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On a factorization of graded Hopf algebras using Lyndon words

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Abstract

We find a generalization of the restricted PBW basis for pointed Hopf algebras over abelian groups constructed by Kharchenko. We obtain a factorization of the Hilbert series for a wide class of graded Hopf algebras. These factors are parametrized by Lyndon words, and they are the Hilbert series of certain graded Hopf algebras.

Keywords: Graded Hopf algebra; Nichols algebra; Lyndon word

1. Introduction

Hopf algebras [Swe69] are far from being classified. Up to now there are two main directions of study: semisimple and pointed Hopf algebras. This paper is mainly a contribution to the latter, although all considerations are performed in a more general context. Specifically, we work with Hopf algebras H generated by a Hopf subalgebra H_0 and a vector space V satisfying properties (4.2) and (4.3) below. This includes in particular pointed Hopf algebras generated by

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group-like and skew-primitive elements. In Theorem 4.12 we prove a factorization result about the Hilbert series of $\operatorname{gr} H$. Moreover, with Theorem 4.18 we show that to each factor one can associate in a natural way a graded Hopf algebra which projects onto a Nichols algebra. These are the main results of the present paper.

Kharchenko [Kha99] proved that when H_0 is the group algebra of an abelian group and it acts on V by characters, H admits a restricted PBW basis. The PBW generators in this basis are labelled by Lyndon words on an alphabet given by a set of skew-primitive elements. Examples, where H_0 is the group algebra of a nonabelian group [MS00,Gra00b,AG03], indicate that in general one cannot expect Kharchenko's result to hold in its strong form. Nevertheless, by extending his ideas we were able to construct a basis of $\operatorname{gr} H$ using ordered products of subquotients of it. In the particular case of Kharchenko's setting one recovers the PBW basis. This is possible because graded braided Hopf algebras generated by one primitive element are easy to classify. The difficulty in the more general setting arises from the fact that the structure of graded braided Hopf algebras generated by an irreducible Yetter–Drinfel'd module over H_0 is not known.

A generalization of Kharchenko's PBW theorem in a different direction was done by Ufer [Ufe04]. Instead of character Hopf algebras (i.e., Hopf algebras of diagonal type) he considers braided Hopf algebras with "triangular" braidings. On the one hand Ufer is able to give a restricted PBW basis. On the other hand some information about the relations of the Hopf algebra is lost. Although we believe that it is possible to obtain a generalization of Ufer's approach to our context, we stick to a simpler setting for two reasons. First, valuable additional information can be obtained in our starting context. Second, the proofs in the triangular case would be even more technical, obscuring the essential arguments.

The proof of the main results of the present paper was possible due to taking advantage both from the lexicographic and the inverse lexicographic order on the set of monotonic super-words built from an alphabet of Lyndon words. This leads in a natural way to the construction of subquotients of a graded Hopf algebra. In this way Kharchenko's PBW theorem becomes more transparent. Note that Ufer's more technical proof stems from the fact that in his setting the inverse lexicographic order on the set of monotonic super-words cannot be used.

Kharchenko's PBW theorem turned out to be essential in the construction of the Weyl groupoid [Hec06] corresponding to a Nichols algebra of diagonal type. This groupoid played the crucial role in the classification of such Nichols algebras [Hec04]. In turn, the knowledge of these Nichols algebras is important for example for the lifting method of Andruskiewitsch and Schneider [AS98] to classify pointed Hopf algebras. We consider the factorization theorem in this paper to be an important step towards the generalization of the Weyl groupoid to a wider class of Nichols algebras.

In this paper $\mathbf{k}$ is an arbitrary field, and all algebras have $\mathbf{k}$ as their base field. The symbol $\otimes$ refers to tensor product over $\mathbf{k}$. We will write m, Δ, ε and S for the product, coproduct, counit, and the antipode of a Hopf algebra.

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2. Preliminaries

In this section we prove two general results about (braided) bialgebras, for which we did not find references in the literature.

Proposition 2.1. *Let $A = (A, \mathrm{m}, \Delta)$ be a bialgebra, $B \subseteq A$ a subalgebra and $I \subseteq B \cap \ker(\varepsilon)$ an ideal of B. Suppose furthermore that*

$$\Delta(B) \subseteq B \otimes B + I \otimes A, \qquad \Delta(I) \subseteq B \otimes I + I \otimes A. \tag{2.2}$$

Then the bialgebra structure on A induces a bialgebra structure on B/I.

Proof. Notice first that B/I is an algebra. Let us take $\bar{\Delta}: B \to A/I \otimes A/I$ as $\bar{\Delta} = (\pi \otimes \pi)\Delta i$, for $\pi : A \to A/I$ the projection and $i : B \to A$ the inclusion. By using the first formula in (2.2), one gets that $\bar{\Delta}(B) \subseteq B/I \otimes B/I$. By using the second formula in (2.2), one gets that $\bar{\Delta}(I) = 0$. Then $\bar{\Delta}$ induces a map $\tilde{\Delta}: B/I \to B/I \otimes B/I$. Also, ε induces a map $\tilde{\varepsilon}: B/I \to \mathbf{k}$. It is clear that $\tilde{\Delta}$ is coassociative and $\tilde{\varepsilon}$ is a counit for $\tilde{\Delta}$, whence B/I is a coalgebra. It is immediate that $\tilde{\varepsilon}$ is an algebra map. We must prove then that $\tilde{\Delta}$ is an algebra map. For $a, b \in B$ we compute

$$\begin{aligned}\tilde{\Delta}\big(\pi(a)\pi(b)\big) &= \tilde{\Delta}\big(\pi(ab)\big) = (\pi \otimes \pi)\Delta(ab) = (\pi \otimes \pi)(a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}) \\ &= \pi(a_{(1)}b_{(1)}) \otimes \pi(a_{(2)}b_{(2)}) = \pi(a_{(1)})\pi(b_{(1)}) \otimes \pi(a_{(2)}b_{(2)}) \\ &= \pi(a_{(1)})\pi(b_{(1)}) \otimes \pi(a_{(2)})\pi(b_{(2)}).\end{aligned}$$

In the first equality above we used that $\pi|_B : B \to B/I$ is an algebra map, and the fifth one is obtained from $a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \in B \otimes A$. The last equality holds by $\pi(a_{(1)})\pi(b_{(1)}) \otimes a_{(2)}b_{(2)} \in B/I \otimes B$. □

For the notion of braided Hopf algebras one may consult for example [Tak00].

Proposition 2.3. *Let $\pi : R \to T$ be a surjective map of braided $\mathbb{N}_0$-graded Hopf algebras* (*either in the sense of Takeuchi or in a Yetter–Drinfel'd category*) *which is an isomorphism in degree* 0, *and assume that R, T are finite-dimensional in each degree. Further, suppose that* $\ker\pi$ *is a categorical braided subspace of R, that is $c(\ker\pi \otimes R) \subset R \otimes \ker\pi$, $c(R \otimes \ker\pi) \subset \ker\pi \otimes R$. Then the quotient $\eta(R,t)/\eta(T,t)$ of Hilbert series is again a series with nonnegative integer coefficients.*

Proof. Let R_n and T_n, where $n \in \mathbb{N}_0$, denote the homogeneous subspaces of R and T, respectively, of degree n. Since π is a graded map and an isomorphism in degree 0, the ideal $\ker\pi$ is homogeneous and is contained in $\bigoplus_{n=1}^{\infty} R_n$. Moreover, π is an algebra map, and hence $(\ker\pi)^2 \subset \ker\pi$. Thus $(\ker\pi)^{n+1} \subset \ker\pi^n$ for all $n \in \mathbb{N}_0$, where $(\ker\pi)^0 := R$. Set

$$\begin{aligned}&R'(i) := (\ker\pi)^i/(\ker\pi)^{i+1}, \\ &R'_n(i) := \big((\ker\pi)^i \cap R_n\big)/\big((\ker\pi)^{i+1} \cap R_n\big), \quad \text{where } i, n \in \mathbb{N}_0.\end{aligned}$$

Since $(\ker\pi)^n \subset \bigoplus_{i=n}^{\infty} R_i$, one obtains that R and

$$R' := \bigoplus_{i=0}^{\infty} R'(i), \quad \text{where } R'(0) = (\ker\pi)^0/(\ker\pi)^1 = R/(\ker\pi) \simeq T, \tag{2.4}$$

are isomorphic as graded vector spaces. Here the grading of R' is induced by the one of R, that is

$$R'_n = \bigoplus_{i=0}^{n} R'_n(i).$$

The algebra structure of R induces two algebra gradings on R':

$$R'_m R'_n \subset R'_{m+n}, \quad R'(m)R'(n) \subset R'(m+n) \quad \text{for all } m, n \in \mathbb{N}_0.$$

Next we prove that R induces a braided Hopf algebra structure on R'. Let $(c_R, \varDelta_R, S_R)$ and $(c_T, \varDelta_T, S_T)$ denote the triples consisting of the braiding, the coproduct, and the antipode of R and T, respectively. Let $\rho_n : R \to R/(\ker\pi)^{n+1}$, where $n \in \mathbb{N}_0$, be the canonical projections. Define

$$\varDelta_{R',n} : R \to \bigoplus_{i=0}^{n} \rho_i(R) \otimes \rho_{n-i}(R)$$

by setting

$$\varDelta_{R',n} := \bigoplus_{i=0}^{n} (\rho_i \otimes \rho_{n-i})\varDelta_R.$$

Since $\ker\pi$ is a coideal of R, that is $\varDelta_R(\ker\pi) \subset R \otimes \ker\pi + \ker\pi \otimes R$, one gets

$$\varDelta_{R',n}\big((\ker\pi)^{n+1}\big) = 0,$$
$$\varDelta_{R',n}\big((\ker\pi)^{n}\big) \subset \bigoplus_{i=0}^{n} R'(i) \otimes R'(n-i).$$

Thus the family of maps $\varDelta_{R',n}$ induces a coproduct $\varDelta_{R'}$ on R' (the coassociativity and compatibility with the counit being obvious) via the definition $\varDelta_{R'}|_{R'(n)} := \varDelta_{R',n}$. We will show that $\varDelta_{R'}$ is an algebra homomorphism, but to do so we have to consider first the braiding on R'.

