{g}(x, s) = \begin{cases} (u_0 + s)^{q-1} - u_0^{q-1} & s \geqslant 0, \\ 0 & s < 0. \end{cases}$$

With these functions $\bar{f}$ and $\bar{g}$ we define

$$\bar{\mathcal{F}}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + |u|^2 \, dx - \int_\Omega \bar{F}(x, u) \, dx - \lambda \int_{\partial\Omega} \bar{G}(x, u) \, d\sigma,$$

where $\bar{F}$ and $\bar{G}$ are primitives of $\bar{f}$ and $\bar{g}$ respectively.

Lemma 6.4. *Given $\lambda \in (0, \Lambda)$, the functional $\bar{\mathcal{F}}$ has a local minimum at $v \equiv 0$ in the $H^1(\Omega)$ topology.*

Proof. First, notice that $v=0$ is a local minimum in C^0 topology. Then, we will follow the ideas by Brezis–Nirenberg in [6], proving that $v=0$ has to be also a local minimum in $H^1(\Omega)$.

Suppose, by contradiction, that there exists a sequence $\{v_\varepsilon\} \subset H^1(\Omega) - \{0\}$ such that $||v_\varepsilon||_{H^1} \leqslant \varepsilon$ and $\bar{\mathscr{F}}(v_\varepsilon) < \bar{\mathscr{F}}(0)$. We can assume that

$$\bar{\mathscr{F}}(v_\varepsilon) = \min_{v \in B_\varepsilon} \bar{\mathscr{F}}(v),$$

where B_ε is the ball of radius ε in $H^1(\Omega)$.

In particular, there exists a Lagrange multiplier μ_ε such that v_ε satisfies

$$\langle \bar{\mathscr{F}}'(v_\varepsilon), \psi \rangle = \mu_\varepsilon \langle v_\varepsilon, \psi \rangle \tag{6.6}$$

(notice that the left-hand side means duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$, while the right-hand side means scalar product in $H^1(\Omega)$).

Moreover, since we are assuming that v_ε is a minimum in B_ε, then

$$\mu_\varepsilon = \frac{\langle \bar{\mathscr{F}}'(v_\varepsilon), v_\varepsilon \rangle}{||v_\varepsilon||^2_{H^1}} \leqslant 0.$$

Eq. (6.6) is the weak form of problem

$$\begin{cases} -\Delta v_\varepsilon + v_\varepsilon = \frac{1}{1-\mu_\varepsilon} \bar{f}(x, v_\varepsilon), \\ \frac{\partial v_\varepsilon}{\partial \nu} = \frac{\lambda}{1-\mu_\varepsilon} \bar{g}(x, v_\varepsilon). \end{cases}$$

Then, since $\mu_\varepsilon \leqslant 0$, $||v_\varepsilon||_{H^1}(\Omega) \leqslant \varepsilon$, and

$$|\bar{f}(x,s)| \leqslant C(1+|s|^{p-1}),$$

$$|\bar{g}(x,s)| \leqslant C(1+|s|^{q-1})$$

using the regularity results and methods in Section 4, we can conclude uniform C^α estimates for the sequence $\{v_\varepsilon\}$. Hence, by Ascoli–Arzela, for a subsequence we get uniform convergence, and since $v_\varepsilon \to 0$ in $H^1(\Omega)$, this implies that $v_\varepsilon \to 0$ uniformly. And this is a contradiction, because 0 has to be a local minimum in C^0. $\square$

Lemma 6.5. *The functional $\bar{\mathscr{F}}$ verifies the Palais–Smale condition.*

Proof. It follows as in Lemma 2.3. $\square$

Now let us prove that there exists a second solution using the mountain pass lemma.

Lemma 6.6. *Given $\lambda \in (0, \Lambda)$, there exists a critical point $v \neq 0$ of the functional $\bar{\mathscr{F}}$.*

Proof. Since $p>2$ we get that are a large t and some v such that

$$\bar{\mathscr{F}}(tv)<0.$$

Hence, since $\bar{\mathscr{F}}$ verifies the Palais–Smale condition, and satisfies the right geometric conditions, the existence of a critical point $v\neq 0$ follows from the mountain pass lemma (in the improved version by Ghoussoub–Preiss (see [17])). $\square$

Let us prove that for $\lambda = \Lambda$ there exists at least one nontrivial positive solution. The idea is to take the limit as $\lambda \nearrow \Lambda$ of $u_{0,\lambda}$, the sequence of minimums of the energy functional $\mathscr{F}$ provided by Lemma 6.3. By remark 6.1 we have that the energy is negative, $\mathscr{F}(u_{0,\lambda})\leqslant 0$, hence

$$\begin{aligned} 0 \geqslant \mathscr{F}(u_{0,\lambda}) &= \mathscr{F}(u_{0,\lambda}) - \frac{1}{p}\langle \mathscr{F}'(u_{0,\lambda}), u_{0,\lambda}\rangle \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)||u_{0,\lambda}||^2_{H^1(\Omega)} - \lambda\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\partial\Omega}(u_{0,\lambda})^q. \end{aligned}$$

Therefore, there exits a constant C such that, for $\Lambda/2<\lambda<\Lambda$,

$$||u_{0,\lambda}||^2_{H^1(\Omega)} \leqslant C||u_{0,\lambda}||^q_{L^q(\partial\Omega)}.$$

Using the Sobolev trace theorem we get

$$||u_{0,\lambda}||_{H^1(\Omega)} \leqslant C,$$

and hence we have a converging subsequence with a weak limit u_Λ in $H^1(\Omega)$. We can pass to the limit in the weak form of the equation and obtain that u_Λ is a solution of (1.1) with $\lambda = \Lambda$. To see that u_Λ is nontrivial, we only have to observe that $u_{0,\lambda} \geqslant u_\lambda > 0$, the sequence of minimal solutions that increases with λ.

