



The cohomology of monogenic extensions in the noncommutative setting

Marco Farinati¹, Jorge A. Guccione^{*,2}, Juan J. Guccione²

*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Pabellón 1, Ciudad Universitaria,
(1428) Buenos Aires, Argentina*

Received 2 August 2007

Available online 22 April 2008

Communicated by Michel Van den Bergh

Abstract

We extend the notion of monogenic extension to the noncommutative setting, and we study the Hochschild cohomology ring of such an extension. As an application we complete the computation of the cohomology ring of the rank one Hopf algebras begun in [S.M. Burciu, S.J. Witherspoon, Hochschild cohomology of smash products and rank one Hopf algebras, *math.RA/0608762*, 2006].

© 2008 Elsevier Inc. All rights reserved.

Keywords: Hochschild cohomology ring; Monogenic extensions

Contents

Introduction	5102
1. Preliminaries	5103
2. The resolution	5104
2.1. Comparison maps	5106
3. Hochschild cohomology of monogenic extensions	5108
3.1. Cup product	5109

* Corresponding author.

E-mail addresses: mfarinat@dm.uba.ar (M. Farinati), vander@dm.uba.ar (J.A. Guccione), jgucchi@dm.uba.ar (J.J. Guccione).

¹ Supported by UBACYT X169 and CONICET: PIP 5099.

² Supported by PICT 12330, UBACYT X0294 and CONICET.

3.2.	Explicit computations	5111
3.2.1.	Cup product	5113
3.2.2.	Gerstenhaber structure	5113
4.	Cohomology of some extensions of a group algebra	5117
4.0.3.	A concrete example	5120
5.	An application	5120
6.	A final example	5121
6.1.	Computation of A^{α^f}	5123
6.2.	Computation of the boundary maps	5123
6.3.	Computation of the cohomology	5124
	Acknowledgment	5124
	References	5124

Introduction

Let k be a field, G a finite group whose order is relatively prime to the characteristic of k and $C = k[x]/\langle x^n \rangle$, where $n \geq 2$. Let $\chi : G \rightarrow k^\times$ be a character. The group G acts on C via ${}^g x = \chi(g)x$ for all $g \in G$. Consider the corresponding skew algebra $H = C \# k[G]$. As a vector space $H = C \otimes k[G]$ and the multiplication is

$$(a \otimes g)(b \otimes h) = a^g b \otimes gh \quad \text{for all } a, b \in C \text{ and } g, h \in G.$$

Assume there is a central element $g_1 \in G$ such that $\chi(g_1)$ is a primitive n th root of 1. Then H is a Hopf algebra with coproduct Δ , counit ϵ and antipode S defined by

$$\begin{aligned} \Delta(x) &= x \otimes 1 + g_1 \otimes x, & \Delta(g) &= g \otimes g, \\ \epsilon(x) &= 0, & \epsilon(g) &= 1, \\ S(x) &= -g_1^{-1}x, & S(g) &= g^{-1}, \end{aligned}$$

for all $g \in G$. These algebras are examples of Rank one Hopf algebras, a generalization of Taft algebras defined in [A-S] for abelian groups G and generalized further in [K-R] (but with the opposite coproduct). The cohomology ring of H was computed in [B-W], where was also posed the problem of compute the cohomology ring of the other rank one Hopf algebras. That is, those ones in which the relation $x^n = 0$ is replaced by $x^n = g_1^n - 1$ (see [K-R]).

The aim of this paper is to solve this problem. We carry out this task by computing the cohomology ring of the monogenic extensions $K[x, \alpha]/\langle f \rangle$ of a separable k -algebra K (here $K[x, \alpha]$ is an endomorphism type Ore extension and $f \in K[x, \alpha]$ is a monic polynomial satisfying suitable hypothesis) and noting that each rank one Hopf algebra is such an extension (as associative algebra).

For many of the results the fact that k is a field is not essential. So we fix a commutative ring k with 1, an associative k -algebra K , which we do not assume to be commutative, and a k -algebra endomorphism α of K , and we consider the Ore extension $B = K[x, \alpha]$, namely the algebra generated by K and x subject to the relations

$$x\lambda = \alpha(\lambda)x \quad \text{for all } \lambda \in K.$$

Let $f = x^n + \sum_{i=1}^n \lambda_i x^{n-i}$ be a monic polynomial of degree $n \geq 2$, where each coefficient $\lambda_i \in K$ satisfies $\alpha(\lambda_i) = \lambda_i$ and $\lambda_i \lambda = \alpha^i(\lambda) \lambda_i$ for every $\lambda \in K$. Sometimes we will write $f = \sum_{i=0}^n \lambda_i x^{n-i}$, assuming that $\lambda_0 = 1$. For instance, the above conditions hold if the λ_i 's are in the center of K , $\alpha(\lambda_i) = \lambda_i$ for all i and $\alpha^i = \text{id}$ for all i such that $\lambda_i \neq 0$. Finally let $A = B/\langle f \rangle$. We call A the *monogenic extension* of K associated with α and f . Notice that under these assumptions,

$$fx = xf \quad \text{and} \quad f\lambda = \alpha^n(\lambda)f \quad \text{for all } \lambda \in K,$$

and so $fB \subseteq Bf = \langle f \rangle$.

Motivated by the problem mentioned above, we study the Hochschild cohomology ring $\text{HH}_K^*(A)$ of A relative to K . If the k -algebra K is separable, then $\text{HH}_K^*(A) = \text{HH}^*(A)$ and so, our results apply to this case. As we said above, based on this study we were able to completely compute the cohomology ring of all the rank one Hopf algebras, which was our original objective. Our methods extend to the noncommutative setting those of [B].

The paper is organized as follows: Section 1 is devoted to establish some notations and basic results. In Section 2, we obtain a small resolution of A as an A -bimodule relative to K , and we build comparison maps between this resolution and the normalized canonical one. Sections 3 and 4 are the core of the paper. In the first part of Section 3 we use these results to obtain a small cochain complex $C_S(A, M)$ given the Hochschild cohomology $\text{H}_K^*(A, M)$, of A relative to K , with coefficients in M , for each A -bimodule M . When $M = A$ we will write $C_S(A)$ instead of $C_S(A, A)$ and $\text{HH}_K^*(A)$ instead of $\text{H}_K^*(A, A)$. Moreover, we obtain a map $C_S(A) \times C_S(A) \rightarrow C_S(A)$ inducing the cup product in $\text{HH}_K^*(A)$. Then, in Section 3.2, applying these results we compute the cohomology ring of A , under suitable hypothesis. In Section 4 we apply the results obtained in Section 3 to study closely the cohomology of $A = K[x, \alpha]/\langle f \rangle$, when $K = k[G]$ is the group algebra of a finite group G and α is the automorphism defined by $\alpha(g) = \chi(g)g$, where $\chi : G \rightarrow k^\times$ is a character satisfying suitable hypothesis. Finally, in Section 5 we solve the problem posed in [B-W].

1. Preliminaries

In this section we fix the general terminology and notation used in the following, and establish some basic formulas.

Let K, α, B, f and A be as in the introduction. We remark that, from the condition $fB \subseteq Bf$ and the fact that f monic, it follows that $\{1, x, \dots, x^{n-1}\}$ is a left K -basis of the algebra A . More precisely, given $P \in B$, there exist unique \bar{P} and \ddot{P} in B such that

$$P = \bar{P}f + \ddot{P} \quad \text{and} \quad \ddot{P} = 0 \quad \text{or} \quad \deg \ddot{P} < n.$$

In this paper, unadorned tensor product \otimes means \otimes_K , all the maps are k -linear and all k -bimodules are symmetric. Given a K -bimodule M , we let $M \otimes$ denote the quotient $M/[M, K]$, where $[M, K]$ is the k -module generated by the commutators $m\lambda - \lambda m$ with $\lambda \in K$ and $m \in M$. Given a k -algebra extension C/K , let $C_{\alpha^r}^2 := C_{\alpha^r} \otimes C$, where C_{α^r} is C endowed with the regular left C -module structure and with the right K -module structure twisted by α^r , namely, if $c \in C_{\alpha^r}$ and $\lambda \in K$, then $c \cdot \lambda = c\alpha^r(\lambda)$. We define

$$\frac{T}{Tx} : B \rightarrow B_\alpha^2$$

as the unique K -derivation such that $\frac{T_x}{T_x} = 1 \otimes 1$. Notice that

$$\frac{T_x^i}{T_x} = \sum_{\ell=0}^{i-1} x^\ell \otimes x^{i-\ell-1}.$$

Composing with the canonical projection $B_\alpha^2 \rightarrow A_\alpha^2$ we also obtain a well-defined derivation $\frac{T}{T_x} : B \rightarrow A_\alpha^2$.

