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Universal deformation formulas and braided module algebras

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We study formal deformations of a crossed product $S(V)\#_f G$, of a polynomial algebra with a group, induced from a universal deformation formula introduced by Witherspoon. These deformations arise from braided actions of Hopf algebras generated by automorphisms and skew derivations. We show that they are non-trivial in the characteristic free context, even if G is infinite, by showing that their infinitesimals are not coboundaries. For this we construct a new complex which computes the Hochschild cohomology of $S(V)\#_f G$.

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Introduction

In [G-Z] Giaquinto and Zhang develop the notion of a universal deformation formula based on a bialgebra H, extending earlier formulas based on universal enveloping algebras of Lie algebras. Each one of these formulas is called universal because it provides a formal deformation for any H-module algebra. In the same paper the authors construct the first family of such formulas based on non-commutative bialgebras, namely the enveloping algebras of central extensions of a Heisenberg Lie

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algebra L. Another of these formulas, based on a Hopf algebra H_q over $\mathbb{C}$, where $q \in \mathbb{C}^\times$ is a parameter, generated by group like elements $\sigma^{\pm 1}$ and two skew primitive elements D_1, D_2, were obtained in the generic case by the same authors, but were not published. In [W] the author generalizes this formula to include the case where q is a root of unity, and she uses it to construct formal deformations of a crossed product $S(V)\#_f G$, where $S(V)$ is the polynomial algebra and the group G acts linearly on V. More precisely, she deals with deformations whose infinitesimal sends $V \otimes V$ to $S(V)w_g$, where g is a central element of G.

In this paper we prove that some results established in [W] under the hypothesis that G is a finite group, remain valid for arbitrary groups, and with $\mathbb{C}$ replaced by an arbitrary field. For instance we show that the determinant of the action of g on V is always 1. Moreover, we do not only consider standard H_q-module algebra structures on $S(V)\#_f G$, but also the more general ones introduced in [G-G1], and we work with actions which depend on two central elements g_1 and g_2 of G and two polynomials P_1 and P_2. When the actions are the standard ones, $g_1 = 1$ and $P_1 = 1$, we obtain the case considered in [W]. Finally, in Section 3.2 we show how to extend the explicit formulas obtained previously, to non-central g_1 and g_2. As was noted by Witherspoon, these formulas necessarily involve all components of $S(V)\#_f G$ corresponding to the elements of a union of conjugacy classes of G.

The paper is organized as follows: in the first section we review the concept of braided module algebra introduced in [G-G1], we adapt the notion of universal deformation formula (UDF) to the braided context, and we show that each one of these formulas produces a deformation on any braided H-module algebra whose transposition (see Definition 1.6) satisfy a suitable hypothesis. We remark that, when the bialgebra H is standard, the use of braided module algebra gives rise to more deformations than the ones obtained using only module algebras, because the transposition can be different from the flip. With this in mind, although we are going to work with the standard Hopf algebra H_q, we establish the basic properties of UDF's in the braided case, because it is the most appropriate setting to deal with arbitrary transpositions. In the second section we recall the definitions of the Hopf algebra H_q and of the UDF $\exp_q$ considered in [W, Section 3], which we are going to study. We also introduce the concept of a good transposition of H_q on an algebra A, and we study some of its properties. Perhaps the most important result in this section is Theorem 2.4, in which we obtain a description of all the H_q-module algebras (A, s), with s a good transposition. This is the first of several results in which we give a systematic account of the necessary and sufficient conditions that an algebra (in general a crossed product $S(V)\#_f G$) must satisfy in order to support a braided H_q-module algebra structure satisfying suitable hypothesis. In Section 4 of [W], using the UDF $\exp_q$ the author constructs a large family of deformations whose infinitesimal sends $V \otimes V$ to $S(V)w_g$, where g is a central element of G. Using cohomological methods she proves that if G is finite, these deformations are non-trivial, that the action of g on V has determinant 1 and that the codimension of ${}^gV$ is 0 or 2. In the first part of Section 3 we study a larger family of deformations and we prove that the last two results hold for this family even if G is infinite and the characteristic of k is non-zero. Finally, in Section 4 we show that, under very general hypothesis, the deformations constructed in the previous section are non-trivial. Once again, we do not assume characteristic zero, nor that the group G is finite. One of the interesting points in this paper is the method developed to deal with the cohomology of $S(V)\#_f G$ when $k[G]$ is non-semisimple. As far as we know it is the first time that this type of cochain complexes is used to prove the non-triviality of a Hochschild cocycle.

1. Preliminaries

After introducing some basic notations we recall briefly the concepts of braided bialgebra and braided Hopf algebra following the presentation given in [T1] (see also [T2,L1,F-M-S,A-S,D,So] and [B-K-L-T]). Then we review the notion of braided module algebra introduced in [G-G1], we recall the concept of universal deformation formula based on a bialgebra H, due to Giaquinto and Zhang, and we show that such a UDF produces a formal deformation when it is applied to an H-braided module algebra, satisfying suitable hypothesis, generalizing slightly a result in [G-Z].

In this paper k is a field, $k^\times = k \setminus \{0\}$, all the vector spaces are over k, and $\otimes = \otimes_k$. Moreover we will use the usual notation $(i)_q = 1 + q + \cdots + q^{i-1}$ and $(i)!_q = (1)_q \cdots (i)_q$, for $q \in k^\times$ and $i \in \mathbb{N}$.

Let V, W be vector spaces and let $c : V \otimes W \to W \otimes V$ be a k-linear map. Recall that:

- If V is an algebra, then c is compatible with the algebra structure of V if $c\circ(\eta \otimes W) = W \otimes \eta$ and $c\circ(\mu \otimes W) = (W \otimes \mu)\circ(c \otimes V)\circ(V \otimes c)$, where $\eta : k \to V$ and $\mu : V \otimes V \to V$ denotes the unit and the multiplication map of V, respectively.
- If V is a coalgebra, then c is compatible with the coalgebra structure of V if $(W \otimes \epsilon)\circ c = \epsilon \otimes W$ and $(W \otimes \Delta)\circ c = (c \otimes V)\circ(V \otimes c)\circ(\Delta \otimes W)$, where $\epsilon : V \to k$ and $\Delta : V \to V \otimes V$ denotes the counit and the comultiplication map of V, respectively.

Of course, there are similar compatibilities when W is an algebra or a coalgebra.

1.1. *Braided bialgebras and braided Hopf algebras*

Definition 1.1. A *braided bialgebra* is a vector space H endowed with an algebra structure, a coalgebra structure and a braiding operator $c \in \mathrm{Aut}_k(H^{\otimes 2})$ (called the *braid* of H), such that c is compatible with the algebra and coalgebra structures of H, $\Delta\circ\mu = (\mu \otimes \mu)\circ(H \otimes c \otimes H)\circ(\Delta \otimes \Delta)$, η is a coalgebra morphism and ϵ is an algebra morphism. Furthermore, if there exists a k-linear map $S : H \to H$, which is the inverse of the identity map for the convolution product, then we say that H is a *braided Hopf algebra* and we call S the *antipode* of H.

Usually H denotes a braided bialgebra, understanding the structure maps, and c denotes its braid. If necessary, we will use notations as c_H, μ_H, etcetera.

Remark 1.2. Assume that H is an algebra and a coalgebra and $c \in \mathrm{Aut}_k(H^{\otimes 2})$ is a solution of the braiding equation, which is compatible with the algebra and coalgebra structures of H. Let $H \otimes_c H$ be the algebra with underlying vector space $H^{\otimes 2}$ and multiplication map given by $\mu_{H\otimes_c H} := (\mu \otimes \mu)\circ(H \otimes c \otimes H)$. It is easy to see that H is a braided bialgebra with braid c if and only if $\Delta : H \to H \otimes_c H$ and $\epsilon : H \to k$ are morphisms of algebras.

Definition 1.3. Let H and L be braided bialgebras. A map $g : H \to L$ is a *morphism of braided bialgebras* if it is an algebra homomorphism, a coalgebra homomorphism and $c\circ(g \otimes g) = (g \otimes g)\circ c$.

Let H and L be braided Hopf algebras. It is well known that if $g : H \to L$ is a morphism of braided bialgebras, then $g\circ S = S\circ g$.

1.2. *Braided module algebras*

Definition 1.4. Let H be a braided bialgebra. A *left H-braided space* (V, s) is a vector space V, endowed with a bijective k-linear map $s : H \otimes V \to V \otimes H$, which is compatible with the bialgebra structure of H and satisfies

$$(s \otimes H)\circ(H \otimes s)\circ(c \otimes V) = (V \otimes c)\circ(s \otimes H)\circ(H \otimes s)$$

(compatibility of s with the braid). Let (V', s') be another left H-braided space. A k-linear map $f : V \to V'$ is said to be a *morphism of left H-braided spaces*, from (V, s) to (V', s'), if $(f \otimes H)\circ s = s'\circ(H \otimes f)$.

We let $\mathcal{LB}_H$ denote the category of all left H-braided spaces. It is easy to check that this is a monoidal category with:

– unit (k,τ), where $\tau: H\otimes k \to k\otimes H$ is the flip,
– tensor product $(V,s_V)\otimes(U,s_U) := (V\otimes U, s_{V\otimes U})$, where $s_{V\otimes U}$ is the map

$$s_{V\otimes U} := (V\otimes s_U)\circ(s_V\otimes U),$$

– the usual associativity and unit constraints.

Definition 1.5. We will say that (A,s) is a *left H-braided algebra* or simply a *left H-algebra* if it is an algebra in $\mathcal{LB}_H$.

We let $\mathcal{ALB}_H$ denote the category of left H-braided algebras.

Definition 1.6. Let A be an algebra. A *left transposition* of H on A is a bijective map $s: H\otimes A\to A\otimes H$, satisfying:

(1) (A,s) is a left H-braided space,
(2) s is compatible with the algebra structure of A.

Remark 1.7. A left H-braided algebra is a pair (A,s) consisting of an algebra A and a left transposition s of H on A. Let (A',s') be another left H-braided algebra. A map $f: A\to A'$ is a morphism of left H-braided algebras, from (A,s) to (A',s'), if and only if it is a morphism of standard algebras and $(f\otimes H)\circ s = s'\circ(H\otimes f)$.

Note that (H,c) is an algebra in $\mathcal{LB}_H$. Hence, one can consider left and right (H,c)-modules in this monoidal category.

Definition 1.8. We will say that (V,s) is a *left H-braided module* or simply a *left H-module* to mean that it is a left (H,c)-module in $\mathcal{LB}_H$.

We let ${}_H(\mathcal{LB}_H)$ denote the category of left H-braided modules.

Remark 1.9. A left H-braided space (V,s) is a left H-module if and only if V is a standard left H-module and

$$s\circ(H\otimes\rho) = (\rho\otimes H)\circ(H\otimes s)\circ(c\otimes V),$$

where ρ denotes the action of H on V. Furthermore, a map $f: V\to V'$ is a *morphism of left H-modules*, from (V,s) to (V',s'), if and only if it is H-linear and $(f\otimes H)\circ s = s'\circ(H\otimes f)$.

Given left H-modules (V,s_V) and (U,s_U), with actions ρ_V and ρ_U respectively, we let $\rho_{V\otimes U}$ denote the diagonal action

$$\rho_{V\otimes U} := (\rho_V\otimes\rho_U)\circ(H\otimes s_V\otimes U)\circ(\Delta_H\otimes V\otimes U).$$

The following proposition says in particular that (k,τ) is a left H-module via the trivial action and that $(V,s_V)\otimes(U,s_U)$ is a left H-module via $\rho_{V\otimes U}$.

Proposition 1.10. *(See [G-G1].) The category ${}_H(\mathcal{LB}_H)$, of left H-braided modules, endowed with the usual associativity and unit constraints, is monoidal.*

Definition 1.11. We say that (A,s) is a *left H-braided module algebra* or simply a *left H-module algebra* if it is an algebra in ${}_H(\mathcal{LB}_H)$.

We let ${}_H(\mathcal{ALB}_H)$ denote the category of left H-braided module algebras.

Remark 1.12. (A,s) is a left H-module algebra if and only if the following facts hold:

(1) A is an algebra,
(2) s is a left transposition of H on A,
(3) A is a standard left H-module,
(4) $s\circ(H\otimes\rho)=(\rho\otimes H)\circ(H\otimes s)\circ(c\otimes A)$,
(5) $\mu_A\circ(\rho\otimes\rho)\circ(H\otimes s\otimes A)\circ(\Delta_H\otimes A\otimes A)=\rho\circ(H\otimes\mu_A)$,
(6) $h\cdot 1=\epsilon(h)1$ for all $h\in H$,

where ρ denotes the action of H on A. So, (A,s) is a left H-module algebra if and only if it is a left H-algebra, a left H-module and satisfies conditions (5) and (6).

In the sequel, given a map $\rho: H\otimes A\to A$, sometimes we will write $h\cdot a$ to denote $\rho(h\otimes a)$.

Remark 1.13. If X generates H as a k-algebra, then conditions (4), (5) and (6) of the above remark are satisfied if and only if

$$s(h\otimes l\cdot a)=(\rho\otimes H)\circ(H\otimes s)\circ(c\otimes A)(h\otimes l\otimes a),$$
$$h\cdot(ab)=\mu_A\circ(\rho\otimes\rho)\circ(H\otimes s\otimes A)\big(\Delta(h)\otimes a\otimes b\big),$$
$$h\cdot 1=\epsilon(h),$$

for all $a,b\in A$ and $h,l\in X$.

Let (A',s') be another left H-module algebra. A map $f: A\to A'$ is a *morphism of left H-module algebras*, from (A,s) to (A',s'), if and only if it is an H-linear morphism of standard algebras that satisfies $(f\otimes H)\circ s=s'\circ(H\otimes f)$.

1.3. Bialgebra actions and universal deformation formulas

Most of the results of [G-Z, Section 1] remain valid in our more general context, with the same arguments and minimal changes. In particular Theorem 1.15 below holds.

Let H be a braided bialgebra. Given a left H-module algebra (A,s) and an element $F\in H\otimes H$, we let $F_l: A\otimes A\to A\otimes A$ denote the map defined by

$$F_l(a\otimes b):=(\rho\otimes\rho)\circ(H\otimes s\otimes A)(F\otimes a\otimes b),$$

where $\rho: H\otimes A\to A$ is the action of H on A. We let A_F denote A endowed with the multiplication map $\mu_A\circ F_l$.

Definition 1.14. We say that $F\in H\otimes H$ is a *twisting element* (*based on* H) if

(1) $(\epsilon\otimes\mathrm{id})(F)=(\mathrm{id}\otimes\epsilon)(F)=1$,
(2) $[(\Delta\otimes\mathrm{id})(F)](F\otimes 1)=[(\mathrm{id}\otimes\Delta)(F)](1\otimes F)$ in $H\otimes_c H\otimes_c H$,
(3) $(c\otimes H)\circ(H\otimes c)(F\otimes h)=h\otimes F$, for all $h\in H$.

Theorem 1.15. *Let (A,s) be a left H-module algebra. If $F\in H\otimes H$ is a twisting element such that $(s\otimes H)\circ(H\otimes s)(F\otimes a)=a\otimes F$, for all $a\in A$, then A_F is an associative algebra with unit 1_A.*

The notions of braided bialgebra, left H-braided module algebra and twisting element make sense in arbitrary monoidal categories. Here we consider the monoidal category $\mathcal{M}[\![t]\!]$ defined as follows:

– the objects are the $k[\![t]\!]$-modules of the form $M[\![t]\!]$ where M is a k-vector space,
– the arrows are the $k[\![t]\!]$-linear maps,
– the tensor product is the completion

$$M[\![t]\!] \mathbin{\widehat{\otimes}}_{k[\![t]\!]} N[\![t]\!]$$

of the algebraic tensor product $M[\![t]\!] \otimes_{k[\![t]\!]} N[\![t]\!]$ with respect to the t-adic topology,
– the unit and the associativity constrains are the evident ones.

We identify $M[\![t]\!] \mathbin{\widehat{\otimes}}_{k[\![t]\!]} N[\![t]\!]$ with $(M \otimes N)[\![t]\!]$ by the map

$$\Theta : M[\![t]\!] \mathbin{\widehat{\otimes}}_{k[\![t]\!]} N[\![t]\!] \to (M \otimes N)[\![t]\!]$$

given by $\Theta(mt^i \otimes nt^j) := (m \otimes n)t^{i+j}$.

If A is a k-algebra, then $A[\![t]\!]$ is an algebra in $\mathcal{M}[\![t]\!]$ via the multiplication map

$$\begin{array}{ccc} (A \otimes A)[\![t]\!] & \xrightarrow{\ \mu\ } & A[\![t]\!] \\ \sum (a_i \otimes b_i)t^i & \longmapsto & \sum a_i b_i t^i, \end{array}$$

where $a_i b_i = \mu_A(a_i \otimes b_i)$. The unit map is the canonical inclusion $k[\![t]\!] \hookrightarrow A[\![t]\!]$.