7. The critical case I. $p = p^* = 2N/(N-2)$, $2<q<2(N-1)/(N-2)$

In this section we study problem (1.1) with p critical, $p^* = 2N/(N-2)$ and q subcritical and superlinear, $2<q<2(N-1)/(N-2)$.

Let us recall that we are looking for critical points of the functional

$$\mathscr{F}_+(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2\,dx - \frac{1}{p^*}\int_\Omega |u_+|^{p^*}\,dx - \frac{\lambda}{q}\int_{\partial\Omega}|u_+|^q\,d\sigma. \tag{7.1}$$

To prove our existence result, since we have lost the compactness in the inclusion $H^1(\Omega)\hookrightarrow L^{2N/(N-2)}(\Omega)$, we can no longer expect the Palais–Smale condition to hold. Anyway we can prove a *local Palais–Smale condition* that will hold for $\mathscr{F}_+(u)$ below a certain value of energy.

The technical result used here, the concentration compactness method, is mainly due to [21,22]. Let

$$S = \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + |u|^2}{\left(\int_\Omega |u|^{p^*}\right)^{2/p^*}}. \tag{7.2}$$

Now we can prove a local Palais–Smale condition below some energy level, related to the value of S.

Lemma 7.1. *Let $u_j \in H^1(\Omega)$ be a Palais–Smale sequence for the functional $\mathcal{F}_+$ given in* (7.1) *with energy level $c < \frac{1}{N} S^{\frac{N}{2}}$, that is*

$$\mathcal{F}_+(u_j) \to c, \quad \mathcal{F}'_+(u_j) \to 0,$$

then there exists a convergent subsequence $u_{j_k} \to u$ in $H^1(\Omega)$. Here S is given by formula (7.2).

Proof. As before we can prove that u_j is bounded in $H^1(\Omega)$. Hence, using the results of [21,22], there exists a subsequence, that we still call u_j, and some points $x_1, \dots, x_l \in \bar{\Omega}$, such that

$$\begin{aligned}
&u_j \rightharpoonup u, \quad \text{weakly in } H^1(\Omega),\\
&u_j \to u, \quad \text{strongly in } L^q(\partial\Omega),\\
&|\nabla u_j|^2 \rightharpoonup d\mu \geqslant |\nabla u|^2 + \sum_{k=1}^{l} \mu_k \delta_{x_k},\\
&|(u_j)_+|^{p^*} \rightharpoonup d\eta = |u_+|^{p^*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.
\end{aligned}$$

Let $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\phi \equiv 1 \text{ in } B(x_k, \varepsilon), \quad \phi \equiv 0 \text{ in } B(x_k, 2\varepsilon)^c, \quad |\nabla \phi| \leqslant \frac{2}{\varepsilon},$$

where x_k belongs to the support of the singular part of $d\eta$. Notice that x_k may lie at the boundary.

Consider $\{u_j \phi\}$. Obviously this sequence is bounded in $H^1(\Omega)$. As $\mathcal{F}'_+(u_j) \to 0$ in $H^1(\Omega)'$, we obtain that

$$\lim_{j \to \infty} \langle \mathcal{F}'_+(u_j); \phi u_j \rangle = 0.$$

Then we have

$$0 = \lim_{j \to \infty} \int_\Omega \nabla u_j \nabla(\phi u_j)\, dx + \int_\Omega \phi u_j^2 - \int_\Omega |(u_j)_+|^{p^*} \phi - \lambda \int_{\partial\Omega} |(u_j)_+|^q \phi.$$

Hence

$$\lim_{j\to\infty}\int_\Omega \nabla u_j\nabla(\phi u_j)\,dx = -\int_\Omega \phi u^2+\int_\Omega \phi\,d\eta+\lambda\int_{\partial\Omega}|u_+|^q\phi.$$

Therefore,

$$\lim_{j\to\infty}\int_\Omega u_j\nabla u_j\nabla\phi\,dx = -\int_\Omega \phi u^2+\int_\Omega \phi\,d\eta+\lambda\int_{\partial\Omega}|u_+|^q\phi-\int_\Omega \phi\,d\mu.$$