Lemma 1.1. *On A_α^2 the following equality holds for all $0 \leq i \leq n - 1$:*

$$\frac{T(fx^i)}{T_x} = x^i \frac{Tf}{T_x} = \frac{Tf}{T_x} x^i.$$

Proof. Since $\frac{T}{T_x}$ is a derivation,

$$\frac{T(fx^i)}{T_x} = \frac{Tf}{T_x} x^i + f \frac{T_x^i}{T_x} = \frac{Tf}{T_x} x^i,$$

but also

$$\frac{T(fx^i)}{T_x} = \frac{T(x^i f)}{T_x} = x^i \frac{Tf}{T_x},$$

which finishes the proof. \square

2. The resolution

Let K, α and f be as in the introduction and let A be the monogenic extension of K associated with α and f . Let \mathcal{Y} be the family of all A -bimodule epimorphisms which split as K -bimodule maps. It is easy to see that the A_{α^r} 's are \mathcal{Y} -projective. The aim of this section is to obtain an \mathcal{Y} -projective resolution $C'_S(A)$, of A as an A -bimodule, smaller than the normalized canonical one. We also build comparison maps between $C'_S(A)$ and the normalized canonical resolution. We consider the following complex

$$\tilde{C}'_S(A) = \dots \rightarrow A_{\alpha^{2n+1}}^2 \xrightarrow{d'_3} A_{\alpha^{2n}}^2 \xrightarrow{d'_4} A_{\alpha^{n+1}}^2 \xrightarrow{d'_3} A_{\alpha^n}^2 \xrightarrow{d'_2} A_\alpha^2 \xrightarrow{d'_1} A^2 \xrightarrow{m} A,$$

where m is the multiplication map and the differentials d'_* are the A -bimodule maps

$$d'_{2m+1} : A_{\alpha^{mn+1}}^2 \rightarrow A_{\alpha^{mn}}^2 \quad \text{and} \quad d'_{2m} : A_{\alpha^{mn}}^2 \rightarrow A_{\alpha^{(m-1)n+1}}^2,$$

defined by

$$d'_{2m+1}(1 \otimes 1) = x \otimes 1 - 1 \otimes x,$$

$$d'_{2m}(1 \otimes 1) = \frac{Tf}{T_x} = \sum_{i=1}^n \lambda_{n-i} \sum_{\ell=0}^{i-1} x^\ell \otimes x^{i-\ell-1}.$$

We remark that, since we are tensoring over K and not over k , the twisting by powers of α in the modules is necessary for the well-definition of the maps d'_* .

Theorem 2.1. $\tilde{C}'_S(A)$ is contractible as an (A, K) -bimodule complex. A contracting homotopy is given by the maps

$$\sigma_0 : A \rightarrow A \otimes A, \quad \sigma_{2m+1} : A_{\alpha^{mn}}^2 \rightarrow A_{\alpha^{mn+1}}^2 \quad \text{and} \quad \sigma_{2m} : A_{\alpha^{(m-1)n+1}}^2 \rightarrow A_{\alpha^{mn}}^2,$$

defined by

$$\begin{aligned} \sigma_0(a) &= a \otimes 1, \\ \sigma_{2m+1}(a \otimes x^i) &= -a \frac{Tx^i}{Tx}, \\ \sigma_{2m}(a \otimes x^i) &= \begin{cases} 0 & \text{if } i < n - 1, \\ a \otimes 1 & \text{if } i = n - 1. \end{cases} \end{aligned}$$

Consequently, the complex

$$C'_S(A) = \cdots \rightarrow A_{\alpha^{2n+1}}^2 \xrightarrow{d'_5} A_{\alpha^{2n}}^2 \xrightarrow{d'_4} A_{\alpha^{n+1}}^2 \xrightarrow{d'_3} A_{\alpha^n}^2 \xrightarrow{d'_2} A_{\alpha}^2 \xrightarrow{d'_1} A^2$$

is a \mathcal{Y} -projective resolution of A .

Proof. We leave to the reader to check that these maps are well-defined. Let us check that σ is a contracting homotopy. To begin, it is clear that $m\sigma_0 = \text{id}_A$ and $\sigma_0 m(1 \otimes x^i) = x^i \otimes 1$. Moreover

$$\begin{aligned} d'_{2m+1} \sigma_{2m+1}(1 \otimes x^i) &= d'_{2m+1} \left(-\frac{Tx^i}{Tx} \right) \\ &= -\sum_{\ell=0}^{i-1} (x^{\ell+1} \otimes x^{i-\ell-1} - x^\ell \otimes x^{i-\ell}) \\ &= 1 \otimes x^i - x^i \otimes 1. \end{aligned}$$

So, in particular, $(\sigma_0 m + d'_1 \sigma_1)(1 \otimes x^i) = 1 \otimes x^i$. Making the computation we obtain

$$\begin{aligned} \sigma_{2m-1} d'_{2m-1}(1 \otimes x^{n-1}) &= \sigma_{2m-1}(x \otimes x^{n-1} - 1 \otimes x^n) \\ &= \sigma_{2m-1}(x \otimes x^{n-1} - 1 \otimes (x^n - f)) \\ &= -x \frac{Tx^{n-1}}{Tx} + \frac{Tx^n}{Tx} - \frac{Tf}{Tx} \\ &= 1 \otimes x^{n-1} - \frac{Tf}{Tx} \end{aligned}$$

and

$$\begin{aligned} \sigma_{2m-1}d'_{2m-1}(1 \otimes x^i) &= \sigma_{2m-1}(x \otimes x^i - 1 \otimes x^{i+1}) \\ &= \frac{Tx^{i+1}}{Tx} - x \frac{Tx^i}{Tx} \\ &= 1 \otimes x^i, \end{aligned}$$

for $i < n - 1$. Besides,

$$d'_{2m}\sigma_{2m}(1 \otimes x^{n-1}) = \frac{Tf}{Tx} \quad \text{and} \quad d'_{2m}\sigma_{2m}(1 \otimes x^i) = 0, \quad \text{for } i < n - 1.$$

So, $d'_{2m}\sigma_{2m} + \sigma_{2m-1}d'_{2m-1} = \text{id}$. To finish the proof, it remains to check that

$$d'_{2m+1}\sigma_{2m+1} + \sigma_{2m}d'_{2m} = \text{id}.$$

But, by Lemma 1.1 and the fact that σ_{2m} is left A -linear,

$$\sigma_{2m}d'_{2m}(1 \otimes x^i) = \sigma_{2m}\left(x^i \frac{Tf}{Tx}\right) = x^i \otimes 1 = 1 \otimes x^i - d'_{2m+1}\sigma_{2m+1}(1 \otimes x^i),$$

as desired. \square

2.1. Comparison maps

From now on we will use the standard notations

$$\bar{A} = A/K, \quad \bar{A}^{\otimes 0} = K \quad \text{and} \quad \bar{A}^{\otimes n} = \bar{A} \otimes \cdots \otimes \bar{A} \quad (n \text{ times}),$$

for $n \geq 1$.

Proposition 2.2. *The family of A -bimodule maps*

$$\psi'_{2m} : A \otimes \bar{A}^{\otimes 2m} \otimes A \rightarrow A_{\alpha^{mn}}^2 \quad \text{and} \quad \psi'_{2m+1} : A \otimes \bar{A}^{\otimes 2m+1} \otimes A \rightarrow A_{\alpha^{m+1}}^2,$$

recursively defined by $\psi'_0 = \text{id}$ and

$$\psi'_{n+1}(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) = \sigma_{n+1}\psi'_n b'_{n+1}(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1),$$

is a homotopy equivalence, from $(A \otimes \bar{A}^{\otimes *}\otimes A, b')$ to $C'_S(A)$, with homotopy inverse given also recursively by $\phi'_0 = \text{id}$ and

$$\phi'_{n+1}(1 \otimes 1) = \zeta_{n+1}\phi'_n d'_{n+1}(1 \otimes 1),$$

where $\zeta_{n+1}(a_0 \otimes \cdots \otimes a_{n+1}) = (-1)^{n+1}a_0 \otimes \cdots \otimes a_{n+1} \otimes 1$.

Proof. It follows by a standard argument in homological algebra. \square

Recall that \ddot{P} and \bar{P} are defined as the unique polynomials such that

$$P = \bar{P}f + \ddot{P} \quad \text{and} \quad \ddot{P} = 0 \quad \text{or} \quad \deg P < n.$$

Theorem 2.3. *The explicit formulas for ϕ' and ψ' are*

$$\begin{aligned} \phi'_0(1 \otimes 1) &= 1 \otimes 1, \\ \phi'_1(1 \otimes 1) &= 1 \otimes x \otimes 1, \\ \phi'_{2m}(1 \otimes 1) &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_\mathbf{i}} x^{|\mathbf{i}-\ell|-m} \otimes \tilde{\mathbf{x}}^{\ell_{m,1}} \otimes 1, \\ \phi'_{2m+1}(1 \otimes 1) &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_\mathbf{i}} x^{|\mathbf{i}-\ell|-m} \otimes \tilde{\mathbf{x}}^{\ell_{m,1}} \otimes x \otimes 1, \\ \psi'_{2m}(1 \otimes \mathbf{x}^{\mathbf{i}1,2m} \otimes 1) &= \overline{x^{i_1+i_2} x^{i_3+i_4} \dots x^{i_{2m-1}+i_{2m}}} \otimes 1, \\ \psi'_{2m+1}(1 \otimes \mathbf{x}^{\mathbf{i}1,2m+1} \otimes 1) &= \overline{x^{i_1+i_2} x^{i_3+i_4} \dots x^{i_{2m-1}+i_{2m}}} \frac{T(x^{i_{2m+1}})}{Tx}, \end{aligned}$$

where

- $\mathbb{I}_m = \{(i_1, \dots, i_m) \in \mathbb{Z}^m : 1 \leq i_j \leq n \text{ for all } j\}$,
- $\mathbb{J}_\mathbf{i} = \{(l_1, \dots, l_m) \in \mathbb{Z}^m : 1 \leq l_j < i_j \text{ for all } j\}$,
- $\lambda_{\mathbf{n}-\mathbf{i}} = \lambda_{n-i_1} \dots \lambda_{n-i_m}$,
- $\tilde{\mathbf{x}}^{\ell_{m,1}} = x \otimes x^{\ell_m} \otimes \dots \otimes x \otimes x^{\ell_1}$,
- $|\mathbf{i} - \ell| = \sum_{j=1}^m (i_j - \ell_j)$,
- $\mathbf{x}^{\mathbf{i}1r} = x^{i_1} \otimes \dots \otimes x^{i_r}$.

