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Trivial central extensions of Lie bialgebras

Marco A. Farinati *,1, Alejandra Patricia Jancsa 2

Dto. de Matemática, FCEyN, UBA, Pabellón I, Cdad. Universitaria, 1428 Buenos Aires, Argentina

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ABSTRACT

From a Lie algebra $\mathfrak{g}$ satisfying $\mathcal{Z}(\mathfrak{g})=0$ and $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}=0$ (in particular, for $\mathfrak{g}$ semisimple) we describe explicitly all Lie bialgebra structures on extensions of the form $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}$ in terms of Lie bialgebra structures on $\mathfrak{g}$ (not necessarily factorizable nor quasi-triangular) and its biderivations, for any field $\mathbb{K}$ of characteristic different form 2, 3. If moreover, $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, then we describe also all Lie bialgebra structures on extensions $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}^n$. In interesting cases we characterize the Lie algebra of biderivations.

1. Introduction and preliminaries

For an arbitrary $\mathbb{K}$-vector space W we will identify $\Lambda^2 V$ as the subspace of $V\otimes V$ and similarly $\Lambda^3 W\subset W\otimes W\otimes W$, so we ask $\frac{1}{2},\frac{1}{3}\in\mathbb{K}$. Even though one might consider the full exterior algebra $\Lambda^\bullet W$, our computations only involve $\Lambda^k W$ for $k=2,3$, so all general results in this paper hold for an arbitrary field $\mathbb{K}$ of characteristic different from 2 and 3.

Recall [4,5] that a Lie bialgebra over a field $\mathbb{K}$ is a triple $(\mathfrak{g},[-,-],\delta)$ where $(\mathfrak{g},[-,-])$ is a Lie algebra over $\mathbb{K}$ and $\delta:\mathfrak{g}\to\Lambda^2\mathfrak{g}$ is such that

- $\delta:\mathfrak{g}\to\Lambda^2\mathfrak{g}$ satisfies co-Jacobi identity, namely $\mathrm{Alt}((\delta\otimes\mathrm{Id})\circ\delta)=0$,
- $\delta:\mathfrak{g}\to\Lambda^2\mathfrak{g}$ is a 1-cocycle in the Chevalley–Eilenberg complex of the Lie algebra $(\mathfrak{g},[-,-])$ with coefficients in $\Lambda^2\mathfrak{g}$.

In the finite-dimensional case, $\delta:\mathfrak{g}\to\Lambda^2\mathfrak{g}$ satisfies co-Jacobi identity if and only if the bracket defined by $\delta^*:\Lambda^2\mathfrak{g}^*\to\mathfrak{g}^*$ satisfies Jacobi identity. In general, co-Jacobi identity for δ is equivalent to the fact that the unique derivation of degree one $\partial_\delta:\Lambda^\bullet\mathfrak{g}\to\Lambda^\bullet\mathfrak{g}$, whose restriction to $\mathfrak{g}$ agrees with δ, satisfies

* Corresponding author.
E-mail addresses: mfarinat@dm.uba.ar (M.A. Farinati), pjancsa@dm.uba.ar (A.P. Jancsa).

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$\partial_\delta^2 = 0$. We will usually denote a Lie bialgebra, with underlying Lie algebra $\mathfrak{g} = (\mathfrak{g}, [-,-])$, by $(\mathfrak{g}, \delta)$. A Lie bialgebra $(\mathfrak{g}, \delta)$ is called a coboundary Lie bialgebra if there exists $r \in \Lambda^2\mathfrak{g}$ such that $\delta(x) = \mathrm{ad}_x(r)$ $\forall x \in \mathfrak{g}$; i.e. $\delta = \partial r$ is a 1-coboundary in the Chevalley–Eilenberg complex with coefficients in $\Lambda^2\mathfrak{g}$. Coboundary Lie bialgebras are denoted by $(\mathfrak{g}, r)$, although r is in general not unique. We have that r and r' give rise to the same cobracket if and only if $r - r' \in (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$, so r is uniquely determined by δ in the semisimple case, since $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$ for $\mathfrak{g}$ semisimple.

Recall that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the *classical Yang–Baxter equation*, CYBE for short, if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

where the Lie bracket is taken in the repeated index; for example, if $r = \sum_i r_i \otimes r^i$ then $r^{12} := r \otimes 1$, $r^{13} := \sum_i r_i \otimes 1 \otimes r^i$ and $r^{23} := 1 \otimes r \in \mathcal{U}(\mathfrak{g})^{\otimes 3}$, so $[r^{12}, r^{13}] = \sum_{i,j} [r_i, r_j] \otimes r^i \otimes r^j \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})^{\otimes 3}$, and so on for the other terms of CYBE. We denote the left-hand side of CYBE by $\mathrm{CYB}(r)$.

If $r \in \Lambda^2\mathfrak{g}$, then $\delta = \partial r$ satisfies co-Jacobi if and only if $\mathrm{CYB}(r) \in \Lambda^3\mathfrak{g}$ is $\mathfrak{g}$-invariant. If $(\mathfrak{g}, r)$ is a coboundary Lie bialgebra and r satisfies CYBE, $(\mathfrak{g}, r)$ is called *triangular*. A Lie bialgebra is *quasi-triangular* if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$, not necessarily skew symmetric, such that $\delta(x) = \mathrm{ad}_x(r)$ $\forall x \in \mathfrak{g}$ and r satisfies CYBE; if, moreover, the symmetric component of r induces a non-degenerate inner product on $\mathfrak{g}^*$, then $(\mathfrak{g}, \delta)$ is called *factorizable* [12]. Quasi-triangular Lie bialgebras are also denoted by $(\mathfrak{g}, r)$, although r is in general not unique. Nevertheless, in the semisimple case the skew symmetric component r_Λ of r is uniquely determined by δ. A quasi-triangular Lie bialgebra $(\mathfrak{g}, r)$ is, in particular, a coboundary Lie bialgebra, with the coboundary chosen as the skew symmetric component of r.

If $(\mathfrak{g}, \delta)$ is a real Lie bialgebra, then $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie bialgebra with cobracket $\delta \otimes_{\mathbb{R}} \mathrm{Id}_{\mathbb{C}} : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to (\Lambda^2_{\mathbb{R}}\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda^2_{\mathbb{C}}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$. A real Lie bialgebra is coboundary if and only if its complexification is coboundary. On the other hand, it may happen that $(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, \delta \otimes_{\mathbb{R}} \mathrm{Id}_{\mathbb{C}})$ is factorizable but $(\mathfrak{g}, \delta)$ is not; in this case we call it *almost factorizable*.

1.1. The theorem of Belavin and Drinfeld

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ the *Casimir* element corresponding to a fixed non-degenerate, symmetric, invariant, bilinear form $(-,-)$ on $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let Δ be a choice of a set of simple roots. A Belavin–Drinfeld triple (BD-triple for short) is a triple $(\Gamma_1, \Gamma_2, \tau)$, where Γ_1, Γ_2 are subsets of Δ, and $\tau : \Gamma_1 \to \Gamma_2$ is a bijection that preserves the inner product and satisfies the nilpotency condition: for any $\alpha \in \Gamma_1$, there exists a positive integer n for which $\tau^n(\alpha)$ belongs to Γ_2 but not to Γ_1. Let $(\Gamma_1, \Gamma_2, \tau)$ be a BD-triple. Let $\widetilde{\Gamma}_i$ be the set of positive roots lying in the subgroup generated by Γ_i, for $i = 1, 2$. There is an associated partial order on Φ^+ given by $\alpha \prec \beta$ if $\alpha \in \widetilde{\Gamma}_1$, $\beta \in \widetilde{\Gamma}_2$ and $\beta = \tau^n(\alpha)$ for a positive integer n. A continuous parameter for the BD-triple $(\Gamma_1, \Gamma_2, \tau)$ is an element $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ such that $(\tau(\alpha) \otimes \mathrm{Id} + \mathrm{Id} \otimes \alpha) r_0 = 0$ $\forall \alpha \in \Gamma_1$, and $r_0 + r_0^{21} = \Omega_0$, the $\mathfrak{h} \otimes \mathfrak{h}$-component of Ω.

Theorem 1.1 *(Belavin–Drinfeld). (See [2].) Let $(\mathfrak{g}, \delta)$ be a factorizable complex simple Lie bialgebra. Then there exists a non-degenerate, symmetric, invariant, bilinear form on $\mathfrak{g}$ with corresponding Casimir element Ω, a Cartan subalgebra $\mathfrak{h}$, a system of simple roots Δ, a BD-triple $(\Gamma_1, \Gamma_2, \tau)$ and continuous parameter $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ such that $\delta(x) = \mathrm{ad}_x(r)$ for all $x \in \mathfrak{g}$, with r given by*

$$r = r_0 + \sum_{\alpha \in \Phi^+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha \in \Phi^+ :\, \alpha \prec \beta} x_{-\alpha} \wedge x_\beta \tag{1}$$

where $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, $\pm\alpha \in \pm\Phi^+$ are root vectors normalized by $(x_\alpha, x_{-\alpha}) = 1$, $\forall \pm\alpha \in \pm\Phi^+$, clearly, $r + r^{21} = \Omega$.

Reciprocally, any r of the form given above satisfies CYBE and endows the Lie algebra $\mathfrak{g}$ of a factorizable Lie bialgebra structure.

The component $\sum_{\alpha\in\Phi^+} x_{-\alpha}\wedge x_\alpha + \Omega$ is called the *standard part* and it is denoted by r_{st}, so $r = r_{st} + \sum_{\alpha\prec\beta} x_{-\alpha}\wedge x_\beta + \lambda$, if we decompose $r_0 = \lambda + \Omega_0$, $\lambda \in \Lambda^2\mathfrak{h}$.

Remark 1.2. Some authors have considered more general versions of the previous theorem (see [10] and [3] for the semisimple and reductive versions). In this work, we give a new description for the reductive Lie bialgebras without using the previous works but starting from a given Lie bialgebra structure on the semisimple factor $\mathfrak{g}$.

Our point of view is the following: From a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ with char $\mathbb{K}=0$ satisfying $\mathcal{Z}(\mathfrak{g})=0$ and $\Lambda^2(\mathfrak{g})^{\mathfrak{g}}=0$ we describe explicitly all the Lie bialgebra structures on extensions of the form $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}$ in terms of Lie bialgebra structures on $\mathfrak{g}$ and its biderivations. If moreover, $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$, then we describe also all the Lie bialgebra structures on extensions $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}^d$ for any d. In the semisimple factorizable case, the Lie bialgebra structures on $\mathfrak{g}$ are known [2,1,3]; we make a detailed analysis of the biderivations in this case and give an alternative description of the extensions to reductive Lie bialgebras. This characterization includes the reductive factorizable case, but actually we obtain all Lie bialgebra structures on $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}^d$ that restrict to a given Lie bialgebra structure on $\mathfrak{g}$, which include non-factorizable and even non-coboundary ones. The latter were not considered in previous works.

1.2. The center and the derived ideal $[\mathfrak{g},\mathfrak{g}]$

The next statement is straightforward but useful:

Proposition 1.3. *Let $\mathfrak{L}$ be a Lie algebra and $\delta:\mathfrak{L}\to\Lambda^2\mathfrak{L}$ a 1-cocycle, then*

1. *$[\mathfrak{L},\mathfrak{L}]$ is a coideal, i.e. $\delta[\mathfrak{L},\mathfrak{L}]\subseteq[\mathfrak{L},\mathfrak{L}]\wedge\mathfrak{L}$. As a consequence, if $(\mathfrak{L},\delta)$ is a Lie bialgebra then the quotient $\mathfrak{L}/[\mathfrak{L},\mathfrak{L}]$ admits a unique Lie bialgebra structure such that the canonical projection is a Lie bialgebra map. Moreover, if $(\mathfrak{L},\delta_1)\cong(\mathfrak{L},\delta_2)$ as Lie bialgebras, then $(\mathfrak{L}/[\mathfrak{L},\mathfrak{L}],\bar\delta_1)\cong(\mathfrak{L}/[\mathfrak{L},\mathfrak{L}],\bar\delta_2)$.*
2. *If $\mathcal{Z}(\mathfrak{L})$ denotes the center of the Lie algebra $\mathfrak{L}$, then $\delta(\mathcal{Z}(\mathfrak{L}))\subseteq\Lambda^2(\mathfrak{L})^{\mathfrak{L}}$.*

Proof. 1. It is enough to notice that for any $x,y\in\mathfrak{L}$, $\delta[x,y]=\mathrm{ad}_x\,\delta y-\mathrm{ad}_y\,\delta x\in[\mathfrak{L},\mathfrak{L}]\wedge\mathfrak{L}$.