If H is a braided bialgebra over k, then $H[\![t]\!]$ is a braided bialgebra in $\mathcal{M}[\![t]\!]$. The multiplication and unit maps are as above. The comultiplication and counits are the maps

$$\begin{array}{ccc} H[\![t]\!] & \xrightarrow{\ \Delta\ } & (H \otimes H)[\![t]\!] \\ \sum h_i t^i & \longmapsto & \sum \Delta_H(h_i)t^i \end{array} \quad \text{and} \quad \begin{array}{ccc} H[\![t]\!] & \xrightarrow{\ \epsilon\ } & k[\![t]\!] \\ \sum h_i t^i & \longmapsto & \sum \epsilon_H(h_i)t^i, \end{array}$$

and the braid operator is the map

$$\begin{array}{ccc} (H \otimes H)[\![t]\!] & \xrightarrow{\ c[\![t]\!]\ } & (H \otimes H)[\![t]\!] \\ \sum (h_i \otimes l_i)t^i & \longmapsto & \sum c_H(h_i \otimes l_i)t^i. \end{array}$$

If (A, s) is an H-module algebra, then $(A[\![t]\!], s[\![t]\!])$, where $s[\![t]\!]$ is the map

$$\begin{array}{ccc} (H \otimes A)[\![t]\!] & \xrightarrow{\ s[\![t]\!]\ } & (A \otimes H)[\![t]\!] \\ \sum (h_i \otimes a_i)t^i & \longmapsto & \sum s(h_i \otimes a_i)t^i, \end{array}$$

is an $H[\![t]\!]$-module algebra, via

$$\begin{array}{ccc} (H \otimes A)[\![t]\!] & \xrightarrow{\ \rho\ } & A[\![t]\!] \\ \sum (h_i \otimes a_i)t^i & \longmapsto & \sum \rho_A(h_i \otimes a_i)t^i. \end{array}$$

A twisting element based on $H[\![t]\!]$ in $\mathcal{M}[\![t]\!]$ is an element $F \in H[\![t]\!] \mathbin{\widehat{\otimes}}_{k[\![t]\!]} H[\![t]\!]$ satisfying conditions (1)–(3) of Definition 1.14. It is easy to check that a power series $F = \sum F_i t^i \in (H \otimes H)[\![t]\!]$ corresponds via Θ^{-1} to a twisting element if and only if

(1) $(\epsilon \otimes \mathrm{id})(F_0) = (\mathrm{id} \otimes \epsilon)(F_0) = 1$ and $(\epsilon \otimes \mathrm{id})(F_i) = (\mathrm{id} \otimes \epsilon)(F_i) = 0$ for $i \geqslant 1$,
(2) for all $n \geqslant 0$,

$$\sum_{i+j=n} (\Delta \otimes \mathrm{id})(F_i)(F_j \otimes 1) = \sum_{i+j=n} (\mathrm{id} \otimes \Delta)(F_i)(1 \otimes F_j) \quad \text{in } H \otimes_c H \otimes_c H,$$

(3) $(c \otimes H) \circ (H \otimes c)(F_n \otimes h) = h \otimes F_n$, for all $h \in H$ and $n \geqslant 0$.

We will say that F is a *universal deformation formula* (UDF) *based on* H if, moreover, $F_0 = 1 \otimes 1$.

Theorem 1.16. *Let (A, s) be a left H-module algebra. If $F = \sum F_i t^i$ is a UDF based on H, such that*

$$(s \otimes H) \circ (H \otimes s)(F_i \otimes a) = a \otimes F_i \quad \text{for all } i \geqslant 0 \text{ and } a \in A,$$

then, the construction considered in Theorem 1.15*, applied to the left $H[\![t]\!]$-module algebra $(A[\![t]\!], s[\![t]\!])$ introduced above, produces a formal deformation of A.*

Proof. It is immediate. □

2. H_q-module algebra structures and deformations

In this section, we briefly review the construction of the Hopf algebra H_q and the UDF $\exp_q$ based on H_q considered in [W], we introduce the notion of a good transposition of H_q on an algebra A, and we describe all the braided H_q-module algebras whose transposition is good.

Let $q \in k^{\times}$ and let H be the algebra generated by D_1, D_2, $\sigma^{\pm 1}$, subject to the relations

$$D_1 D_2 = D_2 D_1, \qquad \sigma\sigma^{-1} = \sigma^{-1}\sigma = 1 \quad \text{and} \quad q\sigma D_i = D_i \sigma \quad \text{for } i = 1, 2.$$

It is easy to check that H is a Hopf algebra with

$$\begin{aligned} \Delta(D_1) &:= D_1 \otimes \sigma + 1 \otimes D_1, & \epsilon(D_1) &:= 0, & S(D_1) &:= -D_1\sigma^{-1}, \\ \Delta(D_2) &:= D_2 \otimes 1 + \sigma \otimes D_2, & \epsilon(D_2) &:= 0, & S(D_2) &:= -\sigma^{-1} D_2, \\ \Delta(\sigma) &:= \sigma \otimes \sigma, & \epsilon(\sigma) &:= 1, & S(\sigma) &:= \sigma^{-1}. \end{aligned}$$

If q is a primitive l-root of unity with $l \geqslant 2$, then the ideal I of H generated by D_1^l and D_2^l is a Hopf ideal. So, the quotient H/I is also a Hopf algebra. Let

$$H_q := \begin{cases} H/I & \text{if } q \text{ is a primitive } l\text{-root of unity with } l \geqslant 2, \\ H & \text{if } q = 1 \text{ or it is not a root of unity.} \end{cases}$$

The Hopf algebra H_q was considered in the paper [W], where it was proved that

$$\exp_q(tD_1 \otimes D_2) := \begin{cases} \sum_{i=0}^{l-1} \frac{1}{(i)!_q} (tD_1 \otimes D_2)^i & \text{if } q \text{ is a primitive } l\text{-root of unity } (l \geqslant 2), \\ \sum_{i=0}^{\infty} \frac{1}{(i)_q!} (tD_1 \otimes D_2)^i & \text{if } q = 1 \text{ or it is not a root of unity,} \end{cases}$$

is a UDF based on H_q.

2.1. Good transpositions of H_q on an algebra

One of our main purposes in this paper is to construct formal deformation of algebras by using the UDF $\exp_q(tD_1 \otimes D_2)$. By Theorem 1.16, it will be sufficient to obtain examples of H_q-module algebras (A, s), whose underlying transpositions s satisfy

$$(s \otimes H_q)\circ(H_q \otimes s)(D_1 \otimes D_2 \otimes a) = a \otimes D_1 \otimes D_2 \quad \text{for all } a \in A. \tag{2.1}$$

Definition 2.1. A k-linear map $s : H_q \otimes A \to A \otimes H_q$ is *good* if condition (2.1) is fulfilled.

It is evident that $s : H_q \otimes A \to A \otimes H_q$ is good if and only if there exists a bijective k-linear map $\alpha : A \to A$ such that

$$s(D_1 \otimes a) = \alpha(a) \otimes D_1 \quad \text{and} \quad s(D_2 \otimes a) = \alpha^{-1}(a) \otimes D_2 \quad \text{for all } a \in A.$$

Lemma 2.2. *Let $k[\sigma^{\pm 1}]$ denote the sub-Hopf algebra of H_q generated by σ. Each transposition $s : H_q \otimes A \to A \otimes H_q$ takes $k[\sigma^{\pm 1}] \otimes A$ onto $A \otimes k[\sigma^{\pm 1}]$.*

Proof. Let τ be the flip. Since $\tau\circ s^{-1}\circ\tau$ is a transposition, it suffices to prove that $s(\sigma^{\pm 1} \otimes a) \in A \otimes k[\sigma^{\pm 1}]$ for all $a \in A$. Write

$$s(\sigma \otimes a) = \sum_{ijk} \gamma_{ijk}(a) \otimes \sigma^i D_1^j D_2^k.$$

Since $S^2(D_1) = q^{-1}D_1$, $S^2(D_2) = qD_2$ and $S^2(\sigma^{\pm 1}) = \sigma^{\pm 1}$, we have

$$\begin{aligned}\sum_{ijk} \gamma_{ijk}(a) \otimes \sigma^i D_1^j D_2^k &= s(\sigma \otimes a) \\ &= s\circ\big(S^2 \otimes A\big)(\sigma \otimes a) \\ &= \big(A \otimes S^2\big)\circ s(\sigma \otimes a) \\ &= \sum_{ijk} q^{k-j} \gamma_{ijk}(a) \otimes \sigma^i D_1^j D_2^k,\end{aligned}$$

and so $\gamma_{ijk} = 0$ for $j \neq k$. Using now that

$$\begin{aligned}\sum_{ij} \gamma_{ijj}(a) \otimes \Delta(\sigma)^i \Delta(D_1)^j \Delta(D_2)^j &= (A \otimes \Delta)\circ s(\sigma \otimes a) \\ &= (s \otimes H_q)\circ(H_q \otimes s)\circ(\Delta \otimes A)(\sigma \otimes a) \\ &= \sum_{iji'j'} \gamma_{i'j'j'}\big(\gamma_{ijj}(a)\big) \otimes \sigma^{i'} D_1^{j'} D_2^{j'} \otimes \sigma^i D_1^j D_2^j,\end{aligned}$$

it is easy to check that $\gamma_{ijj} = 0$ if $j > 0$ (use that in each term of the right side the exponent of D_1 equals the exponent of D_2). For σ^{-1} the same argument carries over. This finishes the proof. □

In the following result we obtain a characterization of the good transpositions of H_q on an algebra A.

Theorem 2.3. *The following facts hold*:

(1) *If $s : H_q \otimes A \to A \otimes H_q$ is a good transposition, then $s(\sigma^{\pm 1} \otimes a) = a \otimes \sigma^{\pm 1}$ for all $a \in A$ and the map $\alpha : A \to A$, defined by $s(D_1 \otimes a) = \alpha(a) \otimes D_1$, is an algebra homomorphism.*
(2) *Given an algebra automorphism $\alpha : A \to A$, there exists only one good transposition $s : H_q \otimes A \to A \otimes H_q$ such that $s(D_1 \otimes a) = \alpha(a) \otimes D_1$ for all $a \in A$.*

Proof. (1) By Lemma 2.2, we know that s induces by restriction a transposition of $k[\sigma^{\pm 1}]$ on A. Hence, by [G-G1, Theorem 4.14], there is a superalgebra structure $A = A_+ \oplus A_-$ such that

$$s(\sigma^i \otimes a) = \begin{cases} a \otimes \sigma^i & \text{if } a \in A_+, \\ a \otimes \sigma^{-i} & \text{if } a \in A_-. \end{cases}$$

Let $\alpha : A \to A$ be as in the statement. Since σ is a transposition, if $a \in A_-$, then

$$\begin{aligned} \alpha(a) \otimes D_1 \otimes \sigma + \alpha(a) \otimes 1 \otimes D_1 &= (A \otimes \Delta) \circ s(D_1 \otimes a) \\ &= (s \otimes H_q) \circ (H_q \otimes s) \circ (\Delta \otimes A)(D_1 \otimes a) \\ &= \alpha(a) \otimes D_1 \otimes \sigma^{-1} + \alpha(a) \otimes 1 \otimes D_1. \end{aligned}$$

So, $A_- = 0$. Finally, α is an algebra homomorphism, because

$$s(h \otimes 1) = 1 \otimes h \quad \text{for each } h \in H_q \quad \text{and} \quad s \circ (H_q \otimes \mu_A) = (\mu_A \otimes H_q) \circ (A \otimes s) \circ (s \otimes A).$$

(2) By item (1) and the comment preceding Lemma 2.2, it must be

$$s(\sigma^{\pm 1} \otimes a) = a \otimes \sigma^{\pm 1}, \qquad s(D_1 \otimes a) = \alpha(a) \otimes D_1 \quad \text{and} \quad s(D_2 \otimes a) = \alpha^{-1}(a) \otimes D_2.$$

So, necessarily

$$s(\sigma^i D_1^j D_2^k \otimes a) = \alpha^{j-k}(a) \otimes \sigma^i D_1^j D_2^k.$$

We leave to the reader the task to prove that s is a good transposition. □

2.2. Some H_q-module algebra structures

Let A be an algebra. Let us consider k-linear maps $\varsigma, \delta_1, \delta_2 : A \to A$. It is evident that there is a (necessarily unique) action $\rho : H_q \otimes A \to A$ such that

$$\rho(\sigma \otimes a) = \varsigma(a), \qquad \rho(D_1 \otimes a) = \delta_1(a) \quad \text{and} \quad \rho(D_2 \otimes a) = \delta_2(a) \tag{2.2}$$

for all $a \in A$, if and only if the maps ς, δ_1 and δ_2 satisfy the following conditions:

(1) ς is a bijective map,
(2) $\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1$,
(3) $q\varsigma \circ \delta_i = \delta_i \circ \varsigma$ for $i = 1, 2$,
(4) if $q \neq 1$ and $q^l = 1$, then $\delta_1^l = \delta_2^l = 0$.

Let $s : H_q \otimes A \to A \otimes H_q$ be a good transposition and let α be the associated automorphism. Let ς, δ_1 and δ_2 be k-linear endomorphisms of A satisfying (1)–(4). Next, we determine the conditions that ς, δ_1 and δ_2 must satisfy in order that (A, s) becomes an H_q-module algebra via the action ρ defined by (2.2).

Theorem 2.4. *(A,s) is an H_q-module algebra via ρ if and only if*

(5) *ς is an algebra automorphism,*
(6) *$\alpha\circ\delta_i = \delta_i\circ\alpha$ for $i = 1, 2$,*
(7) *$\alpha\circ\varsigma = \varsigma\circ\alpha$,*
(8) *$\delta_i(1) = 0$ for $i = 1, 2$,*
(9) *$\delta_1(ab) = \delta_1(a)\varsigma(b) + \alpha(a)\delta_1(b)$ for all $a, b \in A$,*
(10) *$\delta_2(ab) = \delta_2(a)b + \varsigma(\alpha^{-1}(a))\delta_2(b)$ for all $a, b \in A$.*

Proof. Assume that (A,s) is an H_q-module algebra and let $\tau : H_q \otimes H_q \to H_q \otimes H_q$ be the flip. Evaluating the equality

$$s\circ(H_q \otimes \rho) = (\rho \otimes H_q)\circ(H_q \otimes s)\circ(\tau \otimes A)$$

successively on $D_1 \otimes D_i \otimes a$ and $D_1 \otimes \sigma \otimes a$ with $i \in \{1,2\}$ and $a \in A$ arbitrary, we verify that items (6) and (7) are satisfied. Item (8) follows from the fact that $D_1 \cdot 1 = D_2 \cdot 1 = 0$. Finally, using that $\sigma \cdot 1 = 1$ and evaluating the equality

$$\rho\circ(H_q \otimes \mu_A) = \mu_A\circ(\rho \otimes \rho)\circ(H_q \otimes s \otimes A)\circ(\Delta \otimes A \otimes A)$$

on $\sigma \otimes a \otimes b$ and $D_i \otimes a \otimes b$, with $i = 1, 2$ and $a, b \in A$ arbitrary, we see that items (5), (9) and (10) hold. So, conditions (5)–(10) are necessary. By Remark 1.13, in order to verify that they are also sufficient, it is enough to check that they imply that

$$h \cdot 1 = \epsilon(h),$$
$$s(h \otimes l \cdot a) = (\rho \otimes H_q)\circ(H_q \otimes s)(l \otimes h \otimes a),$$
$$h \cdot (ab) = \mu_A\circ(\rho \otimes \rho)\circ(H_q \otimes s \otimes A)\big(\Delta(h) \otimes a \otimes b\big),$$

for all $a, b \in A$ and $h, l \in \{D_1, D_2, \sigma^{\pm 1}\}$. We leave this task to the reader. □

Note that condition (8) in Theorem 2.4 is redundant since it can be obtained by applying conditions (9) and (10) with $a = b = 1$.

3. H_q-module algebra structures on crossed products

Let G be a group endowed with a representation on a k-vector space V of dimension n. Consider the symmetric k-algebra $S(V)$ equipped with the unique action of G by automorphisms that extends the action of G on V and take $A = S(V)\#_f G$, where $f : G \times G \to k^\times$ is a normal cocycle. By definition the k-algebra A is a free left $S(V)$-module with basis $\{w_g\colon\ g \in G\}$. Its product is given by

$$(P w_g)(Q w_h) := P\,{}^gQ\, f(g,h) w_{gh},$$

where ${}^gQ$ denotes the action of g on Q. This section is devoted to the study of the H_q-module algebras (A,s), with s good, that satisfy

$$s(H_q \otimes V) \subseteq V \otimes H_q, \qquad s(H_q \otimes kw_g) \subseteq kw_g \otimes H_q, \qquad \sigma \cdot v \in V \quad \text{and} \quad \sigma \cdot w_g \in kw_g,$$

for all $v \in V$ and $g \in G$. In Theorem 3.5 we give a general characterization of these module algebras, and in Section 3.1 we consider a specific case which is more suitable for finding concrete examples, and we study it in detail. Finally in Section 3.2 we consider the case where the cocycle involves several not necessarily central elements of G.

In the following proposition we characterize the good transpositions s of H_q on A satisfying the hypothesis mentioned above. By Theorem 2.3 this is equivalent to require that the k-linear map $\alpha : A \to A$ associated with α, takes V to V and kw_g to kw_g for all $g \in G$.

Proposition 3.1. *Let $\hat{\alpha} : V \to V$ be a k-linear map and $\chi_\alpha : G \to k^\times$ a map. There is a good transposition $s : H_q \otimes A \to A \otimes H_q$, such that*

$$s(D_1 \otimes v) = \hat{\alpha}(v) \otimes D_1 \quad \textit{and} \quad s(D_1 \otimes w_g) = \chi_\alpha(g) w_g \otimes D_1$$

for all $v \in V$ and $g \in G$, if and only if $\hat{\alpha}$ is a bijective $k[G]$-linear map and χ_α is a group homomorphism.