Proof. Both formulas for ϕ' and ψ' can be checked by induction on the degree. The computations for ϕ' are easy and straightforward. It is clear that the equality for ψ'_0 is true, since $\psi'_0(1 \otimes 1) = 1 \otimes 1$. To abbreviate, given $1 \leq j \leq l$ we write $\mathbf{x}^{\mathbf{i}j} = x^{i_j} \otimes \dots \otimes x^{i_i}$. Assume the formula for ψ'_{2m} . The recursive definition of ψ'_{2m+1} gives

$$\begin{aligned} \psi'(1 \otimes \mathbf{x}^{\mathbf{i}1,2m+1} \otimes 1) &= \sigma \psi' b'(1 \otimes \mathbf{x}^{\mathbf{i}1,2m+1} \otimes 1) \\ &= \sigma \psi'(b'(1 \otimes \mathbf{x}^{\mathbf{i}1,2m+1}) \otimes 1 - 1 \otimes \mathbf{x}^{\mathbf{i}1,2m+1}) \\ &= \sigma(\dots \otimes 1) - \sigma(\overline{x^{i_1+i_2} \dots x^{i_{2m-1}+i_{2m}}} \otimes x^{i_{2m+1}}) \\ &= \overline{x^{i_1+i_1} \dots x^{i_{2m-1}+i_{2m}}} \frac{T(x^{i_{2m+1}})}{Tx}. \end{aligned}$$

Assume now the formula for ψ'_{2m-1} . By the recursive definition of ψ'_{2m} , we have

$$\begin{aligned} \psi'(1 \otimes \mathbf{x}^{i_1, 2m} \otimes 1) &= \sigma \psi' b'(1 \otimes \mathbf{x}^{i_1, 2m} \otimes 1) \\ &= \sigma \psi'(b'(1 \otimes \mathbf{x}^{i_1, 2m}) \otimes 1 + 1 \otimes \mathbf{x}^{i_1, 2m}) \\ &= \sigma \left(\overline{x^{i_1+i_2} \dots x^{i_{2m-3}+i_{2m-2}}} \frac{T(x^{i_{2m-1}})}{Tx} x^{i_{2m}} \right). \end{aligned}$$

We note now that in $B = k[x, \alpha]$,

$$\begin{aligned} \frac{T(x^{i_{2m-1}})}{Tx} x^{i_{2m}} &= \frac{T(x^{i_{2m-1}+i_{2m}})}{Tx} - x^{i_{2m-1}} \frac{T(x^{i_{2m}})}{Tx} \\ &= \frac{T(x^{i_{2m-1}+i_{2m}} f)}{Tx} + \frac{T(x^{i_{2m-1}+i_{2m}})}{Tx} - x^{i_{2m-1}} \frac{T(x^{i_{2m}})}{Tx}, \end{aligned}$$

and this equality in A becomes

$$\frac{T(x^{i_{2m-1}})}{Tx} x^{i_{2m}} = \overline{x^{i_{2m-1}+i_{2m}}} \frac{T(f)}{Tx} + \frac{T(x^{i_{2m-1}+i_{2m}})}{Tx} - x^{i_{2m-1}} \frac{T(x^{i_{2m}})}{Tx},$$

since $\frac{T}{Tx}$ is a derivation and the class of f in A is zero. So,

$$\begin{aligned} \psi'(1 \otimes \mathbf{x}^{i_1, 2m} \otimes 1) &= \sigma \left(\overline{x^{i_1+i_2} \dots x^{i_{2m-1}+i_{2m}}} \frac{T(f)}{Tx} \right) \\ &= \overline{x^{i_1+i_2} \dots x^{i_{2m-1}+i_{2m}}} \otimes 1, \end{aligned}$$

as desired. \square

3. Hochschild cohomology of monogenic extensions

Let K, α, f and A be as in the introduction. As usual, we let A^e denote the enveloping algebra $A \otimes_k A^{op}$. Given an A -bimodule M , we let M^{α^r} denote the k -submodule

$$M^{\alpha^r} = \{m \in M : m\lambda = \alpha^r(\lambda)m \text{ for all } \lambda \in K\} \subseteq M.$$

Theorem 3.1. *Let M be an A -bimodule. The following facts hold:*

(1) *The cochain complex*

$$C_S(A, M) = \dots \xleftarrow{d^5} M^{\alpha^{2n}} \xleftarrow{d^4} M^{\alpha^{n+1}} \xleftarrow{d^3} M^{\alpha^n} \xleftarrow{d^2} M^{\alpha} \xleftarrow{d^1} M^{\alpha^0},$$

where the coboundary maps are defined by

$$d^{2m+1}(m) = xm - mx \quad \text{and} \quad d^{2m}(m) = \sum_{i=1}^n \sum_{\ell=0}^{i-1} \lambda_{n-i} x^\ell m x^{i-\ell-1},$$

computes $H_K^*(A, M)$.

(2) *The maps*

$$\begin{aligned} \phi^* &: (\text{Hom}_{K^e}(\bar{A}^{\otimes^*}, M), b^*) \rightarrow C_S(A, M), \\ \psi^* &: C_S(A, M) \rightarrow (\text{Hom}_{K^e}(\bar{A}^{\otimes^*}, M), b^*), \end{aligned}$$

defined by

$$\begin{aligned} \phi^0(g) &= g(1), \\ \phi^1(g) &= g(x), \\ \phi^{2m}(g) &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} x^{|\mathbf{i}-\ell|-m} g(\tilde{\mathbf{x}}^{\ell_{m,1}}), \\ \phi^{2m+1}(g) &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} x^{|\mathbf{i}-\ell|-m} g(\tilde{\mathbf{x}}^{\ell_{m,1}} \otimes x), \\ \psi^{2m}(\mathbf{m})(\mathbf{x}^{i_{1,2m}}) &= \overline{x^{i_1+i_2} \dots x^{i_{2m-1}+i_{2m}}} \mathbf{m}, \\ \psi^{2m+1}(\mathbf{m})(\mathbf{x}^{i_{1,2m+1}}) &= \sum_{\ell=0}^{i_{2m+1}-1} \overline{x^{i_1+i_2} \dots x^{i_{2m-1}+i_{2m}}} x^\ell \mathbf{m} x^{i_{2m+1}-\ell-1}, \end{aligned}$$

where we are using the same notations as in Theorem 2.3, are chain morphisms which are inverse one of each other up to homotopy.

Proof. For the first item, apply the functor $\text{Hom}_{A^e}(-, M)$ to the resolution $C'_S(A)$, and use the identification

$$\begin{aligned} \text{Hom}_{A^e}(A_{\alpha^r}^2, M) &\xrightarrow{\cong} M^{\alpha^r}, \\ g &\longmapsto g(1 \otimes 1). \end{aligned}$$

Let ψ_* and ϕ_* be the morphism induced by comparison maps ψ'_* and ϕ'_* , introduced in Proposition 2.2. The second item is a straightforward consequence of Theorem 2.3. \square

Recall from the introduction that when $M = A$ we write $C_S(A)$ instead of $C_S(A, A)$.

3.1. Cup product

In this subsection we compute the cup product of $\text{HH}_K^*(A)$ in terms of the small complex $C_S(A)$. Given $\mathbf{m}_1 \in C_S^p(A)$ and $\mathbf{m}_2 \in C_S^q(A)$, we write $\mathbf{m}_1 \bullet \mathbf{m}_2 = \phi^{p+q}(\psi^p(\mathbf{m}_1) \smile \psi^q(\mathbf{m}_2))$, where $\psi^p(\mathbf{m}_1) \smile \psi^q(\mathbf{m}_2)$ denotes the cup product (at the level of cochains) in the canonical normalized Hochschild cochain complex.

Theorem 3.2. *Let $a_1 \in C_S^p(A)$ and $a_2 \in C_S^q(A)$. The following facts hold:*

- (1) *If p is even or q is even, then $a_1 \bullet a_2 = a_1 a_2$.*

(2) If p and q are odd, then

$$a_1 \bullet a_2 = \sum_{i=2}^n \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = i-2}} \lambda_{n-i} x^{j_1} a_1 x^{j_2} a_2 x^{j_3}.$$

