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A Generalization of the Laguerre–Pólya Class of Entire Functions

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Let Θ be a set of real numbers unbounded on both sides and let B be a finite set of positive integers. We characterize the entire functions that can be uniformly approximated on bounded sets by polynomials of the form $\prod_{j \in B} p_j(z^j)$, where each $p_j(z)$ is a polynomial with zeros in Θ.

1. INTRODUCTION: PRELIMINARIES

A theorem of Weierstrass (see [1, pp. 155–159]) states that every entire function can be written as

$$f(z)=\lambda z^m e^{h(z)} \prod_{l \geqslant 1} \left(1-\frac{z}{a_l}\right) \exp\left(\frac{z}{a_l}+\cdots+\frac{1}{n_l}\left(\frac{z}{a_l}\right)^{n_l}\right),$$

where h is an entire function, m is a nonnegative integer, a_l are nonzero complex numbers, and $n_l \geqslant 0$ are integers so that

$$\sum_{l \geqslant 1} \left(\frac{R}{|a_l|}\right)^{n_l+1} < \infty \qquad \text{for all} \quad R>0.$$

When the sequence $n_l (l \geqslant 1)$ can be chosen to be constant, the smallest integer $n \geqslant 0$ such that $\sum_{l \geqslant 1} |a_l|^{-(n+1)} < \infty$ is called the genus of

$$\pi(z)=\prod_{l \geqslant 1} \left(1-\frac{z}{a_l}\right) \exp\left(\frac{z}{a_l}+\cdots+\frac{1}{n}\left(\frac{z}{a_l}\right)^{n}\right),$$

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and $\pi(z)$ is called a canonical product. Otherwise it is said that the corresponding product has infinite genus. A classical result of Laguerre [4] and Pólya [5] asserts that an entire function f can be uniformly approximated on bounded sets by polynomials with real zeros if and only if $f(z) = \lambda z^m e^{\alpha z + \gamma z^2} \pi(z)$, where $\lambda \in \mathbb{C}$, $m \geqslant 0$ is an integer, $\alpha \in \mathbb{R}$, $\gamma \leqslant 0$ and $\pi(z)$ is a canonical product with real zeros of genus at most 1. See [2] for a proof and several generalizations (see also [3]).

The purpose of this paper is to generalize this theorem in one of many possible directions. Given a finite set of positive integers B and a set of real numbers $\Theta \subset \mathbb{R}$ unbounded on both sides, we characterize the entire functions f that can be approximated by polynomials of the form $p(z) = \prod_{j \in B} p_j(z^j)$, where each $p_j(z)$ is a polynomial with zeros in Θ. The characterization involves a curious numerical function depending on B. We begin by defining this function and studying some of its properties. This will enable us to state the main result, Theorem 1.2. There follows a short discussion comparing our theorem with the Laguerre–Pólya Theorem and with a generalization of it given by Korevaar [2, Thm. 6.1]. Next we give some definitions and an elementary lemma that will be used during the rest of the paper. The proofs of necessity and sufficiency for Theorem 1.2 will be given in Sections 2 and 3, respectively. We have split the proofs into some preparatory lemmas and a final conclusive argument.

Let $\mathbb{N}$ denote the set of positive integers and let $j, s \in \mathbb{N}$. The notation $j \mid s$ will mean "j divides s" in $\mathbb{N}$.

DEFINITION. Let $B \subset \mathbb{N}$ be a finite set. The function $H_B : \mathbb{N} \to \{0, 1, 2\}$ is recursively defined by the following conditions.

(a) $H_B(s) = 1$ if $s \in B$ or there is $j \in B$ such that $j \mid s$, s/j is odd and $H_B(tj) = 1$ for all $t < s/j$.

(b) $H_B(s) = 2$ if $H_B(s) \neq 1$ but there is $j \in B$ such that $j \mid s$ and $H_B(tj) = 1$ for all $t < s/j$.

(c) $H_B(s) = 0$ if s has no divisor in B or for every $j \in B$ with $j \mid s$ there is $t < s/j$ such that $H_B(tj) = 2$.