Proof. By Theorem 2.3 we know that s exists if an only if the k-linear map $\alpha : A \to A$ defined by

$$\alpha(v_1 \cdots v_m w_g) := \hat{\alpha}(v_1) \cdots \hat{\alpha}(v_m) \chi_\alpha(g) w_g,$$

is an automorphism. But, if this happens, then:

a) χ_α is a morphism since

$$\chi_\alpha(g)\chi_\alpha(h) f(g,h) w_{gh} = \alpha(w_g)\alpha(w_h) = \alpha(w_g w_h) = \chi_\alpha(gh) f(g,h) w_{gh}$$

for all $g, h \in G$,

b) $\hat{\alpha}$ is a bijective $k[G]$-linear map, since it is the restriction and corestriction of α to V, and

$$\hat{\alpha}\big({}^g v\big) = \alpha(w_g)\hat{\alpha}(v)\alpha\big(w_g^{-1}\big) = \chi_\alpha(g) w_g \hat{\alpha}(v)\big(\chi_\alpha(g) w_g\big)^{-1} = w_g \hat{\alpha}(v) w_g^{-1} = {}^g\hat{\alpha}(v).$$

Conversely, if $\hat{\alpha}$ is a bijective map then α is also, and if $\hat{\alpha}$ is a $k[G]$-linear map and χ_α is a morphism, then

$$\alpha(w_g)\hat{\alpha}(v) = \chi_\alpha(g) w_g \hat{\alpha}(v) = {}^g\hat{\alpha}(v) \chi_\alpha(g) w_g = \hat{\alpha}\big({}^g v\big)\alpha(w_g)$$

and

$$\alpha(w_g)\alpha(w_h) = \chi_\alpha(g) w_g \chi_\alpha(h) w_h = f(g,h)\chi_\alpha(gh) w_{gh} = \alpha\big(f(g,h) w_{gh}\big),$$

for all $v \in V$ and $g, h \in G$, from which it follows easily that α is a morphism. □

Let $A = S(V)\#_f G$ be as above. Throughout this section we fix a morphism $\chi_\alpha : G \to k^\times$ and a bijective $k[G]$-linear map $\hat{\alpha} : V \to V$, and we let $\alpha : A \to A$ denote the automorphism determined by $\hat{\alpha}$ and χ_α. Moreover we will call

$$s : H_q \otimes A \to A \otimes H_q$$

the good transposition associated with α. Our purpose is to obtain all the H_q-module algebra structures on (A, s) such that

$$\sigma \cdot v \in V \quad \text{and} \quad \sigma \cdot w_g \in k w_g \quad \text{for all } v \in V \text{ and } g \in G. \tag{3.3}$$

Under these restrictions we obtain conditions which allow us to construct all H_q-module structures in concrete examples. Thanks to Theorem 1.16 and the fact that $\exp_q(tD_1 \otimes D_2)$ is a UDF based on H_q, each one of these examples produces automatically a formal deformation of A. First note that given an H_q-module algebra structure on (A, s) satisfying (3.3), we can define k-linear maps

$$\hat{\delta}_1 : V \to A, \qquad \hat{\delta}_2 : V \to A \quad \text{and} \quad \hat{\varsigma} : V \to V$$

and maps

$$\bar{\delta}_1 : G \to A, \qquad \bar{\delta}_2 : G \to A \quad \text{and} \quad \chi_\varsigma : G \to k^\times,$$

by

$$\hat{\delta}_i(v) := D_i \cdot v, \qquad \hat{\varsigma}(v) := \sigma \cdot v, \qquad \bar{\delta}_i(g) := D_i \cdot w_g \quad \text{and} \quad \sigma \cdot w_g := \chi_\varsigma(g) w_g.$$

Lemma 3.2. *Let $\hat{\varsigma} : V \to V$ be a k-linear map and $\chi_\varsigma : G \to k^\times$ be a map. Then, the map $\varsigma : A \to A$ defined by*

$$\varsigma(\mathbf{v}_{1m} w_g) := \hat{\varsigma}(v_1) \cdots \hat{\varsigma}(v_m) \chi_\varsigma(g) w_g,$$

is a k-algebra automorphism if and only if $\hat{\varsigma}$ is a bijective $k[G]$-linear map and χ_ς is a group homomorphism.

Proof. This was checked in the proof of Proposition 3.1. □

Lemma 3.3. *Let $\hat{\delta}_1 : V \to A$ and $\hat{\delta}_2 : V \to A$ be k-linear maps and let $\bar{\delta}_1 : G \to A$ and $\bar{\delta}_2 : G \to A$ be maps.*

(1) *The k-linear map $\delta_1 : A \to A$ given by*

$$\delta_1(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \alpha(\mathbf{v}_{1,j-1}) \hat{\delta}_1(v_j) \varsigma(\mathbf{v}_{j+1,m} w_g) + \alpha(\mathbf{v}_{1m}) \bar{\delta}_1(g),$$

where $\mathbf{v}_{hl} = v_h \cdots v_l$, is well defined if and only if

$$\hat{\delta}_1(v)\hat{\varsigma}(w) + \hat{\alpha}(v)\hat{\delta}_1(w) = \hat{\delta}_1(w)\hat{\varsigma}(v) + \hat{\alpha}(w)\hat{\delta}_1(v) \quad \textit{for all } v, w \in V. \tag{3.4}$$

(2) *The map $\delta_2 : A \to A$ given by*

$$\delta_2(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \varsigma\big(\alpha^{-1}(\mathbf{v}_{1,j-1})\big) \hat{\delta}_2(v_j) \mathbf{v}_{j+1,m} w_g + \varsigma\big(\alpha^{-1}\big)(\mathbf{v}_{1m}) \bar{\delta}_2(g)$$

is well defined if and only if

$$\hat{\delta}_2(v)w + \varsigma\big(\hat{\alpha}^{-1}(v)\big)\hat{\delta}_2(w) = \hat{\delta}_2(w)v + \varsigma\big(\hat{\alpha}^{-1}(w)\big)\hat{\delta}_2(v) \quad \textit{for all } v, w \in V. \tag{3.5}$$

Proof. We prove the first assertion and leave the second one, which is similar, to the reader. The only if part follows immediately by noting that

$$\hat{\delta}_1(v)\hat{\varsigma}(w) + \hat{\alpha}(v)\hat{\delta}_1(w) = \delta_1(vw) = \delta_1(wv) = \hat{\delta}_1(w)\hat{\varsigma}(v) + \hat{\alpha}(w)\hat{\delta}_1(v).$$

In order to prove the if part it suffices to check that

$$\delta_1(v_1 \cdots v_{i-1} v_{i+1} v_i v_{i+2} \cdots v_m w_g) = \delta_1(\mathbf{v}_{1m} w_g) \quad \text{for all } i < m,$$

which follows easily from the hypothesis. □

Lemma 3.4. *Assume that ς is an algebra automorphism and δ_1, δ_2 are well defined. The following facts hold*:

(1) *The map δ_1 satisfies*

$$\delta_1(x_1\cdots x_m)=\sum_{j=1}^{m}\alpha(x_1\cdots x_{j-1})\delta_1(x_j)\varsigma(x_{j+1}\cdots x_m)$$

for all $x_1,\dots,x_m\in k\#_f G\cup V$, if and only if
(a) $\hat{\delta}_1({}^g v)\chi_\varsigma(g)w_g+\hat{\alpha}({}^g v)\bar{\delta}_1(g)=\bar{\delta}_1(g)\hat{\varsigma}(v)+\chi_\alpha(g)w_g\hat{\delta}_1(v)$,
(b) $f(g,h)\bar{\delta}_1(gh)=\bar{\delta}_1(g)\chi_\varsigma(h)w_h+\chi_\alpha(g)w_g\bar{\delta}_1(h)$,
for all $v\in V$ and $g,h\in G$.

(2) *The map δ_2 satisfies*

$$\delta_2(x_1\cdots x_m)=\sum_{j=1}^{m}\varsigma\circ\alpha^{-1}(x_1\cdots x_{j-1})\delta_1(x_j)x_{j+1}\cdots x_m$$

for all $x_1,\dots,x_m\in k\#_f G\cup V$, if and only if
(a) $\hat{\delta}_2({}^g v)w_g+\hat{\varsigma}(\hat{\alpha}^{-1}({}^g v))\bar{\delta}_2(g)=\bar{\delta}_2(g)v+\chi_\varsigma(g)\chi_\alpha^{-1}(g)w_g\hat{\delta}_2(v)$,
(b) $f(g,h)\bar{\delta}_2(gh)=\bar{\delta}_2(g)w_h+\chi_\varsigma(g)\chi_\alpha^{-1}(g)w_g\bar{\delta}_2(h)$,
for all $v\in V$ and $g,h\in G$.

Proof. We prove the first assertion and leave the second one to the reader. For the only if part it suffices to note that

$$\hat{\delta}_1\big({}^g v\big)\varsigma(w_g)+\alpha\big({}^g v\big)\bar{\delta}_1(g)=\delta_1\big({}^g v w_g\big)=\delta_1(w_g v)=\bar{\delta}_1(g)\varsigma(v)+\alpha(w_g)\hat{\delta}_1(v),$$
$$f(g,h)\bar{\delta}_1(gh)=\delta_1(w_g w_h)=\bar{\delta}_1(g)\varsigma(w_h)+\alpha(w_g)\bar{\delta}_1(h),$$

and to use the definitions of $\varsigma(w_g)$ and $\alpha(w_g)$. We prove the sufficient part by induction on $r=m+1-i$, where i is the first index with $x_i\in k\#_f G$ (if $x_1,\dots,x_m\in V$ we set $r:=0$). For $r\in\{0,1\}$ the result follows immediately from the definition of δ_1. Assume that it is true when $r<r_0$ and that $m+1-i=r_0$. If $x_i=w_g$ and $x_{i+1}=v\in V$, then

$$\delta_1(x_1\cdots x_m)=\delta_1(y_1\cdots y_m)\quad\text{where } y_j=\begin{cases}x_j & \text{if } j\notin\{i,i+1\},\\ {}^g v & \text{if } j=i,\\ w_g & \text{if } j=i+1,\end{cases}$$

and hence, by the inductive hypothesis and item (a),

$$\begin{aligned}\delta_1(x_1\cdots x_m)&=\sum_{j=1}^{m}\alpha(y_1\cdots y_{j-1})\delta_1(y_j)\varsigma(y_{j+1}\cdots y_m)\\&=\sum_{j=1}^{m}\alpha(x_1\cdots x_{j-1})\delta_1(x_j)\varsigma(x_{j+1}\cdots x_m).\end{aligned}$$

If $x_i=w_g$ and $x_{i+1}=w_h$, then

$$\delta_1(x_1\cdots x_m)=f(g,h)\delta_1(y_1\cdots y_{m-1})\quad\text{where } y_j=\begin{cases}x_j & \text{if } j<i,\\ w_{gh} & \text{if } j=i,\\ x_{j+1} & \text{if } j>i,\end{cases}$$

and hence, by the inductive hypothesis and item (b),

$$\delta_1(x_1\cdots x_m)=\sum_{j=1}^{m-1} f(g,h)\alpha(y_1\cdots y_{j-1})\delta_1(y_j)\varsigma(y_{j+1}\cdots y_{m-1})$$
$$=\sum_{j=1}^{m}\alpha(x_1\cdots x_{j-1})\delta_1(x_j)\varsigma(x_{j+1}\cdots x_m),$$

as we want. □

Theorem 3.5. *Let $\hat{\delta}_1 : V \to A$, $\hat{\delta}_2 : V \to A$ and $\hat{\varsigma} : V \to V$ be k-linear maps and let $\bar{\delta}_1 : G \to A$, $\bar{\delta}_2 : G \to A$ and $\chi_\varsigma : G \to k^\times$ be maps. There is an H_q-module algebra structure on (A, s), such that*

$$\sigma\cdot v=\hat{\varsigma}(v),\qquad \sigma\cdot w_g=\chi_\varsigma(g)w_g,\qquad D_i\cdot v=\hat{\delta}_i(v)\quad\text{and}\quad D_i\cdot w_g=\bar{\delta}_i(g)$$

for all $v \in V$, $g \in G$ and $i \in \{1, 2\}$, if and only if

(1) *$\hat{\varsigma} : V \to V$ is a bijective $k[G]$-linear map and χ_ς is a group homomorphism,*
(2) *conditions* (3.4) *and* (3.5) *in Lemma* 3.3 *and items* (1)(a), (1)(b), (2)(a) *and* (2)(b) *in Lemma* 3.4 *are satisfied,*
(3) $\hat{\delta}_i\circ\hat{\alpha}=\alpha\circ\hat{\delta}_i$,
(4) *$\chi_\alpha(g)\bar{\delta}_i(g)=\alpha(\bar{\delta}_i(g))$ for all $g \in G$,*
(5) $\hat{\varsigma}\circ\hat{\alpha}=\hat{\alpha}\circ\hat{\varsigma}$,
(6) *the maps $\varsigma : A \to A$, $\delta_1 : A \to A$ and $\delta_2 : A \to A$, introduced in Lemmas* 3.2 *and* 3.3, *satisfy the following properties*:

$$\delta_2\circ\hat{\delta}_1=\delta_1\circ\hat{\delta}_2,\qquad \hat{\delta}_i\circ\hat{\varsigma}=q\varsigma\circ\hat{\delta}_i,\qquad \delta_2\circ\bar{\delta}_1=\delta_1\circ\bar{\delta}_2,$$
$$\chi_\varsigma(g)\bar{\delta}_i(g)=q\varsigma\big(\bar{\delta}_i(g)\big),\qquad \delta_1^l=\delta_2^l=0\quad\text{if } q\neq 1 \text{ and } q^l=1.$$

Proof. By Theorem 2.4 and the discussion above it, we know that to have an H_q-module algebra structure on (A, s) satisfying the requirements in the statement is equivalent to have maps $\varsigma, \delta_1, \delta_2 : A \to A$ satisfying conditions (1)–(10) in Section 2.2 and such that

$$\varsigma(v)=\hat{\varsigma}(v),\qquad \varsigma(w_g)=\chi_\varsigma(g)w_g,\qquad \delta_i(v)=\hat{\delta}_i(v)\quad\text{and}\quad \delta_i(w_g)=\bar{\delta}_i(g)$$

for all $v \in V$, $g \in G$ and $i \in \{1, 2\}$. Now, it is easy to see that:

a) If ς, δ_1 and δ_2 satisfy conditions (5), (9) and (10) in Section 2.2, then

$$\varsigma(\mathbf{v}_{1m}w_g)=\hat{\varsigma}(v_1)\cdots\hat{\varsigma}(v_m)\chi_\varsigma(g)w_g,$$
$$\delta_1(\mathbf{v}_{1m}w_g)=\sum_{j=1}^{m}\alpha(\mathbf{v}_{1,j-1})\hat{\delta}_1(v_j)\varsigma(\mathbf{v}_{j+1,m}w_g)+\alpha(\mathbf{v}_{1m})\bar{\delta}_1(g),$$
$$\delta_2(\mathbf{v}_{1m}w_g)=\sum_{j=1}^{m}\varsigma\big(\alpha^{-1}(\mathbf{v}_{1,j-1})\big)\hat{\delta}_2(v_j)\mathbf{v}_{j+1,m}w_g+\varsigma\big(\alpha^{-1}(\mathbf{v}_{1m})\big)\bar{\delta}_2(g),$$

where $\mathbf{v}_{hl}=v_h\cdots v_l$.

b) By Lemmas 3.2, 3.3 and 3.4, the maps defined in a) satisfy conditions (1), (5), (8), (9) and (10) in Section 2.2 if and only if items (1) and (2) of the present theorem are fulfilled.

So, in order to finish the proof it suffices to check that:

c) Conditions (6) and (7) in Section 2.2 are satisfied if and only if items (3)–(5) of the present theorem are fulfilled.
d) Conditions (2), (3) and (4) in Section 2.2 are satisfied if and only if item (6) of the present theorem is fulfilled.

We leave this task to the reader. □

We are going now to consider several particular cases, with the purpose of obtaining more precise results. This will allow us to give some specific examples of formal deformations of associative algebras.