Now, by Hölder inequality and weak convergence, we obtain

$$\begin{aligned}
0 &\leqslant \lim_{j\to\infty}\left|\int_\Omega u_j\nabla u_j\nabla\phi\,dx\right|\\
&\leqslant \lim_{j\to\infty}\left(\int_\Omega |\nabla u_j|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla\phi|^2|u_j|^2\,dx\right)^{1/2}\\
&\leqslant C\left(\int_{B(x_k,2\varepsilon)\cap\Omega}|\nabla\phi|^2|u|^2\,dx\right)^{1/2}\\
&\leqslant C\left(\int_{B(x_k,2\varepsilon)\cap\Omega}|\nabla\phi|^N\,dx\right)^{1/N}\left(\int_{B(x_k,2\varepsilon)\cap\Omega}|u|^{2N/(N-2)}\,dx\right)^{(N-2)/2N}\\
&\leqslant C\left(\int_{B(x_k,2\varepsilon)\cap\Omega}|u|^{2N/(N-2)}\,dx\right)^{(N-2)/2N}\to 0\quad\text{as } \varepsilon\to 0.
\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0}\left[-\int_\Omega \phi u^2+\int_\Omega \phi\,d\eta+\lambda\int_{\partial\Omega}|u_+|^q\phi-\int_\Omega \phi\,d\mu\right]=\eta_k-\mu_k=0.$$

We conclude that

$$\eta_k=\mu_k.$$

On the other hand, by (7.2) it holds that

$$||v||_{L^{2N/(N-2)}(\Omega)}S^{1/2}\leqslant ||v||_{H^1(\Omega)}$$

for all $v\in H^1(\Omega)$. Then

$$\left(\int_\Omega |u_j\phi|^{2N/(N-2)}\right)^{(N-2)/2N}S^{1/2}\leqslant\left(\int_\Omega |\nabla(u_j\phi)|^2+|u_j\phi|^2\right)^{1/2}.$$

Hence

$$\left(\int_\Omega |\phi|^{2N/(N-2)}\,d\eta\right)^{(N-2)/2N}S^{1/2}\leqslant\left(\int_\Omega \phi^2\,d\mu+|\nabla\phi|^2u^2+|u\phi|^2\right)^{1/2}.$$

Then

$$\eta_k^{(N-2)/N} S \leqslant \mu_k. \tag{7.3}$$

Therefore we must have

$$\mu_k = \eta_k = 0 \quad \text{or} \quad \eta_k \geqslant S^{N/2}.$$

If we have the last possibility, that is $\eta_k \geqslant S^{N/2}$ for some k, then

$$\begin{aligned} c = \lim_{j\to\infty} \mathscr{F}_+(u_j) &= \lim_{j\to\infty} \mathscr{F}_+(u_j) - \frac{1}{2}\langle \mathscr{F}'_+(u_j); \phi u_j\rangle \\ &= \frac{1}{N}\int_\Omega |u_+|^{2N/(N-2)} + \frac{1}{N}\int_\Omega d\eta + \lambda\left(\frac{1}{2} - \frac{1}{q}\right)\int_{\partial\Omega} |u_+|^q \\ &\geqslant \frac{1}{N} S^{N/2}, \end{aligned}$$

a contradiction that proves that all the η_k vanishes. It follows that

$$\int_\Omega |(u_j)_+|^{2N/(N-2)} \to \int_\Omega |u_+|^{2N/(N-2)}$$

and therefore $(u_j)_+ \to u_+$ in $L^{2N/(N-2)}(\Omega)$. Now the proof finishes using the continuity of the operator $\mathscr{A} = (-\Delta + I)^{-1}$. □

Proof of Theorem 1.3. In view of the previous result, we seek for critical values below level c. For that purpose, we want to use the Mountain Pass Lemma (see [4]). Hence we have to check the following conditions:

(1) There exist constants $R, r > 0$ such that if $||u||_{H^1(\Omega)} = R$, then $\mathscr{F}_+(u) > r$.

(2) There exists $v_0 \in H^1(\Omega)$ such that $||v_0||_{H^1(\Omega)} > R$ and $\mathscr{F}_+(v_0) < r$.

Let us first check (1). By the Sobolev embedding Theorem we have,

$$\begin{aligned} \mathscr{F}_+(u) &= \frac{1}{2}||u||^2_{H^1(\Omega)} - \frac{(N-2)}{2N}\int_\Omega |u_+|^{2N/(N-2)} - \frac{\lambda}{q}\int_{\partial\Omega} |u_+|^q \\ &\geqslant \frac{1}{2}||u||^2_{H^1(\Omega)} - \frac{(N-2)}{2N} C||u||^{2N/(N-2)}_{H^1(\Omega)} - \frac{\lambda S}{q}||u||^q_{H^1(\Omega)}. \end{aligned}$$

Let

$$g(t) = \frac{1}{2}t^2 - \frac{(N-2)}{2N} Ct^{2N/(N-2)} - \frac{\lambda S}{q} Ct^q.$$

It is easy to check that $g(R) > r$ for some $R, r > 0$.

Condition (2) is immediate as for a fixed $w \in H^1(\Omega)$ with $w|_{\partial\Omega} \not\equiv 0$ we have

$$\lim_{t\to\infty} \mathcal{F}_+(tw) = -\infty.$$

Now the candidate for critical value according to the Mountain Pass Theorem is

$$c = \inf_{\phi\in\mathscr{C}} \sup_{t\in[0,1]} \mathcal{F}_+(\phi(t)),$$

where $\mathscr{C} = \{\phi : [0,1] \to H^1(\Omega);\ \phi$ is continuous and $\phi(0) = 0,\ \phi(1) = v_0\}$. The problem is to show that $c < \frac{1}{N} S^{\frac{N}{2}}$ in order to apply the local Palais–Smale condition.