Proof. We use the same notations as in Theorem 3.1. We recall that the cup product in terms of the normalized resolution relative to K is given by

$$(g \smile h)(a_1 \otimes \cdots \otimes a_{p+q}) = g(\mathbf{a}_{1p})h(\mathbf{a}_{p+1,p+q})$$

for

$$g \in \text{Hom}_{K^e}(\bar{A}^{\otimes p}, A) \quad \text{and} \quad h \in \text{Hom}_{K^e}(\bar{A}^{\otimes q}, A),$$

where $\mathbf{a}_{1p} = a_1 \otimes \cdots \otimes a_p$ and $\mathbf{a}_{p+1,p+q} = a_{p+1} \otimes \cdots \otimes a_{p+q}$. Assume first that $p = 2u$ and $q = 2v$. We have:

$$\begin{aligned} a_1 \bullet a_2 &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} x^{|\mathbf{i}-\ell|-m} (\psi^p(a_1) \smile \psi^q(a_2)) (\bar{\mathbf{x}}^{\ell_{m,1}}) \\ &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} x^{|\mathbf{i}-\ell|-m} a_1 \overline{x^{1+\ell_m}} \cdots \overline{x^{1+\ell_{v+1}}} a_2 \overline{x^{1+\ell_v}} \cdots \overline{x^{1+\ell_1}}, \end{aligned}$$

where $m = u + v$. But $\overline{x^{\ell+1}} = 0$ unless $\ell = n - 1$, and in this case $\bar{x}^n = 1$. If one wants to consider non-vanishing terms, then all ℓ_j must be equal to $n - 1$ and this also forces that all i_j are equal to n . So, the sum reduces to the single term $\lambda_0^m a_1 a_2 = a_1 a_2$. If $p = 2u + 1$ and $q = 2v$, or $p = 2u$ and $q = 2v + 1$, then a similar argument as above shows that $a_1 \bullet a_2 = a_1 a_2$. Finally, if $p = 2u + 1$ and $q = 2v + 1$, then

$$\begin{aligned} a_1 \bullet a_2 &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} x^{|\mathbf{i}-\ell|-m} (\psi^p(a_1) \smile \psi^q(a_2)) (\tilde{\mathbf{x}}^{\ell_{m,1}}) \\ &= \sum_{\mathbf{i} \in \mathbb{I}_m} \lambda_{\mathbf{n}-\mathbf{i}} \sum_{\ell \in \mathbb{J}_i} \sum_{j=0}^{\ell_1-1} x^{|\mathbf{i}-\ell|-m} \Gamma_{\ell_m, v+2} a_1 \Gamma_{\ell_{v+1,2}} x^j a_2 x^{\ell_1-j-1} \\ &= \sum_{i=2}^n \lambda_{n-i} \sum_{\ell=1}^{i-1} \sum_{j=0}^{\ell-1} x^{i-\ell-1} a_1 x^j a_2 x^{\ell-j-1} \\ &= \sum_{i=2}^n \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = i-2}} \lambda_{n-i} x^{j_1} a_1 x^{j_2} a_2 x^{j_3}, \end{aligned}$$

where $m = u + v + 1$,

$$\Gamma_{\ell_m, v+2} = \overline{x^{1+\ell_m}} \cdots \overline{x^{1+\ell_{v+2}}} \quad \text{and} \quad \Gamma_{\ell_{v+1,2}} = \overline{x^{\ell_{v+1}+1}} \cdots \overline{x^{\ell_2+1}}.$$

This finishes the proof. \square

3.2. Explicit computations

Let K , α and f be as in the introduction and let A be the corresponding monogenic extension. In this subsection we are going to compute the cohomology of A with coefficients in A , under suitable hypothesis. In the following theorem $\mathcal{Z}(K)$ denotes the center of K .

Theorem 3.3. *Let $C_S^r(A)$ denote the r th module of $C_S(A)$. If there exists $\check{\lambda} \in \mathcal{Z}(K)$ such that*

- $\alpha^n(\check{\lambda}) = \check{\lambda}$,
- $\check{\lambda} - \alpha^i(\check{\lambda})$ is not a zero divisor of K for $1 \leq i < n$,

then $\lambda_1 = \dots = \lambda_{n-1} = 0$ and

$$C_S^r(A) = \begin{cases} K^{\alpha^{mn}} & \text{if } r = 2m, \\ K^{\alpha^{mn}}x & \text{if } r = 2m + 1. \end{cases}$$

Proof. Since $\check{\lambda}\lambda_i = \lambda_i\check{\lambda} = \alpha^i(\check{\lambda})\lambda_i$ and $\check{\lambda} - \alpha^i(\check{\lambda})$ is not a zero divisor of K for $1 \leq i < n$, we have that $\lambda_1 = \dots = \lambda_{n-1} = 0$. By Theorem 3.1 we know that

$$C_S^r(A) = \begin{cases} A^{\alpha^{mn}} & \text{if } r = 2m, \\ A^{\alpha^{mn+1}} & \text{if } r = 2m + 1. \end{cases}$$

Moreover it is immediate that $a = \sum_{i=0}^{n-1} \lambda'_i x^i \in A$ satisfies $a\lambda = \alpha^r(\lambda)a$ for all $\lambda \in K$ if and only if each $\lambda'_i x^i$ satisfies the same condition. Hence, in order to prove the theorem it will be sufficient to check that $\lambda' x^i \in A^{\alpha^{mn}}$ if and only if $i = 0$ and $\lambda' \in K^{\alpha^{mn}}$, and that $\lambda' x^i \in A^{\alpha^{mn+1}}$ if and only if $i = 1$ and $\lambda' \in K^{\alpha^{mn}}$. If $\lambda' x^i \in A^{\alpha^{mn}}$, then

$$\lambda' x^i \check{\lambda} = \alpha^{mn}(\check{\lambda})\lambda' x^i = \check{\lambda}\lambda' x^i = \lambda' \check{\lambda} x^i,$$

since $\alpha^n(\check{\lambda}) = \check{\lambda}$ and $\check{\lambda} \in \mathcal{Z}(K)$. On the other hand, $\lambda' x^i \check{\lambda} = \lambda' \alpha^i(\check{\lambda}) x^i$. So,

$$\lambda' (\check{\lambda} - \alpha^i(\check{\lambda})) x^i = 0,$$

which implies $\lambda' = 0$ when $1 \leq i < n$, since $\check{\lambda} - \alpha^i(\check{\lambda})$ is not a zero divisor of K in this case. Moreover, it is clear that $\lambda' \in K \cap A^{\alpha^{mn}}$ if and only if $\lambda' \in K^{\alpha^{mn}}$. If $\lambda' x^i \in A^{\alpha^{mn+1}}$, then

$$\lambda' x^i \alpha^{n-1}(\check{\lambda}) = \alpha^{(m+1)n}(\check{\lambda})\lambda' x^i = \check{\lambda}\lambda' x^i = \lambda' \check{\lambda} x^i,$$

and $\lambda' x^i \alpha^{n-1}(\check{\lambda}) = \lambda' \alpha^{n-1+i}(\check{\lambda}) x^i$. So

$$\lambda' (\check{\lambda} - \alpha^{i+n-1}(\check{\lambda})) x^i = 0,$$

which implies that $\lambda' = 0$ or $i = 1$. Moreover, it is easy to check that $\lambda' x \in A^{\alpha^{mn+1}}$ if and only if $\lambda' \in K^{\alpha^{mn}}$. \square

Theorem 3.4. *Under the hypothesis of Theorem 3.3, the coboundary maps of $C^S(A)$ are given by*

$$d^{2m+1}(\lambda) = (\alpha(\lambda) - \lambda)x \quad \text{and} \quad d^{2m+2}(\lambda x) = - \sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)\lambda_n.$$

Consequently, if $\lambda_n = 0$, then the even coboundary maps are zero.

Proof. The first assertion is immediate. Let us check the second one. Let $\lambda x \in C_S^{2m+1}(A) = K^{\alpha^{mn}}x$. By item (1) of Theorem 3.1 and Theorem 3.3

$$\begin{aligned} d^{2m+2}(\lambda x) &= \sum_{\ell=0}^{n-1} x^\ell \lambda x^{n-\ell} \\ &= \sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)x^n \\ &= - \sum_{\ell=0}^{n-1} \sum_{i=0}^{n-1} \alpha^\ell(\lambda)\lambda_{n-i}x^i \\ &= - \sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)\lambda_n, \end{aligned}$$

where the last equality follows from Theorem 3.3. \square

Theorem 3.4 implies that $\alpha^n(\lambda)\lambda_n = \lambda\lambda_n$ for all $\lambda \in K^{\alpha^{mn}}$. Indeed, this can be proved directly from the hypothesis at the beginning of this paper and then it is true with full generality. In fact,

$$\lambda\lambda_n = \alpha^{mn}(\lambda_n)\lambda = \lambda_n\lambda = \alpha^n(\lambda)\lambda_n,$$

where the first equality follows from the fact that $\lambda \in K^{\alpha^{mn}}$.

Corollary 3.5. *Under the hypothesis of Theorem 3.3,*

$$\begin{aligned} \text{HH}_K^0(A) &= \ker(\alpha - \text{id}) \cap \mathcal{Z}(K), \\ \text{HH}_K^{2m+1}(A) &= \frac{\{\lambda x \in K^{\alpha^{mn}}x : \sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)\lambda_n = 0\}}{(\alpha - \text{id})(K^{\alpha^{mn}})x}, \\ \text{HH}_K^{2m+2}(A) &= \frac{\ker(\alpha - \text{id}) \cap K^{\alpha^{(m+1)n}}}{\{\sum_{\ell=0}^{n-1} \alpha^\ell(\lambda)\lambda_n : \lambda \in K^{\alpha^{mn}}\}}. \end{aligned}$$

3.2.1. Cup product

It is easy to refine Corollary 3.5 by describing the cup product in the Hochschild cohomology of A relative to K . By item (1) of Theorem 3.2 we know that the product of two homogeneous elements is induced by the multiplication map in K , whenever at least one of them has even degree. On the other hand, if $\lambda x \in \text{HH}_K^{2m+1}(A)$ and $\lambda' x \in \text{HH}_K^{2m'+1}(A)$, then, by item (2) of Theorem 3.2,

$$\begin{aligned} \lambda x \bullet \lambda' x &= \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = n-2}} x^{j_1} \lambda x x^{j_2} \lambda' x x^{j_3} \\ &= \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = n-2}} \alpha^{j_1}(\lambda) \alpha^{j_1 + j_2 + 1}(\lambda') x^n \\ &= - \sum_{i=0}^{n-1} \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = n-2}} \alpha^{j_1}(\lambda) \alpha^{j_1 + j_2 + 1}(\lambda') \lambda_{n-i} x^i \\ &= - \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = n-2}} \alpha^{j_1}(\lambda) \alpha^{j_1 + j_2 + 1}(\lambda') \lambda_n, \end{aligned}$$

where the last equality follows from Theorem 3.3.