3.1. First case

Let $\hat{\alpha}$, χ_α, α and s be as in the discussion following Proposition 3.1. Let $\hat{\delta}_1 : V \to A$, $\hat{\delta}_2 : V \to A$ and $\hat{\varsigma} : V \to V$ be k-linear maps and let $\chi_\varsigma : G \to k^\times$ be a map. Assume that the kernels of $\hat{\delta}_1$ and $\hat{\delta}_2$ have codimension 1, $\ker\hat{\delta}_1 \neq \ker\hat{\delta}_2$ and there exist $x_i \in V \setminus \ker\hat{\delta}_i$, such that $\hat{\delta}_i(x_i) = P_i w_{g_i}$ with $P_i \in S(V)$ and $g_i \in G$. Without loss of generality we can assume that $x_1 \in \ker\hat{\delta}_2$ and $x_2 \in \ker\hat{\delta}_1$ (and we do it). For $g \in G$ and $i \in \{1,2\}$, let $\lambda_{ig}, \omega_i, \nu_i \in k$ be the elements defined by the following conditions:

$$ {}^g x_i - \lambda_{ig} x_i \in \ker\hat{\delta}_i, \qquad \hat{\varsigma}(x_i) - \omega_i x_i \in \ker\hat{\delta}_i \quad \text{and} \quad \hat{\alpha}(x_i) - \nu_i x_i \in \ker\hat{\delta}_i. $$

Theorem 3.6. *There is an H_q-module algebra structure on (A,s), satisfying*

$$ \sigma\cdot v = \hat{\varsigma}(v), \qquad \sigma\cdot w_g = \chi_\varsigma(g) w_g, \qquad D_i\cdot v = \hat{\delta}_i(v) \quad \textit{and} \quad D_i\cdot w_g = 0 $$

for all $v \in V$, $g \in G$ and $i \in \{1,2\}$, if and only if

(1) *$\hat{\varsigma}$ is a bijective $k[G]$-linear map and χ_ς is a group homomorphism,*
(2) *$\hat{\varsigma}(v) = {}^{g_1^{-1}}\hat{\alpha}(v)$ for all $v \in \ker\hat{\delta}_1$ and $\hat{\varsigma}(v) = {}^{g_2}\hat{\alpha}(v)$ for all $v \in \ker\hat{\delta}_2$,*
(3) *g_1 and g_2 belong to the center of G,*
(4) *$\ker\hat{\delta}_1$ and $\ker\hat{\delta}_2$ are G-submodules of V,*
(5) *${}^g P_1 = \lambda_{1g}\chi_\alpha^{-1}(g)\chi_\varsigma(g) f^{-1}(g,g_1) f(g_1,g) P_1$ for all $g \in G$,*
(6) *${}^g P_2 = \lambda_{2g}\chi_\alpha(g)\chi_\varsigma^{-1}(g) f^{-1}(g,g_2) f(g_2,g) P_2$ for all $g \in G$,*
(7) *$\hat{\alpha}(\ker\hat{\delta}_i) = \ker\hat{\delta}_i$ for $i \in \{1,2\}$,*
(8) *$P_1 \in \ker\delta_2$ and $P_2 \in \ker\delta_1$, where δ_1 and δ_2 are the maps defined by*

$$ \delta_1(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \alpha(\mathbf{v}_{1,j-1})\hat{\delta}_1(v_j)\varsigma(\mathbf{v}_{j+1,m} w_g), $$

$$ \delta_2(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \varsigma\big(\alpha^{-1}(\mathbf{v}_{1,j-1})\big)\hat{\delta}_2(v_j)\mathbf{v}_{j+1,m} w_g, $$

in which $\mathbf{v}_{hl} = v_h \cdots v_l$,

(9) $\varsigma(P_i) = q^{-1}\omega_i \chi_\varsigma^{-1}(g_i) P_i$ *and* $\alpha(P_i) = \nu_i \chi_\alpha^{-1}(g_i) P_i$ *for* $i \in \{1,2\}$, *where* ς *is the map given by*

$$\varsigma(\mathbf{v}_{1m} w_g) = \hat{\varsigma}(v_1) \cdots \hat{\varsigma}(v_m) \chi_\varsigma(g) w_g,$$

(10) *if* $q \neq 1$ *and* $q^l = 1$, *then* $\delta_1^l = \delta_2^l = 0$.

In order to prove this result we first need to establish some auxiliary results.

Lemma 3.7. *The following facts hold*:

(1) *Condition* (3.4) *of Lemma* 3.3 *is satisfied if and only if* ${}^{g_1}\hat{\varsigma}(v) = \hat{\alpha}(v)$ *for all* $v \in \ker \hat{\delta}_1$.
(2) *Condition* (3.5) *of Lemma* 3.3 *is satisfied if and only if* ${}^{g_2}v = \hat{\varsigma}(\hat{\alpha}^{-1}(v))$ *for all* $v \in \ker \hat{\delta}_2$.

Proof. We prove item (1) and we leave item (2), which is similar, to the reader. We must check that

$$\hat{\delta}_1(v)\hat{\varsigma}(w) + \hat{\alpha}(v)\hat{\delta}_1(w) = \hat{\delta}_1(w)\hat{\varsigma}(v) + \hat{\alpha}(w)\hat{\delta}_1(v) \quad \text{for all } v, w \in V \tag{3.6}$$

if and only if $\hat{\varsigma}_1(v) = {}^{g_1^{-1}}\hat{\alpha}(v)$ for all $v \in \ker \hat{\delta}_1$. It is clear that we can suppose that $v, w \in \{x_1\} \cup \ker \hat{\delta}_1$. When $v, w \in \ker \hat{\delta}_1$ or $v = w = x_1$ the equality (3.6) is trivial. Assume $v = x_1$ and $w \in \ker \hat{\delta}_1$. Then,

$$\hat{\delta}_1(v)\hat{\varsigma}(w) + \hat{\alpha}(v)\hat{\delta}_1(w) = P_1 w_{g_1} \hat{\varsigma}(w) = P_1 {}^{g_1}\hat{\varsigma}(w) w_{g_1}$$

and

$$\hat{\delta}_1(w)\hat{\varsigma}(v) + \hat{\alpha}(w)\hat{\delta}_1(v) = \hat{\alpha}(w) P_1 w_{g_1} = P_1 \hat{\alpha}(w) w_{g_1}.$$

So, in this case, the result is true. Case $v \in \ker \hat{\delta}_1$ and $w = x_1$ can be treated in a similar way. □

Lemma 3.8. *The following facts hold*:

(1) *Items* (1)(a) *and* (1)(b) *of Lemma* 3.4 *are satisfied if and only if*
  (a) $\ker \hat{\delta}_1$ *is a* G*-submodule of* V,
  (b) g_1 *belongs to the center of* G,
  (c) ${}^gP_1 = \lambda_{1g} \chi_\alpha^{-1}(g) \chi_\varsigma(g) f^{-1}(g, g_1) f(g_1, g) P_1$, *for all* $g \in G$.
(2) *Items* (2)(a) *and* (2)(b) *of Lemma* 3.4 *are satisfied if and only if*
  (a) $\ker \hat{\delta}_2$ *is a* G*-submodule of* V,
  (b) g_2 *belongs to the center of* G,
  (c) ${}^gP_2 = \lambda_{2g} \chi_\alpha(g) \chi_\varsigma^{-1}(g) f^{-1}(g, g_2) f(g_2, g) P_2$, *for all* $g \in G$.

Proof. We prove item (1) and we leave item (2) to the reader. Since $\bar{\delta}_1 = 0$, it is sufficient to prove that

$$\hat{\delta}_1\left({}^gv\right) \chi_\varsigma(g) w_g = \chi_\alpha(g) w_g \hat{\delta}_1(v) \quad \text{for all } v \in V \text{ and } g \in G, \tag{3.7}$$

if and only if conditions (1)(a), (1)(b) and (1)(c) are satisfied. We can assume that $v \in \{x_1\} \cup \ker \hat{\delta}_1$. When $v \in \ker \hat{\delta}_1$, then equality (3.7) is true if and only if ${}^gv \in \ker \hat{\delta}_1$. Now, since

$$\hat{\delta}_1\left({}^gx_1\right) \chi_\varsigma(g) w_g = \lambda_{1g} P_1 w_{g_1} \chi_\varsigma(g) w_g = \lambda_{1g} P_1 \chi_\varsigma(g) f(g_1, g) w_{g_1 g}$$

and

$$\chi_\alpha(g) w_g \hat{\delta}_1(x_1) = \chi_\alpha(g) w_g P_1 w_{g_1} = \chi_\alpha(g)\, {}^g P_1 f(g, g_1) w_{gg_1},$$

equality (3.7) is true for $v = x_1$ and $g \in G$ if and only if conditions (1)(b) and (1)(c) are satisfied. □

Proof of Theorem 3.6. First note that item (1) coincide with item (1) of Theorem 3.5 and that, by Lemmas 3.7 and 3.8, item (2) of Theorem 3.5 is equivalent to items (2)–(6). Item (4) of Theorem 3.5 and two of the equalities in item (6) of the same theorem, are trivially satisfied because $\bar{\delta}_1 = \bar{\delta}_2 = 0$. Since

$$\hat{\delta}_i\big(\hat{\alpha}(x_i)\big) = \nu_i \hat{\delta}_i(x_i) = \nu P_i w_{g_i} \quad \text{and} \quad \alpha\big(\hat{\delta}_i(x_i)\big) = \alpha(P_i w_{g_i}) = \alpha(P_i)\chi_\alpha(g_i) w_{g_i},$$

item (3) of Theorem 3.5 is true if and only if item (7) and the second equality in item (9) hold. Since $\hat{\alpha}$ is $k[G]$-linear, item (5) of Theorem 3.5 is an immediate consequence of item (2) of Theorem 3.6. Finally we consider the non-trivial equalities in item (6) of Theorem 3.5. It is easy to see that $\hat{\delta}_i(\hat{\varsigma}(x_i)) = q\varsigma(\hat{\delta}_i(x_i))$ if and only if the first equality in item (9) holds. On the other hand $\hat{\delta}_i(\hat{\varsigma}(v)) = q\varsigma(\hat{\delta}_i(v))$ for all $v \in \ker \hat{\delta}_i$ if and only if $\hat{\varsigma}(\ker \hat{\delta}_i) \subseteq \ker \hat{\delta}_i$, which follows from items (2), (4) and (7). The equality $\delta_2(\hat{\delta}_1(v)) = \delta_1(\hat{\delta}_2(v))$ is trivially satisfied for $v \in \ker \hat{\delta}_1 \cap \ker \hat{\delta}_2$, and for $v \in \{x_1, x_2\}$ it is equivalent to item (8). Lastly, the remaining equality coincides with item (10). □

Remark 3.9. The following facts hold:

– Since $\hat{\alpha}$ and $\hat{\varsigma}$ are bijective $k[G]$-linear maps, from item (2) of Theorem 3.6 it follows that

$$ {}^{g_1^{-1}} v = {}^{g_2} v \quad \text{for all } v \in \ker \hat{\delta}_1 \cap \ker \hat{\delta}_2. \tag{3.8}$$

– Since $x_1 \in \ker \hat{\delta}_2$ and $\ker \hat{\delta}_2$ is G-stable, ${}^g x_1 - \lambda_{1g} x_1 \in \ker \hat{\delta}_1 \cap \ker \hat{\delta}_2$. Similarly ${}^g x_2 - \lambda_{1g} x_2 \in \ker \hat{\delta}_1 \cap \ker \hat{\delta}_2$.

– Since $\ker \hat{\delta}_i$ is a G-submodule of V and the k-linear map

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto {}^g v \end{aligned}$$

is an isomorphism for each $g \in G$, it is impossible that ${}^g x_i \in \ker \hat{\delta}_i$. Consequently, $\lambda_{ig} \in k^\times$ for each $g \in G$. Moreover, using again that $\ker \hat{\delta}_i$ is a G-submodule of V, it is easy to see that the map $g \mapsto \lambda_{ig}$ is a group homomorphism. Items (1), (2), (4), (7) and the fact that $\hat{\alpha}$ is bijective imply that also $\omega_1, \omega_2, \nu_1, \nu_2 \in k^\times$.

– Since

$$\hat{\varsigma}(x_1) = \hat{\alpha}\big({}^{g_2}x_1\big) \equiv \lambda_{1g_2} \hat{\alpha}(x_1) \equiv \lambda_{1g_2} \nu_1 x_1 \pmod{\ker \hat{\delta}_1},$$

we have $\omega_1 = \lambda_{1g_2} \nu_1$. A similar argument shows that $\nu_2 = \lambda_{2g_1} \omega_2$.

Corollary 3.10. *Assume that the conditions above Theorem* 3.6 *are fulfilled and that there exists an H_q-module algebra structure on (A, s) satisfying*

$$\sigma \cdot v = \hat{\varsigma}(v), \qquad \sigma \cdot w_g = \chi_\varsigma(g) w_g, \qquad D_i \cdot v = \hat{\delta}_i(v) \quad \textit{and} \quad D_i \cdot w_g = 0$$

for all $v \in V$, $g \in G$ and $i \in \{1, 2\}$. If $P_1 \in S(\ker \hat{\delta}_1)$ and $P_2 \in S(\ker \hat{\delta}_2)$, then

$$\lambda_{1g_1} \lambda_{1g_2} = q \quad \textit{and} \quad \lambda_{2g_1} \lambda_{2g_2} = q^{-1}.$$

Moreover $g_0 := g_1 g_2$ has determinant 1 *as an operator on V.*

Proof. By items (9), (2) and (5) of Theorem 3.6,

$$q^{-1}\omega_1\chi_\varsigma^{-1}(g_1)P_1 = \varsigma(P_1) = {}^{g_1^{-1}}\hat{\alpha}(P_1) = \nu_1\chi_\alpha^{-1}(g_1)\,{}^{g_1^{-1}}P_1 = \nu_1\lambda_{1g_1}^{-1}\chi_\varsigma^{-1}(g_1)P_1.$$

Hence $\lambda_{1g_1}\lambda_{1g_2} = q$ as we want, since $\omega_1 = \nu_1\lambda_{1g_2}$. The proof that $\lambda_{2g_1}\lambda_{2g_2} = q^{-1}$ is similar. It remains to check that $\det(g_0) = 1$. Since $\ker\hat{\delta}_1$ and $\ker\hat{\delta}_2$ are G-invariant, we have

$${}^g x_1 \in \ker\hat{\delta}_2 \quad \text{and} \quad {}^g x_2 \in \ker\hat{\delta}_1 \quad \text{for all } g \in G,$$

and so

$${}^{g_0}x_1 \in \lambda_{1g_1}\lambda_{1g_2}x_1 + W \quad \text{and} \quad {}^{g_0}x_2 \in \lambda_{2g_1}\lambda_{2g_2}x_1 + W,$$

where $W = \ker\hat{\delta}_1 \cap \ker\hat{\delta}_2$. Moreover, by Remark 3.9 we know that g_0 acts as the identity map on W and hence $\det(g_0) = \lambda_{1g_1}\lambda_{1g_2}\lambda_{2g_1}\lambda_{2g_2} = 1$. □

Remark 3.11. A particular case is the H_q-module algebra A considered in [W, Section 4], in which $P_1 = 1$, $g_1 = 1$ and $\hat{\alpha}$ is the identity map. Our P_2, g_2 and f correspond in [W] to s, g and α, respectively. Our computations show that the condition that $h(s) = x_1(h)x_2(h)\alpha(g,h)\alpha^{-1}(h,g)s$, which appears as informed by the cohomology of finite groups in [W], is in fact necessary for the existence of the H_q-module algebra structure of A, and it does not depend on cohomological considerations. In particular we need this condition for any group G, finite or not. Similarly the conditions that g is central and $\det(g) = 1$ are necessary even for infinite groups.

Let G, V, $f : G \times G \to k^\times$ and A be as at the beginning of this section. Let $\hat{\alpha} : V \to V$ be a bijective $k[G]$-linear map, $\chi_\alpha : G \to k^\times$ a group homomorphism, $\alpha : A \to A$ the algebra automorphism induced by $\hat{\alpha}$ and χ_α, and s the good transposition associated with α. Let

a) $V_1 \neq V_2$ subspaces of codimension 1 of V such that V_1 and V_2 are $\hat{\alpha}$-stable G-submodules of V,
b) g_1 and g_2 central elements of G such that ${}^{g_1^{-1}}v = {}^{g_2}v$ for all $v \in V_1 \cap V_2$,
c) $\chi_\varsigma : G \to k^\times$ a group homomorphism and $\hat{\varsigma} : V \to V$ the map defined by

$$\hat{\varsigma}(v) := \begin{cases} \hat{\alpha}({}^{g_1^{-1}}v) & \text{if } v \in V_1, \\ \hat{\alpha}({}^{g_2}v) & \text{if } v \in V_2, \end{cases}$$

d) $x_1 \in V_2 \setminus V_1$, $x_2 \in V_1 \setminus V_2$, $P_1 \in S(V_1)$, $P_2 \in S(V_2)$ and $\hat{\delta}_1, \hat{\delta}_2 : V \to A$ the maps defined by

$$\ker\hat{\delta}_i := V_i \quad \text{and} \quad \hat{\delta}_i(x_i) := P_i w_{g_i}.$$

For $g \in G$ and $i \in \{1,2\}$, let $\lambda_{ig}, \nu_i, \omega_i \in k^\times$ be the elements defined by the conditions ${}^g x_i - \lambda_{ig}x_i \in V_i$, $\hat{\alpha}(x_i) - \nu_i x_i \in V_i$ and $\hat{\varsigma}(x_i) - \omega_i x_i \in V_i$.

The following result is a sort of a reformulation of Theorem 3.6, more appropriate to construct explicit examples. The only new hypothesis that we need is that $P_i \in S(V_i)$.