2. If z is central, then $[z,x]=0$ for all $x\in\mathfrak{L}$, so for a 1-cocycle δ we get

$$0=\delta\big([z,x]\big)=\big[z,\delta(x)\big]+\big[\delta(z),x\big]=\big[\delta(z),x\big],$$

and hence, $\mathrm{ad}_x\,\delta(z)=0$ for all $x\in\mathfrak{L}$. $\square$

Corollary 1.4. *If $\mathfrak{L}$ is a Lie bialgebra such that $(\Lambda^2\mathfrak{L})^{\mathfrak{L}}=0$ then $\mathcal{Z}(\mathfrak{L})$ is a coideal.*

1.3. 1-cocycles in product algebras

Let $\mathfrak{L}=\mathfrak{g}\times V$, where $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{K}$ and V is a $\mathbb{K}$-vector space, considered as abelian Lie algebra. The second exterior power of $\mathfrak{L}$ can be computed as

$$\Lambda^2\mathfrak{L}=\Lambda^2(\mathfrak{g}\times V)\cong\big(\Lambda^2\mathfrak{g}\otimes\Lambda^0V\big)\oplus\big(\Lambda^1\mathfrak{g}\otimes\Lambda^1V\big)\oplus\big(\Lambda^0\mathfrak{g}\otimes\Lambda^2V\big)$$
$$\cong\Lambda^2\mathfrak{g}\oplus\mathfrak{g}\otimes V\oplus\Lambda^2V.$$

Notice that this is an $\mathfrak{L}$-module decomposition, so

$$H^1\big(\mathfrak{L},\Lambda^2\mathfrak{L}\big)\cong H^1\big(\mathfrak{L},\Lambda^2\mathfrak{g}\big)\oplus H^1(\mathfrak{L},\mathfrak{g}\otimes V)\oplus H^1\big(\mathfrak{L},\Lambda^2V\big).$$

Now we recall the Künneth formula

$$\begin{aligned} H^1(\mathfrak{g}\times V, M_1\otimes M_2) &\cong H^1(\mathfrak{g}, M_1)\otimes H^0(V, M_2)\oplus H^0(\mathfrak{g}, M_1)\otimes H^1(V, M_2)\\ &\cong H^1(\mathfrak{g}, M_1)\otimes M_2\oplus M_1^{\mathfrak{g}}\otimes \operatorname{Hom}(V, M_2)\end{aligned}$$

where we use the equality $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$ for any $\mathfrak{g}$-module M. We assume that M_2 is a trivial representation of V (e.g. $M_2 = \mathbb{K}$, V^{ad}, or $\Lambda^2 V$), so $M_2^V = M_2$ and $H^1(V, M_2) = \operatorname{Hom}(V, M_2)$. If we apply the Künneth formula in our case, we get

$$\begin{aligned} H^1\big(\mathfrak{L}, \Lambda^2\mathfrak{L}\big) = {}& H^1\big(\mathfrak{g}, \Lambda^2\mathfrak{g}\big)\oplus \big(\Lambda^2\mathfrak{g}\big)^{\mathfrak{g}}\otimes V^*\oplus H^1(\mathfrak{g},\mathfrak{g})\otimes V\oplus (\mathfrak{g})^{\mathfrak{g}}\otimes \operatorname{Hom}(V, V)\\ &\oplus H^1(\mathfrak{g},\mathbb{K})\otimes \Lambda^2 V\oplus \operatorname{Hom}\big(V, \Lambda^2 V\big).\end{aligned}$$

Recalling that $H^1(\mathfrak{g}, M) = \operatorname{Der}(\mathfrak{g}, M)/\operatorname{InnDer}(\mathfrak{g}, M)$ and, in particular,

$$H^1(\mathfrak{g},\mathbb{K}) = \operatorname{Der}(\mathfrak{g},\mathbb{K})\cong \big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^*,$$

we get the final formula:

$$\begin{aligned} H^1\big(\mathfrak{L}, \Lambda^2\mathfrak{L}\big) = {}& H^1\big(\mathfrak{g}, \Lambda^2\mathfrak{g}\big)\oplus \big(\Lambda^2\mathfrak{g}\big)^{\mathfrak{g}}\otimes V^*\oplus \operatorname{Der}(\mathfrak{g},\mathfrak{g})/\operatorname{InnDer}(\mathfrak{g},\mathfrak{g})\otimes V\\ &\oplus \mathcal{Z}(\mathfrak{g})\otimes \operatorname{End}(V)\oplus \big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)^*\otimes \Lambda^2 V\oplus \operatorname{Hom}\big(V, \Lambda^2 V\big).\end{aligned}$$

We have the following special, favorable cases:

Lemma 1.5. *Let $\mathfrak{L} = \mathfrak{g}\times V$ as before.*

1. *If* $\dim V = 1$ *then*

$$H^1\big(\mathfrak{L}, \Lambda^2\mathfrak{L}\big)\cong H^1\big(\mathfrak{g}, \Lambda^2\mathfrak{g}\big)\oplus \big(\Lambda^2\mathfrak{g}\big)^{\mathfrak{g}}\oplus \operatorname{Der}(\mathfrak{g},\mathfrak{g})/\operatorname{InnDer}(\mathfrak{g},\mathfrak{g})\oplus \mathcal{Z}(\mathfrak{g}).$$

2. *If $\mathfrak{g}$ is semisimple, then $H^1(\mathfrak{L}, \Lambda^2\mathfrak{L})\cong \operatorname{Hom}(V, \Lambda^2 V)$.*
3. *If $\dim V = 1$ and $\mathfrak{g}$ is semisimple, then $H^1(\mathfrak{L}, \Lambda^{\mathfrak{L}}) = 0$, in particular, every Lie bialgebra structure on $\mathfrak{L}$ is coboundary.*

Example 1.6. If $\mathfrak{g} = \mathfrak{su}(2)$ or $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, then every 1-cocycle in $\mathfrak{g}\times\mathbb{R}$ is coboundary. But this property does not hold for instance in $\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}^2$, or $\mathfrak{gl}(2,\mathbb{R})\times\mathfrak{gl}(2,\mathbb{R})$.

1.4. Extensions of scalars

Let $\mathbb{K}\subset\mathbb{E}$ be a field extension, if $\mathfrak{g}$ is a Lie (bi)algebra over $\mathbb{K}$, then $\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E}$ is naturally a Lie (bi)algebra over $\mathbb{E}$ and $\Lambda^2_{\mathbb{E}}(\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E})\cong(\Lambda^2_{\mathbb{K}}\mathfrak{g})\otimes_{\mathbb{K}}\mathbb{E}$. Let us denote by $H^\bullet_{\mathbb{K}}(\mathfrak{g}, -)$ and $H^\bullet_{\mathbb{E}}(\mathfrak{g}\otimes_K\mathbb{E}, -)$ the Lie algebra cohomology of $\mathfrak{g}$ as $\mathbb{K}$-Lie algebra and of $\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E}$ as $\mathbb{E}$-Lie algebra, respectively. Since Lie cohomology extends scalars, i.e. if M is a $\mathfrak{g}$-module and we consider $M\otimes_{\mathbb{K}}\mathbb{E}$ as $(\mathfrak{g}\otimes_K\mathbb{E})$-module then $H^\bullet_{\mathbb{E}}(\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E}, M\otimes_{\mathbb{K}}\mathbb{E}) = H^\bullet_{\mathbb{K}}(\mathfrak{g}, M)\otimes_{\mathbb{K}}\mathbb{E}$, we have $H^\bullet_{\mathbb{E}}(\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E}, M\otimes_{\mathbb{K}}\mathbb{E}) = 0 \Leftrightarrow H^\bullet_{\mathbb{K}}(\mathfrak{g}, M) = 0$ and $H^\bullet_{\mathbb{K}}(\mathfrak{g}, M)$ identifies with a $\mathbb{K}$-vector subspace of $H^\bullet_{\mathbb{E}}(\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{E}, M\otimes_{\mathbb{K}}\mathbb{E})$. In particular, if $(\mathfrak{g},\delta)$ is an $\mathbb{R}$-Lie bialgebra, then it is coboundary if and only if its complexification is coboundary.

2. Biderivations

For a Lie bialgebra $(\mathfrak{g}, \delta)$, a map $D: \mathfrak{g} \to \mathfrak{g}$ which is at the same time a derivation and a coderivation is called a *biderivation*. The set of all biderivations of $(\mathfrak{g}, \delta)$ is denoted by $\mathrm{BiDer}(\mathfrak{g})$. For an inner biderivation we understand a biderivation which is inner as a derivation.

Definition 2.1. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra; we consider the *characteristic map* $\mathcal{D}_{\mathfrak{g}}: \mathfrak{g} \to \mathfrak{g}$ defined by $\mathcal{D}_{\mathfrak{g}}(x) := [\ ,\ ](\delta x) = [x_1, x_2]$ for any $x \in \mathfrak{g}$, where we denote $\delta x = x_1 \wedge x_2$ in Sweedler-type notation.

This map contains much information of the Lie bialgebra and it will be useful along this work. When it is clear from the context, $\mathcal{D}_{\mathfrak{g}}$ will be denoted by $\mathcal{D}$. Due to the next proposition, we will call $\mathcal{D}_{\mathfrak{g}}$ the *characteristic biderivation* of $\mathfrak{g}$.

Proposition 2.2. *If $(\mathfrak{g}, \delta)$ is a Lie bialgebra then its characteristic map $\mathcal{D}$ is both a derivation and a coderivation.*

Proof. Let us see that $\mathcal{D}$ is a derivation. If $x, y \in \mathfrak{g}$, then

$$\begin{aligned}
\mathcal{D}\big([x, y]\big) &= [\ ,\ ]\big(\delta[x, y]\big) = [\ ,\ ](\mathrm{ad}_x \delta y - \mathrm{ad}_y \delta x) \\
&= [\ ,\ ]\big([x, y_1] \wedge y_2 + y_1 \wedge [x, y_2] + [x_1, y] \wedge x_2 + x_1 \wedge [x_2, y]\big) \\
&= \big[[x, y_1], y_2\big] + \big[y_1, [x, y_2]\big] + \big[[x_1, y], x_2\big] + \big[x_1, [x_2, y]\big] = \big[x, [y_1, y_2]\big] + \big[[x_1, x_2], y\big] \\
&= [x, \mathcal{D}y] + [\mathcal{D}x, y].
\end{aligned}$$

Notice that for a finite-dimensional Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$, once we know that $\mathcal{D}_{\mathfrak{g}}$ is a derivation in $(\mathfrak{g}, [\ ,\ ])$, $\mathcal{D}_{\mathfrak{g}^*}$ is a derivation in $(\mathfrak{g}^*, \delta^*)$, thus $\mathcal{D}_{\mathfrak{g}}$ is a coderivation in $(\mathfrak{g}, \delta)$, since $\mathcal{D}_{\mathfrak{g}^*} = (\mathcal{D}_{\mathfrak{g}})^*$. Alternatively, one may prove it directly:

$$\begin{aligned}
\delta\big(\mathcal{D}(x)\big) &= \delta\big([x_1, x_2]\big) = [\delta x_1, x_2] + [x_1, \delta x_2] = [x_{11} \wedge x_{12}, x_2] + [x_1, x_{21} \wedge x_{22}] \\
&= [x_{11}, x_2] \wedge x_{12} + x_{11} \wedge [x_{12}, x_2] + [x_1, x_{21}] \wedge x_{22} + x_{21} \wedge [x_1, x_{22}].
\end{aligned} \tag{2}$$

On the other hand, co-Jacobi identity for δ implies

$$0 = (\delta \otimes 1 - 1 \otimes \delta)\delta(x) = x_{11} \wedge x_{12} \wedge x_2 - x_1 \wedge x_{21} \wedge x_{22}$$

then $x_{11} \wedge x_{12} \wedge x_2 = x_1 \wedge x_{21} \wedge x_{22}$ and $x_{11} \wedge x_2 \wedge x_{12} = x_1 \wedge x_{22} \wedge x_{21}$; hence

$$[x_{11}, x_2] \wedge x_{12} = [x_1, x_{22}] \wedge x_{21}.$$

So, the first and the last terms of the four terms in formula (2) cancel and we get

$$\delta\big(\mathcal{D}(x)\big) = \delta\big([x_1, x_2]\big) = x_{11} \wedge [x_{12}, x_2] + [x_1, x_{21}] \wedge x_{22};$$

using co-Jacobi identity again, the last formula equals

$$= x_1 \wedge [x_{21}, x_{22}] + [x_{11}, x_{12}] \wedge x_2 = x_1 \wedge \mathcal{D}(x_2) + \mathcal{D}(x_1) \wedge x_2 = (1 \otimes \mathcal{D} + \mathcal{D} \otimes 1)\delta(x). \qquad \square$$

Proposition 2.3. *Let $\mathfrak{g}$ be a coboundary Lie bialgebra and $r \in \Lambda^2\mathfrak{g}$ such that $\delta(x) = \mathrm{ad}_x(r)$; consider $H_r := [-,-](r) \in \mathfrak{g}$ and $\mathcal{D}_{\mathfrak{g}}$ the characteristic biderivation of $\mathfrak{g}$, then $\mathcal{D}_{\mathfrak{g}} = -\mathrm{ad}_{H_r}$.*

Proof. Write $r = r_1 \otimes r_2$ in Sweedler-type notation, so for any $x \in \mathfrak{g}$

$$\begin{aligned}\mathcal{D}_{\mathfrak{g}}(x) = [-,-] \circ \delta(x) = [-,-]\big(\mathrm{ad}_x(r_1 \otimes r_2)\big) &= \big[[x,r_1],r_2\big] + \big[r_1,[x,r_2]\big] \\ &= \big[x,[r_1,r_2]\big] = [x,H_r] \\ &= -\,\mathrm{ad}_{H_r}(x). \qquad \square\end{aligned}$$

Proposition 2.4. *Let* $\mathfrak{g}$ *be a Lie bialgebra and* $\mathcal{D}_{\mathfrak{g}}$ *its characteristic biderivation. If* $E \in \mathrm{BiDer}(\mathfrak{g})$ *then* $[\mathcal{D}, E] = 0$.