Additionally, we define a function $V_B : B \to \mathbb{N}$ as

$$V_B(j) = \min\{t \in \mathbb{N} : H_B(tj) = 2\}, \qquad j \in B.$$

Remark 1.1. We can recover H_B from V_B by the following rules. $H_B(s) = 0$ if and only if s has no divisor in B or $jV_B(j) < s$ for every $j \in B$ that divides s. The complementary case arises when there is some $j \in B$ such that $j \mid s$ and $s \leqslant jV_B(j)$, that is, when $s = tj$ for some $j \in B$ and $t \leqslant V_B(j)$. We then have

$$H_B(s) = H_B(tj) = \begin{cases} 1 & \text{if} \quad t < V_B(j) \\ 2 & \text{if} \quad t = V_B(j) \end{cases}. \tag{1.1}$$

An immediate consequence is that $H_B(s) \neq 0$ only for finitely many values of s. More precisely, if $m = 2 \prod_{j \in B} j$ then every $j \in B$ divides m and m/j is even. Thus $H_B(m) \neq 1$, meaning that $H_B(m) = 2$ or 0. By (1.1) then $V_B(j) \leqslant m/j$ for every $j \in B$, and consequently $H_B(s) = 0$ for every $s > m$. An analogous application of (1.1) shows that $H_B(s) = 2$ if and only if there is some $j_0 \in B$ such that $j_0 V_B(j_0) = s$. In the latter case we also have that $jV_B(j) \leqslant s$ for every $j \in B$ such that $j \mid s$. Finally, observe that (1.1) and condition (a) in the definition of H_B imply that $V_B(j)$ is an even number for every $j \in B$.

Now we can establish the result that motivates this paper. If f is an entire function denote by $Z(f)$ the family of its zeros, where repetitions are allowed according to multiplicities. Also, if $B \subset \mathbb{N}$ and $\Theta \subset \mathbb{R}$, let $\mathscr{A}_B(\Theta)$ denote the class of entire functions that can be uniformly approximated on bounded sets by polynomials of the form $\prod_{j \in B} p_j(z^j)$, where each $p_j(z)$ has its zeros in Θ. If $\bar{\Theta}$ denotes the closure of Θ then it is clear that $\mathscr{A}_B(\Theta) = \mathscr{A}_B(\bar{\Theta})$. Thus, we can restrict our analysis to closed sets $\Theta \subset \mathbb{R}$.

THEOREM 1.2. *Let $B \subset \mathbb{N}$ be a finite set and let $\Theta \subset \mathbb{R}$ be a closed set unbounded on both sides. Then $f \in \mathscr{A}_B(\Theta)$ if and only if f has the form*

$$f(z) = \lambda z^m \exp\left(\sum_{s \geqslant 1} d_s z^s \right) \prod_{j \in B} \pi_j(z^j), \tag{1.2}$$

where λ is a complex constant, $m = \sum_{j \in B} n_j j$ for some integers $n_j \geqslant 0$ if $0 \in \Theta$ and $m = 0$ if $0 \notin \Theta$, $d_s = 0$ if $H_B(s) = 0$, $d_s \leqslant 0$ if $H_B(s) = 2$, d_s is any real number if $H_B(s) = 1$, and each $\pi_j(z)$ is a canonical product of genus $< V_B(j)$ with $Z(\pi_j) \subset \Theta$.

If $B = \{1\}$ and $\Theta = \mathbb{R}$ then $H_B(1) = 1$, $H_B(2) = 2$, $H_B(s) = 0$ for $s \geqslant 3$ and $V_B(1) = 2$. Therefore Theorem 1.2 says that $\mathscr{A}_{\{1\}}(\mathbb{R})$ is the Laguerre–Pólya class. For a general B we cannot expect to write every $f \in \mathscr{A}_B(\mathbb{R})$ as

$$f(z) = \prod_{j \in B} f_j(z^j) \qquad \text{with} \quad f_j(z) \in \mathscr{A}_{\{1\}}(\mathbb{R}). \tag{1.3}$$

However, the theorem shows that this is the case for several particular choices of $B \subset \mathbb{N}$, for instance, when B has only odd numbers, when B has a single element, for $B = \{1, 4, 5\}$, etc. In general, the class $\mathscr{A}_B(\mathbb{R})$ could be much wider than the class given by (1.3). Take, for instance, $B = \{1, 2, 3, 4\}$. Then $H_B(s) = 1$ for $s = 1, \ldots, 7, 9$, $H_B(s) = 2$ for $s = 8, 12$ and $H_B(s) = 0$ otherwise. Consequently $V_B(1) = 8$, $V_B(2) = 4 = V_B(3)$, and $V_B(4) = 2$. So, the theorem asserts that $\mathscr{A}_B(\mathbb{R})$ consists of the functions of the form

$$\lambda z^m \exp(a_1 z + \cdots + a_7 z^7 + b_8 z^8 + a_9 z^9 + b_{12} z^{12}) \prod_{j=1}^{4} \pi_j(z^j), \tag{1.4}$$

where λ is a complex constant, m is some nonnegative integer, a_s are arbitrary real numbers, $b_s \leqslant 0$, and $\pi_j(z)$ are canonical products with real zeros of genus $< V_B(j)$. Hence, the function $e^{-z^{12}}$ is in $\mathscr{A}_B(\mathbb{R})$ but it is not of the form (1.3), because the exponential of largest order in (1.3) is e^{bz^8} with $b < 0$.