Corollary 3.12. *There is an H_q-module algebra structure on (A,s), satisfying*

$$\sigma\cdot v = \hat{\varsigma}(v), \qquad \sigma\cdot w_g = \chi_\varsigma(g)w_g, \qquad D_i\cdot v = \hat{\delta}_i(v) \quad \textit{and} \quad D_i\cdot w_g = 0$$

for all $v \in V$, $g \in G$ and $i \in \{1,2\}$, if and only if

(1) $q = \lambda_{1g_1}\lambda_{1g_2}$ *and* $q^{-1} = \lambda_{2g_1}\lambda_{2g_2}$,
(2) ${}^g P_1 = \lambda_{1g}\chi_\alpha^{-1}(g)\chi_\varsigma(g)f^{-1}(g,g_1)f(g_1,g)P_1$,

(3) ${}^{g}P_2 = \lambda_{2g}\chi_\alpha(g)\chi_\varsigma^{-1}(g)f^{-1}(g,g_2)f(g_2,g)P_2$,
(4) $\alpha(P_i) = \nu_i\chi_\alpha^{-1}(g_i)P_i$,
(5) $P_1 \in \ker\delta_2$ *and* $P_2 \in \ker\delta_1$, *where* $\delta_1, \delta_2 : A \to A$ *are the maps defined in item* (8) *of Theorem* 3.6,
(6) *if* $q \neq 1$ *and* $q^l = 1$, *then* $\delta_1^l = \delta_2^l = 0$.

Proof. $\Leftarrow$) By a), b), c) and d), it is obvious that items (1), (2), (3), (4) and (7) of Theorem 3.6 are satisfied. Moreover items (2), (3), (5) and (6) are items (5), (6), (8) and (10) of Theorem 3.6. So, we only must to check that item (9) of Theorem 3.6 is satisfied. But the second equality in this item is exactly the one required in item (4) of the present corollary, and we are going to check that the first one is true with $q = \lambda_{1g_1}\lambda_{1g_2}$. Arguing as in Remark 3.9, and using item (2) with $g = g_1$, items (1) and (4), we obtain

$$\begin{aligned} q^{-1}\omega_1\chi_\varsigma^{-1}(g_1)P_1 &= q^{-1}\lambda_{1g_2}\nu_1\chi_\varsigma^{-1}(g_1)P_1 \\ &= q^{-1}\lambda_{1g_1}\lambda_{1g_2}\nu_1\chi_\alpha^{-1}(g_1)\,{}^{g_1^{-1}}P_1 \\ &= \nu_1\chi_\alpha^{-1}(g_1)\,{}^{g_1^{-1}}P_1 \\ &= {}^{g_1^{-1}}\alpha(P_1) \\ &= \varsigma(P_1), \end{aligned}$$

where the last equality is true since $P_1 \in S(V_1)$. Again arguing as in Remark 3.9, and using item (3) with $g = g_2$, items (1) and (4), we obtain

$$\begin{aligned} q^{-1}\omega_2\chi_\varsigma^{-1}(g_2)P_2 &= q^{-1}\lambda_{2g_1}^{-1}\nu_2\chi_\varsigma^{-1}(g_2)P_2 \\ &= q^{-1}\lambda_{2g_1}^{-1}\lambda_{2g_2}^{-1}\nu_2\chi_\alpha^{-1}(g_2)\,{}^{g_2}P_2 \\ &= \nu_2\chi_\alpha^{-1}(g_2)\,{}^{g_2}P_2 \\ &= {}^{g_2}\alpha(P_2) \\ &= \varsigma(P_2), \end{aligned}$$

where the last equality is true since $P_2 \in S(V_2)$.

$\Rightarrow$) Items (2), (3), (5) and (6) are items (5), (6), (8) and (1) of Theorem 3.6, and item (4) is the first equality in item (9) of that theorem. Finally item (1) follows from Corollary 3.10. □

The following result shows that if x_1 and x_2 are eigenvectors of the maps $v \mapsto {}^{g_1}v$ and $v \mapsto {}^{g_2}v$, then item (5) in the statement of Corollary 3.12 can be easily tested and item (6) can be removed from the hypothesis.

Proposition 3.13. *Assume that conditions* a), b), c) *and* d) *above Corollary* 3.12 *are fulfilled. Let* δ_1 *and* δ_2 *be the maps introduced in item* (8) *of Theorem* 3.6. *If*

$$\lambda_{1g_1}\lambda_{1g_2} = q, \qquad \lambda_{2g_1}\lambda_{2g_2} = q^{-1} \quad \text{and} \quad {}^{g_i}x_j = \lambda_{jg_i}x_j \quad \text{for } 1 \leqslant i, j \leqslant 2,$$

then:

(1) $\delta_1^l = \delta_2^l = 0$, *whenever* $q \neq 1$ *and* $q^l = 1$.
(2) *If* $q = 1$ *or it is not a root of unity, then* $P_1 \in \ker\delta_2$ *and* $P_2 \in \ker\delta_1$ *if and only if* $P_1, P_2 \in S(V_1 \cap V_2)$.
(3) *If* $q \neq 1$ *is a primitive* l*-root of unity, then* $P_1 \in \ker\delta_2$ *and* $P_2 \in \ker\delta_1$ *if and only if* $P_1 \in S(kx_2^l \oplus (V_1 \cap V_2))$ *and* $P_2 \in S(kx_1^l \oplus (V_1 \cap V_2))$.

Proof. The proposition is a direct consequence of the following formulas:

$$\delta_1^s\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)=\begin{cases} c\alpha^s(x_1^{r_1-s}x_2^{r_2}\cdots x_n^{r_n})w_{g_1^s g} & \text{for } s\leqslant r_1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\delta_2^s\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)=\begin{cases} dx_2^{r_2-s}\,{}^{g_2^s}(x_1^{r_1}x_3^{r_3}\cdots x_n^{r_n})w_{g_2^s g} & \text{for } s\leqslant r_2,\\ 0 & \text{otherwise,}\end{cases}$$

where α^s denotes the s-fold composition of α,

$$c=\chi_\varsigma^s(g)\chi_\varsigma^{s(s-1)/2}(g_1)\chi_\alpha^{s(s-1)/2}(g_1)\left(\prod_{k=0}^{s-1}(r_1-k)_q\right)\left(\prod_{k=0}^{s-1}f\big(g_1,g_1^k g\big)\right)\alpha^{s-1}\big(P_1^s\big),$$

$$d=\lambda_{2g_2}^{sr_2-s(s+1)/2}\left(\prod_{k=0}^{s-1}(r_2-k)_q\right)\left(\prod_{k=0}^{s-1}f\big(g_2,g_2^k g\big)\right)\left(\prod_{k=0}^{s-1}{}^{g_2^k}P_2\right).$$

We will prove the formula for δ_1^s and we will leave the other one to the reader. We begin with the case $s=1$. Since $x_2,\dots,x_n\in\ker\hat{\delta}_1$ and $\hat{\delta}_1(x_1)=P_1w_{g_1}$, from the definition of δ_1 it follows that

$$\delta_1\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)=\sum_{j=0}^{r_1-1}\alpha\big(x_1^j\big)P_1w_{g_1}\varsigma\big(x_1^{r_1-j-1}x_2^{r_2}\cdots x_n^{r_n}w_g\big).$$

Thus, using the definition of ς, item c) above Corollary 3.12, the fact that α is G-linear and the hypothesis, we obtain

$$\begin{aligned}\delta_1\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)&=\sum_{j=0}^{r_1-1}\alpha\big(x_1^j\big)P_1w_{g_1}\alpha\big({}^{g_2}x_1^{r_1-j-1}\big)^{g_1^{-1}}\alpha\big(x_2^{r_2}\cdots x_n^{r_n}\big)\chi_\varsigma(g)w_g\\&=\sum_{j=0}^{r_1-1}\alpha\big(x_1^j\big)P_1\alpha\big({}^{g_1g_2}x_1^{r_1-j-1}\big)\alpha\big(x_2^{r_2}\cdots x_n^{r_n}\big)\chi_\varsigma(g)f(g_1,g)w_{g_1g}\\&=\chi_\varsigma(g)(r_1)_qf(g_1,g)P_1\alpha\big(x_1^{r_1-1}x_2^{r_2}\cdots x_n^{r_n}\big)w_{g_1g}.\end{aligned}$$

Assume that $s\leqslant r_1$ and that the formula for δ_1^s holds. Since c depends on s, r_1 and g, it will be convenient for us to use the more precise notation $c_{s,r_1}(g)$ for c. From items (3) and (5) of Theorem 3.5 and item (9) of Theorem 2.4. It follows easily that $\alpha\circ\delta_1=\delta_1\circ\alpha$ on $S(V)$. Using this fact, item (9) of Theorem 2.4 and the inductive hypothesis, we obtain

$$\delta_1^{s+1}\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)=\alpha\big(c_{sr_1}(g)\big)\alpha^s\big(\delta_1\big(x_1^{r_1-s}x_2^{r_2}\cdots x_n^{r_n}\big)\big)\varsigma(w_{g_1^s g}).$$

If $s=r_1$, then $\delta_1(x_1^{r_1-s}x_2^{r_2}\cdots x_n^{r_n})=0$. Otherwise,

$$\begin{aligned}\delta_1^{s+1}\big(x_1^{r_1}\cdots x_n^{r_n}w_g\big)&=\bar{c}\alpha^s\big(\alpha\big(x_1^{r_1-s-1}x_2^{r_2}\cdots x_n^{r_n}\big)w_{g_1}\big)\varsigma(w_{g_1^s g})\\&=\bar{c}\alpha^{s+1}\big(x_1^{r_1-s-1}x_2^{r_2}\cdots x_n^{r_n}\big)\chi_\alpha^s(g_1)\chi_\varsigma^s(g_1)\chi_\varsigma(g)f\big(g_1,g_1^s g\big)w_{g_1^{s+1}g},\end{aligned}$$

where $\bar{c}=\alpha(c_{s,r_1}(g))\alpha^s(c_{1,r_1-s}(1))$. The formula for δ_1^{s+1} follows immediately from this fact. □

Example 3.14. Let $G=\langle g\rangle$ be an order r cyclic group, ξ an element of $k^\times$ and $f_\xi : G\otimes G\to k$ the cocycle defined by

$$f_\xi\big(g^u,g^v\big):=\begin{cases}1 & \text{if } u+v<r,\\ \xi & \text{otherwise.}\end{cases}$$

Of course, if $r=\infty$, then for any ξ this is the trivial cocycle. Let V be a vector space endowed with an action of G and let A be the crossed product $A=S(V)\#_{f_\xi}G$. Let $\{x_1,\dots,x_n\}$ be a basis of V. Let us V_1 and V_2 denote the subspaces of V generated by $\{x_2,\dots,x_n\}$ and $\{x_1,x_3,\dots,x_n\}$, respectively. Let $\hat\alpha : V\to V$ be a bijective $k[G]$-linear map. Assume that V_1 and V_2 are $\hat\alpha$-stable G-submodules of V and that there exist $\lambda_1,\lambda_2\in k^\times$ such that ${}^g x_1=\lambda_1x_1$ and ${}^g x_2=\lambda_2x_2$. Let $m_1,m_2\in\mathbb{Z}$. Assume that ${}^{g^{m_1+m_2}}v=v$ for all $v\in V_1\cap V_2$ (if $r<\infty$ we can take $0\leqslant m_1,m_2<r$). Let $\hat\varsigma : V\to V$ be the map defined by

$$\hat\varsigma(v):=\begin{cases}\hat\alpha({}^{g^{-m_1}}v) & \text{if } v\in V_1,\\ \hat\alpha({}^{g^{m_2}}v) & \text{if } v\in V_2,\end{cases}$$

and let $\chi_\alpha,\chi_\varsigma : G\to k^\times$ be two morphisms. Consider the automorphism of algebras $\alpha : A\to A$ given by $\alpha(v):=\hat\alpha(v)$ for $v\in V$ and $\alpha(w_g)=\chi_\alpha(g)w_g$, and define $\hat\delta_1,\hat\delta_2 : V\to A$ by

$$\hat\delta_1(x_2)=\cdots=\hat\delta_1(x_n):=0,\qquad \hat\delta_1(x_1):=P_1w_{g^{m_1}},$$
$$\hat\delta_2(x_1)=\hat\delta_2(x_3)=\cdots=\hat\delta_1(x_n):=0,\qquad \hat\delta_2(x_2):=P_2w_{g^{m_2}},$$

where $P_1\in S(V_1)\setminus\{0\}$ and $P_2\in S(V_2)\setminus\{0\}$. Let s be the transposition of H_q with A associated with α. There is an H_q-module algebra structure over (A,s) satisfying

$$\sigma\cdot v=\hat\varsigma(v),\qquad \sigma\cdot w_g=\chi_\varsigma(g)w_g,\qquad D_i\cdot v=\hat\delta_i(v)\quad\text{and}\quad D_i\cdot w_g=0\quad\text{for all } v\in V,$$

if and only if

(1) $q=\lambda_1^{m_1+m_2}$ and $q^{-1}=\lambda_2^{m_1+m_2}$,
(2) ${}^gP_1=\lambda_1\chi_\alpha^{-1}(g)\chi_\varsigma(g)P_1$ and ${}^gP_2=\lambda_2\chi_\alpha(g)\chi_\varsigma^{-1}(g)P_2$,
(3) $\alpha(P_1)=\nu_1\chi_\alpha^{-m_1}(g)P_1$ and $\alpha(P_2)=\nu_2\chi_\alpha^{-m_2}(g)P_2$,
(4) if $q=1$ or q is not a root of unity, then $P_1,P_2\in k[x_3,\dots,x_n]$,
(5) if $q\neq1$ is a primitive l-root of unity, then

$$P_1\in k\big[x_2^l,x_3,\dots,x_n\big]\quad\text{and}\quad P_2\in k\big[x_1^l,x_3,\dots,x_n\big].$$

Consequently, in order to obtain explicit examples of braided H_q-module algebra structures on an algebra A of the shape $S(V)\#_{f_\xi}G$, where V is a k-vector space with basis $\{x_1,\dots,x_n\}$ and $G=\langle g\rangle$ is a cyclic group of order $r\leqslant\infty$, we proceed as follows:

First: We define an action of G on V. For this we choose
- a k-linear automorphism γ of $V_{12}:=\langle x_3,\dots,x_n\rangle$, whose order divides r if $r<\infty$,
- $\lambda_1,\lambda_2\in k^\times$ such that $\lambda_1^r=\lambda_2^r=1$ if $r<\infty$,

and we set

$${}^gx_i:=\begin{cases}\lambda_1x_1 & \text{if } i=1,\\ \lambda_2x_2 & \text{if } i=2,\\ \gamma(x_i) & \text{if } i\geqslant 3.\end{cases}$$

Second: We construct the algebra A. For this we choose $\xi \in k^{\times}$ and we define $A = S(V)\#_{f_\xi} G$, where f_ξ is the cocycle associate with ξ.

Third: We endow A with a k-algebra automorphism α. For this we take $\nu_1, \nu_2, \eta \in k^{\times}$ such that $\eta^r = 1$ if $r < \infty$, a k-linear automorphism α' of V_{12} and $v_1, v_2 \in V_{12}$, and we define

$$\alpha(w_g) := \eta w_g \quad \text{and} \quad \alpha(x_i) := \begin{cases} \nu_1 x_1 + v_1 & \text{if } i = 1, \\ \nu_2 x_2 + v_2 & \text{if } i = 2, \\ \alpha'(x_i) & \text{if } i \geqslant 3. \end{cases}$$

Fourth: We choose $m_1, m_2 \in \mathbb{Z}$ and $\zeta \in k^{\times}$ such that

$$\gamma^{m_1+m_2} = \mathrm{id}, \qquad (\lambda_1\lambda_2)^{m_1+m_2} = 1 \quad \text{and} \quad \zeta^r = 1 \quad \text{if } r < \infty,$$

and we define

$$\varsigma(w_g) := \zeta w_g \quad \text{and} \quad \varsigma(x_i) := \begin{cases} \lambda_1^{m_2}(\nu_1 x_1 + v_1) & \text{if } i = 1, \\ \lambda_2^{-m_1}(\nu_2 x_2 + v_2) & \text{if } i = 2, \\ \alpha'(\gamma^{m_2}(x_i)) & \text{if } i \geqslant 3. \end{cases}$$

Fifth: We set $q := \lambda_1^{m_1+m_2}$ and we choose $P_1, P_2 \in S(V) \setminus \{0\}$ such that

- if q is not a root of unity, then $P_1, P_2 \in k[x_3, \dots, x_n]$,
- if q is a primitive l-root of unity, then

$$P_1 \in k\big[x_2^l, x_3, \dots, x_n\big] \quad \text{and} \quad P_2 \in k\big[x_1^l, x_3, \dots, x_n\big],$$

- ${}^g P_1 = \lambda_1 \eta^{-1} \zeta P_1$ and ${}^g P_2 = \lambda_2 \eta \zeta^{-1} P_2$,
- $\alpha(P_1) = \nu_1 \eta^{-m_1} P_1$ and $\alpha(P_2) = \nu_2 \eta^{-m_2} P_2$.

Now, by the discussion at the beginning of this example, there is an H_q-module algebra structure on (A, s), where $s: H_q \otimes A \to A \otimes H_q$ is the good transposition associated with α, such that

$$\sigma \cdot x_j = \varsigma(x_j), \qquad \sigma \cdot w_g = \zeta w_g, \qquad D_i \cdot w_g = 0 \quad \text{and} \quad D_i(x_j) = \begin{cases} 0 & \text{if } i \neq j, \\ P_i w_{g^{mi}} & \text{if } i = j, \end{cases}$$

where $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$.

Remark 3.15. If $P_1(0) \neq 0$ and $P_2(0) \neq 0$, then the conditions in the first step are fulfilled if and only if $\lambda_1\lambda_2 = 1$, $\eta = \lambda_1\zeta$, $\nu_1 = \eta^{m_1}$, $\nu_2 = \eta^{m_2}$, P_1 and P_2 are G-invariants, $\alpha(P_1) = P_1$ and $\alpha(P_2) = P_2$.