Proof. The definition of coderivation says that E satisfies $(E \otimes \mathrm{Id} + \mathrm{Id} \otimes E)\delta = \delta E$; on the other hand, since E is a derivation, $E[x,y] = [Ex,y] + [x,Ey]$, in other words,

$$[-,-](E \otimes \mathrm{Id} + \mathrm{Id} \otimes E) = E[-,-].$$

Both properties together imply

$$\mathcal{D}_{\mathfrak{g}} E = [-,-]\delta E = [-,-](E \otimes \mathrm{Id} + \mathrm{Id} \otimes E)\delta = E[-,-]\delta = E\mathcal{D}_{\mathfrak{g}}. \qquad \square$$

Corollary 2.5. *Let* $\mathfrak{g}$ *be a Lie bialgebra such that* $\mathcal{D}_{\mathfrak{g}}$ *is an inner biderivation; write* $\mathcal{D}_{\mathfrak{g}} = \mathrm{ad}_{H_0}$ *for some* $H_0 \in \mathfrak{g}$.

1. *If* E *is a biderivation, then* $E(H_0) \in \mathcal{Z}\mathfrak{g}$.
2. *If* $E = \mathrm{ad}_x \in \mathrm{BiDer}$ *then* $[x, H_0] \in \mathcal{Z}\mathfrak{g}$; *if also* $\mathcal{Z}\mathfrak{g} = 0$ *then* x *commutes with* H_0.

Proof. 1. We know $[E, \mathcal{D}_{\mathfrak{g}}] = 0$, then for any $x \in \mathfrak{g}$,

$$\begin{aligned}0 = [E,\mathcal{D}_{\mathfrak{g}}](x) = [E,\mathrm{ad}_{H_0}](x) &= E\big(\mathrm{ad}_{H_0}(x)\big) - \mathrm{ad}_{H_0}\big(E(x)\big) \\ &= E\big([H_0,x]\big) - \big[H_0,E(x)\big] \\ &= \big[E(H_0),x\big] + \big[H_0,E(x)\big] - \big[H_0,E(x)\big] \\ &= \big[E(H_0),x\big] = \mathrm{ad}_{E(H_0)}(x)\end{aligned}$$

hence $E(H_0) \in \mathrm{Ker}(\mathrm{ad}) = \mathcal{Z}\mathfrak{g}$. The second statement is a direct consequence of the first. $\square$

Remark 2.6. Because of the interesting properties and applications of $\mathcal{D}_{\mathfrak{g}}$, one may wander for an analogous map in the associative case. For a Hopf algebra H with multiplication $m : H \otimes H \to H$ and comultiplication $\Delta : H \to H \otimes H$ one may define $m \circ \Delta : H \to H$. This map has some similarities but also many differences with the Lie case. First, it is clear that it commutes with any Hopf algebra map $f : H \to H$ (in analogy with Proposition 2.4), but in contrast, if H is not commutative, m is not an algebra map, and if H is not cocommutative, Δ is not a coalgebra map, so it is not expected for $m \circ \Delta$ to be a Hopf algebra map, and in fact it is not, except for a very small family of Hopf algebras. Nevertheless, maps similar to this one were considered by Etingof and Gelaki (see [6]) with very useful applications. On the other hand, for a Lie algebra $\mathfrak{g}$, the obvious Hopf algebra to look at is $U(\mathfrak{g})$, the universal enveloping algebra, with comultiplication determined by $\Delta x = x \otimes 1 + 1 \otimes x$. If in addition $(\mathfrak{g}, \delta)$ is a Lie *bialgebra*, one may consider the ring $A = \mathbb{K}[t]/t^2$ and define a Hopf algebra structure on $H := U(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[t]/t^2$ over A declaring $\Delta_\delta(x) = x \otimes 1 + 1 \otimes x + t\delta(x)$ $(x \in \mathfrak{g})$. It is well known that the cocycle condition says that Δ_δ is well defined and gives an A-algebra map $\Delta_\delta : H \to H \otimes_A H$, and co-Jacobi for δ gives coassociativity for Δ_δ. In this case, the antipode S is given by $S(x) = -x + \frac{1}{2}t\mathcal{D}_g(x)$, and $S^2(x) = x - t\mathcal{D}_g(x)$. We conclude that $S^2 = \mathrm{Id}$ if and only if $\mathcal{D}_g = 0$, and that is the reason why a Lie bialgebra with $\mathcal{D}_g = 0$ is called *involutive*. Also, this example shows that

S^2 is a possible candidate for an analogue to the "exponential" of $-\mathcal{D}_\mathfrak{g}$ in the abstract setting of Hopf algebras. Notice that S^2 also commutes with any Hopf algebra map $f : H \to H$ (since S does) but in addition S^2 is also a Hopf algebra map itself, in analogy with Proposition 2.2.

Going back to Lie bialgebras, we quote a result from [1], which together with Corollary 2.5 implies a very interesting fact.

Proposition 2.7. *If $\mathfrak{g}$ is real or complex semisimple and $(\mathfrak{g}, r)$ is an (almost) factorizable Lie bialgebra, then $H_r := [-,-](r)$ is a regular element and so $\mathfrak{h} := \mathcal{Z}_\mathfrak{g}(H_r)$, the centralizer of H_r, is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. This statement is proved for both, real and complex, simple cases in [1], but the proof remains valid *mutatis mutandis* for the semisimple case. □

Corollary 2.8. *Any biderivation of a factorizable semisimple Lie bialgebra $(\mathfrak{g}, r)$ is of the form ad_H with $H \in \mathfrak{h} = \mathcal{Z}_\mathfrak{g}(H_r)$. In particular, $\mathrm{BiDer}(\mathfrak{g}, \delta)$ is an abelian Lie algebra.*

Proof. If $\mathfrak{g}$ is semisimple then every derivation is inner and $\mathcal{Z}\mathfrak{g} = 0$, so $E = \mathrm{ad}_{x_0}$ and x_0 commutes with H_r. In particular, x_0 belongs to the centralizer of H_r. □

Another characterization of inner biderivations is the following.

Proposition 2.9. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $D = \mathrm{ad}_{x_0}$ an inner derivation, then D is a coderivation if and only if $\delta x_0 \in (\Lambda^2\mathfrak{g})^\mathfrak{g}$. In particular, if $(\Lambda^2\mathfrak{g})^\mathfrak{g} = 0$, then the map $x_0 \mapsto \mathrm{ad}_{x_0}$ induces an isomorphism of Lie algebras $\mathrm{Ker}\,\delta/(\mathcal{Z}(\mathfrak{g}) \cap \mathrm{Ker}\,\delta) \cong \mathrm{InnDer}(\mathfrak{g}) \cap \mathrm{CoDer}(\mathfrak{g})$.*

Proof. By definition, D is a coderivation if and only if $(D \otimes \mathrm{Id} + \mathrm{Id} \otimes D) \circ \delta = \delta \circ D$. Since $D = \mathrm{ad}_{x_0}$, we have $(D \otimes \mathrm{Id} + \mathrm{Id} \otimes D)(x \otimes y) = \mathrm{ad}_{x_0}(x \otimes y)$. So, the coderivation condition reads

$$\delta[x_0, z] = \mathrm{ad}_{x_0}\,\delta(z)$$

for all $z \in \mathfrak{g}$. On the other hand, δ is a 1-cocycle, namely

$$\delta[x_0, z] = \mathrm{ad}_{x_0}\,\delta(z) - \mathrm{ad}_z\,\delta(x_0).$$

Hence, D is a coderivation if and only if $\mathrm{ad}_z\,\delta(x_0) = 0$ for all $z \in \mathfrak{g}$. □

Corollary 2.10. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra such that every derivation is inner, $\mathcal{Z}\mathfrak{g} = 0$ and $(\Lambda^2\mathfrak{g})^\mathfrak{g} = 0$, then $\mathrm{BiDer}(\mathfrak{g}, \delta) \cong \mathrm{Ker}\,\delta$. In particular, if $\mathfrak{g}$ is semisimple then the result holds.*

Example 2.11. The non-commutative two-dimensional Lie algebra $\mathfrak{g} = \mathfrak{aff}_2(\mathbb{K})$, verifies $\mathrm{Der}(\mathfrak{g}) = \mathrm{InnDer}(\mathfrak{g})$, $\mathcal{Z}(\mathfrak{g}) = 0$ and $(\Lambda^2\mathfrak{g})^\mathfrak{g} = 0$ but it is not semisimple. In fact, this is the "$\mathfrak{sl}_2$-case" of the general and classical result (see for instance [9]) that a Borel subalgebra $\mathfrak{b}$ of a semisimple Lie algebra satisfies $\mathrm{Der}(\mathfrak{b}) = \mathrm{InnDer}(\mathfrak{b})$, $\mathcal{Z}(\mathfrak{b}) = 0$ and $(\Lambda^2\mathfrak{b})^\mathfrak{b} = 0$.

2.1. Biderivations in the real or complex semisimple case

Let $(\mathfrak{g}, r)$ be an (almost) factorizable semisimple Lie bialgebra, with r a BD classical r-matrix, i.e. r of the form as in Eq. (1) of Theorem 1.1, that is, for a fixed non-degenerate, symmetric, invariant, bilinear form on $\mathfrak{g}$, a certain Cartan subalgebra $\mathfrak{h}$, an election of positive and simple roots $\Phi^+ \subset \Phi(\mathfrak{h})$ and $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, respectively, a pair of discrete and continuous parameters $(\Gamma_1, \Gamma_2, \tau)$ and $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$, respectively, with $r_0 = \lambda + \Omega_0$, $\lambda \in \Lambda^2\mathfrak{h}$, $\lambda = \sum_{1 \leqslant i < j \leqslant \ell} \lambda_{ij} h_i \wedge h_j$, where $h_i := h_{\alpha_i}$, the antisymmetric component r_Λ of such an r-matrix is of the form

$$r_\Lambda = \sum_{\alpha\in\Phi^+} x_{-\alpha}\wedge x_\alpha + \sum_{\alpha\prec\beta} x_{-\alpha}\wedge x_\beta + \lambda. \tag{3}$$

Here $\ell = \dim\mathfrak{h}$ is the rank of the Lie algebra $\mathfrak{g}$. Notice that $\mathrm{ad}_x r = \mathrm{ad}_x r_\Lambda$ since the symmetric component of r is $\mathfrak{g}$-invariant. If we are in the real almost factorizable case $(\mathfrak{g},\delta)$, namely $(\mathfrak{g}\otimes_\mathbb{R}\mathbb{C}, \delta\otimes_\mathbb{R}\mathbb{C})$ is factorizable, so there exists $r\in(\mathfrak{g}\otimes_\mathbb{R}\mathbb{C})\otimes_\mathbb{C}(\mathfrak{g}\otimes_\mathbb{R}\mathbb{C}) = (\mathfrak{g}\otimes_\mathbb{R}\mathfrak{g})\otimes_\mathbb{R}\mathbb{C}$ with $\delta\otimes_\mathbb{R}\mathrm{Id}_\mathbb{C}(x) = \mathrm{ad}_x(r)$ for all $x\in\mathfrak{g}\otimes_\mathbb{R}\mathbb{C}$; suppose that r is of the form as above, then necessarily $\delta(x) = \mathrm{ad}_x(r) = \mathrm{ad}_x(r_\Lambda)$, for all $x\in\mathfrak{g}$. In particular $r_\Lambda\in\Lambda^2_\mathbb{R}\mathfrak{g}$.

The goal of this section is to prove the result given in the following theorem. In fact, we exhibit two different proofs of it, namely, one is the application of Corollary 2.8, which gives Proposition 2.16. The second is longer but direct and follows in this section, being Proposition 2.16 the most subtle part of the proof.

Theorem 2.12. *Let $(\mathfrak{g}, r)$ be an (almost) factorizable semisimple Lie bialgebra, with r as in the previous paragraph. If $D:\mathfrak{g}\to\mathfrak{g}$ is a biderivation, then $D = \mathrm{ad}_H$ for a (unique) $H\in\mathfrak{h}$ satisfying*

$$\alpha(H) = (\tau\alpha)(H), \quad \text{for all } \alpha\in\Gamma_1.$$

In particular, if there are no discrete parameters, then any $H\in\mathfrak{h}$ determines a biderivation and all biderivations are of this type.