Recall that $\ker\pi$ is a coideal, and by assumption a categorical braided subspace of R. Hence by induction on m and n one gets

$$c_R\big((\ker\pi)^m \otimes (\ker\pi)^n\big) \subset (\ker\pi)^n \otimes (\ker\pi)^m, \quad m, n \in \mathbb{N}_0.$$

Having this, one shows, with the technique used in the construction of $\varDelta_{R'}$, that c_R induces a braiding $c_{R'}$ on R' with the property

$$c_{R'}\big(R'(m) \otimes R'(n)\big) \subset R'(n) \otimes R'(m), \quad m, n \in \mathbb{N}_0.$$

Now we conclude that $\varDelta_{R'}$ is an algebra map. Indeed, for $x \in (\ker\pi)^m$, $y \in (\ker\pi)^n$ one gets

$$
\begin{aligned}
\Delta_{R'}\big(\rho_m(x)\rho_n(y)\big) &= \Delta_{R',m+n}(xy)\\
&= \sum_{i=0}^{m+n} (\rho_i \otimes \rho_{m+n-i})(\mathrm{m}\otimes\mathrm{m})(\mathrm{id}\otimes c_R \otimes \mathrm{id})(x_{(1)}\otimes x_{(2)}\otimes y_{(1)}\otimes y_{(2)})\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} (\mathrm{m}\otimes\mathrm{m})(\mathrm{id}\otimes c_{R'} \otimes \mathrm{id})\big(\rho_i(x_{(1)})\otimes \rho_{m-i}(x_{(2)})\otimes \rho_j(y_{(1)})\otimes \rho_{n-j}(y_{(2)})\big)\\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} \big(\rho_i(x_{(1)})\otimes \rho_{m-i}(x_{(2)})\big)\big(\rho_j(y_{(1)})\otimes \rho_{n-j}(y_{(2)})\big)\\
&= \Delta_{R'}\big(\rho_m(x)\big)\Delta_{R'}\big(\rho_n(y)\big).
\end{aligned}
$$

Analogously, S_R induces an antipode $S_{R'}$ of R'. This proves that R' is a graded braided Hopf algebra.

The fundamental theorem for Hopf modules [Mon93, 1.9.4] implies that a Hopf algebra which admits a projection onto a Hopf subalgebra is isomorphic (via the multiplication map) to the tensor product of the right coinvariants and the Hopf subalgebra. The braided version of this statement holds in our setting: $\mathrm{m}: R'^{\,\mathrm{co}\,T} \otimes T \to R'$ is an isomorphism of graded vector spaces (even as right T-module comodules), where T has its own grading and R' has the grading induced by the one of R. Indeed, let π_0 denote the projection of R' to $T = R'(0)$ which maps $R'(m)$ to 0 for all $m \geqslant 1$. Then the map $(\mathrm{id}\otimes\pi_0)\Delta_{R'}$ is a right coaction of T on R', and the map $x \mapsto \mathrm{m}(\mathrm{id}\otimes S_{R'} \circ \pi_0)\Delta_{R'}(x)$ is a surjective map from R' to $R'^{\,\mathrm{co}\,T}$ (because its restriction to $R'^{\,\mathrm{co}\,T}$ is the identity). Further, $R' \simeq R'^{\,\mathrm{co}\,T} \otimes T$ as graded vector spaces, which implies the statement. □

Remark 2.5. Suppose that $\pi : R \to T$ in the preceding proposition is a surjective map of braided $\mathbb{N}_0$-graded Hopf algebras in a Yetter–Drinfel'd category ${}^{H_0}_{H_0}\mathcal{YD}$, where H_0 is a Hopf algebra with bijective antipode, and both R and T are equipped with the canonical braiding (or both with the inverse of the canonical braiding). Then $\ker\pi$ is an object in the Yetter–Drinfel'd category, too, and the inclusion is a map in the category. Hence, $\ker\pi$ is automatically a categorical braided subspace of R.

Remark 2.6. The subspace $R'^{\,\mathrm{co}\,T}$ is a subalgebra of R'. Indeed, this follows from the right coaction of T on R' being an algebra map, which, in turn, follows from $\ker\pi$ being categorical:

$$(\mathrm{id}\otimes\pi_0)\Delta_{R'}(xy) = (\mathrm{id}\otimes\pi_0)\big(\Delta_{R'}(x)\Delta_{R'}(y)\big) = (\mathrm{id}\otimes\pi_0)\Delta_{R'}(x)\cdot(\mathrm{id}\otimes\pi_0)\Delta_{R'}(y)$$

for all $x, y \in R'$.

Remark 2.7. Assume that R and T are graded braided Hopf algebras in a Yetter–Drinfel'd category ${}^{H_0}_{H_0}\mathcal{YD}$, where H_0 is a Hopf algebra with bijective antipode and the assumptions of Proposition 2.3 are fulfilled. Then the algebras R', defined in Eq. (2.4), and R are isomorphic as graded vector spaces, but not necessarily as braided vector spaces, at least if R is not a semisimple Yetter–Drinfel'd module over H_0.

The assumption that $\ker\pi$ is a categorical subspace of R is necessary for the definition of the braided Hopf algebra structure on R'. We want to thank the referee to pointing out this fact. For illustration we give an example.

Example 2.8. Assume that $V = k\{x_0, x_1, x_2\}$ is the braided vector space with braiding c, where

$$c(x_i \otimes x_j) = -x_{2i-j \bmod 3} \otimes x_i, \quad i, j = 0, 1, 2.$$

Let R be the associated Nichols algebra $\mathcal{B}(V)$ [MS00, Example 6.4]. This is a braided Hopf algebra with primitive generators x_0, x_1, x_2, and as an algebra one has

$$R \cong k\langle x_0, x_1, x_2\rangle / \big(x_0^2, x_1^2, x_2^2, x_0x_1 + x_1x_2 + x_2x_0, x_1x_0 + x_2x_1 + x_0x_2\big).$$

Let $T = k[x]/(x^2)$ be the braided Hopf algebra with primitive generator x and braiding c, where $c(x \otimes x) = -x \otimes x$. Then there exists a unique algebra map $\pi : R \to T$ such that

$$\pi(x_0) = x, \qquad \pi(x_1) = 0, \qquad \pi(x_2) = 0.$$

Moreover, π is a surjective map of braided $\mathbb{N}_0$-graded Hopf algebras. Then $\ker\pi$ is the ideal (x_1, x_2), but it is not categorical because of the relations

$$c(R \otimes \ker\pi) \ni c(x_1 \otimes x_2) = -x_0 \otimes x_1 \notin \ker\pi \otimes R.$$

In this case c does not induce a braiding on the algebra R' defined via Eq. (2.4), and R' does not become a braided Hopf algebra. Nevertheless for this particular example the quotient $\eta(R,t)/\eta(T,t)$ of Hilbert series is still a series with nonnegative integer coefficients, see for example [Gra00a, Theorem 3.8], [MS00, Corollary 3.3]. The proofs of the latter statements use a (noncategorical) section of the map $R \to T$ instead of requiring that $\ker\pi$ is categorical.

Following the suggestions of the referee we add two remarks.