3.2.2. Gerstenhaber structure

We are going to compute the Gerstenhaber bracket in $\text{HH}_K^*(A)$, under the hypothesis of Theorem 3.3. We recall first the definition of the Lie bracket on $\text{HH}_K^*(A)$ introduced in [G].

Definition 3.6 (Gerstenhaber). Let C be a ring and V a C -bimodule. Let $f \in \text{Hom}_{C^e}(V^{\otimes r}, V)$ and $g \in \text{Hom}_{C^e}(V^{\otimes r'}, V)$. The composition into the j th place of f and g , is the map

$$f \circ_j g \in \text{Hom}_{C^e}(V^{\otimes r+r'-1}, V),$$

defined by

$$f \circ_j g(\mathbf{v}_{1,r+r'-1}) = f(\mathbf{v}_{1,j-1} \otimes g(\mathbf{v}_{j,j+r'-1}) \otimes \mathbf{v}_{j+r',r+r'-1}),$$

where $\mathbf{v}_{h,l} = v_h \otimes \dots \otimes v_l$ for $h \leq l$. The composition product $f \circ g$ and the bracket $[f, g]$ are defined by

$$f \circ g = \sum_{j=1}^r (-1)^{(j+1)(r'+1)} f \circ_j g,$$

$$[f, g] = f \circ g - (-1)^{(r+1)(r'+1)} g \circ f.$$

Since our comparison maps are between $C^S(A)$ and the relative to K normalized complex $(\text{Hom}_{K^e}(\bar{A}^{\otimes^*}, A), b^*)$, to compute the Gerstenhaber bracket we need to identify $\text{Hom}_{K^e}(\bar{A}^{\otimes^*}, A)$

with the K^e -submodule of $\text{Hom}_{K^e}(A^{\otimes*}, A)$ consisting of all the functions f vanishing on all the simple tensors $a_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_p$. It is well known that in this way one obtain a subcomplex of $(\text{Hom}_{K^e}(A^{\otimes*}, A), b^*)$ and that the canonical inclusion is a quasi-isomorphism. Also, it is clear that this subcomplex is, in fact, a Lie subalgebra. In order to establish the main result of this paragraph we need first to introduce some notations

Definition 3.7. For $a \in C_S^r(A)$, $a' \in C_S^{r'}(A)$ and $j = 1, \dots, r$, we define

$$a \circ_j^s a' = \phi^{r+r'-1}(\psi^r(a) \circ_j \psi^{r'}(a')),$$

$$a \circ^s a' = \sum_{j=1}^r (-1)^{(j+1)(r'+1)} a \circ_j^s a',$$

$$[a, a']_s = a \circ^s a' - (-1)^{(r+1)(r'+1)} a' \circ^s a.$$

Theorem 3.8. Let $\lambda \in K^{\alpha^{mn}}$ and $\mu \in K^{\alpha^{m'n}}$. Assume that we are in the hypothesis of Theorem 3.3. We have:

$$[\lambda, \mu]_s = 0,$$

$$[\lambda, \mu x]_s = \sum_{h=0}^{mn-1} \alpha^h(\mu)\lambda,$$

$$[\lambda x, \mu x]_s = \sum_{h=0}^{mn} \alpha^h(\mu)\lambda - \sum_{h=0}^{m'n} \alpha^h(\lambda)\mu.$$

To prove this theorem we are going to use the following lemmas:

Lemma 3.9. Let $\mu \in K^{\alpha^{m'n}}$. The following equalities are true:

$$\psi^{2m'}(\mu)(\tilde{\mathbf{x}}^{\ell_{m',1}}) = \begin{cases} \mu & \text{if } \ell_1 = \cdots = \ell_{m'} = n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi^{2m'}(\mu)(x^{\ell_{m'}} \otimes \tilde{\mathbf{x}}^{\ell_{m'-1,1}} \otimes x) = \begin{cases} \mu & \text{if } \ell_1 = \cdots = \ell_{m'} = n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi^{2m'+1}(\mu x)(\tilde{\mathbf{x}}^{\ell_{m',1}} \otimes x) = \begin{cases} \mu x & \text{if } \ell_1 = \cdots = \ell_{m'} = n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi^{2m'+1}(\mu x)(x^{\ell_{m'+1}} \otimes \tilde{\mathbf{x}}^{\ell_{m',1}}) = \begin{cases} \delta_{l_1}(\mu)x^{l_1} & \text{if } \ell_2 = \cdots = \ell_{m'+1} = n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_{l_1}(\mu) = \sum_{h=0}^{\ell_1-1} \alpha^h(\mu)$.

Proof. Theses equalities follow by a direct computation, using the formulas for ψ obtained in item (2) of Theorem 3.1. \square

Lemma 3.10. Let $\lambda \in K^{\alpha^{mn}}$ and $\mu \in K^{\alpha^{m'n}}$. Let $\delta(\mu) = \sum_{h=0}^{n-2} \alpha^h(\mu)$. The following equalities are true:

- (1) $\lambda \circ_j^s \mu = 0$,
- (2) $\lambda x \circ_j^s \mu = 0$,
- (3) if j is odd, then $\lambda \circ_j^s \mu x = \alpha^{(j-1)n/2}(\mu)\lambda$,
- (4) if j is even, then $\lambda \circ_j^s \mu x = \alpha^{1+(j-2)n/2}(\delta(\mu))\lambda$,
- (5) if j is odd, then $\lambda x \circ_j^s \mu x = \alpha^{(j-1)n/2}(\mu)\lambda x$,
- (6) if j is even, then $\lambda x \circ_j^s \mu x = \alpha^{1+(j-2)n/2}(\delta(\mu))\lambda x$.

Proof. All the equalities follow in a similar way. We prove the last one. By definition, the formula for $\phi^{2m+2m'+1}$ obtained in item (2) of Theorem 3.1, and Theorem 3.3, we have

$$\lambda x \circ_j^s \mu x = \sum_{\ell \in \mathbb{J}_n} x^{|\mathbf{n}-\ell|-u} (\psi^{2m+1}(\lambda x) \circ_j \psi^{2m'+1}(\mu x)) (\tilde{\mathbf{x}}^{\ell_{u,1}} \otimes x)$$

where $u = m + m'$ and $\mathbf{n} = (n, \dots, n)$ (u times). By the last equality in Lemma 3.10, we know that

$$\lambda x \circ_j^s \mu x = \sum_{\ell \in \mathbb{J}_n} x^{|\mathbf{n}-\ell|-m} \psi^{2m+1}(\lambda x) (\tilde{\mathbf{x}}^{(\ell_{m,1,j})}),$$

where

$$\tilde{\mathbf{x}}^{(\ell_{m,1,j})} = \tilde{\mathbf{x}}^{\ell_{m,m-(j-4)/2}} \otimes x \otimes \delta_{\ell_{m-(j-2)/2}}(\mu) x^{\ell_{m-(j-2)/2}} \otimes \tilde{\mathbf{x}}^{\ell_{m-j/2,1}} \otimes x.$$

But, by the formula for ψ^{2m+1} obtained in item (2) of Theorem 3.1,

$$\psi^{2m+1}(\lambda x) (\tilde{\mathbf{x}}^{(\ell_{m,1,j})}) = \begin{cases} \alpha^{1+(j-2)n/2}(\delta(\mu))\lambda x & \text{if } \ell_h = n - 1 \text{ for all } h, \\ 0 & \text{otherwise.} \end{cases}$$

So, $\lambda x \circ_j^s \mu x = \alpha^{1+(j-2)n/2}(\delta(\mu))\lambda x$, as desired. \square

Proof of Theorem 3.8. The first equality follows immediately from items (1) and (2) of Lemma 3.10. We check the third one and left the second one to the reader. By definition and items (5) and (6) of Lemma 3.10

$$\begin{aligned} \lambda x \circ^s \mu x &= \sum_{h=0}^m \lambda x \circ_{2h+1}^s \mu x + \sum_{h=1}^m \lambda x \circ_{2h}^s \mu x \\ &= \sum_{h=0}^m \alpha^{hn}(\mu)\lambda x + \sum_{h=0}^{m-1} \alpha^{hn}(\delta(\mu))\lambda x \\ &= \sum_{h=0}^{mn} \alpha^h(\mu)\lambda x. \end{aligned}$$

Similarly

$$\mu x \circ^s \lambda x = \sum_{h=0}^{m/n} \alpha^h(\lambda) \mu x.$$

The third equality follows from these facts. \square

Corollary 3.11. *Under the hypothesis of Theorem 3.3,*

$$[\mathrm{HH}_K^{\mathrm{even}}, \mathrm{HH}_K^{\mathrm{even}}] = 0.$$

In particular $[\mathrm{HH}_K^2(A), \mathrm{HH}_K^2(A)] = 0$ and so every infinitesimal deformation of A may be completed into a formal deformation.

Corollary 3.12. *Let $C_n = \langle w \rangle$ be the cyclic group with n elements. Assume that the hypothesis of Theorem 3.3 are satisfied. If λ_n is invertible and $\alpha^n = \mathrm{id}$, then*

$$\mathrm{HH}_K^*(A) = H^*(C_n, \mathcal{Z}(K)(\alpha)),$$

where $\mathcal{Z}(K)(\alpha)$ denotes $\mathcal{Z}(K)$ endowed with the structure of C_n -module given by $w \cdot \lambda = \alpha(\lambda)$.