First, let us prove the easiest case: an existence result for λ large.

We fix $w \in H^1(\Omega)$ with $||w||_{L^{2N/(N-2)}(\Omega)} = 1$, and define $h(t) = \mathcal{F}_+(tw)$. We want to study the maximum of h. As $\lim_{t\to\infty} h(t) = -\infty$ it follows that there exists a $t_\lambda > 0$ such that $\sup_{t>0} \mathcal{F}_+(tw) = h(t_\lambda)$. Differentiating we obtain,

$$0 = h'(t_\lambda) = t_\lambda ||w||^2_{H^1(\Omega)} - t_\lambda^{(N+2)/(N-2)} - \lambda t_\lambda^{q-1} ||w||^q_{L^q(\partial\Omega)},$$

from where it follows that

$$||w||^2_{H^1(\Omega)} = t_\lambda^{4/(N-2)} + \lambda t_\lambda^{q-2} ||w||^q_{L^q(\partial\Omega)}.$$

Hence

$$t_\lambda \leqslant ||w||^{(N-2)/2}_{H^1(\Omega)}.$$

As $t_\lambda^{\frac{4}{N-2}-q+2} + \lambda ||w||^q_{L^q(\partial\Omega)} \to +\infty$ when $\lambda \to +\infty$, we obtain that

$$\lim_{\lambda\to\infty} t_\lambda = 0. \tag{7.4}$$

On the other hand, it is easy to check that if $\lambda > \tilde{\lambda}$ it must be $\mathcal{F}_+(t_{\tilde{\lambda}} w) \geqslant \mathcal{F}_+(t_\lambda w)$, so by (7.4) we get

$$\lim_{\lambda\to\infty} \mathcal{F}_+(t_\lambda w) = 0.$$

But this identity means that there exists a constant $\lambda_1 > 0$ such that if $\lambda \geqslant \lambda_1$, then

$$\sup_{t\geqslant 0} \mathcal{F}_+(tw) < \frac{1}{N} S^{\frac{N}{2}},$$

and the proof is finished if we choose $v_0 = t_0 w$ with t_0 large in order to have $\mathcal{F}_+(t_0 w) < 0$.

Now we deal with the more delicate problem: we prove that there exists a solution for every $\lambda > 0$. We recall that we have

$$S = \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + |u|^2}{\left(\int_\Omega |u|^{p^*}\right)^{2/p^*}}, \tag{7.5}$$

and as before we want to find a path with energy $c < \frac{1}{N} S^{\frac{N}{2}}$ in order to apply the local Palais–Smale condition.

Let

$$S_{\text{sob}} = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p^*}\right)^{2/p^*}}, \tag{7.6}$$

that is the well known Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

First we state the following result, the proof can be found in [2].

Lemma 7.2. *Let S the constant given by* (7.5) *and S_{sob} the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{p^*}(\Omega)$, then there exists $\eta > 0$ depending on Ω such that*

$$S < \frac{S_{\text{sob}}}{2^{2/N}} - \eta.$$

Proof. This follows concentrating a family of Sobolev minimizers at a point $x_0 \in \partial\Omega$. See [2] for the details. $\square$

This lemma, jointly with a careful reading of the concentration compactness results by Lions (see [21–23]), allows us to improve estimate (7.3):

Lemma 7.3. *Let u_j be a sequence in $H^1(\Omega)$ such that*

$$u_j \rightharpoonup u, \quad \textit{weakly in } H^1(\Omega),$$

$$u_j \to u, \quad \textit{strongly in } L^q(\partial\Omega),$$

$$|\nabla u_j|^2 \rightharpoonup d\mu \geqslant |\nabla u|^2 + \sum_{k=1}^{l} \mu_k \delta_{x_k},$$

$$|u_j|^{p^*} \rightharpoonup d\eta = |u|^{p^*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.$$

Then

$$\mu_k \geqslant S_{\text{sob}}\ \eta_k^{2/p^*}, \quad x_k \in \ int\ \Omega$$

$$\mu_k \geqslant \frac{S_{\text{sob}}}{2^{2/N}}\ \eta_k^{2/p^*}, \quad x_k \in \partial\Omega.$$

Proof. First, (7.3) implies that there are a finite number of points x_k.

Notice that if $x_k \in \mathrm{int}\Omega$ the conclusion follows by the results of [21,22]. Hence let us assume that $x_k \in \partial\Omega$.