Remark 2.9. Suppose that R and T are graded connected cocommutative Hopf algebras over a field of characteristic zero. By the Cartier–Kostant–Milnor–Moore theorem one knows that $R = U(L_R)$ and $T = U(L_T)$ are (isomorphic to) the universal enveloping algebras of the (graded) Lie algebras L_R and L_T of primitive elements in R and T, respectively. The PBW theorem implies that

$$\ker\pi = L_0U(L_R) = U(L_R)L_0,$$

where $L_0 = L_R \cap \ker\pi$ is an ideal of the Lie algebra L_R. In this case one has by definition $R'(n) \simeq (L_0^n U(L_0)/L_0^{n+1}U(L_0)) \otimes U(L_R/L_0)$, where L_0^n has to be interpreted as a subspace of $U(L_0) \subset U(L_R)$. Note that $\bigcap_{n\in\mathbb{N}_0} L_0^n U(L_0) \subset \bigcap_{n\in\mathbb{N}_0}(\ker\pi) = \{0\}$, and hence the freeness of R' over $R'(0)$ is equivalent to the isomorphism $U(L_R) \simeq U(L_0) \otimes U(L_R/L_0)$.

Remark 2.10. There are two natural universal graded braided Hopf algebras associated to a Hopf algebra H_0 with bijective antipode and a Yetter–Drinfel'd module $V \in {}^{H_0}_{H_0}\mathcal{YD}$. These are the tensor Hopf algebra TV and the cotensor Hopf algebra T^cV [Nic78]. On the one hand,

$\mathcal{B}(V)$ is a quotient of TV containing V, and hence generally R' is not isomorphic to TV as a graded algebra, because R' is not generated by V. On the other hand, $\mathcal{B}(V)$ is a braided Hopf subalgebra of both R' and T^cV. However R' is not isomorphic to T^cV as a graded coalgebra. Indeed, R' may contain nonzero primitive elements of degree at least 2, while T^cV does not contain such elements. Therefore the method given in Proposition 2.3 gives a construction of graded (braided) Hopf algebras which are neither tensor nor cotensor Hopf algebras.

3. Lyndon words

Let $A = \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{d}\}$ be a totally ordered set by $\mathbf{1} < \mathbf{2} < \cdots < \mathbf{d}$. We think of A as an *alphabet* and $\mathbf{1}, \dots, \mathbf{d}$ as the *letters* of A. Let $\mathbb{A}$ be the set of nonempty words in this alphabet, and let $\emptyset$ denote the empty word. For a word $u = a_1a_2\cdots a_r$ with $a_i \in A$, $1 \leqslant i \leqslant r$, we say that r is the *length* of u and we write $r = |u|$. We consider on $\mathbb{A}$ the lexicographic order $<$. This means that $u < v$ if and only if $v = uu'$ for some $u' \in \mathbb{A}$ or $u = w\mathbf{i}u'$ and $v = w\mathbf{j}v'$, where $\mathbf{i} < \mathbf{j}$ and $w, u', v' \in \{\emptyset\} \cup \mathbb{A}$.

A word $u \in \mathbb{A}$ is called a *Lyndon word* if $u = u_1u_2$ with $u_1, u_2 \in \mathbb{A}$ implies that $u < u_2$. For example: letters are Lyndon words, $\mathbf{ij}$ is a Lyndon word for $\mathbf{i} < \mathbf{j}$, $\mathbf{12122}$ is a Lyndon word, and $\mathbf{1212}$ is not a Lyndon word. We write $L = \{u \in \mathbb{A} \mid u \text{ is a Lyndon word}\}$.

Proposition 3.1. *(See [Lot83], Proposition 5.1.3.) A word u is a Lyndon word if and only if $u \in A$ or $u = vw$ with $v, w \in L$ and $v < w$. More precisely, if w is the proper right factor of maximal length of $u = vw \in L$ that belongs to L, then also $v \in L$ and $v < vw < w$.*

If $u \in L$, $|u| \geqslant 2$, then the decomposition $u = vw$ in Proposition 3.1 with w of maximal length is called the *Shirshov decomposition* of u. We write $\text{Ш}u = (v, w)$.

Lemma 3.2. *Let $u, v \in L$, $u < v$, $|u| \geqslant 2$, and let $\text{Ш}u = (u_1, u_2)$. Then exactly one of the following possibilities occurs*:

(1) $\text{Ш}uv = (u, v)$ *and* $u_2 \geqslant v$.
(2) $\text{Ш}uv = (u_1', u_1''u_2v)$ *for some words* u_1', u_1'' *such that* $u_1 = u_1'u_1''$ *and* $u_2 < v$ (*here* u_1'' *may be empty*).

Proof. This is equivalent to [Lot83, Proposition 5.1.4]. □

Lemma 3.3. *(See [Kha99], Lemma 4.) Let $u, v \in L$, $u = u_1u_2$ for $u_1, u_2 \in \mathbb{A}$ and suppose that $u_2 < v$. Then $uv < u_1v$.*

We take on L the lexicographic order. Thus, L is a new alphabet containing the original alphabet A, and following Kharchenko [Kha99] we say that the elements of L are *super-letters*. Words in super-letters are called *super-words*. The *length* $|w|$ of a super-word w is the sum of the lengths of its super-letters. A *monotonic super-word* is a non-increasing word on the set of super-letters, i.e., a (possibly empty) word $v_1\cdots v_n$ such that $v_i \in L$ and $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n$. Let M denote the set of monotonic super-words. In what follows the notation $v_1\cdots v_n \in M$ will mean that $v_1, \dots, v_n \in L$ and $v_1 \geqslant \cdots \geqslant v_n$. Sometimes we also write $v_1^{m_1}\cdots v_n^{m_n} \in M$, in which case we mean that $v_1, \dots, v_n \in L$, $v_1 > \cdots > v_n$, and $m_1, \dots, m_n \geqslant 1$. Monotonic super-words are lexicographically ordered on the alphabet of super-letters. Notice that the empty super-word is the smallest super-word. For a super-letter u, we shall write

$$\begin{aligned} L_{>u} &= \{v \in L \mid v > u\}, \qquad L_{\geqslant u} = \{v \in L \mid v \geqslant u\}, \\ M_{>u} &= \{v_1 \cdots v_r \in M \mid r \geqslant 1, v_1 \geqslant \cdots \geqslant v_r > u\}, \\ M_{\geqslant u} &= \{v_1 \cdots v_r \in M \mid r \geqslant 1, v_1 \geqslant \cdots \geqslant v_r \geqslant u\}. \end{aligned} \tag{3.4}$$

Theorem 3.5. *(Lyndon, see [Lot83, Theorem 5.1.5 and Proposition 5.1.6].) A word in $\mathbb{A}$ can be written in a unique way as a monotonic super-word. Moreover, if $u = v_1 \cdots v_n \in M$ then v_n is the smallest right factor of u (smallest with respect to lexicographic order in $\mathbb{A}$).*

As an example, the word **1231233122123** is decomposed as a monotonic super-word as

$$\mathbf{1231233122123} = (\mathbf{1231233})(\mathbf{122123}),$$

and in turn, $\text{Ш}\mathbf{1231233} = (\mathbf{123}, \mathbf{1233})$.

Lemma 3.6. *Let $w = w_1 \cdots w_n$ be a super-word with $w_i \in L$ for $1 \leqslant i \leqslant n$. Then the decomposition of w as a monotonic super-word, $w = v_1 \cdots v_m$, satisfies the relation $v_1 \cdots v_m \geqslant w_1 \cdots w_n$ with respect to the lexicographic order on super-words.*

Proof. We proceed by induction on the number of super-letters of w. If w is a super-letter then we are done. Otherwise $w = w_1 \cdots w_n$, where $n \geqslant 2$ and $w_i \in L$ $\forall i$. Again, if $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$ then we are done. On the other hand, if $w_i < w_{i+1}$ for some i, then $w_i' := w_i w_{i+1} \in L$ by Proposition 3.1, and one has $w = w_1 \cdots w_{i-1} w_i' w_{i+2} \cdots w_n$ with $w_i' > w_i$. Hence the claim follows from the induction hypothesis. □

Lemma 3.7. *(See [Kha99], Lemma 5.) Let $w = w_1 \cdots w_m$ and $v = v_1 \cdots v_n$ be monotonic super-words. Then $w < v$ (with respect to the lexicographic order on M) if and only if $w_1 \cdots w_m < v_1 \cdots v_n$ with respect to the lexicographic order on $\mathbb{A}$.*

The following technical lemma will be needed in the proof of Theorem 4.12.

Lemma 3.8. *Let $w = w_1 \cdots w_m$ and $v = v_1 \cdots v_n$ be nonempty monotonic super-words with $w \geqslant v$ and assume that $v_1 = \cdots = v_r > v_{r+1}$, where $r \leqslant n$. For all $i \leqslant m$ let $(w_i', w_i'') \in M \times M$ such that either $w_i' \geqslant w_i$ or $w_i' = \emptyset$, $w_i'' = w_i$. Then the pair $(w_1' \cdots w_m', w_1'' \cdots w_m'')$ satisfies one of the following relations.*

(1) $w_1' \cdots w_m' > v_{r+1} \cdots v_n$,
(2) $w_1'' \cdots w_m'' > v_1^r$,
(3) $w = v$ *and* $(w_1' \cdots w_m', w_1'' \cdots w_m'') = (v_{r+1} \cdots v_n, v_1^r)$.