Theorem 3.13. *Suppose k is a field and that the hypothesis of Theorem 3.3 are satisfied. If K is finite dimensional over k and α is a diagonalizable epimorphism, then*

$$\begin{aligned} \mathrm{HH}_K^0(A) &= \ker(\alpha - \mathrm{id}) \cap \mathcal{Z}(K), \\ \mathrm{HH}_K^{2m+1}(A) &= (\ker(\alpha - \mathrm{id}) \cap K^{\alpha^{mn}} \cap \mathrm{Ann}(n\lambda_n))x, \\ \mathrm{HH}_K^{2m+2}(A) &= \frac{\ker(\alpha - \mathrm{id}) \cap K^{\alpha^{(m+1)n}}}{n\lambda_n(\ker(\alpha - \mathrm{id}) \cap K^{\alpha^{mn}})}, \end{aligned}$$

where $\mathrm{Ann}(n\lambda_n) = \{\lambda \in K : n\lambda\lambda_n = 0\}$.

Proof. Using that α is an epimorphism it follows easily that each $K^{\alpha^{mn}}$ is α -invariant. So each $K^{\alpha^{mn}}$ decomposes as a direct sum

$$K^{\alpha^{mn}} = (K^{\alpha^{mn}} \cap \ker(\alpha - \mathrm{id})) \oplus (K^{\alpha^{mn}} \cap \ker(\alpha - \mathrm{id}))^\perp,$$

where $(K^{\alpha^{mn}} \cap \ker(\alpha - \mathrm{id}))^\perp$ is the direct sum of the eigenspaces of α with eigenvalue different from one. Clearly

$$(K^{\alpha^{mn}} \cap \ker(\alpha - \mathrm{id}))^\perp = (\alpha - \mathrm{id})(K^{\alpha^{mn}}),$$

and so

$$\sum_{\ell=0}^{n-1} \alpha^\ell(\lambda) \lambda_n = 0 \quad \text{on } (K^{\alpha^{mn}} \cap \ker(\alpha - \mathrm{id}))^\perp,$$

since $\alpha^n(\lambda)\lambda_n = \lambda\lambda_n$ for all $\lambda \in K^{\alpha^{mn}}$. The result follows immediately from this fact and Corollary 3.5. \square

Remark 3.14. Assume that the hypothesis of Theorem 3.13 are satisfied. From the formula for the cup product it follows that if $\lambda x \in \text{HH}_K^{2m+1}(A)$ and $\lambda'x \in \text{HH}_K^{2m'+1}(A)$, then

$$\lambda x \bullet \lambda'x = - \sum_{\substack{j_1, j_2, j_3 \geq 0 \\ j_1 + j_2 + j_3 = n-2}} \alpha^{j_1}(\lambda)\alpha^{j_1+j_2+1}(\lambda')\lambda_n = -\binom{n}{2}\lambda\lambda'\lambda_n,$$

where the last equality is true since $\alpha(\lambda) = \lambda$ and $\alpha(\lambda') = \lambda'$. Since $\text{HH}_K^{2m+1}(A) \subseteq \text{Ann}(n\lambda_n)$, this implies that the product of two elements of odd degree is zero if the characteristic of k is different from 2, and also if the characteristic of k is 2, but n is odd or 4 divides n . If the characteristic of k is 2, n is even and $n/2$ is odd, then $\lambda x \bullet \lambda'x = \lambda\lambda'\lambda_n$.

Next we consider another situation in which the cohomology of A can be computed. The following results are very close to the ones valid in the commutative setting.

Theorem 3.15. *If α is the identity map, then*

$$C_S^r(A) = \mathcal{Z}(K) \oplus \mathcal{Z}(K)x \oplus \dots \oplus \mathcal{Z}(K)x^{n-1} = \frac{\mathcal{Z}(K)[x]}{\langle f \rangle},$$

where $\mathcal{Z}(K)$ is the center of K . Moreover, the odd coboundary maps d^{2m+1} of $C_S(A)$ are zero, and the even coboundary maps d^{2m} are the multiplication by the derivative f' of f .

Proof. This is immediate. \square

Corollary 3.16. *If α is the identity map, then*

$$\begin{aligned} \text{HH}_K^0(A) &= \frac{\mathcal{Z}(K)[x]}{\langle f \rangle}, \\ \text{HH}_K^{2m+1}(A) &= \text{Ann}(f'), \\ \text{HH}_K^{2m+2}(A) &= \frac{\mathcal{Z}(K)[x]}{\langle f, f' \rangle}, \end{aligned}$$

where $\text{Ann}(f') = \{a \in \frac{\mathcal{Z}(K)[x]}{\langle f \rangle} : af' = 0\}$.

4. Cohomology of some extensions of a group algebra

Let k be a field, $K = k[G]$ the group k -algebra of a finite group G and $\chi : G \rightarrow k^\times$ a character. Let $\alpha : K \rightarrow K$ be the automorphism defined by $\alpha(g) = \chi(g)g$ and let $f \in K[x]$ be a monic polynomial whose coefficients satisfy the hypothesis required in the introduction. Assume that there exists $g_1 \in \mathcal{Z}(G)$ such that $\chi(g_1)$ is a primitive n th root of 1. In this section we apply the results obtained in Section 3 to compute the cohomological ring of $A = K[x, \alpha]/\langle f \rangle$. Note that the hypothesis of Theorem 3.3 are fulfilled, taking $\check{\lambda} = g_1$. Since α is diagonalizable, Theorem 3.13

and Remark 3.14 apply. In order to use the former we need first to make some computations. By definition,

$$\ker(\alpha - \text{id}) = k[N], \quad \text{where } N = \ker(\chi : G \rightarrow k^\times)$$

and

$$K^{\alpha^{mn}} = \left\{ \sum_{g \in G} \gamma_g g \in k[G] : \sum_{g \in G} \gamma_g g = \sum_{g \in G} \gamma_g \chi^{mn}(h) h g h^{-1} \forall h \in G \right\}.$$

Note that $\mathcal{Z}(K) \cap k[N] = k[N]^G$, where G acts on $k[N]$ via conjugation. By Theorem 3.13 and the first equality, we have the following result:

Theorem 4.1. *Let $A = K[x, \alpha]/\langle f \rangle$ be as in the beginning of this section. The Hochschild cohomology of A relative to K is given by:*

$$\begin{aligned} \text{HH}_K^0(A) &= K[N]^G, \\ \text{HH}_K^{2m+1}(A) &= (K[N] \cap K^{\alpha^{mn}} \cap \text{Ann}(n\lambda_n))x, \\ \text{HH}_K^{2m+2}(A) &= \frac{K[N] \cap K^{\alpha^{(m+1)n}}}{n(K[N] \cap K^{\alpha^{mn}})\lambda_n}. \end{aligned}$$

It is easy to check now that

$$\sum_{g \in G} \gamma_g g \in K^{\alpha^r} \iff \gamma_{hgh^{-1}} = \gamma_g \chi^r(h) \quad \text{for all } h \in G.$$

Consequently, if there exists $h \in G$ such that $\chi^r(h) \neq 1$ and $hg = gh$, then $\gamma_g = 0$. Let $\mathbb{X}(r) = \{X_1, \dots, X_l\}$ be the set of the conjugacy classes X of G , which satisfy the following property: if $h \in G$ commutes with an element of X , then $\chi^r(h) = 1$. It is easy to see that for each X_j there exist elements $\gamma_g \in k \setminus \{0\}$, where g runs on X_j , satisfying

$$\gamma_{hgh^{-1}} = \gamma_g \chi^r(h) \quad \text{for all } h \in G.$$

Clearly the family $\{a_1, \dots, a_l\}$, where $a_j = \sum_{g \in X_j} \gamma_g g$, is a basis of K^{α^r} . Thus,

$$K^{\alpha^r} = \bigoplus_{X_j \in \mathbb{X}(r)} ka_j \quad \text{and} \quad K^{\alpha^r} \cap \ker(\alpha - \text{id}) = \bigoplus_{X_j \in \mathbb{X}'(r)} ka_j,$$

where $\mathbb{X}'(r) = \{X_j \in \mathbb{X}(r) : X_j \subseteq N\}$. From this discussion it follows that f satisfies the conditions required in the introduction if and only if

$$\lambda_i \in \bigoplus_{X_j \in \mathbb{X}'(i)} ka_j \quad \text{for all } i.$$

Moreover, by Theorem 4.1, we have:

$$\begin{aligned} \mathrm{HH}_K^0(A) &= \bigoplus_{X_j \in \mathbb{X}'(0)} ka_j, \\ \mathrm{HH}_K^{2m+1}(A) &= \left(\mathrm{Ann}(n\lambda_n) \cap \bigoplus_{X_j \in \mathbb{X}'(mn)} ka_j \right) x, \\ \mathrm{HH}_K^{2m+2}(A) &= \frac{\bigoplus_{X_j \in \mathbb{X}'((m+1)n)} ka_j}{n(\bigoplus_{X_j \in \mathbb{X}'(mn)} ka_j)\lambda_n}. \end{aligned}$$

Remark 4.2. Let X be a conjugacy class of G . It is easy to check that there exists $m_0 \geq 0$ such that $X \in \mathbb{X}(mn)$ if and only if m_0 divides m . Consequently, $\mathbb{X}(mn) \subseteq \mathbb{X}(m'n)$ whenever m divides m' .