3.2. Second case

Let $\hat{\alpha}$, χ_α, α and s be as in the discussion following Proposition 3.1, let $\chi_\varsigma : G \to k^{\times}$ be a map and let $\hat{\delta}_1 : V \to A$, $\hat{\delta}_2 : V \to A$ and $\hat{\varsigma} : V \to V$ be k-linear maps such that $\ker \hat{\delta}_1 \neq \ker \hat{\delta}_2$ are subspaces of codimension 1 of V. Here we are going to consider a more general situation that the one studied in the previous subsection. Assume that for each $i \in \{1, 2\}$ there exist

- an element $x_i \in V \setminus \ker(\hat{\delta}_i)$,
- different elements $g_{i1}, \dots, g_{in_i}$ of G,
- polynomials $P^{(i)}_{g_{i1}}, \dots, P^{(i)}_{g_{in_i}} \in S(V) \setminus \{0\}$,

such that

$$\hat{\delta}_i(x_i) = \sum_{j=1}^{n_i} P^{(i)}_{g_{ij}} w_{g_{ij}}.$$

(The reason for the notation $P^{(i)}_{g_{ij}}$ instead of the more simple P_{ij} will became clear in items (5) and (6) of the following theorem.) Without loss of generality we can assume that $x_1 \in \ker\hat{\delta}_2$ and $x_2 \in \ker\hat{\delta}_1$ (and we do it). For $g \in G$ and $i \in \{1,2\}$, let $\lambda_{ig}, \omega_i, \nu_i \in k$ be the elements defined by the following conditions:

$$^{g}x_i - \lambda_{ig}x_i \in \ker\hat{\delta}_i, \qquad \hat{\varsigma}(x_i) - \omega_i x_i \in \ker\hat{\delta}_i \quad \text{and} \quad \hat{\alpha}(x_i) - \nu_i x_i \in \ker\hat{\delta}_i.$$

Lemma 3.16. *The following facts hold*:

(1) *Condition* (3.4) *of Lemma* 3.3 *is satisfied if and only if* $^{g_{1j}}\hat{\varsigma}(v) = \hat{\alpha}(v)$ *for all* $j \leqslant n_1$ *and* $v \in \ker\hat{\delta}_1$.
(2) *Condition* (3.5) *of Lemma* 3.3 *is satisfied if and only if* $^{g_{2j}}v = \hat{\varsigma}(\hat{\alpha}^{-1}(v))$ *for all* $j \leqslant n_2$ *and* $v \in \ker\hat{\delta}_2$.

Proof. Mimic the proof of Lemma 3.7. □

Lemma 3.17. *The following facts hold*:

(1) *Items* (1)(a) *and* (1)(b) *of Lemma* 3.4 *are satisfied if and only if*
  (a) $\ker\hat{\delta}_1$ *is a G-submodule of V*,
  (b) $\{g_{1j}\colon 1 \leqslant j \leqslant n_1\}$ *is a union of conjugacy classes of G*,
  (c) $^{g}P^{(1)}_{g_{1j}} = \lambda_{1g}\chi_\alpha^{-1}(g)\chi_\varsigma(g)f^{-1}(g, g_{1j})f(gg_{1j}g^{-1}, g)P^{(1)}_{gg_{1j}g^{-1}}$ *for* $j \leqslant n_1$.
(2) *Items* (2)(a) *and* (2)(b) *of Lemma* 3.4 *are satisfied if and only if*
  (a) $\ker\hat{\delta}_2$ *is a G-submodule of V*,
  (b) $\{g_{2j}\colon 1 \leqslant j \leqslant n_2\}$ *is a union of conjugacy classes of G*,
  (c) $^{g}P^{(2)}_{g_{2j}} = \lambda_{2g}\chi_\alpha(g)\chi_\varsigma^{-1}(g)f^{-1}(g, g_{2j})f(gg_{2j}g^{-1}, g)P^{(2)}_{gg_{2j}g^{-1}}$ *for* $j \leqslant n_2$.

Proof. Mimic the proof of Lemma 3.8. □

Theorem 3.18. *There is an* H_q*-module algebra structure on* (A, s)*, satisfying*

$$\sigma \cdot v = \hat{\varsigma}(v), \qquad \sigma \cdot w_g = \chi_\varsigma(g)w_g, \qquad D_i \cdot v = \hat{\delta}_i(v) \quad \textit{and} \quad D_i \cdot w_g = 0$$

for all $v \in V$, $g \in G$ *and* $i \in \{1,2\}$, *if and only if*

(1) $\hat{\varsigma}$ *is a bijective* $k[G]$*-linear map and* χ_ς *is a group homomorphism,*
(2) $\hat{\varsigma}(v) = {}^{g_{1j}^{-1}}\hat{\alpha}(v)$ *for* $j \leqslant n_1$ *and all* $v \in \ker\hat{\delta}_1$, *and* $\hat{\varsigma}(v) = {}^{g_{2j}}\hat{\alpha}(v)$ *for* $j \leqslant n_2$ *and all* $v \in \ker\hat{\delta}_2$,
(3) $\{g_{ij}\colon 1 \leqslant j \leqslant n_i\}$ *is a union of conjugacy classes of G for* $i \in \{1,2\}$,
(4) $\ker\hat{\delta}_1$ *and* $\ker\hat{\delta}_2$ *are G-submodules of V*,
(5) $^{g}P^{(1)}_{g_{1j}} = \lambda_{1g}\chi_\alpha^{-1}(g)\chi_\varsigma(g)f^{-1}(g, g_{1j})f(gg_{1j}g^{-1}, g)P^{(1)}_{gg_{1j}g^{-1}}$ *for* $j \leqslant n_1$,
(6) $^{g}P^{(2)}_{g_{2j}} = \lambda_{2g}\chi_\alpha(g)\chi_\varsigma^{-1}(g)f^{-1}(g, g_{2j})f(gg_{2j}g^{-1}, g)P^{(2)}_{gg_{2j}g^{-1}}$ *for* $j \leqslant n_2$,
(7) $\hat{\alpha}(\ker\hat{\delta}_i) = \ker\hat{\delta}_i$ *for* $i \in \{1,2\}$,

(8) $\sum_{j=1}^{n_1} P^{(1)}_{g_{1j}} w_{g_{1j}} \in \ker\delta_2$ *and* $\sum_{j=1}^{n_2} P^{(2)}_{g_{2j}} w_{g_{2j}} \in \ker\delta_1$, *where* δ_1 *and* δ_2 *are the maps defined by*

$$\delta_1(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \alpha(\mathbf{v}_{1,j-1})\hat{\delta}_1(v_j)\varsigma(\mathbf{v}_{j+1,m} w_g),$$

$$\delta_2(\mathbf{v}_{1m} w_g) := \sum_{j=1}^{m} \varsigma\big(\alpha^{-1}(\mathbf{v}_{1,j-1})\big)\hat{\delta}_2(v_j)\mathbf{v}_{j+1,m} w_g,$$

in which $\mathbf{v}_{hl} = v_h \cdots v_l$,

(9) $\varsigma(P^{(i)}_{g_{ij}}) = q^{-1}\omega_i \chi_\varsigma^{-1}(g_{ij})P^{(i)}_{g_{ij}}$ *and* $\alpha(P^{(i)}_{g_{ij}}) = \nu_i \chi_\alpha^{-1}(g_{ij})P^{(i)}_{g_{ij}}$ *for* $i \in \{1,2\}$ *and* $j \leqslant n_i$, *where* ς *is the map given by*

$$\varsigma(\mathbf{v}_{1m} w_g) := \hat{\varsigma}(v_1)\cdots\hat{\varsigma}(v_m)\chi_\varsigma(g) w_g,$$

(10) *if* $q \neq 1$ *and* $q^l = 1$, *then* $\delta_1^l = \delta_2^l = 0$.

Proof. Mimic the proof of Theorem 3.6, but using Lemmas 3.16 and 3.17 instead of Lemmas 3.7 and 3.8, respectively. □

Remark 3.19. Since α and ς are bijective $k[G]$-linear maps, from item (2) it follows that

$$^{g_{1j}}v = {}^{g_{1h}}v \quad \text{for } 1 \leqslant j, h \leqslant n_1 \text{ and all } v \in \ker\hat{\delta}_1, \tag{3.9}$$

$$^{g_{2j}}v = {}^{g_{2h}}v \quad \text{for } 1 \leqslant j, h \leqslant n_2 \text{ and all } v \in \ker\hat{\delta}_2, \tag{3.10}$$

$$^{g_{1j}^{-1}}v = {}^{g_{2h}}v \quad \text{for } 1 \leqslant j \leqslant n_1,\ 1 \leqslant h \leqslant n_2 \text{ and all } v \in \ker\hat{\delta}_1 \cap \ker\hat{\delta}_2. \tag{3.11}$$

On the other hand, arguing as in Remark 3.9 we can check that

- $^{g}x_i - \lambda_{1g}x_i \in \ker\hat{\delta}_1 \cap \ker\hat{\delta}_2$ for all $g \in G$,
- $\lambda_{ig} \in k^\times$ for all $g \in G$,
- the maps $g \mapsto \lambda_{ig}$ are morphisms,
- $\omega_1, \omega_2, \nu_1, \nu_2 \in k^\times$.

Finally, since

$$\hat{\varsigma}(x_1) = \hat{\alpha}\big({}^{g_{2j}}x_1\big) \equiv \lambda_{1g_{2j}}\hat{\alpha}(x_1) \pmod{\ker\hat{\delta}_1},$$

we have $\omega_1 = \lambda_{1g_{2j}}\nu_1$ for $j \leqslant n_2$. Similarly, $\nu_2 = \lambda_{2g_{1j}}\omega_2$ for $j \leqslant n_1$. Consequently,

$$\lambda_{1g_{21}} = \cdots = \lambda_{1g_{2n_2}} \quad \text{and} \quad \lambda_{2g_{11}} = \cdots = \lambda_{2g_{1n_1}},$$

which also follows from (3.9) and (3.10).

Corollary 3.20. *Assume that the conditions at the beginning of the present subsection are fulfilled and that there exists an* H_q*-module algebra structure on* (A, s)*, satisfying*

$$\sigma \cdot v = \hat{\varsigma}(v), \qquad \sigma \cdot w_g = \chi_\varsigma(g) w_g, \qquad D_i \cdot v = \hat{\delta}_i(v) \quad \textit{and} \quad D_i \cdot w_g = 0$$

for all $v \in V$, $g \in G$ and $i \in \{1,2\}$. If $P^{(1)}_{g_{1j}} \in S(\ker \hat{\delta}_1)$ and $P^{(2)}_{g_{2h}} \in S(\ker \hat{\delta}_2)$ for all $j \leqslant n_1$ and $h \leqslant n_2$, then

$$\lambda_{1g_{1j}} \lambda_{1g_{2h}} = q \quad \text{and} \quad \lambda_{2g_{1j}} \lambda_{2g_{2h}} = q^{-1}.$$

Moreover $g_{1j} g_{2h}$ has determinant 1 as an operator on V.

Proof. This result generalizes Corollary 3.10, and its proof is similar. □

Let G, V, $f : G \times G \to k^\times$, A, $\hat{\alpha} : V \to V$, $\chi_\alpha : G \to k^\times$, $\alpha : A \to A$ and s be as below of Remark 3.11. Assume we have

a) subspaces $V_1 \neq V_2$ of codimension 1 of V such that V_1 and V_2 are $\hat{\alpha}$-stable G-submodules of V, and vectors $x_1 \in V_2 \setminus V_1$ and $x_2 \in V_1 \setminus V_2$,
b) different elements $g_{i1}, \dots, g_{in_i}$ of G, where $i \in \{1,2\}$, such that:
   - $\{g_{11}, \dots, g_{1n_1}\}$ and $\{g_{21}, \dots, g_{2n_2}\}$ are unions of conjugacy classes of G,
   - ${}^{g_{1j}}v = {}^{g_{1h}}v$ for $1 \leqslant j, h \leqslant n_1$ and all $v \in V_1$,
   - ${}^{g_{2j}}v = {}^{g_{2h}}v$ for $1 \leqslant j, h \leqslant n_2$ and all $v \in V_2$,
   - ${}^{g_{1j}^{-1}}v = {}^{g_{2h}}v$ for $1 \leqslant j \leqslant n_1$, $1 \leqslant h \leqslant n_2$ and all $v \in V_1 \cap V_2$,
c) a morphism $\chi_\varsigma : G \to k^\times$,
d) non-zero polynomials $P^{(1)}_{g_{1j}} \in S(V_1)$ and $P^{(2)}_{g_{2h}} \in S(V_2)$, where $1 \leqslant j \leqslant n_1$ and $1 \leqslant h \leqslant n_2$.

Let $\hat{\varsigma} : V \to V$ and $\hat{\delta}_1, \hat{\delta}_2 : V \to A$ be the maps defined by

$$\hat{\varsigma}(v) := \begin{cases} \hat{\alpha}({}^{g_{11}^{-1}}v) & \text{if } v \in V_1, \\ \hat{\alpha}({}^{g_{21}}v) & \text{if } v \in V_2, \end{cases} \qquad \ker \hat{\delta}_i := V_i \quad \text{and} \quad \hat{\delta}_i(x_i) := \sum_{j=1}^{n_i} P^{(i)}_{g_{ij}} w_{g_{ij}}.$$

For $g \in G$ and $i \in \{1,2\}$, let $\lambda_{ig}, \nu_i \in k^\times$ be the elements defined by the following conditions: ${}^g x_i - \lambda_{ig} x_i \in V_i$ and $\alpha(x_i) - \nu_i x_i \in V_i$. Note that, by item b),

$$\lambda_{2g_{11}} = \cdots = \lambda_{2g_{1n_1}} \quad \text{and} \quad \lambda_{1g_{21}} = \cdots = \lambda_{1g_{2n_2}}.$$

Corollary 3.21. *There is an H_q-module algebra structure on (A, s), satisfying*

$$\sigma \cdot v = \hat{\varsigma}(v), \qquad \sigma \cdot w_g = \chi_\varsigma(g) w_g, \qquad D_h \cdot v = \hat{\delta}_h(v) \quad \text{and} \quad D_h \cdot w_g = 0,$$

for all $v \in V$, $g \in G$ and $i \in \{1,2\}$, if and only if for all $j \leqslant n_1$ and $h \leqslant n_2$ the following facts hold:

(1) $q = \lambda_{1g_{1j}} \lambda_{1g_{21}}$ *and* $q^{-1} = \lambda_{2g_{11}} \lambda_{2g_{2h}}$,
(2) ${}^g P^{(1)}_{g_{1j}} = \lambda_{1g} \chi_\alpha^{-1}(g) \chi_\varsigma(g) f^{-1}(g, g_{1j}) f(g g_{1j} g^{-1}, g) P^{(1)}_{g g_{1j} g^{-1}}$,
(3) ${}^g P^{(2)}_{g_{2h}} = \lambda_{2g} \chi_\alpha(g) \chi_\varsigma^{-1}(g) f^{-1}(g, g_{2h}) f(g g_{2h} g^{-1}, g) P^{(2)}_{g g_{2h} g^{-1}}$,
(4) $\alpha(P^{(1)}_{g_{1j}}) = \nu_1 \chi_\alpha^{-1}(g_{1j}) P^{(1)}_{g_{1j}}$ *and* $\alpha(P^{(2)}_{g_{2h}}) = \nu_2 \chi_\alpha^{-1}(g_{2h}) P^{(2)}_{g_{2h}}$,
(5) $\sum_{j=1}^{n_1} P^{(1)}_{g_{1j}} w_{g_{1j}} \in \ker \delta_2$ *and* $\sum_{h=1}^{n_2} P^{(2)}_{g_{2h}} w_{g_{2h}} \in \ker \delta_1$, *where $\delta_1, \delta_2 : A \to A$ are the maps defined in item (8) of Theorem 3.18,*
(6) *if $q \neq 1$ and $q^l = 1$, then $\delta_1^l = \delta_2^l = 0$.*

Proof. It is similar to the proof of Corollary 3.12, using Theorem 3.18 instead of Theorem 3.6. The proof that ς is G-linear requires additionally the fact that ${}^{g g_{ij} g^{-1}}v = {}^{g_{ij}}v$ for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant n_i$, which is true by b). □

Remark 3.22. Assume that the hypotheses of Corollary 3.21 are fulfilled. Then, as it was note above this corollary,

$$\lambda_{2g_{11}} = \cdots = \lambda_{2g_{1n_1}} \quad \text{and} \quad \lambda_{1g_{21}} = \cdots = \lambda_{1g_{2n_2}}.$$

Moreover, by item (1) it is clear that

$$\lambda_{1g_{11}} = \cdots = \lambda_{1g_{1n_1}} \quad \text{and} \quad \lambda_{2g_{21}} = \cdots = \lambda_{2g_{2n_2}}.$$