It is useful to recall the notion of *level* or *height* of a root; if $\alpha = \sum_{i=1}^{\ell} n_i\alpha_i$ define the height of α as the integer

$$\mathrm{height}(\alpha) = \sum_{i=1}^{\ell} n_i.$$

The same definition can be extended for any weight $\mu\in\mathfrak{h}^*$, namely, if $\mu = \sum_{i=1}^{\ell}\mu_i\alpha_i$, define $\mathrm{height}(\mu) := \sum_{i=1}^{\ell}\mu_i\in\mathbb{C}$. The adjoint representation of $\mathfrak{g}$ decomposes in *height spaces*, explicitly given by

$$\mathfrak{g}_{(n)} := \bigoplus_{\alpha\in\Phi\cup\{0\}:\ \mathrm{height}(\alpha)=n} \mathfrak{g}_\alpha$$

where we include $\mathfrak{g}_{(0)} = \mathfrak{g}_0 = \mathfrak{h}$, so $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_{(n)}$. Analogously, the representation $\Lambda^2\mathfrak{g}$ decomposes in weight spaces $(\Lambda^2\mathfrak{g})_\mu$, with weights of integer levels. For each μ in the $\mathbb{Z}$-span of Φ, we have

$$\left(\Lambda^2\mathfrak{g}\right)_\mu = \bigoplus_{\alpha,\beta\in\Phi\cup\{0\}:\ \alpha+\beta=\mu} (\mathfrak{g}_\alpha\wedge\mathfrak{g}_\beta).$$

This decomposition can be rearranged as a decomposition in height spaces as follows:

$$\Lambda^2\mathfrak{g} = \bigoplus_{\mu\in\mathfrak{h}^*}\left(\Lambda^2\mathfrak{g}\right)_\mu = \bigoplus_{n\in\mathbb{Z}}\left(\bigoplus_{\mu\in\mathfrak{h}^*:\ \mathrm{height}(\mu)=n}\left(\Lambda^2\mathfrak{g}\right)_\mu\right) = \bigoplus_{n\in\mathbb{Z}}\left(\Lambda^2\mathfrak{g}\right)_{(n)}$$

where $(\Lambda^2\mathfrak{g})_{(n)} := \bigoplus_{\mu\in\mathfrak{h}^*:\ \mathrm{height}(\mu)=n}(\Lambda^2\mathfrak{g})_\mu$ is said to be the *component of height n*. Notice that $\Lambda^2\mathfrak{h}\subset(\Lambda^2\mathfrak{g})_{(0)}$ but also $x_{-\alpha}\wedge x_\alpha\in(\Lambda^2\mathfrak{g})_{(0)}$; moreover, conditions on the BD-triple (see [2] and [1]) force that $\alpha\prec\beta$ implies $\mathrm{height}(\alpha) = \mathrm{height}(\beta)$ and so, $x_{-\alpha}\wedge x_\beta\in(\Lambda^2\mathfrak{g})_{(0)}$, then, all the terms which appear in r_Λ are in $(\Lambda^2\mathfrak{g})_{(0)}$. Hence,

$$\boxed{r_\Lambda \in \left(\Lambda^2\mathfrak{g}\right)_{(0)}}$$

Also, from $[\mathfrak{g}_\mu, \Lambda^2\mathfrak{g}_\nu] \subseteq \Lambda^2\mathfrak{g}_{\mu+\nu}$, it is clear that $[\mathfrak{g}_{(n)}, (\Lambda^2\mathfrak{g})_{(k)}] \subseteq (\Lambda^2\mathfrak{g})_{(n+k)}$. This discussion implies the following lemma.

Lemma 2.13. *Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the triangular decomposition related to $\mathfrak{h}$ and a to an election of positive roots; if we write $x = x_+ + x_\mathfrak{h} + x_-$ then $\mathrm{ad}_x(r) = 0$ if and only if $\mathrm{ad}_{x_\pm}(r) = \mathrm{ad}_{x_\mathfrak{h}}(r) = 0$. Moreover, if $x = \sum_n x_{(n)}$ with $x_{(n)} \in \mathfrak{g}_{(n)}$ corresponding to the decomposition of $\mathfrak{g}$ in height spaces, then $\delta x = \mathrm{ad}_x(r) = 0$ if and only if $\delta x_{(n)} = \mathrm{ad}_{x_{(n)}}(r) = 0$ for all $n \in \mathbb{Z}$.*

The proof of the theorem relies in a close observation of the adjoint action, explicitly stated in the following lemma.

Lemma 2.14. *Let $x_\gamma \in \mathfrak{g}_\gamma$ with $\gamma \in \Phi^+$, then*

$$\begin{aligned}\mathrm{ad}_{x_\gamma} r = h_\gamma \wedge x_\gamma &+ \sum_{\gamma\neq\alpha\in\Phi^+} (c_{\gamma,-\alpha} x_{-\alpha+\gamma} \wedge x_\alpha + c_{\gamma,\alpha} x_{-\alpha} \wedge x_{\gamma+\alpha}) \\ &+ \sum_{\alpha\prec\beta:\ \gamma\neq\alpha,\beta} (c_{\gamma,\alpha} x_{-\alpha+\gamma} \wedge x_\beta + c_{\gamma,\beta} x_{-\alpha} \wedge x_{\beta+\gamma}) + \sum_{\beta:\ \gamma\prec\beta} (h_\gamma \wedge x_\beta + c_{\gamma,\beta} x_{-\gamma} \wedge x_{\beta+\gamma}) \\ &+ \sum_{\alpha:\ \alpha\prec\gamma} c_{\gamma,-\alpha} x_{-\alpha+\gamma} \wedge x_\gamma + [x_\gamma, \lambda]\end{aligned}$$

where $c_{\gamma,\pm\alpha} \in \mathbb{C}$ are the structure constants such that $[x_\gamma, x_{\pm\alpha}] = c_{\gamma,\pm\alpha} x_{\gamma\pm\alpha}$. In addition, if we write $\lambda = \sum_{1\leqslant i<j\leqslant\ell} \lambda_{ij} h_i \wedge h_j = \frac{1}{2}\sum_{i,j=1}^{\ell} \lambda_{ij} h_i \wedge h_j$ with $\lambda_{ji} = -\lambda_{ij}$ then

$$\begin{aligned}[x_\gamma, \lambda] &= -\sum_{1\leqslant i<j\leqslant\ell} \lambda_{ij}\big(\gamma(h_i) x_\gamma \wedge h_j + \gamma(h_j) h_i \wedge x_\gamma\big) = \Bigg(\sum_{1\leqslant i<j\leqslant\ell} \lambda_{ij}\big(\gamma(h_i) h_j - \gamma(h_j) h_i\big)\Bigg) \wedge x_\gamma \\ &= \frac{1}{2}\sum_{i,j=1}^{\ell} \lambda_{ij}\big(\gamma(h_i) h_j - \gamma(h_j) h_i\big) \wedge x_\gamma = \sum_{i,j=1}^{\ell} \lambda_{ij} \gamma(h_i) h_j \wedge x_\gamma.\end{aligned}$$

Proof. Straightforward. □

In order to deal with this formula, we simplify it by considering the following decomposition of $\Lambda^2\mathfrak{g}$ induced by the triangular decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{n}_+ \oplus \mathfrak{n}_-)$, namely

$$\Lambda^2\mathfrak{g} = \Lambda^2\mathfrak{h} \oplus \big(\mathfrak{h} \wedge (\mathfrak{n}_+ \oplus \mathfrak{n}_-)\big) \oplus \Lambda^2(\mathfrak{n}_+ \oplus \mathfrak{n}_-).$$

Define $p : \Lambda^2\mathfrak{g} \to \Lambda^2\mathfrak{h} \oplus (\mathfrak{h} \wedge (\mathfrak{n}_+ \oplus \mathfrak{n}_-))$, the canonical projection associated to the above decomposition. The formula of Lemma 2.14 implies the following:

$$p(\mathrm{ad}_{x_\gamma} r) = h_\gamma \wedge \Bigg(x_\gamma + \sum_{\beta:\ \gamma\prec\beta} x_\beta\Bigg) + \Bigg(\sum_{1\leqslant i<j\leqslant\ell} \lambda_{ij}\big(\gamma(h_i) h_j - \gamma(h_j) h_i\big)\Bigg) \wedge x_\gamma.$$

It is convenient to introduce the element $H_\lambda^\gamma := h_\gamma + \sum_{i,j=1}^{\ell} \lambda_{ij} \gamma(h_i) h_j$. Write $\gamma = \sum_{i=1}^{\ell} n_i \alpha_i$ then

$$H_\lambda^\gamma = \sum_{j=1}^{\ell}\left(n_j + \sum_{i=1}^{\ell}\lambda_{ij}\gamma(h_i)\right)h_j = \sum_{j=1}^{\ell}\left(n_j + \sum_{i,k}\lambda_{ij}n_k\alpha_k(h_i)\right)h_j,$$

under this notation we have

$$p(\mathrm{ad}_{x_\gamma} r) = H_\lambda^\gamma \wedge x_\gamma + h_\gamma \wedge \left(\sum_{\beta:\, \gamma\prec\beta} x_\beta\right).$$

Lemma 2.15. *For any $\gamma \in \Phi$ and any $\lambda \in \Lambda^2\mathfrak{h}$, the element $H_\lambda^\gamma \in \mathfrak{h}$ is nonzero.*

Proof. Recall we write $\gamma = \sum_{i=1}^{\ell} n_i\alpha_i$; if $H_\lambda^\gamma = 0$ then, in particular, $\alpha(H_\lambda^\gamma) = 0$ for all $\alpha \in \mathfrak{h}^*$, so

$$0 = \sum_{m=1}^{\ell}\alpha_m\big(H_\lambda^\gamma\big)n_m = \sum_{j,m=1}^{\ell} n_m\alpha_m(h_j)n_j + \sum_{j,k,m=1}^{\ell} n_k\alpha_k(h_i)\lambda_{ij}\alpha_m(h_j)n_m.$$

It is convenient to use matrix notation. Let us denote by K the matrix with entries $\kappa_{ij} = \alpha_i(h_j)$, Λ the matrix with coefficients λ_{ij} and $\underline{n} = (n_1, \ldots, n_\ell)$. Notice that K is the matrix of the Killing form restricted to $\mathfrak{h}$. The formula above can be written as

$$0 = \underline{n}\cdot K\cdot \underline{n}^t + \underline{n}\cdot K\cdot\Lambda\cdot K\cdot\underline{n}^t.$$

The second term is easily seen to be zero since

$$\underline{n}\cdot K\cdot\Lambda\cdot K\cdot\underline{n}^t = \big(\underline{n}\cdot K\cdot\Lambda\cdot K\cdot\underline{n}^t\big)^t = \underline{n}\cdot K^t\cdot\Lambda^t\cdot K^t\cdot\underline{n}^t = -\underline{n}\cdot K\cdot\Lambda\cdot K\cdot\underline{n}^t$$

where the first equality holds because we are transposing a complex number, the second is valid for any product of matrices, and the last uses the fact that K is symmetric and Λ antisymmetric. Besides, in the basis $\{h_1, \ldots, h_\ell\}$, the matrix K is *real symmetric and positive defined* (see for instance [11, Corollary 2.38]), hence

$$\sum_m \alpha_m(H)n_m = \sum_{j,m} n_m\kappa_{mj}n_j = \underline{n}\cdot K\cdot\underline{n}^t > 0 \quad \forall\underline{n}\in\mathbb{R}^n\setminus 0;$$

in particular, it gives a nonzero real number for any $0 \neq (n_1, \ldots, n_\ell) \in \mathbb{Z}^n$. □

Proposition 2.16. *Let $x \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$, then $\mathrm{ad}_x(r) = 0 \Leftrightarrow x = 0$.*

Proof. Let $x = \sum_{\gamma\in\Phi} c_\gamma x_\gamma$ with c_γ arbitrary, and suppose $\mathrm{ad}_x r = 0$. Since $\mathrm{ad}_-(r)$ preserves the height (see Lemma 2.13), we can consider different heights separately. Since we will not need such refinement in all its strength, we will only consider separately the cases height$(\gamma) > 0$ or height$(\gamma) < 0$, namely γ a positive or negative root.

So let us consider an element $x = \sum_{\gamma\in\Phi^+} c_\gamma x_\gamma$, the case in Φ^- is analogous. We have

$$0 = p(\mathrm{ad}_x r) = \sum_{\gamma\in\Phi^+} c_\gamma H_\lambda^\gamma \wedge x_\gamma + \sum_{\gamma\in\Phi^+} c_\gamma h_\gamma \wedge \left(\sum_{\beta:\,\gamma\prec\beta} x_\beta\right).$$

Denote $\widetilde{\Gamma}$ the $\mathbb{Z}$-span of the discrete parameter; we claim that if $\gamma \in \widetilde{\Gamma}\cap\Phi^+$, then $c_\gamma = 0$. To see this, consider $\Gamma_0 \subset \Phi^+$ the set of minimal elements $\gamma \in \widetilde{\Gamma}$ such that $c_\gamma \neq 0$ (minimal with respect to $\prec$).