For general closed sets $\Theta \subset \mathbb{R}$ (unbounded on both sides) and $B \subset \mathbb{N}$ finite, the zeros of a function $f \in \mathscr{A}_B(\Theta)$ are contained in

$$\mathscr{R} \stackrel{\text{def}}{=} \{\omega \in \mathbb{C} : \omega^j \in \Theta \text{ for some } j \in B\}.$$

Let $C(\mathscr{R})$ denote the class of entire functions that can be uniformly approximated on bounded sets by polynomials with zeros in $\mathscr{R}$. Obviously $\mathscr{A}_B(\Theta) \subset C(\mathscr{R})$, and the inclusion is proper except for $B = \{1\}$. Indeed, suppose that there is $j \in B \backslash \{1\}$ and take $x \in \Theta$ with $x < 0$. If ω_l $(1 \leqslant l \leqslant j)$ are the j-roots of x then the polynomials $z - \omega_l$ belong to $C(\mathscr{R})$ for $l = 1, \ldots, j$, but at least one of them does not belong to $\mathscr{A}_B(\Theta)$.

In general, the difference between both classes of entire functions is much more significant than the one expressed by the above polynomials. In [2, Thm. 6.1] Korevaar obtained a result that, as a particular case, provides a characterization of $C(\mathscr{R})$ in terms of Weierstrass's decomposition. So, a direct comparison between his theorem and Theorem 1.2 shows how different $C(\mathscr{R})$ could be from $\mathscr{A}_B(\Theta)$.

Let k be the smallest positive integer such that $\mathscr{R}^k \stackrel{\text{def}}{=} \{\omega^k : \omega \in \mathscr{R}\}$ is contained in an angular sector of opening $< \pi$. It is easy to see that $k = 2m$, where m is the least common multiple of the elements of B.

THEOREM (Korevaar). *The class $C(\mathscr{R})$ consists of the entire functions of the form*

$$\lambda z^m \exp\left(\sum_{1 \leqslant s \leqslant k} c_s z^s \right) \pi(z),$$

where $\lambda \in \mathbb{C}$, $m \geqslant 0$ if $0 \in \Theta$, and $m = 0$ otherwise; $\pi(z)$ is a canonical product of genus $\leqslant k - 1$ with zeros in $\mathscr{R} \backslash \{0\}$, $c_s \in \mathbb{C}$ for $1 \leqslant s \leqslant k - 1$ and $c_k \leqslant 0$.

Going back to our example $\Theta = \mathbb{R}$ and $B = \{1, 2, 3, 4\}$, the theorem says that $C(\mathscr{R})$ is formed by the functions

$$\lambda z^m \exp(c_1 z + \cdots + c_{23} z^{23} + b_{24} z^{24})\, \pi(z),$$

where $\lambda \in \mathbb{C}$, $m \geqslant 0$, genus $\pi(z) \leqslant 23$ (with $Z(\pi) \subset \mathscr{R} \setminus \{0\}$), $c_s \in \mathbb{C}$, and $b_{24} \leqslant 0$. The difference with (1.4) is clear.

It will be enough to prove Theorem 1.2 for a set Θ not containing the value 0. When $0 \notin \Theta$, by a dilation of the variable z we can assume without loss of generality that $|\theta| > 1$ for every $\theta \in \Theta$. So, for the rest of the paper Θ will denote a fixed closed set of real numbers of modulus > 1 which is unbounded on both sides.