Proof. Assume first that $w_1 > v_1$. Then either $w_1' > v_1 > v_{r+1} \cdots v_n$ or $w_1' = \emptyset$, $w_1'' = w_1 > v_1$. In the first case we have relation (1) and in the second one relation (2) is fulfilled. On the other hand, if $w_1 \leqslant v_1$ then because of $w \geqslant v$ and w is monotonic, one has $m \geqslant r$ and $w_i = v_1$ for all $i \leqslant r$. Suppose first that there exists $i \leqslant r$ such that $w_i' \neq \emptyset$ and $w_j' = \emptyset$ for all $j < i$. In this case

$w'_i \geqslant v_1 > v_{r+1} \cdots v_n$ and hence (1) holds. It remains to consider the case $w'_i = \emptyset$ for all $i \leqslant r$. Then one has $w''_1 \cdots w''_r = v^r_1$. Therefore, if $w''_i \neq \emptyset$ for some $i > r$ then again relation (2) holds. Otherwise $w'_i \geqslant w_i$ for all $i > r$. Then one has either $w'_1 \cdots w'_m = w'_{r+1} \cdots w'_m > v_{r+1} \cdots v_n$, in which case (1) holds, or $m = n$, $w = v$, and $w'_i = w_i$ for all $i > r$. The latter relations imply (3). □

Let H_0 be a Hopf algebra with bijective antipode and let $V = \bigoplus_{\mathbf{i}=\mathbf{1}}^{\mathbf{d}} V_{\mathbf{i}}$ be a direct sum of Yetter–Drinfel'd modules over H_0. Let TV be the tensor algebra of V. For simplicity, we will omit the $\otimes$ symbol in products of elements of TV. Let . and δ denote the left action and the left coaction of H_0 on TV, respectively. We will use Sweedler notation $\delta(x) = x_{(-1)} \otimes x_{(0)}$ for $x \in TV$. The algebra TV has a braiding $c : TV \otimes TV \to TV \otimes TV$. Note that one has

$$c(x \otimes y) = x_{(-1)}.y \otimes x_{(0)}, \qquad c^{-1}(x \otimes y) = y_{(0)} \otimes S^{-1}(y_{(-1)}).x$$

for all $x, y \in TV$. In particular, equations

$$c(V_{\mathbf{i}} \otimes V_{\mathbf{j}}) = V_{\mathbf{j}} \otimes V_{\mathbf{i}}, \quad \mathbf{i}, \mathbf{j} \in \{\mathbf{1}, \ldots, \mathbf{d}\}$$

hold. We define

$$[x, y] = xy - \mathrm{m}c^{-1}(x \otimes y), \qquad [\![x, y]\!] = xy - \mathrm{m}c(x \otimes y).$$

where m is the multiplication in TV.

Definition 3.9. Let $a_1, \ldots, a_m \in A$ and $u = a_1 \cdots a_m \in \mathbb{A}$. We write $V^u = V_{a_1} V_{a_2} \cdots V_{a_m} \simeq V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_m}$. The elements in V^u will be called *u-vectors*. If $x_u \in V^u$ and $u = vw$, then we write $x_u = x_v \otimes x_w \in V^v \otimes V^w$ (which is in general a sum of tensors) using the canonical isomorphism $V^u \simeq V^v \otimes V^w$.

We shall inductively define bracket operations $[\,] : \bigoplus_{n \geqslant 0} V^{\otimes n} \to TV$ and $[\![\,]\!] : \bigoplus_{n \geqslant 0} V^{\otimes n} \to TV$ as follows. Let x_u be a u-vector.

(1) If u has length 0 or 1, then $[x_u] = [\![x_u]\!] = x_u$.
(2) If the word u is a Lyndon word and $Шu = (v, w)$, then $[x_u] = [[x_v], [x_w]]$ and $[\![x_u]\!] = [\![[\![x_v]\!], [\![x_w]\!]]\!]$ (see Definition 3.9).
(3) If the word u is decomposed as a monotonic super-word by $u = v_1 \cdots v_r$, then $[x] = [x_{v_1}] \cdot [x_{v_2}] \cdots [x_{v_r}]$ and $[\![x]\!] = [\![x_{v_1}]\!] \cdot [\![x_{v_2}]\!] \cdots [\![x_{v_r}]\!]$.

Remark 3.10. Recall that the braided antipode S_{TV} of the braided Hopf algebra TV satisfies $S_{TV}(x) = x_{(-1)} S(x_{(0)})$, where S is the antipode of $TV\#H_0$. Moreover, $S_{TV}(xy) = \mathrm{m}c(S_{TV}(x) \otimes S_{TV}(y)) = (x_{(-1)}.S_{TV}(y)) S_{TV}(x_{(0)})$. With these formulas it is easy to see that for any $u \in L$ and any u-vector x one has $S_{TV}([x]) = (-1)^{|u|-1} [\![x]\!]$. Therefore most of the following considerations can be performed without difficulties with $[\![\,]\!]$'s instead of $[\,]$'s. Even if $[\![\,]\!]$'s seem to be more natural, we will follow the tradition of Kharchenko [Kha99] with $[\,]$'s.

Definition 3.11. If u is a Lyndon word and x is a u-vector, then $[x]$ will be called *a u-$[\,]$-letter*. If u is a monotonic super-word and x is a u-vector then $[x]$ will be called *a u-$[\,]$-word*. Let

$V^{[u]} = [V^u]$ denote the space of u-[ ]-words. A [ ]-letter ([ ]-word) is a u-[ ]-letter (u-[ ]-word) for some super-letter (monotonic super-word) u.

Lemma 3.12. *Let u be a monotonic super-word. For $x, y \in TV$, let $c^{-1}(x \otimes y) =: \sum {}^x y \otimes x^y$. Then*

(1) *if x is a u-vector and $h \in H_0$ then one has $[h.x] = h.[x]$ and $[x]_{(-1)} \otimes [x]_{(0)} = x_{(-1)} \otimes [x_{(0)}]$;*
(2) *if x is a u-vector and y is a v-vector for a monotonic super-word v, then ${}^{[x]}y \otimes [x]^y = {}^x y \otimes [x^y]$ and ${}^x[y] \otimes x^{[y]} = [{}^x y] \otimes x^y$;*
(3) *for all $x, y, z \in TV$,*

$$[x, yz] = [x, y]z + \sum {}^x y\big[x^y, z\big];$$

(4) *for all $x, y, z \in TV$,*

$$\big[[x, y], z\big] - \big[x, [y, z]\big] = \sum \big[x, {}^y z\big]y^z - \sum {}^x y\big[x^y, z\big].$$

Proof. (1) follows from the fact that the brading is a map of Yetter–Drinfel'd modules. (2) follows from (1) and the braid relation. (3) and (4) are straightforward calculations using the definition of [ ] and the braid relation for c^{-1}. □

We prove now a variant of [Kha99, Lemma 6].

Lemma 3.13. *Let X, Y be a u-[ ]-letter and a v-[ ]-letter respectively, where $u, v \in L$ and $u < v$. Then $[X, Y]$ is a homogeneous linear combination of products of [ ]-letters corresponding to super-letters in $L_{\geqslant uv}$.*

Proof. Let $z = uv$. If $Ш z = (u, v)$, then $[X, Y]$ is a z-[ ]-letter and we are done. We proceed by induction on $|uv|$, since we already know the lemma for the case $|u| = |v| = 1$.

Let $m = |uv|$ and suppose that the lemma holds for $u', v' \in L$, $u' < v'$, $X \in V^{[u']}$, $Y \in V^{[v']}$, where either $|u'v'| < m$ or $|u'v'| = m$, $u' < u$ (notice that there are only finitely many Lyndon words of a given length). As noted above, we may suppose that $Ш uv \neq (u, v)$. Then, if $Ш u = (u_1, u_2)$, we must have $u_2 < v$ because of Lemma 3.2. Let $X = [x]$ for $x \in V^u$, $Y = [y]$ for $y \in V^v$, and let $x = x_u = x_{u_1} \otimes x_{u_2}$, $X_{u_1} \otimes X_{u_2} = [x_{u_1}] \otimes [x_{u_2}]$. Thanks to Lemma 3.12(4), (2), we have

$$\begin{aligned}[X, Y] &= \big[[X_{u_1}, X_{u_2}], Y\big] \\ &= \big[X_{u_1}, [X_{u_2}, Y]\big] + \sum \big[X_{u_1}, {}^{X_{u_2}}Y\big](X_{u_2})^Y - \sum {}^{X_{u_1}}X_{u_2}\big[X_{u_1}^{X_{u_2}}, Y\big] \\ &= \big[X_{u_1}, [X_{u_2}, Y]\big] + \sum \big[X_{u_1}, \big[{}^{x_{u_2}}y\big]\big] \cdot \big[x_{u_2}^y\big] - \sum \big[{}^{x_{u_1}}x_{u_2}\big] \cdot \big[\big[x_{u_1}^{x_{u_2}}\big], Y\big]. \quad (3.14)\end{aligned}$$

We start by considering the first summand on the right-hand side. By the induction hypothesis, $[X_{u_2}, Y]$ is a homogeneous linear combination of products of [ ]-letters corresponding to super-letters in $L_{\geqslant u_2 v}$, and the degree of these products is $|u_2 v|$. By Lemma 3.12(3), and since $u_1 < u_2 < u_2 v$, the element $[X_{u_1}, [X_{u_2}, Y]]$ is a combination of products of [ ]-letters corresponding to super-letters in $L_{\geqslant u_2 v} \subset L_{>uv}$ on one hand, and a bracket $[h.X_{u_1}, X']$ on the other hand, where

$h \in H_0$ and X' is a w-[ ]-letter with $w \geqslant u_2 v$. By induction hypothesis again, the latter is a linear combination of products of [ ]-letters corresponding to super-letters in $L_{\geqslant u_1 u_2 v} = L_{\geqslant uv}$.