Notice that the terms with x_β where $\gamma \prec \beta$ cannot cancel any term with x_γ for $\gamma \in \Gamma_0$, because if $c_\beta \neq 0$ then γ could not be minimal. Hence, if we consider only elements in Γ_0, necessarily

$$0 = \sum_{\gamma \in \Gamma_0} c_\gamma H_\lambda^\gamma \wedge x_\gamma$$

since the $\{x_\gamma\}_{\gamma \in \Gamma_0}$ are linearly independent, then

$$0 = c_\gamma H_\lambda^\gamma \quad \forall \gamma \in \Gamma_0.$$

But $H^\gamma \neq 0$ implies $c_\gamma = 0$, which is absurd because $\gamma \in \Gamma_0$. We conclude that $c_\gamma = 0$ for all γ in $\widetilde{\Gamma}$. Hence the equality $\mathrm{ad}_x(r) = 0$ implies

$$0 = p(\mathrm{ad}_x r) = \sum_{\gamma \in \Phi^+} c_\gamma H_\lambda^\gamma \wedge x_\gamma.$$

Now we can repeat word by word the same argument as in case $\gamma \in \Gamma_0$, namely the linear independence of the x_γ implies $c_\gamma H_\lambda^\gamma = 0$ $\forall \gamma \in \Phi^+$, but $H_\lambda^\gamma \neq 0$ implies $c_\gamma = 0$ $\forall \gamma \in \Phi^+$. □

Proof of Theorem 2.12. In order to conclude the proof, almost all work is done. We know that if $\mathrm{ad}_x(r) = 0$ then $x \in \mathfrak{g}_{(0)} = \mathfrak{g}_0 = \mathfrak{h}$. Notice that the standard component and the continuous parameter have total weight equal to zero, i.e. $\mathrm{ad}_H(r_{st} + \lambda) = 0$ for $H \in \mathfrak{h}$, then the only terms surviving in $\mathrm{ad}_H(r)$ are

$$\mathrm{ad}_H(r) = \sum_{\alpha \prec \beta} \mathrm{ad}_H(x_{-\alpha} \wedge x_\beta) = \sum_{\alpha \prec \beta} \big(\beta(H) - \alpha(H)\big) x_{-\alpha} \wedge x_\beta$$

so $\alpha(H) = \beta(H)$ for all $\alpha \prec \beta$, and that is equivalent to $\alpha(H) = (\tau\alpha)(H)$ for all $\alpha \in \Gamma_1$. At this stage, we have finished the description of $\mathrm{Ker}(\delta)$, but in virtue of Corollary 2.10, this implies as well a description of the biderivations in $(\mathfrak{g}, \delta)$. Notice that in the real case, even if $r \in (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}) \setminus \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}$, we know that $r_\Lambda \in \Lambda^2_{\mathbb{R}} \mathfrak{g}$, so the proof of the complexified Lie algebra descends to the real form $\mathfrak{g}$. □

Remark 2.17. Corollary 2.8 says that if $\mathrm{ad}_x \in \mathrm{BiDer}(\mathfrak{g}, r)$ then $x \in \mathfrak{h}$, so Corollary 2.8 together with the very last argument above gives an alternative proof of Theorem 2.12.

2.2. Extension of scalars

For a given Lie bialgebra, it is possible to define a (double) complex of the form $C^{p,q}\mathfrak{g} = \Lambda^p \mathfrak{g}^* \otimes \Lambda^q \mathfrak{g}$, where the vertical differentials are the Chevalley–Eilenberg differential of $\mathfrak{g}$ with coefficients in $\Lambda^q \mathfrak{g}$, and horizontal differentials are the dual of the Chevalley–Eilenberg differential corresponding to the Lie coalgebra structure. This complex was first described in [8]. In particular, for $p = q = 1$, if one identifies $\mathfrak{g}^* \otimes \mathfrak{g} = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}) = \mathrm{End}(\mathfrak{g})$, we get that the kernel of the vertical differential consists of derivations (the image of the preceding differential are the inner ones), and the kernel of horizontal differential consists of coderivations, so the kernel of both differentials is precisely the set of biderivations. As a consequence, the set of biderivations extends scalars in the sense that if $\mathbb{K} \subset \mathbb{E}$ is a field extension, then $\mathrm{BiDer}_{\mathbb{E}}(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{E}) = \mathrm{BiDer}_{\mathbb{K}}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{E}$, and a given biderivation D of a $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is inner if and only if $D \otimes_{\mathbb{K}} \mathrm{Id}_{\mathbb{E}}$ is inner as biderivation of $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{E}$.

3. Main construction for trivial abelian extensions

Along this section, we denote by V a d-dimensional vector space over a field $\mathbb{K}$, $\{t_1, \dots, t_d\}$ a basis of V and $\{t_1^*, \dots, t_d^*\}$ the associated dual basis of V^*.

Theorem 3.1. *Let $(\mathfrak{g}, \delta_{\mathfrak{g}})$ be a Lie bialgebra, (V, δ_V) a d-dimensional Lie coalgebra, V^* the dual Lie algebra and $\mathbb{D} : V^* \to \mathrm{BiDer}(\mathfrak{g})$ a Lie algebra map, then the following map defines a Lie bialgebra structure on $\mathfrak{L} = \mathfrak{g} \times V$, for all $x \in \mathfrak{g}$ and $v \in V$:*

$$\delta(x + v) = \delta_{\mathfrak{g}}(x) + 2\sum_{i=1}^{d} D_i(x) \wedge t_i + \delta_V(v)$$

where $\{t_1, \dots, t_d\}$ is a basis of V, $\{t_1^*, \dots, t_d^*\}$ the dual basis of V^* and $D_i = \mathbb{D}(t_i^*)$, $1 \leqslant i \leqslant d$.

Proof. We need to prove co-Jacobi and the 1-cocycle condition. In order to prove co-Jacobi for δ, for any linear function $f : \mathfrak{g} \to \Lambda^2(\mathfrak{g})$, denote by $\partial_f : \Lambda^2(\mathfrak{g}) \to \Lambda^3(\mathfrak{g})$ the map given by $\partial_f(x \wedge y) = f(x) \wedge y - x \wedge f(y)$. So, under this notation, δ satisfies co-Jacobi if and only if $\partial_\delta \circ \delta = 0$. Notice that $\partial_{f+g} = \partial_f + \partial_g$, so

$$\partial_\delta = \partial_{\delta_{\mathfrak{g}}} + 2\sum_{i=1}^{d} \partial_{D_i(-)\wedge t_i} + \partial_{\delta_V}.$$

Let us prove first that $\partial_\delta \circ \delta(x) = 0$ for any $x \in \mathfrak{g}$,

$$\begin{aligned}
\partial_\delta\big(\delta(x)\big) &= \partial_\delta\big(\delta_{\mathfrak{g}}(x)\big) + 2\sum_{i=1}^{d} \partial_\delta(D_i x \wedge t) \\
&= \partial_{\delta_{\mathfrak{g}}}\big(\delta_{\mathfrak{g}}(x)\big) + 2\sum_{i=1}^{d} \partial_{D_i \wedge t_i}\big(\delta_{\mathfrak{g}}(x)\big) + 2\sum_{i=1}^{d}\big(\delta(D_i x) \wedge t_i - D_i x \wedge \delta(t_i)\big) \\
&= A + B + C
\end{aligned}$$

where these three terms are computed separately as follows. The first term, $A = \partial_{\delta_{\mathfrak{g}}}(\delta_{\mathfrak{g}}(x))$ has to be zero since $\delta_{\mathfrak{g}}$ satisfies co-Jacobi. For the second term,

$$\begin{aligned}
\frac{1}{2}B = \sum_{i=1}^{d} \partial_{D_i \wedge t_i}\big(\delta_{\mathfrak{g}}(x)\big) &= \sum_{i=1}^{d}\big(D_i x_1^{\mathfrak{g}} \wedge t_i \wedge x_2^{\mathfrak{g}} - x_1^{\mathfrak{g}} \wedge D_i x_2^{\mathfrak{g}} \wedge t_i\big) \\
&= -\sum_{i=1}^{d}\big((D_i \otimes \mathrm{Id} + \mathrm{Id} \otimes D_i)\big(x_1^{\mathfrak{g}} \wedge x_2^{\mathfrak{g}}\big) \wedge t_i\big) \\
&= -\sum_{i=1}^{d}\big((D_i \otimes \mathrm{Id} + \mathrm{Id} \otimes D_i)(\delta_{\mathfrak{g}} x) \wedge t_i\big) \in \Lambda^2\mathfrak{g} \wedge V
\end{aligned}$$

where we used the Sweedler-type notation $\delta_{\mathfrak{g}} x = x_1^{\mathfrak{g}} \wedge x_2^{\mathfrak{g}}$. Half of the third term equals

$$\frac{1}{2}C = \sum_{i=1}^{d}\big(\delta(D_i x)\wedge t_i - D_i x\wedge\delta(t_i)\big)$$
$$= \sum_{i=1}^{d}\delta_{\mathfrak g}(D_i x)\wedge t_i + 2\sum_{i,j=1}^{d} D_j(D_i x)\wedge t_j\wedge t_i - \sum_{i=1}^{d} D_i x\wedge\delta_V(t_i).$$

Notice that, in C, only the first sum belongs to $\Lambda^2\mathfrak g\wedge V$, and it cancels with B because D_i are coderivations. It only remains to verify that the second and third terms of C cancel each other, or, equivalently, that the following identity holds

$$2\sum_{i,j=1}^{d} D_j(D_i x)\wedge t_j\wedge t_i = \sum_{i=1}^{d} D_i x\wedge\delta_V(t_i). \tag{4}$$

Observe that in the left-hand side we have

$$2\sum_{i,j=1}^{d} D_j(D_i x)\wedge t_j\wedge t_i = \sum_{i,j=1}^{d}[D_j, D_i](x)\wedge t_j\wedge t_i$$

because $t_j\wedge t_i$ is antisymmetric in the indexes i, j. On the right-hand side of (4), we may write $\delta_V(t_k)$ as a linear combination of the $t_j\wedge t_i$, explicitly $\delta_V(t_k)=\sum_{i,j=1}^{d} c_k^{j,i} t_j\wedge t_i$. So, identity (4) is also equivalent to

$$[D_j, D_i] = \sum_{k=1}^{d} c_k^{j,i} D_k$$

which holds because the map $\mathbb D: V^*\to \mathrm{BiDer}(\mathfrak g)$, $t_i^*\mapsto D_i$, is a Lie algebra map.

Finally, $\delta|_V=\delta_V$ and $\delta_V(V)\subseteq\Lambda^2 V$ since by construction (V,δ_V) is a Lie subcoalgebra. Hence, $\partial_\delta\delta(v)=\partial_\delta\delta_V(v)=\partial_{\delta_V}\delta_V(v)=0$ for any $v\in V$. □

Example 3.2. As a toy example, consider $\mathfrak g=\mathfrak{aff}_2(\mathbb K)$ the non-abelian 2-dimensional Lie algebra, with basis $\{h,x\}$ and bracket $[h,x]=x$, and $V=\mathbb K t$. All possible cobrackets in $\mathfrak{aff}_2(\mathbb K)$ up to isomorphism of Lie bialgebras are (see [7]) as follows:

1. $\delta^0(h)=h\wedge x$, $\delta^0(x)=0$, in this case $\mathcal D=-\mathrm{ad}_x$ and $\mathrm{BiDer}(\mathfrak{aff}_2,\delta^0)=\mathbb K\,\mathrm{ad}_x$; or
2. the 1-parameter family $\delta_\mu(h)=0$, $\delta_\mu(x)=\mu h\wedge x$, $\mu\in\mathbb K$, so $\mathcal D=\mu\,\mathrm{ad}_h$. In this case, $\mathrm{BiDer}(\mathfrak{aff}_2,\delta_\mu)=\mathbb K\,\mathrm{ad}_h$ if $\mu\neq 0$ and $\mathrm{BiDer}(\mathfrak{aff}_2,\delta_\mu)=\mathrm{Der}(\mathfrak{aff}_2)$ if $\mu=0$. Notice that $\mathrm{Der}(\mathfrak{aff}_2)=\mathrm{InnDer}(\mathfrak{aff}_2)$.

The biderivations given above were easily obtained by means of Corollary 2.10. The procedure described in Theorem 3.1 says that if $D\in\mathrm{BiDer}(\mathfrak{aff}_2(\mathbb K),\delta_{\mathfrak{aff}_2})$,

$$\delta(t)=0,\qquad \delta(u)=\delta_{\mathfrak{aff}_2}(u)+D(u)\wedge t,\quad \forall u\in\mathfrak{aff}_2(\mathbb K)$$

is a Lie cobracket on $\mathfrak{aff}_2(\mathbb K)\times\mathbb K$. We obtain the whole list of possible such choices (see table below).

Notice that the Lie bialgebra of the case (i) is isomorphic to the one of case (ii) by means of the map $x\mapsto x$; $h\mapsto h+t$; $t\mapsto t$. Analogously, the Lie bialgebra of the case (iii) with parameter μ is isomorphic to the one of case (iv) with parameter $-\mu$, by means of the map $x\mapsto x$; $h\mapsto h+\frac{1}{\mu}t$; $t\mapsto t$. In case (v) if the derivation $D=\mathrm{ad}_{\alpha x+\beta h}$ then $D: x\mapsto\beta x$, $h\mapsto -\alpha x$, so the cobracket has the

Possible Lie cobrackets on $\mathfrak{aff}_2(\mathbb{K}) \times \mathbb{K}$.

