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Braided module and comodule algebras, Galois extensions and elements of trace 1

Mauricio Da Rocha [a,1], Jorge A. Guccione [b,2], Juan J. Guccione [b,*,2]

[a] *Cíclo Básico Común, Pabellón 3, Ciudad Universitaria, (1428) Buenos Aires, Argentina*
[b] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Pabellón 1, Ciudad Universitaria, (1428) Buenos Aires, Argentina*

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Abstract

Let k be a field and let H be a rigid braided Hopf k-algebra. In this paper we continue the study of the theory of braided Hopf crossed products began in [J.A. Guccione, J.J. Guccione, Theory of braided Hopf crossed products, J. Algebra 261 (2003) 54–101]. First we show that to have an H-braided comodule algebra is the same that to have an $H^{\dagger}$-braided module algebra, where $H^{\dagger}$ is a variant of H^*, and then we study the maps $[\,,\,]$ and $(\,,\,)$, that appear in the Morita context introduced in the above cited paper.

Keywords: Braided Hopf algebras; Crossed products; Galois extensions

0. Introduction

Let k be a field, H a finite-dimensional Hopf k-algebra and H^* the dual Hopf algebra of H. It is well known that to have a right H-comodule is "the same" that to have a left H^*-module. A similar duality exists between the notions of right H-comodule algebra and left H^*-module algebra. More generality, these duality results are also satisfied by rigid Hopf algebras in a braided category (see, for instance, [T2, Proposition 2.7]). The main purpose of this paper is to extend

* Corresponding author.
E-mail addresses: vander@dm.uba.ar (J.A. Guccione), jjgucci@dm.uba.ar (J.J. Guccione).
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them to the context introduced in [G-G], and to show that most of the results that appears in [C-F-M,C-F], remain valid in this setting.

Let H be a braided Hopf k-algebra. Recall from [G-G] that a left H-braided space (V,s) is a k-vector space V together with a bijective map $s: H\otimes V\to V\otimes H$, which is compatible with the operations of H and satisfies

$$(s\otimes H)\circ(H\otimes s)\circ(c\otimes V)=(V\otimes c)\circ(s\otimes H)\circ(H\otimes s),$$

where c is the braid of H (compatibility of s with c). When V is replaced by a k-algebra A and s is also compatible with the operations of A we say that (A,s) is a left H-braided algebra and s is a left transposition on H on A. Assume that H is rigid and let H^* be the dual of H. Following V. Lyubashenko, given a bijective map $s: H\otimes V\to V\otimes H$ we define a map $(s^{-1})^\flat: H^*\otimes V\to V\otimes H^*$ by

$$\left(s^{-1}\right)^\flat=\left(\mathsf{ev}_H\otimes V\otimes H^*\right)\circ\left(H^*\otimes s^{-1}\otimes H^*\right)\circ\left(H^*\otimes V\otimes \mathsf{coev}_H\right),$$

where $\mathsf{ev}_H: H^*\otimes H\to k$ and $\mathsf{coev}_H: k\to H\otimes H^*$ are the evaluation and coevaluation maps. Then, we show that (V,s) is a left H-braided space if and only if $(V,(s^{-1})^\flat)$ is a left H^*-braided space. Similarly, we show that if A is a k-algebra, then $s: H\otimes A\to A\otimes H$ is a transposition if and only if $(s^{-1})^\flat: H^*\otimes A\to A\otimes H^*$ is. Let (V,s) be a left H-braided space and let (A,s) be a left H-braided algebra. Recall from [G-G] that:

- (V,s) is a left H-module if V is a left H-module in the standard sense and the action $\rho: H\otimes V\to V$ satisfies $s\circ(H\otimes\rho)=(\rho\otimes H)\circ(H\otimes s)\circ(c\otimes V)$.
- (V,s) is a right H-comodule if V is a right H-comodule in the standard sense and the coaction $\nu: V\to V\otimes H$ satisfies $(\nu\otimes H)\circ s=(V\otimes c)\circ(s\otimes H)\circ(H\otimes\nu)$.
- (A,s) is a left H-module algebra if (A,s) is a left H-module and the action $\rho: H\otimes A\to A$ satisfies:
  (a) $\rho(h\otimes 1)=\epsilon(h)1$,
  (b) $\rho\circ(H\otimes\mu)=\mu\circ(\rho\otimes\rho)\circ(H\otimes s\otimes A)\circ(\Delta\otimes A\otimes A)$.
- (A,s) is a right H-comodule algebra if (A,s) is a right H-comodule and the coaction $\nu: A\to A\otimes H$ satisfies:
  (a) $\nu(1)=1\otimes 1$,
  (b) $\nu\circ\mu=(\mu\otimes\mu)\circ(A\otimes s\otimes A)\circ(\nu\otimes\nu)$.

Assume that H is rigid and let $H^\dagger=H^{*\mathsf{op\,cop\,op\,cop}}$. Given a map $\nu: V\to V\otimes H$ we define a map $\rho_\nu: H^\dagger\otimes V\to V$ by

$$\rho_\nu:=(V\otimes\mathsf{ev}_H)\circ\left(\left(s^{-1}\right)^\flat\otimes H\right)\circ\left(H^\dagger\otimes\nu\right).$$

We show that (V,s) is a right H-comodule via ν if and only if $(V,(s^{-1})^\flat)$ is a left $H^\dagger$-module via ρ_ν. Furthermore we show that if we have a left H-braided algebra (A,s) instead of a left H-braided space (V,s), then (A,s) is a right H-comodule algebra via ν if and only if $(V,(s^{-1})^\flat)$ is a left $H^\dagger$-module algebra via ρ_ν. Note that this does not work if we use H^* instead $H^\dagger$, as was pointed by Takeuchi in [T2, Proposition 2.7].

Let H be a braided Hopf algebra and (A, s) a left H-module algebra. There are two associative k-algebras associated with (A, s). The ring of invariants

$$^{H}A = \{a \in A\colon\ h \cdot a = \epsilon(h)a\}$$

and the smash product $A \# H$, which is the vector space $A \otimes H$, endowed with the product

$$(a \# h)(b \# l) = \sum_i ah_{(1)} \cdot b_i \# h_i l,$$

where $\sum_i b_i \otimes h_i = s(h \otimes b)$ and, as usual, $a \# h$ denotes $a \otimes h$, etcetera. Recall from [G-G] that if H is rigid, then A has a structure of $({}^{H}A, A \# H)$-bimodule and a structure of $(A \# H, {}^{H}A)$-bimodule, such that

$$[\,,\,]: N \otimes_{{}^{H}A} M \to A \# H, \qquad \text{given by } [a, b] = aTb,$$
$$(\,,\,): M \otimes_{A\#H} N \to {}^{H}A, \qquad \text{given by } (a, b) = T \cdot (ab),$$

is a Morita context relating ${}^{H}A$ and $A \# H$, where $T \in H^{\dagger}$ is a fixed nonzero left integral, $M = {}_{{}^{H}A}A_{A\#H}$ and $N = {}_{A\#H}A_{{}^{H}A}$.

Using the results mentioned above we establish conditions for any or both of the maps $[\,,\,]$ and $(\,,\,)$ be surjective and we give some applications. In particular we generalize Theorems 1.2 and 1.2′ of [C-F-M] and Theorems 1.8, 1.11 and 1.15 of [C-F].

1. Preliminaries

In this article we work in the category of vector spaces over a field k. Then we assume implicitly that all the maps are k-linear and all the algebras and coalgebras are over k. The tensor product over k is denoted by $\otimes$, without any subscript, and the category of k-vector spaces is denoted by $\mathcal{V}ect$. Given a vector space V and $n \geqslant 1$, we let V^n denote the n-fold tensor power $V \otimes \cdots \otimes V$. Given vector spaces U, V, W and a map $f: V \to W$ we write $U \otimes f$ for $\mathrm{id}_U \otimes f$ and $f \otimes U$ for $f \otimes \mathrm{id}_U$. We assume that the algebras are associative unitary and the coalgebras are coassociative counitary. Given an algebra A and a coalgebra C, we let $\mu: A \otimes A \to A$, $\eta: k \to A$, $\Delta: C \to C \otimes C$ and $\epsilon: C \to k$ denote the multiplication, the unit, the comultiplication and the counit, respectively, specified with a subscript if necessary.