We continue with the second and third summands in (3.14). Concerning the u_2-[ ]-letters appearing there, notice that $u_2 > uv$. Indeed, $u_2 > u_1 u_2$ and, since $u_1 u_2$ is not the beginning of u_2, then we still have $u_2 > u_1 u_2 v = uv$. On the other factors, which are brackets between u_1-[ ]-letters and v-[ ]-letters, we can apply the induction hypothesis since $|u_1 v| < |uv|$ and $u_1 < u_1 u_2 < u_2 < v$. These factors are then linear combinations of products of [ ]-letters corresponding to super-letters in $L_{\geqslant u_1 v}$, and $u_1 v > uv$ by Lemma 3.3. □

Lemma 3.15. *Let $u \in L$. Any product of* [ ]*-letters corresponding to super-letters in $L_{\geqslant u}$ is a linear combination of* (*monotonic*) [ ]*-words corresponding to super-words in $M_{\geqslant u}$.*

Proof. Let $t = v_1 \cdot v_2 \cdots v_n$ be a super-word, where $v_i \in L_{\geqslant u}$ $\forall i$. Let $x = x_t = x_{v_1} \cdots x_{v_n}$ be a t-vector, where here t is considered as a word on A. We call $x_{[t]} := [x_{v_1}] \cdots [x_{v_n}]$ *a t-*[ ]*-vector* (notice that the term "t-[ ]-word" is reserved for when t is monotonic). We proceed by induction. We take on the set of super-words the lexicographic order $\prec$ given by the order $<$ on L. Suppose that the result is true for w-[ ]-vectors when $|w| < |t|$ or when $|w| = |t|$ and $w \succ t$. If $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n$, there is nothing to prove. Otherwise, let i be such that $1 \leqslant i < n$ and $v_i < v_{i+1}$. Let $t' = v_1 \cdots v_{i+1} v_i \cdots v_n$ and $t'' = v_1 \cdots (v_i v_{i+1}) \cdots v_n$, where the factor $(v_i v_{i+1})$ stands for the Lyndon word $v_i v_{i+1}$. Let

$$x_{[t']} := [x_{v_1}] \otimes \cdots \otimes c^{-1}\big([x_{v_i}] \otimes [x_{v_{i+1}}]\big) \otimes \cdots \otimes [x_{v_n}],$$

$$x_{[t'']} := [x_{v_1}] \otimes \cdots \otimes \big[[x_{v_i}], [x_{v_{i+1}}]\big] \otimes \cdots \otimes [x_{v_n}].$$

Notice that $x_{[t']}$ is a t'-[ ]-vector where t' is a super-word in the same super-letters as t, and $t' \succ t$. Also, by Lemma 3.13, and since $v_i v_{i+1} > v_i$, $x_{[t'']}$ is a linear combination of w-[ ]-vectors, where w runs on super-words $\succ t$ having only super-letters in $L_{\geqslant u}$. The last thing to notice is that $x_{[t]} = x_{[t']} + x_{[t'']}$. Therefore, the induction hypothesis implies the claim. □

Definition 3.16. Let $x \in TV \setminus \{0\}$, $x = \sum_n x_n$, where $x_n \in V^{\otimes n}$. The greatest n such that $x_n \neq 0$ will be called the *degree* of x. Let m be the degree of x and let $x_m = \sum_{u \in \mathbb{A}_m} x_u$, where $\mathbb{A}_m$ is the set of words of length m and x_u is a u-vector for all u. Let v be the least u (with respect to the order in $\mathbb{A}_m$) such that $x_u \neq 0$. Then x_v will be called *the leading vector* of x.

Lemma 3.17. *Let $x \in TV$ be a nonzero u-vector for a monotonic super-word u. Then x is the leading vector of* $[x]$.

Proof. We prove first the lemma for Lyndon words u by induction on $|u|$. If $|u| = 1$, the result is clear. If $\text{Ш}u = (v, w)$, then

$$[x] = \big[[x_v], [x_w]\big] = \big(\mathrm{m} - \mathrm{m}c^{-1}\big)\big([x_v] \otimes [x_w]\big).$$

By the induction hypothesis, $[x_v][x_w]$ is a sum of products of v'-vectors and w'-vectors, where v' runs on words $\geqslant v$ and w' runs on words $\geqslant w$. For such v', w' we have $u = vw \leqslant v'w'$, whence $[x_v][x_w]$ is a sum of u'-vectors with $u' \geqslant u$. Furthermore, the equality holds if and only if $v' = v$ and $w' = w$. Thus, by using induction hypothesis, the leading vector of $[x_v][x_w]$ is $x_v x_w$. For the term $\mathrm{m}c^{-1}([x_v]\otimes[x_w])$ we reason similarly: as u is a Lyndon word, $u=vw<w<wv$, from where $\mathrm{m}c^{-1}([x_v]\otimes[x_w])$ is a sum of u'-vectors where u' runs on words $\geqslant wv>u$. Therefore, such u'-vectors do not contribute to the leading vector of $[x_v x_w]$.

If u is not a super-letter, then by Theorem 3.5 we have $u=v_1\cdots v_n\in M$. By definition $[x]=[x_{v_1}]\cdots[x_{v_n}]$, which, by the previous step, is a sum of $v'_1\cdots v'_n$-vectors with $v'_i\geqslant v_i$ $\forall i$, and its leading vector is $x_{v_1}\cdots x_{v_n}=x$. □

Recall that M is the set of monotonic super-words.

Corollary 3.18. *One has* $TV=\bigoplus_{u\in M}V^{[u]}$.

Proof. Since letters in A are also super-letters, the spaces $V^{[u]}$ generate TV thanks to Lemma 3.15. The linear independence of these spaces follows immediately from Lemma 3.17. □

Corollary 3.19. *Let $u\in\mathbb{A}$ and x be a u-vector. Then $x-[x]$ is a linear combination of w-[ ]-words with $w>u$.*

Proof. This assertion can be proven with the help of bases of $\bigoplus_{|v|=|u|}V^v$ and $\bigoplus_{|v|=|u|}V^{[v]}$ which are obtained from each other using triangular matrices. Alternatively, Lemmas 3.7 and 3.17 imply that $x-[x]$ is a linear combination of w-vectors with $w>u$, and proceed by induction. □

Corollary 3.19 allows us to give a description of products of [ ]-letters, which is different from the one in Lemma 3.15.

Corollary 3.20. *Let $n\in\mathbb{N}$, $v_1,\dots,v_n\in L$ and let X_i be a v_i-[ ]-letter for all $i\in\{1,\dots,n\}$. Then $X_1\cdots X_n$ is a linear combination of w-[ ]-words, where w runs over monotonic super-words $\geqslant v'_1\cdots v'_{n'}$, and $v'_1\cdots v'_{n'}$ is the decomposition of $v_1\cdots v_n$ as a monotonic super-word.*

Definition 3.21. Let K be a totally ordered set and let there be vector spaces W_k for each $k\in K$. We define $\bigotimes^{>}_{k\in K}W_k$ to be the direct sum of vector spaces $W_{k_1}\otimes\cdots\otimes W_{k_r}$ where $k_1>k_2>\cdots>k_r$.

For a vector space W, we write $T^+W=\bigoplus_{n\geqslant 1}W^{\otimes n}$. Notice that for each $u\in L$, $T^+V^{[u]}$ is a non-unital subalgebra of T^+V in the category ${}^{H_0}_{H_0}\mathcal{YD}$. Further, the $\mathbb{Z}$-grading of TV induces a $\mathbb{Z}$-grading on its subalgebras $TV^{[u]}$. In particular, the Hilbert series $\eta(TV^{[u]},t)$ of $TV^{[u]}$ is a series in the variable $t^{|u|}$.

Theorem 3.22. *One has*

$$T^+V\simeq\bigotimes^{>}_{u\in L}T^+V^{[u]}.$$

More precisely, the map

$$\mu:\bigotimes^{>}_{u\in L}T^+V^{[u]}\to T^+V,$$

which is the multiplication map in each summand, is an isomorphism in the category ${}^{H_0}_{H_0}\mathcal{YD}$. *In particular, the Hilbert series of* TV *is*

$$\eta(TV,t)=\prod_{u\in L}\eta\big(TV^{[u]},t\big).$$

Proof. This is a reformulation of Corollary 3.18. □

4. Hopf algebras generated by a Hopf subalgebra and a vector space

Our aim is now to apply the results of the previous section to arbitrary Hopf algebras. In many (though not all) cases we will be able to provide some new structure results.