LEMMA 1.3. *Let $f(z) = \lambda e^{b(z)} \prod_{j \in B} \pi_j(z^j)$, where $b(z) = \sum_{k \geqslant 1} (b_k/k) z^k$ is an entire function and $\pi_j(z)$ is a canonical product of genus g_j, with $Z(\pi_j) \subset \Theta$ for every $j \in B$. Then for $s \geqslant 1$ integer,*

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} z^{-s}\, dz = b_s - \sum_{j \in B,\ j \mid s,\ g_j < s/j} \left[\sum_{a \in Z(\pi_j)} j a^{-s/j} \right]. \tag{1.5}$$

Proof. By the theorem of residues,

$$\begin{aligned}
\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} z^{-s}\, dz &= \frac{1}{2\pi i} \int_{|z|=1} (b_1 + \cdots + b_r z^{r-1} + \cdots)\, z^{-s}\, dz \\
&\quad + \frac{1}{2\pi i} \sum_{j \in B} \left[\int_{|z|=1} j z^{j-1} \sum_{a \in Z(\pi_j)} \frac{z^{j g_j}}{a^{g_j}(z^j - a)} z^{-s}\, dz \right] \\
&= b_s + \sum_{j \in B} \sum_{a \in Z(\pi_j)} j \operatorname{Res} \left(\frac{z^{j-1} z^{j g_j} z^{-s}}{a^{g_j}(z^j - a)} \right) \\
&= b_s - \sum_{j \in B} \sum_{a \in Z(\pi_j)} j \operatorname{Res} \left(a^{-g_j - 1} \sum_{d \geqslant 0} a^{-d} z^{j(d + g_j + 1) - s - 1} \right).
\end{aligned} \tag{1.6}$$

Here $\operatorname{Res}(h(z))$ denotes the residue of $h(z)$ at $z = 0$. The residue vanishes unless $j(d + g_j + 1) = s$. If this equality holds then j divides s and $d = s/j - g_j - 1 \geqslant 0$. Hence, the residue is zero if j does not divide s or $g_j \geqslant s/j$, and it is

$$a^{-g_j - 1} a^{-s/j + g_j + 1} = a^{-s/j} \qquad \text{if} \quad j \mid s \qquad \text{and} \qquad g_j < s/j.$$

Inserting this in (1.6) we obtain (1.5). ∎

Corollary 1.4. *Let $p(z)=\prod_{j\in B} p_j(z^j)$, where each $p_j(z)$ is a polynomial with $Z(p_j)\subset\Theta$. Then*

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{p'(z)}{p(z)}z^{-s}\,dz=-\sum_{j\in B,\,j\,|\,s}\left[\sum_{\alpha\in Z(p_j)} j\alpha^{-s/j}\right] \tag{1.7}$$

for every integer $s\geqslant 1$.

Proof. This is a particular case of (1.5), where $b_s=0$ for all $s\in\mathbb{N}$ and $g_j=0$ for all $j\in B$. ▮

Definition. Let $B\subset\mathbb{N}$ be a finite set and for each $j\in B$ let $P^j\subset\Theta$ be a finite family (i.e., with repetitions allowed). For $s\in\mathbb{N}$ we write

$$\beta(\{P^j : j\in B\}, s)\overset{\text{def}}{=}\sum_{j\in B,\,j\,|\,s}\left[\sum_{\alpha\in P^j} j\alpha^{-s/j}\right].$$

Observe that with this notation Corollary 1.4 asserts that $\beta(\{Z(p_j): j\in B\}, s)=(-1/2\pi i)\int_{|z|=1}(p'/p)(z)\,z^{-s}\,dz$.

2. PROOF OF NECESSITY: $f\in\mathscr{A}_B(\Theta)$ IMPLIES (1.2)

Lemma 2.1. *Let $P_n^j\subset\Theta$ (for $j\in B$ and $n\geqslant 1$) be finite families such that for every $s\in\mathbb{N}$ there is $C(s)>0$ with $|\beta(\{P_n^j : j\in B\}, s)|\leqslant C(s)$ for all $n\geqslant 1$. Then there is $K>0$ such that*

$$\sum_{\alpha\in P_n^j}|\alpha|^{-V_B(j)}<K \qquad \text{for all}\quad j\in B \qquad \text{and} \qquad n\geqslant 1.$$

Proof. By Remark 1.1, $H_B^{-1}(2)\overset{\text{def}}{=}\{s\in\mathbb{N} : H_B(s)=2\}=\{jV_B(j) : j\in B\}$. Let us write its elements as $s_1<s_2<\cdots<s_k$. We will prove by induction on i that if $j\in B$ is such that $jV_B(j)=s_i$ then there is $K_i>0$ such that

$$\sum_{\alpha\in P_n^j} j\,|\alpha|^{-V_B(j)}<K_i \qquad \text{for all}\quad n\geqslant 1. \tag{2.1}$$