Some of the results of this paper are valid in the context of monoidal categories. In fact we use the nowadays well-known graphic calculus for monoidal and braided categories. As usual, morphisms will be composed from up to down and tensor products will be represented by horizontal concatenation in the corresponding order. The identity map of a vector space will be represented by a vertical line. Given an algebra A, the diagrams

, and 

stand for the multiplication map, the unit and the action of A on a left A-module, respectively. Given a coalgebra C, the comultiplication, the counit and the coaction of C on a right C-comodule will be represented by the diagrams

, and ,

respectively. The maps c and s, which appear in Definition 1.1 and at the beginning of Section 2, will be represented by the diagrams

 and ,

respectively. The inverse maps of c and s will be represented by

 and .

Finally, any other map $g : V \to W$ will be geometrically represented by the diagram

.

Let V, W be vector spaces and let $c : V \otimes W \to W \otimes V$ be a map. Recall that:

- If V is an algebra, then c is compatible with the algebra structure of V if $c\circ(\eta \otimes W) = W \otimes \eta$ and $c\circ(\mu \otimes W) = (W \otimes \mu)\circ(c \otimes V)\circ(V \otimes c)$.
- If V is a coalgebra, then c is compatible with the coalgebra structure of V if $(W \otimes \epsilon)\circ c = \epsilon \otimes W$ and $(W \otimes \Delta)\circ c = (c \otimes V)\circ(V \otimes c)\circ(\Delta \otimes W)$.

Of course, there are similar compatibilities when W is an algebra or a coalgebra.

1.1. Braided bialgebras and braided Hopf algebras

Below we recall briefly the concepts of braided bialgebra and braided Hopf algebra following the presentation given in [T1]. For a study of braided Hopf algebras we refer to [T1,T2,L1,F-M-S,A-S,D,So,B-K-L-T].

Definition 1.1. A *braided bialgebra* is a vector space H endowed with an algebra structure, a coalgebra structure and a braiding operator $c \in \mathsf{Aut}_k(H^2)$ (called the *braid* of H), such that c is compatible with the algebra and coalgebra structures of H, $\Delta\circ\mu = (\mu \otimes \mu)\circ(H \otimes c \otimes H)\circ(\Delta \otimes \Delta)$, η is a coalgebra morphism and ϵ is an algebra morphism. Furthermore, if there exists a map $S : H \to H$, which is the inverse of the identity map for the convolution product, then we say that H is a *braided Hopf algebra* and we call S the *antipode* of H.

Usually H denotes a braided bialgebra, understanding the structure maps, and c denotes its braid. If necessary, we will use notations as c_H, μ_H, et cetera.

Remark 1.2. Assume that H is an algebra and a coalgebra and $c \in \mathsf{Aut}_k(H^2)$ is a solution of the braiding equation, which is compatible with the algebra and coalgebra structures of H. Let $H \otimes_c H$ be the algebra with underlying vector space $H \otimes H$ and multiplication map given by $\mu_{H\otimes_c H} := (\mu \otimes \mu)\circ(H \otimes c \otimes H)$. It is easy to see that H is a braided bialgebra with braid c iff $\Delta : H \to H \otimes_c H$ and $\epsilon : H \to k$ are morphisms of algebras.

Definition 1.3. Let H and L be braided bialgebras. A map $g : H \to L$ is a *morphism of braided bialgebras* if it is a morphism of algebras, a morphism of coalgebras and $c\circ(g \otimes g) = (g \otimes g)\circ c$.

Let H and L be braided Hopf algebras. It is well known that if $g : H \to L$ is a morphism of braided bialgebras, then $g \circ S_H = S_L \circ g$.

Remark 1.4. Let H be a braided bialgebra. A direct computation shows that $\widehat{H} = (H, \mu \circ \tau, \eta, \tau \circ \Delta, \epsilon)$, where $\tau : H \otimes H \to H \otimes H$ denotes the flip, is a braided bialgebra, with braid $\widehat{c} := \tau \circ c \circ \tau$. Note that if H is a braided Hopf algebra with antipode S, then $\widehat{H}$ also is.