Proposition 3.23. *Let G, V, f, A, α, V_1, V_2, $g_{11},\dots,g_{1n_1}, g_{21},\dots,g_{2n_2}$, $\hat{\varsigma}$, χ_ς, $\hat{\delta}_1$, $\hat{\delta}_2$, x_1, x_2, ν_1, ν_2, λ_{1g} and λ_{2g}, where $g \in G$, be as in the discussion above Corollary* 3.21. *Assume that*

$$\lambda_{2g_{11}} = \cdots = \lambda_{2g_{1n_1}}, \qquad \lambda_{1g_{21}} = \cdots = \lambda_{1g_{2n_2}},$$
$$\lambda_{1g_{11}} = \cdots = \lambda_{1g_{1n_1}}, \qquad \lambda_{2g_{21}} = \cdots = \lambda_{2g_{2n_2}},$$

and that conditions a), b), c) *and* d) *above that corollary are fulfilled. If*

$$\lambda_{1g_{11}}\lambda_{1g_{21}} = q, \qquad \lambda_{2g_{11}}\lambda_{2g_{21}} = q^{-1} \quad \text{and} \quad {}^{g_{ih}}x_j = \lambda_{jg_{ih}}x_j,$$

for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant h \leqslant n_i$, then:

(1) *$\delta_1^l = \delta_2^l = 0$, whenever $q \neq 1$ and $q^l = 1$.*
(2) *If $q = 1$ or q is not a root of unity, then $P^{(1)}_{g_{1j}} \in \ker\delta_2$ and $P^{(2)}_{g_{2h}} \in \ker\delta_1$ if and only if $P^{(1)}_{g_{1j}}, P^{(2)}_{g_{2h}} \in S(V_1 \cap V_2)$.*
(3) *If $q \neq 1$ is a primitive l-root of unity, then $P^{(1)}_{g_{1j}} \in \ker\delta_2$ and $P^{(2)}_{g_{2h}} \in \ker\delta_1$ if and only if $P^{(1)}_{g_{1j}} \in S(kx_2^l \oplus (V_1 \cap V_2))$ and $P^{(2)}_{g_{2h}} \in S(kx_1^l \oplus (V_1 \cap V_2))$.*

Proof. Let $\mathbf{x}^{\mathbf{r}} = x_1^{r_1}\cdots x_n^{r_n}$. Using the hypothesis it is easy to check by induction on s that

$$\delta_1^s(\mathbf{x}^{\mathbf{r}} w_g) = \begin{cases} \sum_{\mathbf{h}\in\mathbb{I}^s_{n_1}} c_{\mathbf{h}}c'_{\mathbf{h}}\alpha^s(x_1^{r_1-s}x_2^{r_2}\cdots x_n^{r_n})w_{g_{1h_s}g_{1h_{s-1}}\cdots g_{1h_1}g} & \text{for } s \leqslant r_1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\delta_2^s(\mathbf{x}^{\mathbf{r}} w_g) = \begin{cases} \sum_{\mathbf{h}\in\mathbb{I}^s_{n_2}} d_{\mathbf{h}}d'_{\mathbf{h}}x_2^{r_2-s}\, {}^{g^s_{21}}(x_1^{r_1}x_3^{r_3}\cdots x_n^{r_n})w_{g_{2h_s}g_{2h_{s-1}}\cdots g_{2h_1}g} & \text{for } s \leqslant r_2,\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\mathbb{I}^s_{n_i} = \underbrace{\mathbb{I}_{n_i}\times\cdots\times\mathbb{I}_{n_i}}_{s\text{ times}}, \quad \text{with } \mathbb{I}_{n_i} = \{1,\dots,n_i\},$$

α^s denotes the s-fold composition of α,

$$c_{\mathbf{h}} = \chi_\varsigma^s(g)\prod_{k=1}^{s-1}\chi_\varsigma^{s-k}(g_{1h_k})\prod_{k=2}^{s}\chi_\alpha^{k-1}(g_{1h_k}),$$

$$c'_{\mathbf{h}} = \left(\prod_{k=0}^{s-1}(r_1-k)_q\right)\left(\prod_{k=1}^{s} f(g_{1h_k}, g_{1h_{k-1}}\cdots g_{1h_1}g)\right)\left(\prod_{k=1}^{s}\alpha^{s-1}\big(P^{(1)}_{g_{1h_k}}\big)\right),$$

$$d_{\mathbf{h}} = \lambda_{2g_{21}}^{sr_2 - s(s+1)/2},$$
$$d'_{\mathbf{h}} = \left(\prod_{k=0}^{s-1}(r_2-k)_q\right)\left(\prod_{k=1}^{s} f(g_{2h_k}, g_{2h_{k-1}}\cdots g_{2h_1}g)\right)\left(\prod_{k=0}^{s-1} {}^{g_{21}^k}P^{(2)}_{g_{2h_{s-k}}}\right).$$

The result follows easily from these formulas. □

Example 3.24. Let D_u be the Dihedral group $D_u := \langle s, t \mid s^2, t^u, stst\rangle$. Then D_u acts on $k[X_1, X_2]$ via

$${}^sX_1 = -X_1, \qquad {}^sX_2 = -X_2, \qquad {}^tX_1 = X_1 \quad \text{and} \quad {}^tX_2 = X_2.$$

Let $A = k[X_1, X_2]\#D_u$. We have:

– Assume u is even. Then, there is an H_1-module algebra structure on A, such that

$$\begin{array}{llll} \sigma\cdot X_1 = X_1, & \sigma\cdot X_2 = X_2, & \sigma\cdot w_{t^i} = w_{t^i}, & \sigma\cdot w_{t^is} = -w_{t^is}, \\ D_1\cdot X_1 = w_t + w_{t^{-1}}, & D_1\cdot X_2 = 0, & D_1\cdot w_{t^i} = 0, & D_1\cdot w_{t^is} = 0, \\ D_2\cdot X_1 = 0, & D_2\cdot X_2 = w_{t^{u/2}}, & D_2\cdot w_{t^i} = 0, & D_2\cdot w_{t^is} = 0. \end{array}$$

– There is an H_{-1}-module algebra structure on A, such that

$$\begin{array}{llll} \sigma\cdot X_1 = X_1, & \sigma\cdot X_2 = -X_2, & \sigma\cdot w_{t^i} = w_{t^i}, & \sigma\cdot w_{t^is} = -w_{t^is}, \\ D_1\cdot X_1 = \sum_{i=0}^{u-1} w_{t^is}, & D_1\cdot X_2 = 0, & D_1\cdot w_{t^i} = 0, & D_1\cdot w_{t^is} = 0, \\ D_2\cdot X_1 = 0, & D_2\cdot X_2 = w_t + w_{t^{-1}}, & D_2\cdot w_{t^i} = 0, & D_2\cdot w_{t^is} = 0. \end{array}$$

– Assume u is even. Let $\alpha : A \to A$ be the k-algebra map defined by

$$\alpha(Q\,w_{t^i}) := Q\,w_{t^i} \quad \text{and} \quad \alpha(Q\,w_{t^is}) := -Q\,w_{t^is},$$

and let $s : H_1 \otimes A \to A \otimes H_1$ be the transposition associated with α. There is an H_1-module algebra structure on A, such that

$$\begin{array}{llll} \sigma\cdot X_1 = X_1, & \sigma\cdot X_2 = X_2, & \sigma\cdot w_{t^i} = w_{t^i}, & \sigma\cdot w_{t^is} = w_{t^is}, \\ D_1\cdot X_1 = w_t + w_{t^{-1}}, & D_1\cdot X_2 = 0, & D_1\cdot w_{t^i} = 0, & D_1\cdot w_{t^is} = 0, \\ D_2\cdot X_1 = 0, & D_2\cdot X_2 = w_{t^{u/2}}, & D_2\cdot w_{t^i} = 0, & D_2\cdot w_{t^is} = 0. \end{array}$$

4. Non-triviality of the deformations

Let $A = S(V)\#_f G$ be as in Section 3. By Theorem 1.16 we know that each H_q-module algebra (A, s), with s a good transposition, produces to a formal deformation A_F of A, which is constructed using the UDF $F = \exp_q(tD_1 \otimes D_2)$. The aim of this section is to prove that if (A, s) satisfies the conditions required in Corollary 3.21 and $P^{(1)}_{g_{1j}}, P^{(2)}_{g_{2h}} \in S(V_1 \cap V_2)$ for $1 \leqslant j \leqslant n_1$ and $1 \leqslant h \leqslant n_2$, then A_F is non-trivial. We will prove this showing that its infinitesimal

$$\Phi(a\otimes b) = \delta_1\big(\alpha^{-1}(a)\big)\delta_2(b),$$

is not a coboundary. For this we use a complex $\overline{X}^*(A)$, giving the Hochschild cohomology of A, which is simpler than the canonical one.

4.1. A simple resolution

Given a symmetric k-algebra $S := S(V)$, we consider the differential graded algebra (Y_*, ∂_*) generated by elements y_v and z_v, of zero degree, and $\bar{v}$, of degree one, where $v \in V$, subject to the relations

$$z_{\lambda v+w} = \lambda z_v + z_w, \qquad y_{\lambda v+w} = \lambda y_v + y_w, \qquad \overline{v+w} = \lambda\bar{v} + \bar{w},$$
$$y_v y_w = y_w y_v, \qquad y_v z_w = z_w y_v, \qquad z_v z_w = z_w z_v,$$
$$\bar{v} y_w = y_w \bar{v}, \qquad \bar{v} z_w = z_w \bar{v}, \qquad \bar{v}^2 = 0,$$

where $\lambda \in k$ and $v, w \in V$, and with differential ∂ defined by $\partial(\bar{v}) := \rho_v$, where $\rho_v = z_v - y_v$.

Note that S is a subalgebra of Y_* via the embedding that takes v to y_v for all $v \in V$. This produces a structure of left S-module on Y_*. Similarly we consider Y_* as a right S-module via the embedding of S in Y_* that takes v to z_v for all $v \in V$.

Proposition 4.1. *Let $\widetilde{\mu} : Y_0 \to S$ be the algebra map defined by $\widetilde{\mu}(y_v) = \widetilde{\mu}(z_v) := v$ for all $v \in V$. The S-bimodule complex*

$$S \xleftarrow{\widetilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} \cdots \tag{4.12}$$

is contractible as a left S-module complex.

Proof. Let $\{x_1, \dots, x_n\}$ be a basis of V. We will write y_i, z_i, ρ_i and $\bar{v}_i$ instead of y_{x_i}, z_{x_i}, ρ_{x_i} and $\overline{v_{x_i}}$, respectively. A contracting homotopy

$$\varsigma_0 : S \to Y_0 \quad \text{and} \quad \varsigma_{r+1} : Y_r \to Y_{r+1} \quad (r \geqslant 0),$$

of (4.12) is given by

$$\varsigma(1) := 1,$$
$$\varsigma\big(\rho_{i_1}^{m_1}\bar{v}_{i_1}^{\delta_1}\cdots\rho_{i_l}^{m_l}\bar{v}_{i_l}^{\delta_l}\big) := \begin{cases} (-1)^s \rho_{i_1}^{m_1}\bar{v}_{i_1}^{\delta_1}\cdots\rho_{i_{l-1}}^{m_{l-1}}\bar{v}_{i_{l-1}}^{\delta_{l-1}}\rho_{i_l}^{m_l-1}\bar{v}_{i_l} & \text{if } \delta_l = 0,\\ 0 & \text{if } \delta_l = 1,\end{cases}$$

where we assume that $i_1 < \cdots < i_l$, $\delta_1 + \cdots + \delta_l = s$ and $m_l + \delta_l > 0$. In fact, a direct computation shows that:

- $\widetilde{\mu}\circ\sigma^{-1}(1) = \widetilde{\mu}(1) = 1$.
- $\varsigma\circ\widetilde{\mu}(1) = \varsigma(1) = 1$ and $\partial\circ\varsigma(1) = \partial(0) = 0$.
- If $\mathbf{x} = \mathbf{x}'\rho_{i_l}^{m_l}$, where $m_l > 0$ and $\mathbf{x}' = \rho_{i_1}^{m_1}\cdots\rho_{i_{l-1}}^{m_{l-1}}$ with $i_1 < \cdots < i_l$, then

$$\varsigma\circ\widetilde{\mu}(\mathbf{x}) = \varsigma(0) = 0 \quad \text{and} \quad \partial\circ\varsigma(\mathbf{x}) = \partial\big(\mathbf{x}'\rho_{i_l}^{m_l-1}\bar{v}_{i_l}\big) = \mathbf{x}.$$

- Let $\mathbf{x} = \mathbf{x}'\rho_{i_l}^{m_l}\bar{v}_{i_l}^{\delta_l}$, where $m_l + \delta_l > 0$ and $\mathbf{x}' = \rho_{i_1}^{m_1}\bar{v}_{i_1}^{\delta_1}\cdots\rho_{i_{l-1}}^{m_{l-1}}\bar{v}_{i_{l-1}}^{\delta_{l-1}}$ with $i_1 < \cdots < i_l$ and $\delta_1 + \cdots + \delta_l = s > 0$. If $\delta_l = 0$, then

$$\varsigma\circ\partial(\mathbf{x}) = \varsigma\big(\partial(\mathbf{x}')\rho_{i_l}^{m_l}\big) = (-1)^{s-1}\partial(\mathbf{x}')\rho_{i_l}^{m_l-1}\bar{v}_{i_l},$$
$$\partial\circ\varsigma(\mathbf{x}) = \partial\big((-1)^s\mathbf{x}'\rho_{i_l}^{m_l-1}\bar{v}_{i_l}\big) = (-1)^s\partial(\mathbf{x}')\rho_{i_l}^{m_l-1}\bar{v}_{i_l} + \mathbf{x},$$

and if $\delta_l = 1$, then

$$\varsigma \circ \partial(\mathbf{x}) = \varsigma\big(\partial(\mathbf{x}')\rho_{i_l}^{m_l}\bar{v}_{i_l} + (-1)^{s-1}\mathbf{x}'\rho_{i_l}^{m_l+1}\big) = \mathbf{x},$$
$$\partial \circ \varsigma(\mathbf{x}) = \partial(0) = 0.$$

The result follows immediately from all these facts. □

Let G be a group acting on V. We consider S as a $k[G]$-module algebra via the action induced by the one of G on V. Let $f : k[G] \times k[G] \to k^\times$ be a normal cocycle and let $A = S\#_f k[G]$ be the associated crossed product. In the sequel we will use the following

Notation 4.2. We let $\overline{k[G]}$ denote $k[G]/k$. Moreover:

– Given $g_1, \dots, g_s \in \overline{k[G]}$ and $1 \leqslant i < j \leqslant s$, we set $\mathbf{g}_{ij} := g_i \otimes \cdots \otimes g_j$.
– Given $v_1, \dots, v_r \in V$ and $1 \leqslant i < j \leqslant r$, we set $\bar{\mathbf{v}}_{ij} := \overline{v_i} \cdots \overline{v_j}$.

For all $r, s \geqslant 0$, let

$$Z_s = \big(A \otimes \overline{k[G]}^{\otimes s}\big) \otimes_S A \quad \text{and} \quad X_{rs} = \big(A \otimes \overline{k[G]}^{\otimes s}\big) \otimes_S Y_r \otimes_S A,$$

where we consider $A \otimes \overline{k[G]}^{\otimes s}$ as a right S-module via

$$(a_0 w_{g_0} \otimes \mathbf{g}_{1s}) \cdot a = a_0\, {}^{g_0 \cdots g_s}a w_{g_0} \otimes \mathbf{g}_{1s}.$$

The X_{rs}'s and the Z_s's are A-bimodules in a canonical way. Note that

$$Z_s \simeq A \otimes \overline{k[G]}^{\otimes s} \otimes k[G] \quad \text{and} \quad X_{rs} \simeq A \otimes \overline{k[G]}^{\otimes s} \otimes \Lambda^r V \otimes A.$$

In particular, X_{rs} is a free A-bimodule. Consider the diagram of A-bimodules and A-bimodule maps

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\downarrow{\scriptstyle -\delta_2} & & & & & & \\
Z_2 & \xleftarrow{\mu_2} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & \cdots \\
\downarrow{\scriptstyle -\delta_2} & & & & & & \\
Z_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \cdots \\
\downarrow{\scriptstyle -\delta_1} & & & & & & \\
Z_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \cdots,
\end{array}$$

where

– each δ_s is defined by

$$\delta(1 \otimes \mathbf{g}_{1s} \otimes_S 1) := w_{g_1} \otimes \mathbf{g}_{2s} \otimes_S 1 + \sum_{i+1}^{s-1} (-1)^i f(g_i, g_{i+1}) \otimes \mathbf{g}_{1,i-1} \otimes g_i g_{i+1} \otimes \mathbf{g}_{i+2,s} \otimes_S 1$$
$$+ (-1)^s 1 \otimes \mathbf{g}_{1,s-1} \otimes_S w_{g_s},$$

– for each $s \geqslant 0$, the complex (X_{*s}, d_{*s}) is $(-1)^s$ times (Y_*, ∂_*), tensored over S, on the right with A and on the left with $A \otimes \overline{k[G]}^{\otimes s}$,
– for each $s \geqslant 0$, the map μ_s is defined by

$$\mu(1 \otimes \mathbf{g}_{1s} \otimes 1) := 1 \otimes \mathbf{g}_{1s} \otimes_S 1.$$

Each row in this diagram is contractible as a left A-module. A contracting homotopy

$$\varsigma^0_{0s} : Z_s \to X_{0s} \quad \text{and} \quad \varsigma^0_{r+1,s} : X_{rs} \to X_{r+1,s} \quad (r \geqslant 0),$$

is given by

$$\varsigma^0(1 \otimes \mathbf{g}_{1s} \otimes_S 1) := 1 \otimes \mathbf{g}_{1s} \otimes 1,$$
$$\varsigma^0(1 \otimes \mathbf{g}_{1s} \otimes_S \mathbf{P} \otimes_S 1) := (-1)^s 1 \otimes \mathbf{g}_{1s} \otimes_S \varsigma(\mathbf{P}) \otimes_S 1.$$

For $r \geqslant 0$ and $1 \leqslant l \leqslant s$, we define A-bimodule maps $d^l_{rs} : X_{rs} \to X_{r+l-1,s-l}$, recursively on l and r, by

$$d^l(\mathbf{x}) := \begin{cases} \varsigma^0 \circ \delta \circ \mu(\mathbf{x}) & \text{if } l = 1 \text{ and } r = 0, \\ -\varsigma^0 \circ d^1 \circ d^0(\mathbf{x}) & \text{if } l = 1 \text{ and } r > 0, \\ -\sum_{j=1}^{l-1} \varsigma^0 \circ d^{l-j} \circ d^j(\mathbf{x}) & \text{if } 1 < l \text{ and } r = 0, \\ -\sum_{j=0}^{l-1} \varsigma^0 \circ d^{l-j} \circ d^j(\mathbf{x}) & \text{if } 1 < l \text{ and } r > 0, \end{cases}$$

for $\mathbf{x} = 1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1$.