Suppose first that $j_0\in B$ is such that $j_0V_B(j_0)=s_1$. Since $s_1=\min H_B^{-1}(2)$, for every $j\in B$ that divides s_1 we must have $H_B(tj)=1$ for $t<s_1/j$, and s_1/j must be even. Then

$$\alpha^{-s_1/j}=|\alpha|^{-s_1/j} \qquad \text{for all}\quad \alpha\in P_n^j,\quad j\in B,\quad j\,|\,s_1.$$

In particular, this holds for $s_1/j_0 = V_B(j_0)$. Hence

$$\sum_{\alpha \in P_n^{j_0}} j_0 |\alpha|^{-V_B(j_0)} \leqslant \sum_{j \in B,\, j \mid s_1} \left[\sum_{\alpha \in P_n^j} j\, |\alpha|^{-s_1/j} \right] = \beta(\{P_n^j : j \in B\}, s_1) \leqslant C(s_1),$$

which proves the first inductive step with $K_1 = C(s_1)$. Now suppose that (2.1) holds when $jV_B(j) = s_l$ for $l = 1, ..., i-1$, and let $j_0 \in B$ be such that $j_0 V_B(j_0) = s_i$. Remark 1.1 tells us that $jV_B(j) \leqslant s_i$ for all $j \in B$ that divide s_i. Then

$$\begin{aligned} &\beta(\{P_n^j : j \in B\}, s_i) \\ &\quad = \sum_{j \in B,\, j \mid s_i,\, jV_B(j) < s_i} \left[\sum_{\alpha \in P_n^j} j\alpha^{-s_i/j} \right] + \sum_{j \in B,\, jV_B(j) = s_i} \left[\sum_{\alpha \in P_n^j} j\alpha^{-s_i/j} \right] \\ &\quad = I_1(n) + I_2(n). \end{aligned}$$

When $j \in B$ is such that $jV_B(j) = s_i$ then $s_i/j = V_B(j)$ is even and consequently $\alpha^{-s_i/j} = |\alpha|^{-s_i/j}$ for $\alpha \in P_n^j$. In particular, this is the case for j_0. Therefore

$$\sum_{\alpha \in P_n^{j_0}} j_0 |\alpha|^{-V_B(j_0)} \leqslant I_2(n) = \beta(\{P_n^j : j \in B\}, s_i) - I_1(n) \leqslant C(s_i) + |I_1(n)|. \tag{2.2}$$

When $j \in B$ is such that $jV_B(j) < s_i$ then $jV_B(j) \in \{s_1, ..., s_{i-1}\}$, and $|\alpha|^{-s_i/j} < |\alpha|^{-V_B(j)}$ for all $\alpha \in P_n^j$ (because $|\alpha| > 1$). Therefore by the inductive hypothesis

$$\begin{aligned} |I_1(n)| &\leqslant \sum_{j \in B,\, j \mid s_i,\, jV_B(j) < s_i} \left[\sum_{\alpha \in P_n^j} j\, |\alpha|^{-s_i/j} \right] \\ &< \sum_{j \in B,\, j \mid s_i,\, jV_B(j) < s_i} \left[\sum_{\alpha \in P_n^j} j\, |\alpha|^{-V_B(j)} \right] \\ &\leqslant \sum_{j \in B} \max\{K_1, ..., K_{i-1}\} = \sharp(B) \max\{K_1, ..., K_{i-1}\}, \end{aligned} \tag{2.3}$$

where $\sharp(B)$ denotes the number of elements in B. By (2.2) and (2.3) then (2.1) holds for i with $K_i = C(s_i) + \sharp(B) \max\{K_1, ..., K_{i-1}\}$. The lemma now follows if one takes $K = \max\{K_l : 1 \leqslant l \leqslant k\}$. ∎

We are ready to prove that every $f \in \mathscr{A}_B(\Theta)$ has the form (1.2). Let $p_n \in \mathscr{A}_B(\Theta)$ be a sequence of polynomials that tends to f uniformly on bounded sets. If we write $\{\omega_k\}$ for the zero sequence of f, it is clear that for every ω_k there is some $j \in B$ such that ω_k is a j-root of some element in Θ. Since we are assuming that every element of Θ has modulus bigger than