Let $\phi \in C^\infty(\mathbb{R}^N)$ such that

$$\phi \equiv 1 \quad \text{in } B(x_k, \varepsilon), \qquad \phi \equiv 0 \quad \text{in } B(x_k, 2\varepsilon)^c, \qquad |\nabla\phi| \leqslant \frac{2}{\varepsilon},$$

where x_k belongs to the support of $d\eta$, and ε is small, in such a way that x_k is the unique singular point contained in the support of ϕ. And let

$$u_j - u = \tilde{u}_j, \quad v_j = \phi\tilde{u}_j.$$

We have that

$$\int_\Omega |\nabla v_j|^2 = \int_\Omega |\nabla(\phi\tilde{u}_j)|^2 = \int_\Omega \phi^2|\nabla\tilde{u}_j|^2 + \int_\Omega |\nabla\phi|^2(\tilde{u}_j)^2 + \int_\Omega \phi\tilde{u}_j\nabla\tilde{u}_j\nabla\phi.$$

The first term converges as $j \to \infty$ to μ_k,

$$\int_\Omega \phi^2|\nabla\tilde{u}_j|^2 \to \int_\Omega \phi^2 d\mu = \mu_k.$$

The second term goes to zero as $j \to \infty$,

$$\int_\Omega |\nabla\phi|^2(\tilde{u}_j)^2 \to 0.$$

The last term also goes to zero as $j \to \infty$

$$\begin{aligned}\int_\Omega \phi\tilde{u}_j\nabla\tilde{u}_j\nabla\phi &\leqslant \left(\int_\Omega \phi^2\tilde{u}_j^2\right)^{1/2}\left(\int_\Omega |\nabla\phi|^2|\nabla\tilde{u}_j|^2\right)^{1/2} \\ &\leqslant \left(\int_\Omega \phi^2\tilde{u}_j^2\right)^{1/2}\frac{K}{\varepsilon} \to 0.\end{aligned}$$

On the other hand

$$\left(\int_\Omega \phi^{p^*}(\tilde{u}_j)^{p^*}\right)^{2/p^*} \to \eta_k^{2/p^*}, \quad j \to \infty.$$

Now we recall a crucial estimate from [32]. Taking into account that the boundary of the domain is smooth, we have that there exists a function $f(\varepsilon)$ with $f(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that

$$\frac{S_{\text{sob}}}{2^{2/N}} - f(\varepsilon) \leqslant \frac{\int_\Omega |\nabla v_j|^2}{\left(\int_\Omega v_j^{p^*}\right)^{2/p^*}} \to \frac{\mu_k}{\eta_k^{2/p^*}}.$$

We end the proof just taking $\varepsilon\to 0$. $\square$

With this Lemma we can prove that the constant S is attained.

Lemma 7.4. *There exists a positive function $v_S \in H^1(\Omega)$ such that*

$$S = \frac{\int_\Omega |\nabla v_S|^2 + |v_S|^2}{\left(\int_\Omega |v_S|^{p^*}\right)^{2/p^*}}. \tag{7.7}$$

Proof. Let u_j be a minimizing sequence for (7.5). We normalize the sequence imposing that

$$||u_j||_{L^{p^*}(\Omega)} = 1.$$

As u_j is bounded in $H^1(\Omega)$ we have that (up to a subsequence)

$$u_j \rightharpoonup v_S, \quad \text{weakly in } H^1(\Omega),$$

$$|\nabla u_j|^2 \rightharpoonup d\mu \geqslant |\nabla v_S|^2 + \sum_{k=1}^{l} \mu_k \delta_{x_k},$$

$$|u_j|^{p^*} \rightharpoonup d\eta = |v_S|^{p^*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.$$

As $||u_j||_{L^{p^*}(\Omega)} = 1$ we have that

$$\int_\Omega |v_S|^{p^*} + \sum_{k=1}^{l} \eta_k = 1.$$

As

$$\sum_{k=1}^{l} \eta_k \leqslant 1,$$

and since $2/p^* < 1$, we get

$$\sum_{k=1}^{l} \eta_k^{2/p^*} \geqslant \left(\sum_{k=1}^{l} \eta_k \right)^{2/p^*}.$$

Hence the previous Lemma implies that

$$S = \lim_{n \to \infty} \int_\Omega |\nabla u_j|^2 + |u_j|^2 \geqslant \int_\Omega |\nabla v_S|^2 + |v_S|^2 + \sum_{k=1}^{l} \mu_j$$
$$\geqslant S(||v_S||^{p^*})^{2/p^*} + \frac{S_{\text{sob}}}{2^{2/N}} \sum_{k=1}^{l} \eta_j^{2/p^*} \geqslant S\left[(||v_S||^{p^*})^{2/p^*} + \sum_{k=1}^{l} \eta_j^{2/p^*} \right] \geqslant S.$$

If there exists any concentration phenomena then, using Lemma 7.2 the inequality is strict, a contradiction. We conclude that $\eta_j = \mu_j = 0$ and hence

$$S = \int_\Omega |\nabla v_S|^2 + |v_S|^2,$$

with

$$\int_\Omega |v_S|^{p^*} = 1.$$

This proves that S is attained at v_S. Taking absolute value if necessary we may assume that $v_S \geqslant 0$. $\square$

Now let $v_S \in H^1(\Omega)$ be a function where S is attained. We normalize v_S imposing that

$$||v_S||_{L^{p^*}(\Omega)} = 1.$$

Let us prove that

$$||v_S||_{L^q(\partial\Omega)} \geqslant c > 0. \tag{7.8}$$

To see this fact we argue by contradiction. Assume that $v_S \equiv 0$ on $\partial\Omega$, hence $v_S \in H_0^1(\Omega)$. The definition of the Sobolev constant, (7.6), gives

$$S_{\text{sob}} = S_{\text{sob}} \left(\int_\Omega |v_S|^{p^*} \right)^{2/p^*} \leqslant \int_\Omega |\nabla v_S|^2.$$

Therefore

$$S_{\text{sob}} \leqslant \int_\Omega |\nabla v_S|^2 \leqslant \int_\Omega |\nabla v_S|^2 + |v_S|^2 = S,$$

a contradiction with Lemma 7.2 that proves (7.8).