Remark 4.3. Let $v \geq 0$ be the order of χ^n . If m is congruent to m' module v , then $\mathbb{X}'(m'n) = \mathbb{X}'(mn)$ and $\mathbb{X}'((m'+1)n) = \mathbb{X}'((m+1)n)$, and so

$$\mathrm{HH}_K^{2m+1}(A) = \mathrm{HH}_K^{2m'+1}(A) \quad \text{and} \quad \mathrm{HH}_K^{2m+2}(A) = \mathrm{HH}_K^{2m'+2}(A).$$

Hence, except for $\mathrm{HH}_K^0(A)$, the cohomology $\mathrm{HH}_K^*(A)$ is periodic of period $2v$. Moreover, if $n\lambda_n = 0$, then

$$\mathrm{HH}_K^{2v}(A) = \mathrm{HH}_K^0(A) \quad \text{and} \quad \mathrm{HH}_K^{2m+1}(A) \simeq \mathrm{HH}_K^{2m}(A),$$

for all $m \geq 0$. We now are going to study the cup product in $\mathrm{HH}_K^*(A)$. We already know that if p or q is even, then the multiplication map

$$\mathrm{HH}_K^p(A) \otimes_k \mathrm{HH}_K^q(A) \rightarrow \mathrm{HH}_K^{p+q}(A)$$

is induced by the multiplication in A . We assert that the multiplication of two odd degree elements is zero. In fact, since $\chi(g_1)$ has order n , the characteristic of k is relative prime to n . Consequently, by Remark 3.14, the assertion is true still when the characteristic of k is 2.

Remark 4.4. If the characteristic of k is relative prime to the order of G , then $k[G]$ is a separable k -algebra. Hence, by [G-S, Theorem 1.2], $\mathrm{HH}^*(A) = \mathrm{HH}_K^*(A)$. From this and the discussion above, the computations made out in [B-W, Section 2], follow immediately.

Remark 4.5. Let v be the order of χ^n . From Remark 4.3 it follows that $\mathrm{HH}_K^*(A)$ is generated as a k -algebra by

- $k[N]^G$ in degree 0,
- an arbitrary set of generators of $\mathrm{HH}_K^{2m+1}(A)$ as an $\mathrm{HH}_K^0(A)$ -module in degree $2m + 1$ for $0 \leq m < v$,
- an arbitrary set of generators of $\mathrm{HH}_K^{2m}(A)$ as an $\mathrm{HH}_K^0(A)$ -module in degree $2m$ for $0 < m < v$, and
- the class of 1 in degree $2p$.

If $n\lambda_n = 0$, then the situation is simpler. In this case $\mathrm{HH}_K^*(A)$ is generated by

- $k[N]^G$ in degree 0,
- x in degree 1,
- an arbitrary set of generators of $\text{HH}_K^{2m}(A)$ as an $\text{HH}_K^0(A)$ -module in degree $2m$ for $0 < m < v$, and
- the unit $1 \in k[N]^G$ in degree $2v$.

Consequently, if $\text{HH}_K^{2m}(A) = 0$ for $0 < m < v$ (for instance if $v = 1$), then the algebra $\text{HH}_K^*(A)$ is isomorphic to $k[N]^G \otimes_k k[y, x]/\langle x^2 \rangle$, where the degree of x is 1 and the degree of y is $2v$. From this it follows immediately that if the characteristic of k is relative prime to the order of G , then $\text{HH}^*(A)$ is isomorphic to $k[N]^G \otimes_k k[y, x]/\langle x^2 \rangle$. When $f = x^n$ this gives Theorem 3.4 of [B-W].

4.0.3. A concrete example

Let $G = \langle g, h: g^u = 1 = h^4, hg = g^{-1}h \rangle$. Clearly $G = \{g^j h^l: 0 \leq j < u, 0 \leq l < 4\}$. Let $k = \mathbb{C}$ and let $\chi: G \rightarrow \mathbb{C}^\times$ be the character defined by $\chi(g^j h^l) = i^l$. Consider $f = x^2$. The hypothesis of Theorem 3.3 are satisfied with $\tilde{\lambda} = h^2$. By Remark 4.3, we know that the cohomology of A is periodic of period 4, and, by a direct computation,

$$\begin{aligned} \text{HH}^0(A) &= \left\{ \sum_{j=0}^{u-1} \gamma_j g^j: \gamma_j = \gamma_{u-j} \text{ for all } j > 0 \right\}, \\ \text{HH}^1(A) &= \text{HH}^0(A)x, \\ \text{HH}^2(A) &= \left\{ \sum_{j=1}^{u-1} \gamma_j g^j: \gamma_j = -\gamma_{u-j} \text{ for all } j \right\}, \\ \text{HH}^3(A) &= \text{HH}^2(A)x. \end{aligned}$$

As a \mathbb{C} -algebra, $\text{HH}^*(A)$ is generated by $a = g + g^{-1}$ in degree zero, x in degree one, $b = g - g^{-1}$ in degree two, and $c = 1$ in degree four. The 4-periodicity is given by the multiplication by c . Notice that if r is even, then $\dim_k(\text{HH}^2(A)) < \dim_k(\text{HH}^0(A))$ and so $\text{HH}^*(A)$ is not isomorphic to $\text{HH}^0(A) \otimes_k V$ for any k -vector space V .

5. An application

Let k be a field, G a finite group whose order is relative prime to the characteristic of k and $\chi: G \rightarrow k^\times$ a character. Assume that there exists $g_1 \in \mathcal{Z}(G)$ such that $\chi(g_1)$ is a primitive n th root of 1. In particular n is coprime relative to the characteristic of k . In this section we compute the cohomology of $A = k[G][x, \alpha]/\langle x^n - \xi(g_1^n - 1) \rangle$ over k , where $\xi \in k^\times$ and $\alpha \in \text{Aut}(k[G])$ is defined by $\alpha(g) = \chi(g)g$. Recall that $\ker(\alpha - \text{id}) = k[N]$, where $N = \ker(\chi)$. We consider two different cases. The second one solves the problem posed in [B-W], mentioned in the introduction.

$\chi^n \neq \mathbf{1}$. Let $g \in G$ such that $\chi^n(g) \neq 1$. Since

$$g^{-1}(x^n - \xi(g_1^n - 1))g = \chi^n(g)x^n - \xi(g_1^n - 1),$$

we conclude that $\langle x^n - \xi(g_1^n - 1) \rangle = \langle x^n, g_1^n - 1 \rangle$. So, the algebra A is equal to $k[G/\langle g_1^n \rangle][x, \tilde{\alpha}]/\langle x^n \rangle$, where $\tilde{\alpha}$ is the automorphism induced by α . We consider now $K := k[G/\langle g_1^n \rangle]$ and $f = x^n$. These data satisfy the hypothesis of Theorem 4.1, with the character from $G/\langle g_1^n \rangle$ to k^\times induced by χ and the class of g_1 in $G/\langle g_1^n \rangle$ playing the role of g_1 . Moreover $K := k[G/\langle g_1^n \rangle]$ is separable over k and so, $\text{HH}^*(A) = \text{HH}_K^*(A)$. Thus,

$$\begin{aligned} \text{HH}^0(A) &= K^{G/\langle g_1^n \rangle}, \\ \text{HH}^{2m+1}(A) &= (k[N/\langle g_1^n \rangle] \cap K^{\tilde{\alpha}^{mn}})x, \\ \text{HH}^{2m+2}(A) &= k[N/\langle g_1^n \rangle] \cap K^{\tilde{\alpha}^{(m+1)n}}. \end{aligned}$$

The cup product of two homogeneous elements is zero if both of them have odd degree, and it is induced by the multiplication in K , otherwise.

$\chi^n = \mathbf{1}$. In this case $f = x^n - \xi(g_1^n - 1)$ satisfies the hypothesis required in the introduction (that is $\alpha(\xi(g_1^n - 1)) = \xi(g_1^n - 1)$ and $\xi(g_1^n - 1)\lambda = \alpha^n(\lambda)\xi(g_1^n - 1)$). Moreover $K = k[G]$ is separable over k and so, $\text{HH}^*(A) = \text{HH}_K^*(A)$. Since $\chi^n = \text{id}$, we have

$$K^{\alpha^{mn}} = \mathcal{Z}(K) \quad \text{and so} \quad K[N] \cap K^{\alpha^{mn}} = K[N]^G.$$

Hence, by Theorem 4.1,

$$\begin{aligned} \text{HH}^0(A) &= k[N]^G, \\ \text{HH}^{2m+1}(A) &= (k[N]^G \cap \text{Ann}(g_1^n - 1))x, \\ \text{HH}^{2m+2}(A) &= \frac{k[N]^G}{(g_1^n - 1)k[N]^G}, \end{aligned}$$

since $n\xi \in k^\times$. By Remark 4.3 the cup product of two homogeneous elements is zero if both of them have odd degree, and it is induced by the multiplication in K , otherwise. Finally let $\tilde{A} = k[G/\langle g_1^n \rangle][x, \tilde{\alpha}]/\langle x^n \rangle$, where $\tilde{\alpha}$ is the automorphism induced by α . Using the formulas obtained in Section 4, it is easy to see that $\text{HH}^m(A) = \text{HH}^m(\tilde{A})$, for all $m > 0$.

6. A final example

Here we consider an example in order to show that the cochain complex introduced in Theorem 3.1 can be used to perform explicit computations still when the hypothesis introduced in Section 3.2 are not satisfied.