1 then $|\omega_k| > 1$ for all k. If $m = 2\prod_{j \in B} j$, the zeros of $p_n(z)$ are contained in the rays $\arg z = 2\pi l/m$ for $l = 1, ..., m$, implying that $\gamma^{-m} = |\gamma|^{-m}$ for every $\gamma \in Z(p_n)$. Then for an arbitrary $N \in \mathbb{N}$ the theorem of residues gives

$$\sum_{1 \leqslant k \leqslant N} |\omega_k|^{-m} \leqslant \limsup_n \sum_{\gamma \in Z(p_n)} |\gamma|^{-m} = \limsup_n \sum_{\gamma \in Z(p_n)} \gamma^{-m}$$

$$= \lim_n \frac{-1}{2\pi i} \int_{|z|=1} \frac{p_n'}{p_n}(z)\, z^{-m}\, dz$$

$$= \frac{-1}{2\pi i} \int_{|z|=1} \frac{f'}{f}(z)\, z^{-m}\, dz.$$

Therefore $\sum_{k \geqslant 1} |\omega_k|^{-m} < \infty$, and by the theorem of Weierstrass we can factorize $f(z) = \lambda e^{b(z)} \pi(z)$, where λ is constant, $b(z)$ is an entire function such that $b(0) = 0$, and $\pi(z)$ is a canonical product. Moreover, the geometry of $Z(p_n)$ makes it clear that we can arrange the zeros of $\pi(z)$ in such a way that $\pi(z) = \prod_{j \in B} \pi_j(z^j)$, where each $\pi_j(z)$ is a canonical product of finite genus with zeros in Θ.

Let $j_0 \in B$. Among the several possible factorizations of the type $p_n(z) = \prod_{j \in B} p_{j,n}(z^j)$ with $Z(p_{j,n}) \subset \Theta$, choose one for every n that maximizes the number of zeros of $p_{j_0,n}(z)$. If $\{a_k\}$ is the zero sequence of $\pi_{j_0}(z)$ then Lemma 2.1 implies that

$$\sum_{1 \leqslant k \leqslant N} |a_k|^{-V_B(j_0)} \leqslant \limsup_n \sum_{\alpha \in Z(p_{j_0,n})} |\alpha|^{-V_B(j_0)} < \infty$$

for all $N \geqslant 1$. So, $g_{j_0} = \text{genus}\ \pi_{j_0}(z) < V_B(j_0)$.

If $r > 0$ and h is an entire function such that $h(0) \neq 0$, let $T_r h$ denote the polynomial defined by the conditions $(T_r h)(0) = 1$ and $Z(T_r h) = \{\omega \in Z(h) : |\omega| \leqslant r\}$, where multiplicities are taken into account. For $r > 0$ consider $q_n(z) = p_n(z)/T_r p_n(z)$ and $g(z) = f(z)/T_r f(z)$. So, $q_n(z)$ and $g(z)$ are zero free on $|z| \leqslant r$, and $q_n(z) \to g(z)$ uniformly on bounded sets if $f(z)$ has no zeros on $|z| = r$. Furthermore,

$$g(z) = \lambda e^{b(z)} e^{\sum_{j \in B} h_j(z)} \prod_{j \in B} \tilde{\pi}_j(z^j),$$

where

$$\tilde{\pi}_j(z^j) = \prod_{a \in Z(\pi_j),\, |a| > r^j} \left(1 - \frac{z^j}{a}\right) \exp\left[\left(\frac{z^j}{a}\right) + \cdots + \frac{1}{g_j}\left(\frac{z^j}{a}\right)^{g_j}\right]$$

and

$$h_j(z) = \sum_{a \in Z(\pi_j),\ |a| \leqslant r^j} \left[\left(\frac{z^j}{a} \right) + \cdots + \frac{1}{g_j} \left(\frac{z^j}{a} \right)^{g_j} \right].$$

In the above two formulas the expression between square brackets reduces to 0 if $g_j = 0$. Write also $b(z) = \sum_{k \geqslant 1} (b_k/k)\, z^k$.

Suppose that s is a positive integer such that $H_B(s) = 2$ or 0. Then either there is no $j \in B$ that divides s or for every $j \in B$ that divides s we have $jV_B(j) \leqslant s$. Since we have proved that $g_j < V_B(j)$ then $jg_j < s$ for every divisor $j \in B$ of s. This means that for every such j the polynomial $h_j(z)$ has degree less than s. Hence, by (1.5),

$$\begin{aligned} -b_s &+ \sum_{j \in B,\ j \mid s,\ g_j < s/j} \left[\sum_{a \in Z(\pi_j),\ |a| > r^j} j a^{-s/j} \right] \\ &= \frac{-1}{2\pi i} \int_{|z|=1} \frac{g'}{g}(z)\, z^{-s}\, dz = \lim_n \frac{-1}{2\pi i} \int_{|z|=1} \frac{q_n'}{q_n}(z)\, z^{-s}\, dz \\ &= \lim_n \sum_{j \in B,\ j \mid s} \left[\sum_{\alpha \in Z(p_{j,n}),\ |\alpha| > r^j} j \alpha^{-s/j} \right]. \end{aligned} \tag{2.4}$$