Now we choose $h(t) = \mathscr{F}_+(tv_S)$. We want to study the maximum of h. As $\lim_{t\to\infty} h(t) = -\infty$ it follows that there exists $t_\lambda > 0$ such that $\sup_{t>0} h(t) = h(t_\lambda)$. Differentiating we obtain,

$$0 = h'(t_\lambda) = t_\lambda ||v_S||^2_{H^1(\Omega)} - t_\lambda^{(N+2)/(N-2)} - \lambda t_\lambda^{q-1} ||v_S||^q_{L^q(\partial\Omega)}$$

from where it follows that

$$S = ||v_S||^2_{H^1(\Omega)} = t_\lambda^{4/(N-2)} + \lambda t_\lambda^{q-2} ||v_S||^q_{L^q(\partial\Omega)}.$$

Hence

$$t_\lambda \leqslant ||v_S||^{(N-2)/2}_{H^1(\Omega)} = S^{(N-2)/4}.$$

Then,

$$\begin{aligned} h(t_\lambda) &= \mathscr{F}_+(t_\lambda v_S) = \frac{t_\lambda^2}{2} \int_\Omega |\nabla v_S|^2 + |v_S|^2 \, dx - \frac{t_\lambda^{p^*}}{p^*} \int_\Omega v_S^{p^*} \, dx - \frac{\lambda t_\lambda^q}{q} \int_{\partial\Omega} v_S^q \, d\sigma \\ &= \frac{t_\lambda^2}{2} S - \frac{1}{p^*} \left(t_\lambda^2 S - \lambda t_\lambda^q \int_{\partial\Omega} |v_S|^q \, d\sigma \right) - \frac{\lambda t_\lambda^q}{q} \int_{\partial\Omega} |v_S|^q \, d\sigma \\ &\leqslant t_\lambda^2 \left(\frac{1}{N} S - C t_\lambda^{q-2} \lambda \right) \leqslant \frac{1}{N} S^{N/2} - C. \end{aligned}$$

But this means that

$$\sup_{t\geqslant 0} \mathscr{F}_+(tv_S) < \frac{1}{N} S^{\frac{N}{2}},$$

and the proof is finished if we choose $v_0 = t_0 v_S$ with t_0 large in order to have $\mathscr{F}_+(t_0 v_S) < 0$. $\square$

8. The critical case II. $p = p^* = 2N/(N-2)$, $1 < q < 2$

In this section we prove that, in the critical case $p = p^*$ with a concave boundary term, i.e., $1 < q < 2$, there exists Λ such that there are at least two positive solutions for $\lambda < \Lambda$, at least one positive solution for $\lambda = \Lambda$ and no positive solution for $\lambda > \Lambda$.

As in Section 5 we consider the functional,

$$\mathscr{F}_+(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + |u|^2 \, dx - \frac{1}{p^*} \int_\Omega u_+^{p^*} \, dx - \frac{\lambda}{q} \int_{\partial\Omega} u_+^q \, d\sigma. \tag{8.1}$$

As before, nontrivial critical points are weak solutions of

$$\begin{cases} -\Delta u + u = u_+^{p^*-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u_+^{q-1} & \text{on } \partial\Omega. \end{cases}$$

The minimum principle and the Hopf Lemma say that if u is nontrivial the minimum of u must be positive, therefore nontrivial critical points of (8.1) are in fact positive weak solutions of (1.1). Moreover, let

$$\Lambda = \sup\{\lambda \text{ such that (1.1) has a nontrivial positive solution}\}.$$

We recall that in Section 4 we have proved that for λ large there is no positive solution, therefore Λ is finite.

Also, we recall from Section 6, that Lemmas 6.1 and 6.2 are valid if $p = p^*$ hence we have that

Lemma 8.1. *Let $1 < q < 2 < p = p^*$ and $\lambda < \Lambda$ such that there exists a positive solution of* (1.1). *Then there exists a positive minimal solution of* (1.1). *Moreover if u_1 and u_2 are two minimal solutions corresponding to $\lambda_1 < \lambda_2$, respectively, then*

$$u_1 < u_2$$

in $\bar{\Omega}$.

Moreover, also Lemma 6.3 holds true and therefore we have that,

Lemma 8.2. *For every $\lambda < \Lambda$ the functional $\mathscr{F}_+$ has a solution, u_0, that is a local minimum in the C^0 topology.*

As before, let us look for a second positive solution of the form

$$u = u_0 + v,$$

with $v > 0$. The problem that v satisfies is the following:

$$\begin{cases} -\Delta v + v = (u_0 + v)^{p^*-1} - u_0^{p^*-1} & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \lambda (u_0 + v)^{q-1} - \lambda u_0^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{8.2}$$

Let

$$\bar{f}(x,s) = \begin{cases} (u_0 + s)^{p^*-1} - u_0^{p^*-1} & s \geqslant 0, \\ 0 & s < 0, \end{cases}$$

and

$$\bar{g}(x,s) = \begin{cases} \lambda(u_0+s)^{q-1} - \lambda u_0^{q-1} & s \geqslant 0, \\ 0 & s < 0. \end{cases}$$

With these functions $\bar{f}$ and $\bar{g}$ we define

$$\bar{\mathscr{F}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2\,dx - \int_\Omega \bar{F}(u)\,dx - \int_{\partial\Omega} \bar{G}(u)\,d\sigma,$$

where $\bar{F}$ and $\bar{G}$ are primitives of $\bar{f}$ and $\bar{g}$, respectively.