Let \mathbb{H} be the skew field of the quaternions over \mathbb{R} . Recall that \mathbb{H} is the four dimensional real algebra with basis $\{1, i, j, k\}$, unit 1 and multiplication defined by $i^2 = j^2 = -1$ and $ij = -ji = k$. Let $\alpha : \mathbb{H} \rightarrow \mathbb{H}$ be the \mathbb{R} -algebra automorphism, defined by

$$\begin{aligned} \alpha(1) &= 1, & \alpha(i) &= \cos \theta i + \sin \theta j, \\ \alpha(k) &= k, & \alpha(j) &= -\sin \theta i + \cos \theta j, \end{aligned}$$

where $0 < \theta < 2\pi$. So α acts as the rotation of angle θ with axis k on the pure imaginary quaternions. Take a monic polynomial with coefficients in \mathbb{H}

$$f = x^n + \lambda_1 x^{n-1} + \dots + \lambda_n$$

and write

$$\lambda_u = \lambda_{u1} + \lambda_{ui}i + \lambda_{uj}j + \lambda_{uk}k \quad \text{with } \lambda_{u1}, \lambda_{ui}, \lambda_{uj}, \lambda_{uk} \in \mathbb{R}.$$

Next we ask for the conditions in order that f satisfies the hypothesis required in the introduction. That is

$$\alpha(\lambda_u) = \lambda_u \quad \text{and} \quad \lambda_u \lambda = \alpha^u(\lambda) \lambda_u \quad \text{for all } \lambda \in \mathbb{H}.$$

By definition

$$\alpha(\lambda_u) = \lambda_{u1} + (\lambda_{ui} \cos \theta - \lambda_{uj} \sin \theta)i + (\lambda_{ui} \sin \theta + \lambda_{uj} \cos \theta)j + \lambda_{uk}k.$$

Consequently,

$$\alpha(\lambda_u) = \lambda_u \quad \Leftrightarrow \quad \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \lambda_{ui} \\ \lambda_{uj} \end{pmatrix} = \begin{pmatrix} \lambda_{ui} \\ \lambda_{uj} \end{pmatrix}.$$

So,

$$\alpha(\lambda_u) = \lambda_u \quad \text{if and only if} \quad \lambda_{ui} = \lambda_{uj} = 0.$$

Then we assume that this condition is satisfied. Again by definition

$$\begin{aligned} \lambda_u 1 &= \alpha^u(1) \lambda_u, \\ \lambda_u k &= \alpha^u(k) (\lambda_{u1} + \lambda_{uk}k), \\ \lambda_u i &= (\lambda_{u1} + \lambda_{uk}k)i = \lambda_{u1}i + \lambda_{uk}j, \\ \alpha^u(i) \lambda_u &= (\cos(u\theta)i + \sin(u\theta)j) (\lambda_{u1} + \lambda_{uk}k) \\ &= (\cos(u\theta)\lambda_{u1} + \sin(u\theta)\lambda_{uk})i + (\sin(u\theta)\lambda_{u1} - \cos(u\theta)\lambda_{uk})j, \\ \lambda_u j &= (\lambda_{u1} + \lambda_{uk}k)j = \lambda_{u1}j - \lambda_{uk}i, \\ \alpha^u(j) \lambda_u &= (-\sin(u\theta)i + \cos(u\theta)j) (\lambda_{u1} + \lambda_{uk}k) \\ &= (-\sin(u\theta)\lambda_{u1} + \cos(u\theta)\lambda_{uk})i + (\cos(u\theta)\lambda_{u1} + \sin(u\theta)\lambda_{uk})j. \end{aligned}$$

Hence, $\alpha^u(\lambda) \lambda_u = \lambda_u \lambda$ for all $\lambda \in \mathbb{H}$ if and only if

$$\begin{pmatrix} \cos(u\theta) & \sin(u\theta) \\ -\sin(u\theta) & \cos(u\theta) \end{pmatrix} \begin{pmatrix} \lambda_{u1} \\ \lambda_{uk} \end{pmatrix} = \begin{pmatrix} \lambda_{u1} \\ -\lambda_{uk} \end{pmatrix}.$$

Write

$$\lambda_u = \lambda_{u1} + \lambda_{uk}k = \rho e^{k\beta} = \rho(\cos \beta + k \sin \beta)$$

with $\rho \geq 0$ and $0 \leq \beta < 2\pi$. The above matrix equality is equivalent to $\beta = -u\theta/2 \pmod{\pi}$. Thus the following assertions are equivalent

- $\alpha(\lambda_u) = \lambda_u$ and $\alpha^u(\lambda)\lambda_u = \lambda_u\lambda$,
- $\lambda_u = \rho e^{-ku\theta/2}$ with $\rho \in \mathbb{R}$.

Next we compute the complex $C_S(A)$ introduced in item (1) of Theorem 3.1.

6.1. Computation of A^{α^r}

Let $\gamma_0 + \gamma_1x + \dots + \gamma_{n-1}x^{n-1} \in A$. Since

$$\gamma_u x^u \lambda = \gamma_u \alpha^u(\lambda) x^u,$$

we have that

$$\gamma_0 + \gamma_1x + \dots + \gamma_{n-1}x^{n-1} \in A^{\alpha^r} \iff \gamma_u \lambda = \alpha^{r-u}(\lambda) \gamma_u$$

for all u and all $\lambda \in \mathbb{H}$. Write $\gamma_u = \gamma_{u1} + \gamma_{ui}i + \gamma_{uj}j + \gamma_{uk}k$. Since,

$$\gamma_u k = \gamma_{u1}k - \gamma_{ui}j + \gamma_{uj}i - \gamma_{uk} \quad \text{and} \quad k\gamma_u = \gamma_{u1}k + \gamma_{ui}j - \gamma_{uj}i - \gamma_{uk},$$

if $\gamma_u k = \alpha^{r-u}(k)\gamma_u = k\gamma_u$, then $\gamma_u = \gamma_{u1} + \gamma_{uk}k$. Arguing now for γ_u as above for λ_u , we obtain that

$$\gamma_u \lambda = \alpha^{r-u}(\lambda) \gamma_u \quad \text{for all } \lambda \in \mathbb{H},$$

if and only if

$$\gamma_{u1} + \gamma_{uk}k = \rho e^{k(u-r)\theta/2} = \rho \left(\cos\left(\frac{(u-r)\theta}{2}\right) + k \sin\left(\frac{(u-r)\theta}{2}\right) \right)$$

with $\rho \in \mathbb{R}$. So, the following conditions are equivalent

- $\gamma_0 + \gamma_1x + \dots + \gamma_{n-1}x^{n-1} \in A^{\alpha^r}$.
- $\gamma_u = \rho e^{k(u-r)\theta/2}$ with $\rho \in \mathbb{R}$.

6.2. Computation of the boundary maps

By the above computations we know that if $\gamma_n + \gamma_{n-1}x + \dots + \gamma_1x^{n-1} \in A^{\alpha^{mn}}$, then $\alpha(\gamma_u) = \gamma_u$. Hence,

$$d^{2m+1}(\gamma_n + \dots + \gamma_1x^{n-1}) = (\alpha(\gamma_n) - \gamma_n)x + \dots + (\alpha(\gamma_1) - \gamma_1)x^n = 0$$

and

$$d^{2m+2}(\gamma_n + \dots + \gamma_1x^{n-1}) = \left(\sum_{i=1}^n i \lambda_{n-i} x^{i-1} \right) (\gamma_n + \dots + \gamma_1x^{n-1}).$$

6.3. Computation of the cohomology

Let $C = \mathbb{R}[x]/\langle g \rangle$, where $g = x^n + \zeta_1 x^{n-1} + \dots + \zeta_n \in \mathbb{R}[x]$ is the polynomial defined by $\lambda_u = \zeta_u e^{-k \frac{u\theta}{2}}$. A direct computation, using the results obtained in Sections 6.1 and 6.2, shows that the map $\vartheta^* : C_S(C) \rightarrow C_S(A)$, defined by

$$\vartheta^{2m}(x^u) = e^{k(u-mn)\theta/2} x^u \quad \text{and} \quad \vartheta^{2m+1}(x^u) = e^{k(u-mn-1)\theta/2} x^u,$$

is an isomorphism of complexes. Since \mathbb{H} is an \mathbb{R} -algebra separable

$$\begin{aligned} \mathrm{HH}^0(A) &= \mathrm{HH}^0(C) = C, \\ \mathrm{HH}^{2m+1}(A) &= \mathrm{HH}^{2m+1}(C) = \{h \in C : g'h = 0\}, \\ \mathrm{HH}^{2m+2}(A) &= \mathrm{HH}^{2m+2}(C) = \frac{C}{\langle g' \rangle}. \end{aligned}$$

Moreover, from the formulas obtained in Theorem 3.2, it follows that ϑ^* induces an isomorphism of algebras from $\mathrm{HH}_{\mathbb{R}}^*(C)$ to $\mathrm{HH}_{\mathbb{R}}^*(A)$. So, the product of two homogeneous elements is induced by the multiplication map in A whenever at least one of them have even degree, and it is zero otherwise.

Acknowledgment

We thank Graciela Carboni for a careful reading of a preliminary version and useful remarks.

References

- [A-S] N. Andruskiewitsch, H.J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3 , *J. Algebra* 209 (1998) 658–691.
- [B] The Buenos Aires cyclic homology group, Cyclic homology of algebras with one generator, *K-Theory* 5 (1991) 51–69.
- [B-W] S.M. Burciu, S.J. Witherspoon, Hochschild cohomology of smash products and rank one Hopf algebras, *math.RA/0608762*, 2006.
- [G] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math. (2)* 78 (1963) 267–288.
- [G-S] M. Gerstenhaber, S.D. Schack, Relative Hochschild cohomology, rigid algebras, and the Bockstein, *J. Pure Appl. Algebra* 43 (1) (1986) 53–74.
- [K-R] L. Krop, D. Radford, Finite dimensional Hopf algebras of rank 1 in characteristic 0, *J. Algebra* 302 (1) (2006) 214–230.