Let $\varepsilon > 0$. The condition $g_j < s/j$ implies that $\sum_{a \in Z(\pi_j)} |a|^{-s/j} < \infty$. So, if r is large enough we obtain

$$\left| \sum_{j \in B,\ j \mid s,\ g_j < s/j} \left[\sum_{a \in Z(\pi_j),\ |a| > r^j} j a^{-s/j} \right] \right| < \varepsilon. \tag{2.5}$$

On the other hand, since $jV_B(j) \leqslant s$ for every $j \in B$ that divides s then we can split the sum under the limit in (2.4) in two sums, according to whether $jV_B(j) < s$ or $jV_B(j) = s$. Lemma 2.1 tells us that there is a constant $K > 0$ not depending on j or n such that $\sum_{\alpha \in Z(p_{j,n})} j\, |\alpha|^{-V_B(j)} < K$. Therefore, for large values of r we also get

$$\begin{aligned} &\left| \sum_{j \in B,\ j \mid s,\ jV_B(j) < s} \left[\sum_{\alpha \in Z(p_{j,n}),\ |\alpha| > r^j} j \alpha^{-s/j} \right] \right| \\ &\leqslant \sum_{j \in B,\ j \mid s,\ jV_B(j) < s} \left[\sum_{\alpha \in Z(p_{j,n}),\ |\alpha| > r^j} j\, |\alpha|^{-V_B(j)}\, |\alpha|^{-s/j + V_B(j)} \right] \\ &\leqslant \sum_{j \in B,\ j \mid s,\ jV_B(j) < s} r^{-s + jV_B(j)} \left[\sum_{\alpha \in Z(p_{j,n})} j\, |\alpha|^{-V_B(j)} \right] \\ &\leqslant r^{-1} \sum_{j \in B,\ j \mid s,\ jV_B(j) < s} K \leqslant r^{-1}\, \sharp(B)\, K < \varepsilon. \end{aligned} \tag{2.6}$$

Let

$$A(n) = \sum_{j \in B,\, jV_B(j)=s} \left[\sum_{\alpha \in Z(p_{j,n}),\, |\alpha| > r^j} j\alpha^{-s/j} \right],$$

where $A(n)$ reduces to 0 if there is no $j \in B$ such that $j\, V_B(j) = s$. Taking estimates (2.5) and (2.6) into equality (2.4) we obtain

$$\limsup_n |b_s + A(n)| \leqslant 2\varepsilon. \tag{2.7}$$

Summing up, (2.7) holds when $H_B(s) = 2$ or 0, $f(z)$ is zero free on $|z| = r$ and r is large enough so that (2.5) and (2.6) hold.

Since by Remark 1.1 $V_B(j)$ is even for every $j \in B$, in the sum defining $A(n)$ we have $\alpha^{-s/j} = \alpha^{-V_B(j)} = |\alpha|^{-V_B(j)}$. Consequently $A(n) \geqslant 0$ for all n, which together with (2.7) gives $b_s \leqslant 2\varepsilon$. Since ε is arbitrary then $b_s \leqslant 0$, as wished.

If $H_B(s) = 0$ there is no $j \in B$ such that $jV_B(j) = s$ (because otherwise $H_B(s) = 2$). Therefore $A(n) = 0$ and (2.7) says that $|b_s| \leqslant 2\varepsilon$. So, actually $b_s = 0$, concluding the proof of necessity. ∎

3. PROOF OF SUFFICIENCY: f AS IN (1.2) IMPLIES $f \in \mathscr{A}_B(\Theta)$.

My original proof was based on three preparatory lemmas. All of them can be replaced by the next simpler and ingenious result of Korevaar (personal communication). Thanks to him this paper is shorter and much easier to read.

Since $\mathscr{A}_B(\Theta)$ is multiplicative we can factorize a general f of the form (1.2) as convenience dictates and prove that each of the factors is in $\mathscr{A}_B(\Theta)$. Notice also that $\mathscr{A}_B(\Theta)$ is closed in the topology of uniform convergence on bounded sets.

LEMMA 3.1. *Let f be an entire function of the form*

$$f(z) = \exp(b_1 z + b_2 z^2 + \cdots + b_s z^s + \cdots),$$

where $b_s = 0$ if $H_B(s) = 0$, $b_s \leqslant 0$ if $H_B(s) = 2$ and b_s is an arbitrary real number if $H_B(s) = 1$. Then f belongs to $\mathscr{A}_B(\Theta)$.