Let H be a Hopf algebra with bijective antipode and a filtration

$$0=\mathcal{F}_{-1}\subset\mathcal{F}_0\subset\mathcal{F}_1\subset\mathcal{F}_2\subset\cdots\subseteq H.$$

The filtration is a *Hopf algebra filtration* if

(1) It is a filtration: $H=\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$,
(2) it is an algebra filtration: $\mathcal{F}_n\mathcal{F}_m\subseteq\mathcal{F}_{n+m}$,
(3) it is a coalgebra filtration: $\Delta(\mathcal{F}_n)\subseteq\sum_{i+j=n}\mathcal{F}_i\otimes\mathcal{F}_j$, and
(4) it behaves well with the antipode: $S(\mathcal{F}_n)\subseteq\mathcal{F}_n$.

Notice that in this case $\mathcal{F}_0$ is a Hopf subalgebra of H. We will consider Hopf algebras with a Hopf algebra filtration satisfying a stronger version of condition (2):

Assumption 4.1. $\mathcal{F}_n\mathcal{F}_m=\mathcal{F}_{n+m}$.

A Hopf algebra with a Hopf algebra filtration satisfying Assumption 4.1 can be presented in the following way.

Suppose that the Hopf algebra H is generated (as an algebra) by a Hopf subalgebra H_0 and a vector space V, such that

$$\Delta V\subseteq V\otimes H_0+H_0\otimes V+H_0\otimes H_0,\quad \text{and} \tag{4.2}$$

$$S(V)\subseteq KVK. \tag{4.3}$$

We define $\mathcal{F}_0=H_0$, $\mathcal{F}_1=H_0+H_0VH_0$, and $\mathcal{F}_n=(\mathcal{F}_1)^n$. Then $\mathcal{F}_*$ is a Hopf algebra filtration which satisfies Assumption 4.1.

As an important example, assume that H is generated by grouplikes and skew-primitive elements and take H_0 to be the subalgebra generated by the grouplikes and V to be the subspace generated by skew-primitives.

For $n\geqslant 0$ let $H_n=\mathcal{F}_n/\mathcal{F}_{n-1}$. Let then

$$\operatorname{gr}_{\mathcal{F}}H=\bigoplus_{n\geqslant 0}\mathcal{F}_n/\mathcal{F}_{n-1}=\bigoplus_{n\geqslant 0}H_n$$

be the associated graded Hopf algebra. Then we can consider the projection $\pi : \operatorname{gr}_{\mathcal{F}} H \to H_0$ and the inclusion $\iota : H_0 \to \operatorname{gr}_{\mathcal{F}} H$, and this allows to write $\operatorname{gr}_{\mathcal{F}} H = R\#H_0$ as the smash product of H_0 and the right coinvariants $R = \{x \in \operatorname{gr}_{\mathcal{F}} H \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$. This is now a standard procedure: the last part is due to Radford [Rad85] and Majid [Maj94], while the first part is a modification of the one due to Andruskiewitsch and Schneider [AS98].

Also, R is a braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$, and $R = \bigoplus_{n \geqslant 0} R_n$, R_0 is the base field and R_1 is a Yetter–Drinfel'd module which generates R because of Assumption 4.1. We let $V := R_1$ and we make the following assumption:

Assumption 4.4. $V = \bigoplus_{i=1}^{d} V_i$ is a direct sum of Yetter–Drinfel'd modules over H_0.

Remark 4.5. If V is irreducible, the methods in this paper do not yield any information on R. Otherwise, V has a nontrivial maximal flag $V = V_1 \supset V_2 \supset \cdots \supset V_d$ of Yetter–Drinfel'd submodules over H_0 (which is not necessarily a full flag of vector subspaces). Consider on the vector space R_n the $\mathbb{Z}$-filtration $R_n = R_{n,n} \supset R_{n,n+1} \supset \cdots$, where $R_{n,m} = \sum_{i_1+\cdots+i_n \geqslant m} V_{i_1} V_{i_2} \cdots V_{i_n}$. Then

$$R' := \bigoplus_{n=0}^{\infty} R'_n, \quad \text{where } R'_n = \bigoplus_{m \geqslant n} R_{n,m}/R_{n,m+1},$$

is a graded braided Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$, and all of the following considerations may be applied to R' instead of R.

Consider the projection $TV \to R$. We want to study the images under this projection of the components $T^+V^{[u]}$ appearing in Theorem 3.22. We begin by considering the comultiplication in $TV\#H_0$.

We write the left H_0-coaction on $r \in R$ by $\delta(r) = r_{(-1)} \otimes r_{(0)}$. Recall that the coproduct in the smash product $R\#H_0$ is given by $\Delta(r\#h) = (r^{(1)}\#{r^{(2)}}_{(-1)}h_{(1)}) \otimes ({r^{(2)}}_{(0)}\#h_{(2)})$, where $\Delta_R(r) =: r^{(1)} \otimes r^{(2)}$ is the coproduct of the braided Hopf algebra R. This notation applies in particular for $R = TV$.

Proposition 4.6. *Let $X \in TV \subseteq TV\#H_0$ be a u-[ ]-letter. Then the coproduct Δ of $TV\#H_0$ satisfies*

$$\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)} + \sum_i \bigl(X'_i h_i\bigr) \otimes X''_i,$$

where $X'_i, X''_i \in T^+V$, $h_i \in H_0$, and each X'_i is a w_i-[ ]-word with $w_i \in M_{>u}$.

Proof. We proceed by induction on $|u|$. If $u \in A$, we get $X \in V$ and then $\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)}$, whence we are done. Assume now that $|u| \geqslant 2$, Ш$u = (v, w)$ and x is a u-vector. Then $x = x_v x_w$, and we write $X = [x]$, $Y = [x_v]$, $Z = [x_w]$. By standard computations,

$$\begin{aligned}\Delta(X) &= \Delta\big([Y,Z]\big) = \Delta\big(YZ - Z_{(0)}\big(S^{-1}(Z_{(-1)}).Y\big)\big)\\ &= \big(Y^{(1)}Y^{(2)}{}_{(-1)} \otimes Y^{(2)}{}_{(0)}\big)\cdot\big(Z^{(1)}Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}\big)\\ &\quad - \big(Z_{(0)}{}^{(1)}Z_{(0)}{}^{(2)}{}_{(-1)} \otimes Z_{(0)}{}^{(2)}{}_{(0)}\big)\\ &\quad \cdot\big(\big(S^{-1}(Z_{(-1)}).Y^{(1)}\big)S^{-1}(Z_{(-2)})Y^{(2)}{}_{(-1)}Z_{(-4)} \otimes S^{-1}(Z_{(-3)}).Y^{(2)}{}_{(0)}\big). \qquad (4.7)\end{aligned}$$

Note that Y is a v-[ ]-letter and Z is a w-[ ]-letter. According to the induction hypothesis and Lemma 3.12(1), for any $h \in H_0$, $h.Y^{(1)}$ can be taken to be either $\varepsilon(h)1$, $h.Y$ or a v'-[ ]-word with $v' \in M_{>v}$. Similarly, $h.Z^{(1)}$ can be taken to be either $\varepsilon(h)1$, $h.Z$ or a w'-[ ]-word with $w' \in M_{>w}$, and $Z_{(0)}{}^{(1)}$ to be either 1, $Z_{(0)}$ or a w'-[ ]-word with $w' \in M_{>w}$.