Theorem 4.3. *There is a resolution of A as an A-bimodule*

$$A \xleftarrow{-\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} \cdots,$$

where $\mu : X_{00} \to A$ is the multiplication map,

$$X_n = \bigoplus_{r+s=n} X_{rs} \quad \textit{and} \quad d_n = \sum_{l=1}^{n} d^l_{0n} + \sum_{r=1}^{n} \sum_{l=0}^{n-r} d^l_{r,n-r}.$$

Proof. See [G-G2, Appendix A]. □

Proposition 4.4. *The maps d^l vanish for all $l \geqslant 2$. Moreover*

$$\begin{aligned} d^1(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) = {} & w_{g_1} \otimes \mathbf{g}_{2s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1 \\ & + \sum_{i=1}^{s-1} (-1)^i f(g_i, g_{i+1}) \otimes \mathbf{g}_{1,i-1} \otimes g_i g_{i+1} \otimes \mathbf{g}_{i+2,s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1 \\ & + (-1)^s 1 \otimes \mathbf{g}_{1,s-1} \otimes \overline{{}^{g_s} v_1} \cdots \overline{{}^{g_s} v_r} \otimes w_{g_s}. \end{aligned}$$

In particular, (X_*, d_*) *is the total complex of the double complex*

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle d^1_{03}} & & \downarrow{\scriptstyle d^1_{13}} & & \downarrow{\scriptstyle d^1_{23}} & & \\
X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & X_{22} & \xleftarrow{d^0_{32}} & \cdots \\
\downarrow{\scriptstyle d^1_{02}} & & \downarrow{\scriptstyle d^1_{12}} & & \downarrow{\scriptstyle d^1_{22}} & & \\
X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & X_{21} & \xleftarrow{d^0_{31}} & \cdots \\
\downarrow{\scriptstyle d^1_{01}} & & \downarrow{\scriptstyle d^1_{11}} & & \downarrow{\scriptstyle d^1_{21}} & & \\
X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & X_{20} & \xleftarrow{d^0_{30}} & \cdots.
\end{array}$$

Proof. The computation of d^1_{rs} can be obtained easily by induction on r, using that

$$d^1(\mathbf{x}) = \varsigma^0 \circ \delta \circ \mu(\mathbf{x}) \quad \text{for } \mathbf{x} = 1 \otimes \mathbf{g}_{1s} \otimes 1,$$

and

$$d^1(\mathbf{x}) = -\varsigma^0 \circ d^1 \circ d^0(\mathbf{x}) \quad \text{for } r \geqslant 1 \text{ and } \mathbf{x} = 1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1.$$

The assertion for d^l_{rs}, with $l \geqslant 2$, follows by induction on l and r, using the recursive definition of d^l_{rs}. □

4.2. A comparison map

Let $\overline{A} = A/k$. In this subsection we introduce and study a comparison map from (X_*, d_*) to the canonical normalized Hochschild resolution $(A \otimes \overline{A}^* \otimes A, b'_*)$. It is well known that there is an A-bimodule homotopy equivalence

$$\theta_* : (X_*, d_*) \to \left(A \otimes \overline{A}^* \otimes A, b'_*\right)$$

such that $\theta_0 = \mathrm{id}_{A \otimes A}$. It can be recursively defined by $\theta_0 := \mathrm{id}_{A \otimes A}$ and

$$\theta(\mathbf{x}) := (-1)^{r+s} \theta \circ d(\mathbf{x}) \otimes 1 \quad \text{for } \mathbf{x} = 1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1 \text{ with } r + s \geqslant 1.$$

Next we give a closed formula for θ_*. In order to establish this result we need to introduce a new notation. We recursively define $(w_{g_1} \otimes \cdots \otimes w_{g_s}) * (P_1 \otimes \cdots \otimes P_r)$ by

- $(w_{g_1} \otimes \cdots \otimes w_{g_s}) * (Q_1 \otimes \cdots \otimes Q_r) := (Q_1 \otimes \cdots \otimes Q_r)$ if $s = 0$,
- $(w_{g_1} \otimes \cdots \otimes w_{g_s}) * (Q_1 \otimes \cdots \otimes Q_r) := (w_{g_1} \otimes \cdots \otimes w_{g_s})$ if $r = 0$,
- if $r, s \geqslant 1$, then $(w_{g_1} \otimes \cdots \otimes w_{g_s}) * (Q_1 \otimes \cdots \otimes Q_r)$ equals

$$\sum_{i=0}^{r} (-1)^i (w_{g_1} \otimes \cdots \otimes w_{g_{s-1}}) * \left({}^{g_s}Q_1 \otimes \cdots \otimes {}^{g_s}Q_i\right) \otimes w_{g_s} \otimes Q_{i+1} \otimes \cdots \otimes Q_r.$$

Proposition 4.5. *We have*

$$\theta(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) = (-1)^r \sum_{\tau \in \mathfrak{S}_r} \mathrm{sg}(\tau) \otimes (w_{g_1} \otimes \cdots \otimes w_{g_s}) * \mathbf{v}_{\tau(1r)} \otimes 1,$$

where $\mathfrak{S}_r$ is the symmetric group in r elements and $\mathbf{v}_{\tau(1r)} = v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)}$.

Proof. We proceed by induction on $n = r + s$. The case $n = 0$ is obvious. Suppose that $r + s = n$ and the result is valid for θ_{n-1}. By the recursive definition of θ and Theorem 4.3,

$$\begin{aligned}
\theta(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) &= (-1)^n \theta \circ d(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) \otimes 1 \\
&= (-1)^n \theta \circ \big(d^0 + d^1\big)(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) \otimes 1 \\
&= \sum_{i=1}^{r} (-1)^{i+r} \theta\big({}^{g_1 \cdots g_s} v_i \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1,i-1} \bar{\mathbf{v}}_{i+1,r} \otimes 1\big) \otimes 1 \\
&\quad - \sum_{i=1}^{r} (-1)^{i+r} \theta(1 \otimes \mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1,i-1} \bar{\mathbf{v}}_{i+1,r} \otimes v_i) \otimes 1 \\
&\quad + (-1)^n \theta(w_{g_1} \otimes \mathbf{g}_{2s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) \otimes 1 \\
&\quad + \sum_{i=1}^{s-1} (-1)^{n+i} \theta(1 \otimes \mathbf{g}_{1,i-1} \otimes g_i g_{i+1} \otimes \mathbf{g}_{i+1,s} \otimes \bar{\mathbf{v}}_{1r} \otimes 1) \otimes 1 \\
&\quad + (-1)^r \theta\big(1 \otimes \mathbf{g}_{1,s-1} \otimes \mathbf{g}_{1,s-1} \otimes \overline{{}^{g_s} v_1} \cdots \overline{{}^{g_s} v_r} \otimes w_{g_s}\big) \otimes 1.
\end{aligned}$$

The desired result follows now from the inductive hypothesis. □

4.3. The Hochschild cohomology

Let M be an A-bimodule and A^e the enveloping algebra of A. Applying the functor $\mathrm{Hom}_{A^e}(-, M)$ to $(X_{**}, d^0_{**}, d^1_{**})$ and using the identifications

$$\mathrm{Hom}_{A^e}(X_{rs}, M) \simeq \mathrm{Hom}_k\big(\overline{k[G]}^{\otimes s} \otimes \Lambda^r V, M\big)$$

we obtain the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow \bar{d}_1^{03} & & \uparrow \bar{d}_1^{13} & & \uparrow \bar{d}_1^{23} & & \\
\bar{X}^{02} & \xrightarrow{\bar{d}_0^{12}} & \bar{X}^{12} & \xrightarrow{\bar{d}_0^{22}} & \bar{X}^{22} & \xrightarrow{\bar{d}_0^{32}} & \cdots \\
\uparrow \bar{d}_1^{02} & & \uparrow \bar{d}_1^{12} & & \uparrow \bar{d}_1^{22} & & \\
\bar{X}^{01} & \xrightarrow{\bar{d}_0^{11}} & \bar{X}^{11} & \xrightarrow{\bar{d}_0^{21}} & \bar{X}^{21} & \xrightarrow{\bar{d}_0^{31}} & \cdots \\
\uparrow \bar{d}_1^{01} & & \uparrow \bar{d}_1^{11} & & \uparrow \bar{d}_1^{21} & & \\
\bar{X}^{00} & \xrightarrow{\bar{d}_0^{10}} & \bar{X}^{10} & \xrightarrow{\bar{d}_0^{20}} & \bar{X}^{20} & \xrightarrow{\bar{d}_0^{30}} & \cdots,
\end{array}$$

where

$$\overline{X}^{rs} = \operatorname{Hom}_k\big(\overline{k[G]}^{\otimes s} \otimes \Lambda^r V, M\big),$$

$$\bar{d}_0(\varphi)(\mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1,r+1}) = \sum_{i=1}^{r+1} (-1)^{s+i+1} \varphi(\mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1,i-1} \bar{\mathbf{v}}_{i+1,r+1}) v_i + \sum_{i=1}^{r+1} (-1)^{s+i}\, {}^{g_1 \cdots g_s} v_i \varphi(\mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1,i-1} \bar{\mathbf{v}}_{i+1,r+1}),$$

$$\bar{d}_1(\varphi)(\mathbf{g}_{1,s+1} \otimes \bar{\mathbf{v}}_{1r}) = w_{g_1} \varphi(\mathbf{g}_{2,s+1} \otimes \bar{\mathbf{v}}_{1r}) + \sum_{i=1}^{s} (-1)^i f(g_i, g_{i+1}) \varphi(\mathbf{g}_{1,i-1} \otimes g_i g_{i+1} \otimes \mathbf{g}_{i+1,s+1} \otimes \bar{\mathbf{v}}_{1r}) + (-1)^{s+1} \varphi\big(\mathbf{g}_{1s} \otimes \overline{{}^{g_{s+1}} v_1} \cdots \overline{{}^{g_{s+1}} v_r}\big) w_{g_{s+1}},$$

whose total complex $\overline{X}^*(M)$ gives the Hochschild cohomology $\mathrm{H}^*(A, M)$ of A with coefficients in M. The comparison map θ_* induces a quasi-isomorphism

$$\bar{\theta}^* : \big(\operatorname{Hom}_k\big(\overline{A}^*, M\big), \bar{b}^*\big) \to \overline{X}^*(M).$$

It is immediate that

$$\bar{\theta}(\varphi)(\mathbf{g}_{1s} \otimes \bar{\mathbf{v}}_{1r}) = (-1)^r \sum_{\tau \in \mathfrak{S}_r} \operatorname{sg}(\tau) \varphi\big((w_{g_1} \otimes \cdots \otimes w_{g_s}) * \mathbf{v}_{\tau(1r)}\big),$$

where $\mathfrak{S}_r$ is the symmetric group in r elements and $\mathbf{v}_{\tau(1r)} = v_{\tau(1)} \otimes \cdots \otimes v_{\tau(r)}$.

From now on we take $M = A$ and we write $\mathrm{HH}^*(A)$ instead of $\mathrm{H}^*(A, A)$.

4.4. *Proof of the main result*

We are ready to prove that the cocycle Φ is non-trivial. For this it is sufficient to show that $\bar{\theta}(\Phi)$ is not a coboundary. Let $x_1, \dots, x_n$, $P^{(1)}_{g_{11}}, \dots, P^{(1)}_{g_{1n_1}}$, $P^{(2)}_{g_{21}}, \dots, P^{(2)}_{g_{2n_2}}$, $g_{11}, \dots, g_{1n_1}$ and $g_{21}, \dots, g_{2n_2}$ be as in Corollary 3.21. A direct computation, using the formulas for δ_1 and δ_2 obtained in the proof of Proposition 3.23, shows that

$$\bar{\theta}(\Phi)(g \otimes \bar{v}) = 0 \quad \text{and} \quad \bar{\theta}(\Phi)(g \otimes h) = 0$$

for $g, h \in G$ and $v \in V$, and that

$$\bar{\theta}(\Phi)(\overline{x_1}\,\overline{x_2}) = \sum_{j=1}^{n_1} \sum_{h=1}^{n_2} \chi_\alpha^{-1}(g_{1j}) f(g_{1j}, g_{2h}) \alpha^{-1} \big(P^{(1)}_{g_{1j}}\big)^{g_{1j}} P^{(2)}_{g_{2h}} w_{g_{1j} g_{2h}}$$

and

$$\bar{\theta}(\Phi)(\overline{x_i}\,\overline{x_j}) = 0 \quad \text{for } 1 \leqslant i < j \leqslant n \text{ with } (i, j) \neq (1, 2).$$

We next prove that $\bar{\theta}(\Phi)$ is not a coboundary. Let $\varphi_0 \in \overline{X}_{01}$ and $\varphi_1 \in \overline{X}_{10}$. By definition

$$\bar{d}_1(\varphi_0)(g\otimes h)=w_g\varphi_0(h)-f(g,h)\varphi_0(gh)+\varphi_0(g)w_h,$$
$$\bar{d}_0(\varphi_0)(g\otimes \bar{v})={}^g v\varphi_0(g)-\varphi_0(g)v,$$
$$\bar{d}_1(\varphi_1)(g\otimes \bar{v})=w_g\varphi_1(\bar{v})-\varphi_1\big(\overline{{}^g v}\big)w_g,$$
$$\bar{d}_0(\varphi_1)(\overline{v_1}\,\overline{v_2})=\varphi_1(\overline{v_2})v_1-v_1\varphi_1(\overline{v_2})+v_2\varphi_1(\overline{v_1})-\varphi_1(\overline{v_1})v_2,$$

and so $\bar{\theta}(\Phi)$ is a coboundary if and only if there exist φ_0 and φ_1 such that

$$w_g\varphi_0(h)-f(g,h)\varphi_0(gh)+\varphi_0(g)w_h=0\quad\text{for all } g,h\in G,$$
$${}^g v\varphi_0(g)-\varphi_0(g)v+w_g\varphi_1(\bar{v})-\varphi_1\big(\overline{{}^g v}\big)w_g=0\quad\text{for all } g\in G \text{ and } v\in V,$$
$$\big[\varphi_1(\overline{x_j}),x_i\big]+\big[x_j,\varphi_1(\overline{x_i})\big]=0\quad\text{for all } i<j \text{ with } (i,j)\neq(1,2),$$

where, as usual, $[a,b]=ab-ba$, and

$$\big[\varphi_1(\overline{x_2}),x_1\big]+\big[x_2,\varphi_1(\overline{x_1})\big]=\sum_{j=1}^{n_1}\sum_{h=1}^{n_2}\chi_\alpha^{-1}(g_{1j})f(g_{1j},g_{2h})\alpha^{-1}\big(P^{(1)}_{g_{1j}}\big)^{g_{1j}}P^{(2)}_{g_{2h}}w_{g_{1j}g_{2h}}.$$

But, since $w_gx_j={}^g x_jw_g$,

$$w_{g_{1j}g_{2h}}x_1=f(g_{1j},g_{2h})^{-1}w_{g_{1j}}w_{g_{2h}}x_1=qx_1\quad\text{and}\quad w_{g_{1j}g_{2h}}x_2=q^{-1}x_2,$$

if

$$\varphi_1(\overline{x_1})=\sum_{g\in G}Q^{(1)}_g w_g\quad\text{and}\quad \varphi_1(\overline{x_2})=\sum_{g\in G}Q^{(2)}_g w_g,$$

then necessarily

$$\sum_{g\in\Upsilon}(q-1)\big(x_1Q^{(2)}_g+q^{-1}x_2Q^{(1)}_g\big)w_g=\sum_{j=1}^{n_1}\sum_{h=1}^{n_2}D_{jh}\alpha^{-1}\big(P^{(1)}_{g_{1j}}\big)^{g_{1j}}P^{(2)}_{g_{2h}}w_{g_{1j}g_{2h}},$$

where

$$D_{jh}=\chi_\alpha^{-1}(g_{1j})f(g_{1j},g_{2h})\quad\text{and}\quad \Upsilon=\{g_{1j}g_{2h}\colon 1\leqslant j\leqslant n_1 \text{ and } 1\leqslant h\leqslant n_2\},$$

which is impossible because $\alpha^{-1}(P^{(1)}_{g_{1j}})^{g_{1j}}P^{(2)}_{g_{2h}}\in k[x_3,\dots,x_n]\setminus\{0\}$.

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