Proof. We will use induction on the set $\mathbb{N}B = \{tj : t \in \mathbb{N} \text{ and } j \in B\}$. Let $s \in B$ and suppose that $b \neq 0$ is a real number. For $r \in \mathbb{R}$ let $[r]$ denote the integer part of r (i.e., $[r] \leqslant r < [r]+1$). If $a \in \Theta$ is such that $ab<0$ then $m = [-ab] \geqslant 0$. Taking limits when $|a| \to \infty$ we get

$$\left(1-\frac{z^s}{a}\right)^m \to \exp(bz^s),$$

with uniform covergence on bounded sets. Since $H_B(s)=1$ for every $s \in B$, this proves the first step of the induction. Suppose now that $s \in \mathbb{N}B$ is not in B and $H_B(s) \neq 0$. Therefore

$$s = vj, \quad \text{where} \quad j \in B, \quad v > 1 \quad \text{and} \quad H_B(tj) = 1 \quad \text{for} \quad 1 \leqslant t < v. \tag{3.1}$$

If $H_B(s)=1$ we can further assume that v is odd, while v is necessarily even if $H_B(s)=2$. The inductive hypothesis says that the lemma holds for every $s' \in \mathbb{N}B$ with $s' < s$. Let $b \neq 0$ be an arbitrary real number if $H_B(s)=1$ and $b<0$ if $H_B(s)=2$. In any of these cases, by choosing $a \in \Theta$ such that $ab<0$ we obtain $m = [-va^v b] \geqslant 0$. Using that $\log(1-\omega) = -\sum_{k \geqslant 1} \omega^k/k$ for $|\omega|<1$, we see that for $|z^j| < |a|$,

$$m \log\left(1-\frac{z^j}{a}\right) + m\left(\frac{z^j}{a} + \frac{z^{2j}}{2a^2} + \cdots + \frac{z^{vj}}{va^v}\right) = -m \sum_{k \geqslant v+1} \frac{z^{kj}}{ka^k}. \tag{3.2}$$

It is clear from our choice of $m = m(a)$ that the right member of (3.2) tends uniformly to 0 on bounded sets when $|a| \to \infty$. Therefore, letting $|a| \to \infty$ with $a \in \Theta$ and $ab<0$, one finds that

$$\exp(bz^{vj}) = \lim \left(1-\frac{z^j}{a}\right)^m \exp\left[m\left(\frac{z^j}{a} + \cdots + \frac{z^{(v-1)j}}{(v-1)\,a^{(v-1)}}\right)\right],$$

with uniform convergence on bounded sets. The polynomial in the above expression is in $\mathscr{A}_B(\Theta)$ and each one of the exponentials $\exp(mz^{tj}/ta^t)$, with $1 \leqslant t < v$, belongs to $\mathscr{A}_B(\Theta)$ by (3.1) and the inductive hypothesis. ∎

Let $j \in B$. In order to finish the proof we must see that if $\pi_j(z)$ is a canonical product with $Z(\pi_j) = \{a_k\} \subset \Theta$ and $g_j = \text{genus}\, \pi_j(z) < V_B(j)$ then $\pi_j(z^j)$ is in $\mathscr{A}_B(\Theta)$. But $\pi_j(z^j)$ is the uniform limit on bounded sets of the functions

$$\begin{aligned} f_n(z) &= \prod_{k=1}^{n} \left(1-\frac{z^j}{a_k}\right) \exp\left[\left(\frac{z^j}{a_k}\right) + \cdots + \frac{1}{g_j}\left(\frac{z^j}{a_k}\right)^{g_j}\right] \\ &= \exp\left\{\sum_{k=1}^{n} \left[\left(\frac{z^j}{a_k}\right) + \cdots + \frac{1}{g_j}\left(\frac{z^j}{a_k}\right)^{g_j}\right]\right\} \prod_{k=1}^{n}\left(1-\frac{z^j}{a_k}\right), \end{aligned} \tag{3.3}$$

where the expressions between square brakcets reduce to 0 if $g_j = 0$. Since $jg_j < jV_B(j)$ then $H_B(j) = H_B(2j) = \cdots = H_B(jg_j) = 1$. Therefore Lemma 3.1 tells us that the exponential in (3.3) is in $\mathscr{A}_B(\Theta)$. Consequently so is $\pi_j(z^j)$.

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