We begin by considering the summand

$$\begin{aligned}&\big(Y^{(1)}Y^{(2)}{}_{(-1)} \otimes Y^{(2)}{}_{(0)}\big)\cdot\big(Z^{(1)}Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}\big)\\ &\quad = Y^{(1)}Y^{(2)}{}_{(-1)}Z^{(1)}Z^{(2)}{}_{(-1)} \otimes Y^{(2)}{}_{(0)}Z^{(2)}{}_{(0)}\\ &\quad = Y^{(1)}\big(Y^{(2)}{}_{(-2)}.Z^{(1)}\big)Y^{(2)}{}_{(-1)}Z^{(2)}{}_{(-1)} \otimes Y^{(2)}{}_{(0)}Z^{(2)}{}_{(0)}.\end{aligned}$$

We consider the summands in which $Y^{(1)}$ is a v'-[ ]-word with $v' \in M_{>v}$. Notice that since these v' are shorter than v, they belong to $M_{>vw}$. Therefore, since $Z^{(1)}$ is either 1 or a t-[ ]-word with $t \in M_{>vw}$, by Lemma 3.15 these summands satisfy the claim of the proposition. The summands in which $Y^{(1)} = 1$ and $Z^{(1)} \neq 1$ also satisfy the claim, because $w > u$. We are thus left with

$$Y_{(-1)}Z_{(-1)} \otimes Y_{(0)}Z_{(0)} + YZ^{(1)}Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}. \qquad (4.8)$$

We consider now the other summand of $\Delta(X)$. By similar reasons, we are left with

$$\begin{aligned}&-Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}\big(S^{-1}(Z_{(-1)}).Y_{(0)}\big)\\ &\quad - Z_{(0)}{}^{(1)}Z_{(0)}{}^{(2)}{}_{(-1)}\big(S^{-1}(Z_{(-1)}).Y\big)S^{-1}(Z_{(-2)})Z_{(-4)} \otimes Z_{(0)}{}^{(2)}{}_{(0)}\big(S^{-1}(Z_{(-3)}).1\big)\\ &= -Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}\big(S^{-1}(Z_{(-1)}).Y_{(0)}\big)\\ &\quad - Z_{(0)}{}^{(1)}Z_{(0)}{}^{(2)}{}_{(-1)}\big(S^{-1}(Z_{(-1)}).Y\big) \otimes Z_{(0)}{}^{(2)}{}_{(0)}\\ &= -Y_{(-1)}Z_{(-2)} \otimes Z_{(0)}\big(S^{-1}(Z_{(-1)}).Y_{(0)}\big)\\ &\quad - Z^{(1)}{}_{(0)}\big(\big(Z^{(2)}{}_{(-2)}S^{-1}\big(Z^{(2)}{}_{(-3)}\big)S^{-1}\big(Z^{(1)}{}_{(-1)}\big)\big).Y\big)Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}\\ &= -Y_{(-1)}Z_{(-1)} \otimes \mathrm{m}c^{-1}(Y_{(0)} \otimes Z_{(0)}) - \mathrm{m}c^{-1}\big(Y \otimes Z^{(1)}\big)Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}. \qquad (4.9)\end{aligned}$$

Adding (4.8) and (4.9) we get

$$Y_{(-1)}Z_{(-1)} \otimes [Y_{(0)}, Z_{(0)}] + \big[Y, Z^{(1)}\big]Z^{(2)}{}_{(-1)} \otimes Z^{(2)}{}_{(0)}.$$

Notice that $Y_{(-1)}Z_{(-1)} \otimes [Y_{(0)}, Z_{(0)}] = X_{(-1)} \otimes X_{(0)}$. Also, when in the second summand we put $Z^{(1)} = Z$, we get $X \otimes 1$. Finally, when $Z^{(1)}$ is a w'-[ ]-word with $w' \in M_{>w}$, by using Lemmas 3.12(3), 3.13 and 3.15 we obtain terms which satisfy the claim. □

Corollary 4.10. *Let $u \in L$ and $v \in M_{\geqslant u}$. Let $X \in TV \subseteq TV\#H_0$ be a v-[ ]-word. Then the coproduct Δ of $TV\#H_0$ satisfies*

$$\Delta(X) = X \otimes 1 + X_{(-1)} \otimes X_{(0)} + \sum_i \bigl(X_i' h_i\bigr) \otimes X_i'',$$

where $X_i', X_i'' \in T^+V$, $h_i \in H_0$, and each X_i' is a w_i-[ ]-word with $w_i \in M_{\geqslant u}$.

Proof. This follows at once from Proposition 4.6 and Lemma 3.15. □

We study the structure of R now. For that, we will use the following notation.

Definition 4.11. For $u \in L$, let $\mathcal{V}^{[\geqslant u]}$ be the subalgebra of TV generated by $(\sum_{v\in L, v\geqslant u} V^{[v]})$. Let $\mathcal{I}^{[\geqslant u]}$ be the ideal of $\mathcal{V}^{[\geqslant u]}$ generated by $(\sum_{v\in L, v> u} V^{[v]})$. We define also $V_{[u]} = \pi(V^{[u]})$, $\mathcal{V}_{[\geqslant u]} = \pi(\mathcal{V}^{[\geqslant u]})$, $\mathcal{V}^+_{[\geqslant u]} = \pi(\mathcal{V}^{[\geqslant u]}) \cap \ker\varepsilon$, and $\mathcal{I}_{[\geqslant u]} = \pi(\mathcal{I}^{[\geqslant u]})$, where $\pi : TV \to R$ is the canonical projection.

The grading on R induces a grading on all of the algebras and ideals defined above. Take $u \in L$. Notice that the graded algebras $\mathcal{V}^{[\geqslant u]}/\mathcal{I}^{[\geqslant u]}$ and $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ have only elements in degrees $m|u|$ where $m \in \mathbb{N}_0$. Moreover, Lemma 3.15 and Corollary 3.20 imply that $\mathcal{V}^{[\geqslant u]}$ is the subspace of TV generated by w-[ ]-words with $w \in M_{\geqslant u}$ and $\mathcal{I}^{[\geqslant u]}$ is the subspace of $\mathcal{V}^{[\geqslant u]}$ generated by w-[ ]-words with $w = w_1 \cdots w_n \in M_{\geqslant u}$ with $w_1 > u$.

Notice that the leading vector of any $X \in \mathcal{V}^{[\geqslant u]} \setminus \mathcal{I}^{[\geqslant u]}$ of degree $m|u|$ is a u^m-vector. Thus, we can choose for any $u \in L$ a graded linear map

$$\iota_u : \mathcal{V}^+_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]} \to T^+V^{[u]} = \bigoplus_{m\in\mathbb{N}} V^{[u^m]} \subset \mathcal{V}^{[\geqslant u]}$$

such that $\pi_u \circ \pi|_{T^+V^{[u]}} \circ \iota_u = \mathrm{id}$, where $\pi : TV \to R$ and $\pi_u : \mathcal{V}_{[\geqslant u]} \to \mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ are the canonical maps.

Recall the notion of $\bigotimes^>$ form Definition 3.21.

Theorem 4.12. *The map $\phi : \bigotimes^>_{u\in L} \mathcal{V}^+_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]} \to R^+$ defined by*

$$\phi(X_{u_1} \otimes \cdots \otimes X_{u_n}) = \pi\bigl(\iota_{u_1}(X_{u_1}) \cdots \iota_{u_n}(X_{u_n})\bigr), \tag{4.13}$$

where $u_1, \ldots, u_n \in L$, $u_1 > \cdots > u_n$, and $X_{u_i} \in \mathcal{V}^+_{[\geqslant u_i]}/\mathcal{I}_{[\geqslant u_i]}$ for all i, is an isomorphism of graded vector spaces.

To prove the theorem we will use the following lemma.

Lemma 4.14. *Let $u = u_1^{n_1} \cdots u_r^{n_r} \in M$, and let $X_i \in V^{[u_i^{n_i}]}$ for all i. Take $Y_i = \pi_{u_i}(\pi(X_i)) \in \mathcal{V}^+_{[\geqslant u_i]}/\mathcal{I}_{[\geqslant u_i]}$. Then*

$$X_1 \cdots X_r - \iota_{u_1}(Y_1) \cdots \iota_{u_r}(Y_r) \in \ker\pi + \sum_{\substack{w>u\\ w\in M}} V^{[w]}.$$

Proof. Since $\pi_{u_i}\pi(X_i-\iota_{u_i}(Y_i))=0$, we have $X_i-\iota_{u_i}(Y_i)\in\ker\pi+\mathcal{I}^{[\geqslant u_i]}$. We then consider

$$\begin{aligned}Z&=X_1\cdots X_r-\iota_{u_1}(Y_1)\cdots\iota_{u_r}(Y_r)\\&=\sum_{j=1}^{r}X_1\cdots X_{j-1}\big(X_j-\iota_{u_j}(Y_j)\big)\iota_{u_{j+1}}(Y_{j+1})\cdots\iota_{u_r}(Y_r)\\&\in\sum_{j=1}^{r}V^{[u_1^{n_1}]}\dots V^{[u_{j-1}^{n_{j-1}}]}\mathcal{I}^{[\geqslant u_j]}V^{[u_{j+1}^{n_{j+1}}]}\dots V^{[u_r^{n_r}]}+\ker\pi.\end{aligned}$$

As mentioned above the theorem, $\mathcal{I}^{[\geqslant u_j]}$ consists of sums of w-[ ]-words where w runs on monotonic super-words $>u_j^{n_j}$. Thus, by Corollary 3.20 and Lemma 3.6, Z is a sum of w-[ ]-words, where w runs on super-words $>u$. □

Proof of Theorem 4.12. Since the set of words of a given length is finite, the surjectivity of ϕ follows easily from the previous lemma.