(i)	$\delta(x) = 0$	$\delta(h) = h \wedge x + x \wedge t$	$\delta(t) = 0$	
(ii)	$\delta(x) = 0$	$\delta(h) = h \wedge x$	$\delta(t) = 0$	
(iii)	$\delta(x) = \mu h \wedge x + x \wedge t$	$\delta(h) = 0$	$\delta(t) = 0$	$\mu \neq 0$
(iv)	$\delta(x) = \mu h \wedge x$	$\delta(h) = 0$	$\delta(t) = 0$	$\mu \neq 0$
(v)	$\delta(x) = D(x) \wedge t$	$\delta(h) = D(h) \wedge t$, $D \in \mathrm{Der}(\mathfrak{aff}_2)$	$\delta(t) = 0$	

form $\delta x = \beta x \wedge t$, $\delta t = 0$ and $\delta h = -\alpha x \wedge t$. In matrix notation, choosing basis $\{x, t, h\}$ of the Lie algebra $\mathfrak{L} = \mathfrak{aff}_2 \times \mathbb{K}t$, and basis $\{x \wedge t, t \wedge h, h \wedge x\}$ of $\Lambda^2\mathfrak{L}$, the cobrackets given by the above construction are

$$\text{(ii)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad \text{(iv)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}; \qquad \text{(v)} \begin{pmatrix} \beta & 0 & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (v) is isomorphic to $\left(\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)$ if $\alpha \neq 0$, $\beta = 0$, simply by considering the transformation $x \leftrightarrow \frac{1}{\alpha}x$. If $\beta \neq 0$, then (v) is isomorphic to $\left(\begin{smallmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)$ by the transformation $x \mapsto x$, $h \mapsto h + \frac{\alpha}{\beta}x$, $t \mapsto \beta t$. If one compares all this possibilities with the classification result in [7] for the Lie algebra $\mathfrak{r}_{3,\lambda=0}$ one sees that we have covered all possibilities. This is not surprising due to the following result.

Next theorem says that with some extra hypothesis, Theorem 3.1 has its converse. See the table in Example 3.5 for non-semisimple examples where next theorem applies.

Theorem 3.3. *Let $\mathfrak{g}$ be a Lie algebra such that $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$ and $\mathcal{Z}(\mathfrak{g}) = 0$; let V be a vector space considered as abelian Lie algebra; assume that either $\dim V > 1$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, or $\dim V = 1$. If $\mathfrak{L} = \mathfrak{g} \times V$ then all Lie cobrackets on $\mathfrak{L}$ are as in Theorem 3.1. Explicitly, if δ defines a Lie bialgebra structure on $\mathfrak{L}$, then*

1. *$\delta(V) \subseteq \Lambda^2(V)$, so, V is a Lie subcoalgebra with $\delta_V = \delta|_V$. In particular, V is an ideal and a coideal, hence, $\mathfrak{L}/V$ inherits a unique Lie bialgebra structure such that $\pi : \mathfrak{L} \to \mathfrak{L}/V$ is a Lie bialgebra map.*
2. *Let $\pi_{\mathfrak{g}} : \mathfrak{L} \to \mathfrak{g}$ be the canonical projection associated to the decomposition $\mathfrak{L} = \mathfrak{g} \times V$, then $\delta_{\mathfrak{g}} := (\pi_{\mathfrak{g}} \wedge \pi_{\mathfrak{g}}) \circ \delta|_{\mathfrak{g}} : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a Lie bialgebra structure on $\mathfrak{g}$ and $\mathfrak{L}/V \cong \mathfrak{g}$ canonically as Lie bialgebras.*
3. *$\delta(\mathfrak{g}) \subseteq \Lambda^2\mathfrak{g} \oplus \mathfrak{g} \wedge V$. If $\{t_i\}_{i=1}^d$ is a basis of V, then for any $x \in \mathfrak{g}$, $\delta(x)$ is of the form*

$$\delta x = \delta_{\mathfrak{g}} x + 2\sum_{i=1}^{d} D_i x \wedge t_i$$

   where $D_i : \mathfrak{g} \to \mathfrak{g}$, $i = 1, \dots, d$, are derivations and coderivations of $(\mathfrak{g}, \delta_{\mathfrak{g}})$. The linear subspace generated by $\{D_1, \dots, D_d\}$ is a Lie subalgebra of $\mathrm{BiDer}(\mathfrak{g})$; moreover, the map $\mathbb{D} : (V^*, \delta_V^*) \to \mathrm{BiDer}(\mathfrak{g})$ defined by $\mathbb{D}(t_i^*) = D_i$, is a Lie algebra map.
4. *Let $(\mathfrak{L}, \delta)$ be the Lie bialgebra associated to a data $(\mathfrak{L}, \delta_{\mathfrak{g}}, \delta_V, D_1, \dots, D_d)$. Let $\Phi = (\phi_{\mathfrak{g}}, \phi_V)$ be a linear automorphism of $\mathfrak{L}$, with $\phi_{\mathfrak{g}}$ a Lie algebra automorphism of $\mathfrak{g}$ and $\phi_V \in \mathrm{GL}(V)$. If we denote by $\widetilde{\delta}_{\mathfrak{g}} = (\phi_{\mathfrak{g}} \wedge \phi_{\mathfrak{g}}) \circ \delta_{\mathfrak{g}} \circ \phi_{\mathfrak{g}}^{-1}$, $\widetilde{\delta}_V = (\phi_V \wedge \phi_V) \circ \delta_V \circ \phi_V^{-1}$, $\widetilde{D}_i := \sum_{j=1}^d A_{ij} \phi_{\mathfrak{g}} \circ D_j \circ \phi_{\mathfrak{g}}^{-1}$, where $\phi_V(t_j) = \sum_{i=1}^d A_{ij} t_i$, $1 \leqslant j \leqslant d$, and $(\mathfrak{L}, \widetilde{\delta})$ the Lie bialgebra associated to the data $(\mathfrak{L}, \widetilde{\delta}_{\mathfrak{g}}, \widetilde{\delta}_V, \widetilde{D}_1, \dots, \widetilde{D}_d)$, then $\Phi : (\mathfrak{L}, \delta) \xrightarrow{\sim} (\mathfrak{L}, \widetilde{\delta})$ is a Lie bialgebra isomorphism. If, moreover, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then any Lie bialgebra isomorphism from $(\mathfrak{L}, \delta)$ to $(\mathfrak{L}, \widetilde{\delta})$ is of this form.*

Proof. Consider the decomposition $\Lambda^2(\mathfrak{L}) = \Lambda^2(\mathfrak{g}) \oplus \mathfrak{g} \wedge V \oplus \Lambda^2(V)$. It is straightforward to see that if $\delta(\mathfrak{g}) \subseteq \Lambda^2\mathfrak{g} \oplus \mathfrak{g} \wedge V$ and $\delta(V) \subseteq \Lambda^2 V$ then the implications in the proof of Theorem 3.1 can be reversed. This will prove items 1, 2, 3. So, let us see $\delta(V) \subseteq \Lambda^2 V$ first:

By Proposition 1.3 together with $\mathcal{Z}(\mathfrak{g})=0$, we have

$$\delta(V)=\delta\big(\mathcal{Z}(\mathfrak{L})\big)\subseteq\big(\Lambda^2\mathfrak{L}\big)^{\mathfrak{L}}=\big(\Lambda^2\mathfrak{g}\big)^{\mathfrak{g}}\oplus\mathcal{Z}(\mathfrak{g})\wedge V\oplus\Lambda^2V=\Lambda^2V.$$

On the other hand, $\delta(\mathfrak{g})\subseteq\Lambda^2\mathfrak{g}\oplus\mathfrak{g}\wedge V$ is trivial in case $\dim V=1$ since in this case, $\Lambda^2V=0$. If $\dim V>1$, assuming $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]$ then also $\mathfrak{g}=[\mathfrak{L},\mathfrak{L}]$, so by Proposition 1.3

$$\delta(\mathfrak{g})=\delta\big([\mathfrak{g},\mathfrak{g}]\big)=\delta\big([\mathfrak{L},\mathfrak{L}]\big)\subset[\mathfrak{L},\mathfrak{L}]\wedge\mathfrak{L}=\mathfrak{g}\wedge\mathfrak{L}=\Lambda^2\mathfrak{g}\oplus\mathfrak{g}\otimes V.$$

4. Notice that $\widetilde{\delta}_{\mathfrak{L}}=(\Phi\wedge\Phi)\circ\delta_{\mathfrak{L}}\circ\Phi^{-1}$ if and only if $\widetilde{\delta}_{\mathfrak{g}}=(\phi_{\mathfrak{g}}\wedge\phi_{\mathfrak{g}})\circ\delta_{\mathfrak{g}}\circ\phi_{\mathfrak{g}}^{-1}$, $\widetilde{\delta}_V=(\phi_V\wedge\phi_V)\circ\delta_V\circ\phi_V^{-1}$ and

$$(\Phi\wedge\Phi)\left(\sum_i D_i\big(\phi_{\mathfrak{g}}^{-1}x\big)\wedge t_i\right)=\sum_i\widetilde{D}_i(x)\wedge t_i.$$

The identities concerning $\widetilde{\delta}_{\mathfrak{g}}$ and $\widetilde{\delta}_V$ are true by hypothesis. For the last, notice that

$$(\Phi\wedge\Phi)\big(D_i\big(\phi_{\mathfrak{g}}^{-1}x\big)\wedge t_i\big)=\big(\phi_{\mathfrak{g}}D_i\phi_{\mathfrak{g}}^{-1}(x)\big)\wedge\phi_V(t_i);$$

write $\phi_V(t_i)=\sum_j A_{ij}t_j$, then

$$(\Phi\wedge\Phi)\big(D_i\big(\phi_{\mathfrak{g}}^{-1}x\big)\wedge t_i\big)=\sum_j A_{ij}\phi_{\mathfrak{g}}\big(D_i\big(\phi_{\mathfrak{g}}^{-1}x\big)\big)\wedge t_j.$$

For the converse, if $(\mathfrak{L},\delta_{\mathfrak{L}})$ and $(\mathfrak{L},\widetilde{\delta}_{\mathfrak{L}})$ are Lie bialgebras, then we have the corresponding $(\delta_{\mathfrak{g}},\delta_V,D_1,\dots,D_d)$ and $(\widetilde{\delta}_{\mathfrak{g}},\widetilde{\delta}_V,\widetilde{D}_1,\dots,\widetilde{D}_d)$. If they are isomorphic Lie bialgebras, then there exists a Lie algebra isomorphism $\Phi:\mathfrak{L}\to\mathfrak{L}$ such that $\widetilde{\delta}_{\mathfrak{L}}=(\Phi\wedge\Phi)\circ\delta_{\mathfrak{L}}\circ\Phi^{-1}$. It is necessary to prove that it induces the existence of $\phi_{\mathfrak{g}}:\mathfrak{g}\to\mathfrak{g}$ and $\phi_V:V\to V$, or, in other words, that $\Phi(\mathfrak{g})\subseteq\mathfrak{g}$ and $\Phi(V)\subseteq V$. This holds because $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ and $\mathcal{Z}(\mathfrak{g})=0$ imply $\Phi(\mathfrak{g})=\Phi([\mathfrak{g},\mathfrak{g}])=\Phi([\mathfrak{L},\mathfrak{L}])=[\mathfrak{L},\mathfrak{L}]=[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ and $\Phi(V)=\Phi(\mathcal{Z}(\mathfrak{L}))=\mathcal{Z}(\mathfrak{L})=V$. □

Specializing the main theorem to the case of $\dim V=1$, we obtain the following.

Corollary 3.4. *Let $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}$ be a Lie bialgebra where the underlying Lie algebra is the product of the Lie algebra $\mathfrak{g}$ and the field $\mathbb{K}=\langle t\rangle$ considered as trivial one-dimensional Lie algebra; suppose that $\mathcal{Z}(\mathfrak{g})=0$ and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}=0$; then the Lie bialgebra structures on $\mathfrak{L}$ are determined by pairs $(\delta_{\mathfrak{g}},D)$, where $\delta_{\mathfrak{g}}$ is a Lie bialgebra structure on $\mathfrak{g}$ and $D\in\mathrm{BiDer}(\mathfrak{g},\delta_{\mathfrak{g}})$. The Lie cobracket on $\mathfrak{L}$ is explicitly given by $\delta(x)=\delta_{\mathfrak{g}}(x)+D(x)\wedge t$, for any $x\in\mathfrak{g}$, and $\delta(t)=0$.*

Example 3.5. The table below exhibits some properties of the non-abelian, real 3-dimensional Lie algebras. We see that there are non-semisimple examples of $\mathfrak{g}$ where $\mathcal{Z}(\mathfrak{g})=0$ and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}=0$; so, the previous result applies in order to describe Lie bialgebra structures on 4-dimensional real Lie algebras of type $\mathfrak{g}\times\mathbb{R}$.

The hypothesis $\mathcal{Z}(\mathfrak{g})=0$ and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}=0$ hold in the semisimple case.

Corollary 3.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra (so all cocycles on $\mathfrak{g}$ are coboundary and every derivation on $\mathfrak{g}$ is inner), then*

Invariants of 3-dimensional real Lie algebras.