Lemma 8.3. *The functional $\bar{\mathscr{F}}$ has a local minimum at $v \equiv 0$ in the $H^1(\Omega)$ topology.*

Proof. It follows as Lemma 6.4. □

Lemma 8.4. *If $u = 0$ is the only critical point of $\bar{\mathscr{F}}$ then the functional $\bar{\mathscr{F}}$ verifies the Palais–Smale condition below the level*

$$c^* = \frac{(S_{\mathrm{sob}})^{N/2}}{2N}.$$

Proof. It follows as in Lemma 7.1. Consider a Palais–Smale sequence u_n at level c, that is a sequence such that

$$\bar{\mathscr{F}}(u_n) \to c, \quad \bar{\mathscr{F}}'(u_n) \to 0.$$

One can prove that u_n is bounded and hence there exists a subsequence such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. Using this weak convergence we obtain that u is a critical point of $\bar{\mathscr{F}}$ and therefore $u = 0$. By the same arguments of Lemma 7.1 we have that u_n concentrates with $\eta_k = \mu_k$. Hence, using Lemma 7.3, we have that if

$$c < \frac{(S_{\mathrm{sob}})^{N/2}}{2N},$$

then u_n converges strongly, and no concentration phenomena takes place. □

Now let us prove that there exists a second solution.

Lemma 8.5. *There exists a critical point $v \neq 0$ of the functional $\bar{\mathscr{F}}$.*

Proof. By contradiction: assume that there is no solution $v \neq 0$. In this case, as in Section 6, we want to find a path with energy under the critical level c^*. To this end we choose v_S, a normalized minimizer of (7.5) in such a way that

$$||v_S||_{L^{p^*}(\Omega)} = 1.$$

We consider

$$h(t) = \bar{\mathscr{F}}(tv_S).$$

We want to study the maximum of h. As $\lim_{t\to\infty} h(t) = -\infty$ it follows that there exists a $t_\lambda > 0$ such that

$$\sup_{t>0} \bar{\mathscr{F}}(tv_S) = h(t_\lambda).$$

Differentiating we obtain,

$$0 = h'(t_\lambda) = t_\lambda ||v_S||^2_{H^1(\Omega)} - \int_\Omega ((u_0 + t_\lambda v_S)^{p^*-1} v_S - u_0^{p^*-1} v_S)\, dx$$
$$- \int_{\partial\Omega} ((u_0 + t_\lambda v_S)^{q-1} v_S - u_0^{q-1} v_S)\, d\sigma.$$

Using the elementary inequality

$$(1+x)^{p^*-1} - 1 \geqslant x^{p^*-1},$$

valid for any $x \geqslant 0$, it follows that

$$(u_0 + t_\lambda v_S)^{p^*-1} v_S - u_0^{p^*-1} v_S \geqslant t_\lambda^{p^*-2} v_S^{p^*},$$

and therefore,

$$||v_S||^2_{H^1(\Omega)} \geqslant t_\lambda^{4/(N-2)}.$$

In particular,

$$t_\lambda \leqslant ||v_S||^{(N-2)/2}_{H^1(\Omega)} = S^{\frac{N-2}{4}}.$$

Hence, using

$$(1+x)^{p^*} - p^* x \geqslant x^{p^*},$$

we conclude

$$\frac{(u_0+t_\lambda v_S)^{p^*}}{p^*}-u_0^{p^*-1}t_\lambda v_S \geqslant \frac{(t_\lambda v_S)^{p^*}}{p^*}$$

which leads to

$$\begin{aligned}\bar{\mathscr{F}}(t_\lambda v_S) &= \frac{t_\lambda^2}{2}\int_\Omega |\nabla v_S|^2+|v_S|^2\,dx-\int_\Omega\left(\frac{1}{p^*}(u_0+t_\lambda v_S)^{p^*}-u_0^{p^*-1}t_\lambda v_S\right)dx\\ &\quad -\lambda\int_{\partial\Omega}\left(\frac{1}{q}(u_0+t_\lambda v_S)^q-u_0^{q-1}t_\lambda v_S\right)d\sigma\\ &\leqslant \frac{1}{2}S^{\frac{N}{2}}-\frac{1}{p^*}t_\lambda^{p^*}-\lambda\int_{\partial\Omega}\left(\frac{1}{q}(u_0+t_\lambda v_S)^q-u_0^{q-1}t_\lambda v_S\right)d\sigma\\ &\leqslant \frac{1}{N}S^{\frac{N}{2}}-\lambda\int_{\partial\Omega}\left(\frac{1}{q}(u_0+t_\lambda v_S)^q-u_0^{q-1}t_\lambda v_S\right)d\sigma.\end{aligned}$$