We now prove injectivity of ϕ. To do so define $\phi':\bigotimes_{u\in L}^{>}\mathcal{V}^{+}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}\to T^{+}V$ by

$$\phi'(X_{u_1}\otimes\cdots\otimes X_{u_n})=\iota_{u_1}(X_{u_1})\cdots\iota_{u_n}(X_{u_n}),$$

where $u_1,\dots,u_n\in L$, $u_1>\cdots>u_n$, and $X_{u_i}\in\mathcal{V}^{+}_{[\geqslant u_i]}/\mathcal{I}_{[\geqslant u_i]}$ for all i. Assume then that there exists a smallest integer m such that ϕ is not injective in degree m. For all $u\in L$ let $B_u=\{b_{u,i}\mid i\in I_u\}$ be a homogeneous basis of $\mathcal{V}^{+}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$, where I_u is an appropriate index set, and let $X_{u,i}:=\iota_u(b_{u,i})$ for all $u\in L$, $i\in I_u$.

Suppose that there exists a nonempty finite subset M' of M with $|w|=m$ for $w\in M'$, and for each $w=w_1^{n_1}\cdots w_r^{n_r}\in M'$ there exist nonzero elements $b_w\in\mathcal{V}^{+}_{[\geqslant w_1]}/\mathcal{I}_{[\geqslant w_1]}\otimes\cdots\otimes\mathcal{V}^{+}_{[\geqslant w_r]}/\mathcal{I}_{[\geqslant w_r]}$ such that $\phi'(\sum_{w\in M'}b_w)\in\ker\pi$. Let $u=u_1^{m_1}\cdots u_s^{m_s}$, where $u_1>\cdots>u_s$, be the minimal element of M', and write $b_u:=\sum_{i_1,\dots,i_s}\lambda_{i_1,\dots,i_s}b_{u_1,i_1}\otimes\cdots\otimes b_{u_s,i_s}$ with $\lambda_{i_1,\dots,i_s}\in\mathbf{k}$. We consider TV as a subalgebra of $TV\#H_0$, and then we have

$$\Delta\bigg(\phi'\bigg(\sum_{w\in M'}b_w\bigg)\bigg)=:\sum_i Z'_i\otimes Z''_i\in(\ker\pi\#H_0)\otimes TV+(TV\#H_0)\otimes\ker\pi.$$

Therefore,

$$\sum_i S^{-1}\big(Z''_{i\,(-1)}\big)Z'_i\otimes Z''_{i\,(0)}\in\ker\pi\otimes TV+TV\otimes\ker\pi.\tag{4.15}$$

We apply Proposition 4.6 to each [ ]-letter in $\phi'(\sum_{w\in M'}b_w)$, and we use Lemma 3.8 to obtain a description of the tensor factors of (4.15). Afterwards, we apply Corollary 3.20 and Lemma 3.6 to rearrange the tensor factors as sums of [ ]-words. This gives

$$\sum_i S^{-1}\big(Z''_{i\,(-1)}\big)Z'_i \otimes Z''_{i\,(0)}$$
$$\in \sum_{i_1,\dots,i_s} \lambda_{i_1,\dots,i_s} S^{-1}(X_{u_1,i_1\,(-1)})X_{u_1,i_1\,(-2)}X_{u_2,i_2}\cdots X_{u_s,i_s} \otimes X_{u_1,i_1\,(0)}$$
$$+ \sum_{\substack{w',w''\in M\\ w'>u_2^{m_2}\cdots u_s^{m_s} \text{ or } w''>u_1^{m_1}}} V^{[w']}\otimes V^{[w'']}.$$

By repeatedly using Lemma 4.14 and since $\phi'(\sum_{w\in M'} b_w)\in \ker\pi$, we get

$$\sum_i S^{-1}\big(Z''_{i\,(-1)}\big)Z'_i \otimes Z''_{i\,(0)} \in \sum_{i_1,\dots,i_s} \lambda_{i_1,\dots,i_s}\phi'(b_{u_2,i_2})\cdots\phi'(b_{u_s,i_s})\otimes\phi'(b_{u_1,i_1})$$
$$+ \sum_{\substack{w',w''\in M,\,|w'|,|w''|<|u|,\\ w'>u_2^{m_2}\cdots u_s^{m_s} \text{ or } w''>u_1^{m_1}}} \big(\operatorname{Im}\phi'\cap V^{[w']}\big)\otimes\big(\operatorname{Im}\phi'\cap V^{[w'']}\big)$$
$$+\ker\pi\otimes TV + TV\otimes\ker\pi.$$

Therefore, (4.15) shows that

$$\sum_{i_1,\dots,i_s} \lambda_{i_1,\dots,i_s}\phi'(b_{u_2,i_2}\cdots b_{u_s,i_s})\otimes\phi'(b_{u_1,i_1})$$
$$\in \sum_{\substack{w',w''\in M,\,|w'|,|w''|<|u|,\\ w'>u_2^{m_2}\cdots u_s^{m_s} \text{ or } w''>u_1^{m_1}}} \big(\operatorname{Im}\phi'\cap V^{[w']}\big)\otimes\big(\operatorname{Im}\phi'\cap V^{[w'']}\big)$$
$$+\ker\pi\otimes TV + TV\otimes\ker\pi. \tag{4.16}$$

By the assumption on m, $\pi\circ\phi' = \phi$ is injective in degrees $< m$ and hence the sums

$$\big(\ker\pi\cap V^{\otimes n}\big) + \Bigg(\bigoplus_{\substack{w'\in M,\,|w'|=n,\\ w'>u_2^{m_2}\cdots u_s^{m_s}}} \operatorname{Im}\phi'\cap V^{[w']}\Bigg) + \big(\operatorname{Im}\phi'\cap V^{[u_2^{m_2}\cdots u_s^{m_s}]}\big)$$
$$\big(\ker\pi\cap V^{\otimes n}\big) + \Bigg(\bigoplus_{\substack{w''\in M,\,|w''|=n,\\ w''>u_1^{m_1}}} \operatorname{Im}\phi'\cap V^{[w'']}\Bigg) + \big(\operatorname{Im}\phi'\cap V^{[u_1^{m_1}]}\big)$$

are direct in TV whenever $1\leqslant n<m$. Thus (4.16) implies that $\lambda_{i_1,\dots,i_s}=0$ for all $i_1,\dots,i_s$, which contradicts to the choice of b_u. □

Corollary 4.17. *The Hilbert series of R factors as*

$$\eta(R,t) = \prod_{u\in L}\eta(\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]},t).$$

The importance of Corollary 4.17 becomes clearer with the following theorem.

Theorem 4.18. *For each $u \in L$, the algebra $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}\#H_0$ is a $|u|\mathbb{Z}$-graded Hopf algebra, where the grading is induced by that of $R\#H_0$. Equivalently, $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ is a $|u|\mathbb{Z}$-graded braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$. Moreover, $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ is generated by $V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]})$ and it projects onto the Nichols algebra $\mathcal{B}(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$. The quotient*

$$\eta(\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}, t)/\eta\big(\mathcal{B}\big(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]})\big), t^{|u|}\big)$$

is a power series with nonnegative integer coefficients.

Proof. Since $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ is graded and its degree 0 part is $\mathbf{k}$, in order to show the first statement it is sufficient to prove that $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}\#H_0$ is a bialgebra (see [Tak71]). This follows from Proposition 2.1, by taking $A = R\#H_0$, $B = \mathcal{V}_{[\geqslant u]}$, and $I = \mathcal{I}_{[\geqslant u]}$. It remains to show that Eq. (2.2) hold in this case. Indeed, it suffices to prove this for generators of B and I, and [ ]-letters satisfy (2.2) thanks to Proposition 4.6.

By the definition of $\mathcal{V}_{[\geqslant u]}$ and $\mathcal{I}_{[\geqslant u]}$, $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ is generated as an algebra by the space $V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]})$. Further, $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ can be considered as the quotient of the tensor algebra $T(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$ by a graded Hopf ideal consisting of elements of degree $\geqslant 2$. Since $\mathcal{B}(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$ is the quotient of $T(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$ by the maximal Hopf ideal consisting of elements of degree $\geqslant 2$, there exists a natural projection $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]} \to \mathcal{B}(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$. The last statement follows from Proposition 2.3 (see also Remark 2.5). □

Remark 4.19. In Theorem 4.18 it is necessary to put $t^{|u|}$ as the variable of the Hilbert series of the Nichols algebra, since in $\mathcal{B}(V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]}))$ the elements of $V_{[u]}/(\mathcal{I}_{[\geqslant u]} \cap V_{[u]})$ are considered to be in degree 1.

Open Problems 4.20.

(1) Assume that R is a Nichols algebra. Are the graded Hopf algebras $\mathcal{V}_{[\geqslant u]}/\mathcal{I}_{[\geqslant u]}$ appearing in Theorem 4.18 again Nichols algebras? This is true in the case where $\operatorname{char}\mathbf{k} = 0$, H_0 is the group algebra of an abelian group, and R is finite-dimensional, by Kharchenko's PBW theorem. More generally, if R has a finite number of PBW generators, the statement follows by using the Weyl groupoid.
(2) Generalize Theorems 4.12 and 4.18 to a more general setting which covers also Ufer's PBW basis.
(3) Is it possible to generalize results of this paper to arbitrary (say finite-dimensional) non-semisimple Hopf algebras?

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