$\mathfrak{g}$	$\mathcal{Z}\mathfrak{g}$	$(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$	$[\mathfrak{g},\mathfrak{g}]$
$\mathfrak{h}_3 : [x,y]=z$	$\mathbb{R}z$	$\mathbb{R}x\wedge z\oplus\mathbb{R}y\wedge z$	$\mathbb{R}z$
$\mathfrak{r}_3$			
$[h,x]=x,\ [h,y]=x+y,\ [x,y]=0$	0	0	$\mathbb{R}x\oplus\mathbb{R}y$
$\mathfrak{r}_{3,\lambda}$			
$[h,x]=x,\ [h,y]=\lambda y,\ [x,y]=0$:			
$\lambda\in(-1,1],\ \lambda\neq 0$	0	0	$\mathbb{R}x\oplus\mathbb{R}y$
$\lambda=-1$	0	$\mathbb{R}x\wedge y$	$\mathbb{R}x\oplus\mathbb{R}y$
$\lambda=0$	$\mathbb{R}y$	0	$\mathbb{R}x$
$\mathfrak{r}'_{3,\lambda},\ \lambda\geqslant 0$			
$[h,x]=\lambda x-y,\ [h,y]=x+\lambda,\ [x,y]=0$	0	0	$\mathbb{R}x\oplus\mathbb{R}y$
$\mathfrak{su}(2)$	0	0	$\mathfrak{su}(2)$
$\mathfrak{sl}(2,\mathbb{R})$	0	0	$\mathfrak{sl}(2,\mathbb{R})$

1. *All Lie bialgebra structures on $\mathfrak{L}=\mathfrak{g}\times\mathbb{K}t$ are coboundary and determined by a Lie bialgebra structure on $\mathfrak{g}$, denoted by $\delta_{\mathfrak{g}}(x)=\mathrm{ad}_x(r)$, with $r\in\Lambda^2\mathfrak{g}$ satisfying $[r,r]\in(\Lambda^3\mathfrak{g})^{\mathfrak{g}}$, and a biderivation $D:(\mathfrak{g},\delta_{\mathfrak{g}})\to(\mathfrak{g},\delta_{\mathfrak{g}})$, which is necessarily of the form ad_H with $H\in\operatorname{Ker}\delta_{\mathfrak{g}}$. The cobracket on $\mathfrak{L}$ is given by*

$$\delta(x)=\mathrm{ad}_x(r)+[H,x]\wedge t=\mathrm{ad}_x(r-H\wedge t)$$

   for any $x\in\mathfrak{g}$ and $\delta(t)=0$. Since we may choose t up to scalar multiple, the element H may be modified by a nonzero scalar without changing the isomorphism class of the Lie bialgebra.
2. *Assume in addition that $(\mathfrak{g},\delta_{\mathfrak{g}})$ is (almost) factorizable, $\delta_{\mathfrak{g}}(x)=\mathrm{ad}_x(r)$ with r given by a BD-data, i.e. a Cartan subalgebra $\mathfrak{h}$, simple roots Δ, a BD-triple (Γ_1,Γ_2,τ) and a continuous parameter with skew symmetric component $\lambda\in\Lambda^2\mathfrak{h}$, with r_Λ as in Eq. (3). Then $H\in\operatorname{Ker}\delta_g$ if and only if $H\in\mathfrak{h}$ and $\tau\alpha(H)=\alpha(H)$ for all $\alpha\in\Gamma_1$.*

Example 3.7. *Lie bialgebra structures on $\mathfrak{gl}(2,\mathbb{R})=\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}t$.* Let δ be any Lie bialgebra structure on $\mathfrak{sl}(2,\mathbb{R})$, which is a simple Lie algebra, $\delta=\partial r$. From [7], we know that there are factorizable, almost factorizable and triangular structures on $\mathfrak{sl}(2,\mathbb{R})$. Let $\{x,h,y\}$ be the usual basis of $\mathfrak{sl}(2,\mathbb{R})$.

Case 1. If $r=h\wedge x$, then $H_r=2x$ (which is not a regular, but a nilpotent element). We get that $ah+bx+cy$ commutes with x if and only if $a=c=0$, so $\mathrm{BiDer}(\mathfrak{sl}(2,\mathbb{R}))=\mathbb{R}\,\mathrm{ad}_x$. In particular, $(\mathfrak{sl}(2,\mathbb{R}),r)$ is a triangular Lie bialgebra.
Case 2. If $r=x\wedge y$ then $H_r=h$, then every biderivation is a multiple of ad_h in this case. In particular, $(\mathfrak{sl}(2,\mathbb{R}),r)$ is a factorizable Lie bialgebra.
Case 3. If $r=h\wedge(x+y)$ then $H_r=x-y$ is semisimple non-diagonalizable. One can easily check that every biderivation is a multiple of ad_{x-y}. In particular, $(\mathfrak{sl}(2,\mathbb{R}),r)$ is an almost factorizable (non-factorizable) Lie bialgebra.

Hence, we obtain the following description.

Corollary 3.8. *An exhaustive list of isomorphism classes of Lie bialgebra structures on $\mathfrak{gl}(2,\mathbb{R})=\mathfrak{sl}(2,\mathbb{R})\times\mathbb{R}t$ is given as follows.*

(a) *With nonzero cobracket on $\mathfrak{sl}(2,\mathbb{R})$:*
   1. *Let $r=\pm h\wedge x$ if $D=0$, or $r=\pm(h\wedge x+x\wedge t)$ if $D\neq 0$; in particular, $(\mathfrak{gl}(2,\mathbb{R}),r)$ is a triangular Lie bialgebra.*
   2. *Let $r=\beta x\wedge y$ if $D=0$, or $r=\beta(x\wedge y+h\wedge t)$ if $D\neq 0$, $\beta\in\mathbb{R}_+$; in particular, $(\mathfrak{gl}(2,\mathbb{R}),r)$ is a factorizable Lie bialgebra.*
   3. *Let $r=\alpha h\wedge(x+y)$ if $D=0$, or $r=\alpha(h\wedge(x+y)+(x-y)\wedge t)$ if $D\neq 0$, $\alpha\in\mathbb{R}\setminus\{0\}$; in particular, $(\mathfrak{gl}(2,\mathbb{R}),r)$ is an almost factorizable Lie bialgebra.*

(b) *With zero cobracket on* $\mathfrak{sl}(2,\mathbb{R})$ *we have* $\mathrm{BiDer}(\mathfrak{sl}(2,\mathbb{R})) = \mathrm{Der}(\mathfrak{g})$, *then there are three nontrivial isomorphism classes and, in each of them, there is a unique derivation up to the action of* $\mathfrak{sl}(2,\mathbb{R})$. *In each case,* $(\mathfrak{gl}(2,\mathbb{R}), r)$ *is a triangular Lie bialgebra.*
   1. *If* $D = \mathrm{ad}_{\widetilde{X}}$ *with* $\widetilde{X}$ *nonzero nilpotent, conjugated to* x, *then* $r = x \wedge t$.
   2. *If* $D = \mathrm{ad}_{\widetilde{H}}$ *with* $\widetilde{H}$ *semisimple diagonalizable, conjugated to any non-negative multiple of* h, *then* $r = h \wedge t$ *or zero.*
   3. *If* $D = \mathrm{ad}_{\widetilde{U}}$ *with* $\widetilde{U}$ *semisimple non-diagonalizable, conjugated to any multiple of* $x - y$, *then* $r = (x - y) \wedge t$.

Remark 3.9. If $\mathfrak{g}$ is a simple Lie algebra, then any Lie bialgebra structure on $\mathfrak{g}$ is either triangular or (almost) factorizable. If $\mathfrak{g}$ is a semisimple Lie algebra, then any Lie bialgebra structure on it may be triangular, (almost) factorizable or none on them, depending on the situation in each component.

Remark 3.10 *(Dimension of the space of solutions).* Let $\mathfrak{L} = \mathfrak{g} \times \mathbb{K} \cdot t$, with $\mathfrak{g}$ a semisimple Lie algebra, and suppose that $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is (almost) factorizable such that $\delta_{\mathfrak{g}} = \partial r_\Lambda$, with r_Λ of BD-form, then Corollary 3.6 applies. For each BD-triple $(\Gamma_1, \Gamma_2, \tau)$, [2] gives the dimension of the space of solutions of the skew symmetric component of all possible continuous parameters $\lambda \in \Lambda^2\mathfrak{h}$, namely $\frac{k(k-1)}{2}$ if $k = |\Delta \setminus \Gamma_1|$. Besides, there are $|\Gamma_1|$ amount of equations for the possible $H \in \mathfrak{h}$ such that $\tau\alpha(H) = \alpha(H)$ for all $\alpha \in \Gamma_1$; this gives in addition $|\Delta \setminus \Gamma_1|$; hence, the set of pairs (λ, H) is an affine space of dimension $\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$ for each BD-triple. Since we may choose t up to scalar multiple, this dimension is, indeed, one unit less.

Example 3.11. The r-matrices corresponding to all the (almost) factorizable Lie bialgebra structures on real forms of complex simple Lie algebras are given in [1]. This, together with the techniques explained in this section, gives an exhaustive list of Lie bialgebra isomorphism classes on real Lie algebras of the form $\mathfrak{s} \times V$, with $\mathfrak{s}$ a real form of a complex simple Lie algebra with a given (almost) factorizable structure. For instance, $\mathfrak{u}(n) = \mathfrak{su}(n) \times \mathbb{R}$, $\mathfrak{u}(p,q) = \mathfrak{su}(p,q) \times \mathbb{R}$.

Example 3.12. The classification of the Lie bialgebra structures on three-dimensional real Lie algebras, both in the (almost) factorizable and in the triangular case, is given in [7]. This, combined with the results of this section, provides all the Lie bialgebra isomorphism classes on real Lie algebras of shape $\mathfrak{g} \times V$, with $\mathfrak{g}$ any three-dimensional real Lie algebra such that $\mathcal{Z}\mathfrak{g} = 0$ and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$. For instance, $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}$, with $0 \neq \lambda \in (-1, 1]$, and $\mathfrak{r}'_{3,\lambda}$ satisfy the hypothesis, so Theorem 3.3 applies for $\mathfrak{L} = \mathfrak{g} \times \mathbb{R}$. However, among them there may be some repetitions, since in general we do not have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ if $\mathfrak{g}$ is solvable.

3.1. Abelian extensions with $\dim V > 1$

If $\dim V > 1$, there are more possibilities than $\mathbb{D} = 0$ or $\mathbb{D} \neq 0$; we can stratify them by the dimension of the image of $\mathbb{D}$. If the image of a linear map $\mathbb{D}: V^* \to \mathrm{BiDer}(\mathfrak{g})$ is d_0-dimensional, $0 \leqslant d_0 \leqslant d$, consider a basis $\{t_1, \dots, t_d\}$ of V and the corresponding dual basis $\{t_1^*, \dots, t_d^*\}$ of V^* such that $\{t_{d_0+1}^*, \dots, t_d^*\}$ is a basis of $\mathrm{Ker}\,\mathbb{D}$, namely, $D_1, \dots, D_{d_0}$ are linearly independent and $D_{d_0+1} = \dots = D_d = 0$. The condition $[D_i, D_j] = \sum_{k=1}^d c_k^{ij} D_k = \sum_{k=1}^{d_0} c_k^{ij} D_k$ determines uniquely c_k^{ij} for $k = 1, \dots, d_0$ in terms of the constant structures of the Lie algebra $\mathrm{Im}(\mathbb{D}) \subseteq \mathrm{BiDer}$. In the case $(\mathfrak{g}, \delta_{\mathfrak{g}})$ semisimple and factorizable, we know that $\mathrm{BiDer}(\mathfrak{g}, \delta_{\mathfrak{g}}) \subseteq \mathfrak{h}$ (Theorem 2.12), which is abelian, so the general Theorem 3.3 specializes in the following result:

Proposition 3.13. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra over* $\mathbb{K}$, $V = \mathbb{K}^d$, *the abelian Lie algebra of dimension* d. *Consider* $\mathfrak{L} = \mathfrak{g} \times V$ *the trivial abelian extension of the Lie algebra* $\mathfrak{g}$ *by* V. *If* $\delta: \mathfrak{L} \to \Lambda^2\mathfrak{L}$ *is a Lie bialgebra structure on* $\mathfrak{L}$ *such that* $(\mathfrak{g}, \delta_{\mathfrak{g}})$ *is an (almost) factorizable Lie bialgebra,* $\delta_{\mathfrak{g}}(x) = \mathrm{ad}_x(r)$ *for all* $x \in \mathfrak{g}$, *with* r *given by a BD-data* $\mathfrak{h}$, Δ, $(\Gamma_1, \Gamma_2, \tau)$, $\lambda \in \Lambda^2\mathfrak{h}$, *then, there exists a basis* $\{t_1, \dots, t_d\}$ *of* V *and* $H_1, \dots, H_{d_0} \in \mathfrak{h}$ *linearly independent elements* $(d_0 \leqslant d)$ *satisfying*

$$\alpha(H_i) = \tau\alpha(H_i) \quad \forall \alpha \in \Gamma_1,\ i = 1, \ldots, d_0,$$

such that for all $x \in \mathfrak{g}$

$$\delta(x) = \delta_{\mathfrak{g}}(x) + \sum_{i=1}^{d_0} [H_i, x] \wedge t_i = \mathrm{ad}_x\left(r - \sum_{i=1}^{d_0} H_i \wedge t_i\right)$$

and a Lie coalgebra structure $\delta_V : V \to \Lambda^2 V$ satisfying

$$\delta_V t_1 = \cdots = \delta_V t_{d_0} = 0.$$

Remark 3.14. In the notation of the above theorem, if $d_0 = d$ then $\delta_V \equiv 0$. Notice that if $\dim V > 1$, the structure on $\mathfrak{L}$ is coboundary if and only if $\delta_V \equiv 0$, which was already predicted in item 2 of Lemma 1.5. The examples with $\delta_V \neq 0$ were not covered in [3], since this work considers only coboundary structures.