Finally, taking into account that $\frac{1}{q}(u_0+t_\lambda v_S)^q-u_0^{q-1}t_\lambda v_S>0$, and using Lemma 7.2, we get

$$\sup_{t\geqslant 0}\bar{\mathscr{F}}(tv_S)<\frac{1}{N}S^{\frac{N}{2}}<\frac{S_{\text{sob}}}{2^{2/N}},$$

and the proof is finished if we choose $v_0=t_0v_S$ with t_0 large in order to have $\mathscr{F}(t_0v_S)<0$. □

Remark 8.1. Notice that Lemma 8.4 is just a step in a proof by contradiction: we prove that if $u\equiv 0$ is the unique critical point of $\bar{\mathscr{F}}$, then $\bar{\mathscr{F}}$ satisfies a Palais–Smale condition, and therefore, since it has the right geometry, it must have a Mountain-Pass solution. Therefore, $u\equiv 0$ is not the unique critical point, a contradiction which implies that in fact there exists a second solution. But we cannot say in general that this second solution is of Mountain-Pass type, nor that $\bar{\mathscr{F}}$ satisfies in general the Palais–Smale condition.

We finish this section observing that the arguments used in Section 6 to obtain a nontrivial solution for $\lambda=\Lambda$ also hold in this case.

9. Perturbative approach. Positive solutions for λ small

In this section we use the implicit function theorem to get the existence of a positive solution for λ small, which corresponds to a branch of solutions that converges to $u_0\equiv 1$ when λ goes to zero. The advantage of this procedure is that there is no restriction on p nor on q. On the other hand this method only gives a solution for λ small and there is also a restriction on the domains that we can deal with.

Let $K_1 : C^\alpha \to C^\alpha$ and $K_2 : C^\alpha \to C^\alpha$ be given by $K_1(f) = u_1$, $K_2(g) = u_2$ where u_1 and u_2 are weak solutions of

$$\begin{cases} -\Delta u_1 + u_1 = f & \text{in } \Omega, \\ \frac{\partial u_1}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u_2 + u_2 = 0 & \text{in } \Omega, \\ \frac{\partial u_2}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases}$$

respectively. The regularity results in Section 4 imply that K_1 and K_2 are well defined. With K_1 and K_2 we can define

$$\Phi : [0, +\infty) \times C^\alpha(\Omega) \to C^\alpha(\Omega)$$

as follows,

$$\Phi(\lambda, u)(v) = u - K_1(|u|^{p-2}u) - \lambda K_2(|u|^{q-2}u).$$

We want to solve

$$\Phi(\lambda, u) = 0.$$

In fact if (λ, u) verifies $\Phi(\lambda, u) = 0$ we get a weak solution of (1.1). We have

$$\Phi(0, 1) = 0.$$

We want to apply the implicit function theorem near the point $(0, 1)$. To this end we compute $Ker(D_u\Phi(0, 1))$. Then if $w \in Ker(D_u\Phi(0, 1))$, w is a weak solution of

$$\begin{cases} -\Delta w + w = (p-1)w & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9.1}$$

Hence we have to assume that $p - 1 \notin \sigma_{\text{Neu}}(-\Delta + I)$ to get that $KerD_u(\Phi)(0, 1) = 0$. Therefore, we can apply the implicit function theorem that provides a curve (λ, u_λ) of solutions of

$$\Phi(\lambda, u_\lambda) = 0,$$

with $u_\lambda \to 1$ in $C^\alpha(\bar{\Omega})$ as $\lambda \to 0$.

10. A bifurcation result. The case $q = 2$ with $2 < p < p^*$

In this section we deal with $q = 2$. In this case we have a bifurcation problem.

First let us see that if there exists a positive solution u then $\lambda < \lambda_1$. To see this fact just take as test function in the weak form of (1.1) φ_1, the positive eigenfunction associated to λ_1. We get

$$\int_\Omega \nabla u \nabla \varphi_1 + \int_\Omega u\varphi_1 = \int_\Omega u^{p-1}\varphi_1 + \lambda \int_{\partial\Omega} u^{q-1}\varphi_1.$$

Using that φ_1 is an eigenfunction associated to λ_1 we obtain

$$(\lambda_1 - \lambda)\int_{\partial\Omega} u^{q-1}\varphi_1 = \int_\Omega u^{p-1}\varphi_1 > 0,$$

and hence $\lambda < \lambda_1$.

We observe that we can apply the global bifurcation theorem of Rabinowitz [26], writing the problem as in the previous section

$$\Phi(\lambda, u)(v) = u - K_1(|u|^{p-2}u) - \lambda K_2(|u|^{q-2}u) = 0,$$

for $u \in C^\alpha$. In this way we obtain a branch of positive solutions emanating from $(\lambda_1, 0)$ to the left, that is with $\lambda < \lambda_1$.

From the previous a priori bound obtained in Section 3 we get that this branch of positive solutions must intersect $\lambda = 0$ proving that there exists a positive solution for every $0 < \lambda < \lambda_1$.

Acknowledgments

This research was started while the third author (JDR) was a visitor at Universidad Autónoma de Madrid. He is grateful to this institution for its hospitality and stimulating atmosphere.

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