Notice that the election of the H_i appearing in the theorem above depends on a choice of a basis for the complement of $\mathrm{Ker}(\mathbb{D}) \subset V^*$. If one fixes a complement (of dimension d_0 in the notations of the theorem), then the action of $\mathrm{GL}(d_0, \mathbb{K})$ acts on the set of basis of this complement, so we see that $\mathrm{GL}(d_0, \mathbb{K})$ acts on the set of d_0-uples $(H_1, \ldots, H_{d_0})$ in the obvious way, without changing the isomorphism class of the Lie bialgebra $\mathfrak{L}$. The case $d_0 = 1$ is Corollary 3.6. The following is an example for $\dim V = 2$.

Example 3.15. Suppose that $\mathfrak{L} = \mathfrak{g} \times V$ is a product of a semisimple Lie algebra $\mathfrak{g}$ and an abelian Lie algebra V with $\dim V = 2$; write $V = \langle t_1, t_2 \rangle$; then the Lie bialgebra structures on $\mathfrak{L}$ are of three possible types:

1. If $\mathbb{D} = 0$ then $\mathfrak{L} = \mathfrak{g} \times V$ is a product Lie bialgebra, i.e.

$$\delta(x + v) = \delta_{\mathfrak{g}}(x) + \delta_V(v)$$

for any $x \in \mathfrak{g}$, $v \in V$. For any fixed Lie bialgebra structure $\delta_{\mathfrak{g}}$ on $\mathfrak{g}$, there are two isomorphism classes, namely, $\delta_V = 0$, or $\delta_V \neq 0$, which is the unique non-coabelian two-dimensional Lie coalgebra.

2. If $\mathrm{Im}\,\mathbb{D} = \mathbb{K}D \neq 0$, then

$$\delta(x) = \delta_{\mathfrak{g}}(x) + [H, x] \wedge t_1, \qquad \delta t_1 = 0, \qquad \delta t_2 = a t_1 \wedge t_2$$

with $\delta_{\mathfrak{g}}$ a Lie cobracket on $\mathfrak{g}$ and $H \in \mathrm{Ker}(\delta_g)$. Changing H by a nonzero scalar multiple, the isomorphism class of the Lie bialgebra does not change. We may also assume $a = 0$ or 1. Notice that if $a = 1$ then the cobracket is *not* coboundary.

3. If $\mathrm{Im}\,\mathbb{D} = \mathbb{K}D_1 \oplus \mathbb{K}D_2$ of dimension two, $D_i = \mathrm{ad}_{H_i}$, $i = 1, 2$, then

$$\delta(x + v) = \delta_{\mathfrak{g}}(x) + [H_1, x] \wedge t_1 + [H_2, x] \wedge t_2 + \delta_V(v)$$

for any $x \in \mathfrak{g}$, $v \in V$, with the following restrictions: there exists $c = 0, 1$ such that $[H_1, H_2] = cH_1$ and the Lie coalgebra structure δ_V is given by $\delta_V t_1 = c t_1 \wedge t_2$, $\delta_V t_2 = 0$. Notice that if the Lie bialgebra structure $\delta_{\mathfrak{g}}$ on $\mathfrak{g}$ is factorizable, then $c = 0$ and hence δ is coboundary.

Example 3.16 *(Cremmer–Gervais).* Consider $\mathfrak{L} = \mathfrak{gl}(n, \mathbb{K}) = \mathfrak{sl}(n, \mathbb{K}) \times \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Fix the Cartan subalgebra $\mathfrak{h}$ of traceless diagonal matrices and the factorizable Lie bialgebra structure on $\mathfrak{sl}(n, \mathbb{C})$ given by an r-matrix r with $r + r^{21} = \Omega$ and skew symmetric component obtained from the discrete parameter $\Gamma_1 = \{\alpha_1, \dots, \alpha_{n-2}\}$, $\Gamma_2 = \{\alpha_2, \dots, \alpha_{n-1}\}$ and $\tau(\alpha_i) = \alpha_{i+1}$, $1 \leqslant i \leqslant n-2$, and any corresponding $\lambda \in \Lambda^2\mathfrak{h}$. As it was proved in [1], this BD-data on $\mathfrak{sl}(n, \mathbb{C})$ gives place to a factorizable Lie bialgebra structure on $\mathfrak{sl}(n, \mathbb{R})$, considered as its split form via the usual sesquilinear involution, if and only if $\lambda \in \Lambda^2_{\mathbb{C}}(\mathfrak{h}) \cap \Lambda^2_{\mathbb{R}}\mathfrak{sl}(n, \mathbb{R}) = \Lambda^2_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}})$, if we denote by $\mathfrak{h}_{\mathbb{R}}$ the Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ consisting of traceless real diagonal matrices.

The equations

$$\alpha(H) = (\tau\alpha)(H)$$

for all $\alpha \in \Gamma_1$, $H \in \mathfrak{h}$, form a system of $n-2$ equations in the $n-1$ variables which are the coefficients of H in the basis $\{H_{\alpha_1}, \dots, H_{\alpha_{n-1}}\}$ of $\mathfrak{h}$; hence the space of solutions has dimension one. In fact, we knew by other means that the regular element

$$H_r := [\ ,\ ](r_\Lambda) = \sum_{\alpha \in \Phi^+} H_\alpha$$

lies in $\mathrm{Ker}(\delta)$, since $\mathcal{D} = [\ ,\ ] \circ \delta = \mathrm{ad}_{H_r}$ is a biderivation, in virtue of Propositions 2.2 and 2.9. As a consequence, all biderivations of $(\mathfrak{g}, r)$ are scalar multiples of ad_{H_r}. On the other hand, analogous result holds in the real case, if we consider the subspace of $\mathfrak{h}_{\mathbb{R}}$ of real solutions. Notice that $H_r = [\ ,\ ](r_\Lambda) \in \mathfrak{sl}(n, \mathbb{R})$.

Both in the complex and in the real case, we conclude that there are exactly two isomorphism classes of Lie bialgebra on $\mathfrak{L}$ such that $\mathfrak{L}/V = (\mathfrak{g}, r)$, given explicitly by

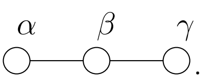
$$\delta_1(x + v) = \delta_{\mathfrak{g}}(x) + D(x) \wedge t = \mathrm{ad}_x(r) + [H_r, x] \wedge t$$

and

$$\delta_2(x + v) = \delta_{\mathfrak{g}}(x) = \mathrm{ad}_x(r).$$

Example 3.17. Let $\mathfrak{L} = \mathfrak{gl}(4, \mathbb{C}) = \mathfrak{sl}(4, \mathbb{C}) \times \mathbb{C}$ and $\mathfrak{L}_0 = \mathfrak{gl}(4, \mathbb{R}) = \mathfrak{sl}(4, \mathbb{R}) \times \mathbb{R}$, denote also $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ and $\mathfrak{g}_0 = \mathfrak{sl}(4, \mathbb{R})$. Let $\Delta = \{\alpha, \beta, \gamma\}$ be a choice of simple roots with respect to a root system for a given Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Recall that a basis of root vectors of $\mathfrak{g}$ is

$$\mathcal{B} = \{x_\alpha, x_\beta, x_\gamma, x_{\alpha+\beta}, x_{\beta+\gamma}, x_{\alpha+\beta+\gamma}, x_{-\alpha}, x_{-\beta}, x_{-\alpha-\beta}, x_{-\beta-\gamma}, x_{-\alpha-\beta-\gamma}\} \cup \{h_\alpha, h_\beta, h_\gamma\},$$

the Cartan matrix is $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ and the Dynkin diagram is 

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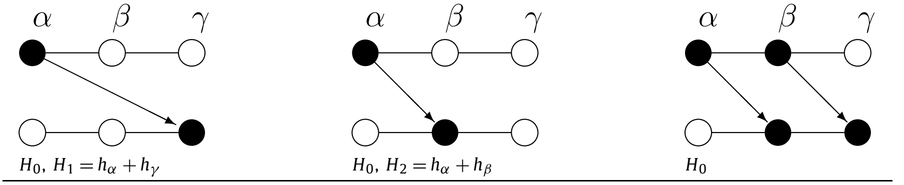
In case of the empty BD-triple, all $H \in \mathfrak{h}$ are solutions of $\tau\alpha(H) = \alpha(H)$. In the following table, we list (up to isomorphism of the Dynkin diagram) all possible nontrivial discrete parameters for $\mathfrak{sl}(4, \mathbb{C})$ and generators of the space of solutions $\{H \in \mathfrak{h}\colon\ \alpha(H) = (\tau\alpha)(H)\ \forall \alpha \in \Gamma_1\}$. Notice that $\mathfrak{h} = \mathcal{Z}_{\mathfrak{g}}(H_0)$ (see Proposition 2.7), i.e. the initial Cartan subalgebra coincides with the centralizer of the regular element $H_0 = [\ ,\ ](r_\Lambda)$ explicitly given by

$$H_0 = 3h_\alpha + 4h_\beta + 3h_\gamma = \sum_{\alpha \in \Phi^+} H_\alpha.$$

Γ_1 and Γ_2 are subsets of Δ represented by the black roots.



Indeed, we knew that the regular element H_0 lies in $\mathrm{Ker}(\delta)$ for δ coming from any choice of BD-triple, because $\mathcal{D} = [\ ,\ ] \circ \delta = -\mathrm{ad}_{H_0}$ is a biderivation, in virtue of Propositions 2.2 and 2.9, and it is independent of the BD-triple by inspection.

On the other hand, for the real case, $H_0 = [\ ,\ ](r_\Lambda) \in \mathfrak{sl}(4,\mathbb{R})$, then in particular, $\mathfrak{h}_0 := \mathcal{Z}_{\mathfrak{g}_0}(H_0)$ is a (real) Cartan subalgebra of $\mathfrak{g}_0$. For each data, it is only left to find the generators of the real space of solutions of $\tau\alpha(H) = \alpha(H)$ for all $H \in \mathfrak{h}_0$. Notice

$$\dim_{\mathbb{R}}\{H \in \mathfrak{h}_0\colon \alpha(H) = (\tau\alpha)(H)\ \forall \alpha \in \Gamma_1\} = \dim_{\mathbb{C}}\{H \in \mathfrak{h}\colon \alpha(H) = (\tau\alpha)(H)\ \forall \alpha \in \Gamma_1\},$$

i.e. this real space is a real form of the complex space of solutions of the same equations viewed in $\mathfrak{h}$.

Example 3.18 *(A non-triangular, non-factorizable and not coboundary example).* Consider $\mathfrak{g} = \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{L} = \mathfrak{g} \times \mathbb{R}^2$, $\{u_1, u_2, u_3\}$ a basis of $\mathfrak{su}(2)$ with brackets $[u_i, u_j] = \sum_k \epsilon_{ijk} u_k$, where ϵ is the totally antisymmetric symbol, and $\{h, x, y\}$ the standard basis of $\mathfrak{sl}(2,\mathbb{R})$. There are no nontrivial triangular structures in $\mathfrak{su}(2)$ (see [7]); moreover, all Lie bialgebra structures on $\mathfrak{su}(2)$ are almost factorizable and isomorphic to some positive multiple of the coboundary associated to $u_1 \wedge u_2$. On the other hand, there are nontrivial triangular structures in $\mathfrak{sl}(2,\mathbb{R})$, all of them isomorphic to the corresponding to $\pm h \wedge x$. So, let us fix $r = u_1 \wedge u_2 + h \wedge x \in \Lambda^2\mathfrak{g}$ and $\delta_{\mathfrak{g}}(w) = \mathrm{ad}_w(r)$, for all $w \in \mathfrak{g}$. In order to list *all* isomorphism classes of Lie bialgebra structures on $\mathfrak{L} = \mathfrak{g} \times \mathbb{R}^2$, we need to compute $\mathrm{BiDer}(\mathfrak{g}, \delta_{\mathfrak{g}})$. Let

$$H_r = [-,-](r) = [u_1, u_2] + [h, x] = u_3 + 2x$$

thus, by Corollaries 2.10 and 2.5, we know that

$$\mathrm{BiDer}(\mathfrak{g}, \delta_{\mathfrak{g}}) \cong \mathrm{Ker}\,\delta_{\mathfrak{g}} \subseteq \{w \in \mathfrak{g}\colon [w, H_r] = 0\}.$$

For any $w = u + s \in \mathfrak{su}(2) \times \mathfrak{sl}(2,\mathbb{R})$, we get $[w, H_r] = 0 \Leftrightarrow [u, u_3] = 0$ and $[s, x] = 0$. We conclude that $\mathrm{BiDer}(\mathfrak{g}, \delta_g)$ is 2-dimensional, with basis $\{\mathrm{ad}_{u_3}, \mathrm{ad}_x\}$. In order to determine all possible Lie bialgebra structures on $\mathfrak{L}$ one may proceed as in Example 3.15. We illustrate it showing only one possibility. Choose $\{t_1, t_2\}$ a basis of $\mathbb{R}^2$; if one defines

$$\delta(w) = \mathrm{ad}_w(r) + [w, c_1 u_3 + c_2 x] \wedge t_1, \qquad \delta t_1 = 0, \qquad \delta t_2 = t_1 \wedge t_2$$

for any $c_1, c_2 \in \mathbb{R}$, then this structure is not coboundary, since $\delta|_{\mathbb{R}^2} \neq 0$. We remark also that all non-coboundary structures on $\mathfrak{L}$, such that induce $\delta_{\mathfrak{g}}$ on $\mathfrak{g}$, are of this form.

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