Remark 1.5. If H is a braided bialgebra, then $H_c^{\mathrm{op}} := (H, \mu \circ c^{-1}, \eta, \Delta, \epsilon)$ and $H_c^{\mathrm{cop}} := (H, \mu, \eta, c^{-1} \circ \Delta, \epsilon)$ are braided bialgebras, with braid c^{-1}. By combining these constructions we obtain the braided bialgebras $H_c^{\mathrm{op\,cop}} := (H, \mu \circ c^{-1}, \eta, c \circ \Delta, \epsilon)$ and $H_c^{\mathrm{cop\,op}} := (H, \mu \circ c, \eta, c^{-1} \circ \Delta, \epsilon)$, with braid c. Furthermore, if S is an antipode for H, then S is also an antipode for $H_c^{\mathrm{op\,cop}}$ and $H_c^{\mathrm{cop\,op}}$, and if S is bijective, then S^{-1} is an antipode for H_c^{op} and H_c^{cop}. For a proof of these facts see [A-G, Proposition 2.2.4].

Let H be a braided Hopf algebra. The antipode S of H is a morphism of braided Hopf algebras from $H_c^{\mathrm{op\,cop}}$ to H, and from H to $H_c^{\mathrm{cop\,op}}$. Furthermore, $(S \otimes H) \circ c = c \circ (H \otimes S)$ and $(H \otimes S) \circ c = c \circ (S \otimes H)$ (compatibility of S with c).

1.2. Rigid braided bialgebras

In this subsection we recall the definition and some properties of rigid braided bialgebras and Hopf algebras, that we will need later.

Let V and W be vector spaces. Assume that W is finite-dimensional. Let $\mathsf{ev}_W : W^* \otimes W \to k$ be the evaluation map and let $\mathsf{coev}_W : k \to W \otimes W^*$ be the coevaluation map. For each map $T : V \otimes W \to W \otimes V$, Lyubashenko [L2] has introduced the map

$$T^\flat : W^* \otimes V \to V \otimes W^*,$$

defined by $T^\flat := (\mathsf{ev}_W \otimes V \otimes W^*) \circ (W^* \otimes T \otimes W^*) \circ (W^* \otimes V \otimes \mathsf{coev}_W)$.

Definition 1.6. A finite-dimensional braided bialgebra H is called *rigid* if the map $c^\flat : H^* \otimes H \to H \otimes H^*$ is bijective. In this case $(c^{-1})^\flat : H^* \otimes H \to H \otimes H^*$ is also a bijective map.

For each rigid braided bialgebra H, let $c_{H^*H} := (c^{-1})^\flat$, $c_{HH^*} := (c^\flat)^{-1}$ and $c_{H^*} := c^{\flat\flat}$.

Theorem 1.7. [T1, Theorem 5.7] *Let H be a rigid braided bialgebra. There exists a Hopf algebra L, with bijective antipode, and a braided bialgebra $\mathfrak{H}$ in the Yetter–Drinfeld category $\mathcal{YD}_L^L$, such that*:

(1) $(F(\mathfrak{H}), F(\mu_{\mathfrak{H}}), F(\eta_{\mathfrak{H}}), F(\Delta_{\mathfrak{H}}), F(\epsilon_{\mathfrak{H}})) = (H, \mu_H, \eta_H, \Delta_H, \epsilon_H)$, *where F is the forgetful functor from $\mathcal{YD}_L^L$ to $\mathcal{V}ect$.*
(2) *If $\mathfrak{c}$ is the braid of $\mathcal{YD}_L^L$, then $F(\mathfrak{c}_{\mathfrak{H}\mathfrak{H}}) = c_H$, $F(\mathfrak{c}_{\mathfrak{H}\mathfrak{H}^*}) = c_{HH^*}$, $F(\mathfrak{c}_{\mathfrak{H}^*\mathfrak{H}}) = c_{H^*H}$ and $F(\mathfrak{c}_{\mathfrak{H}^*\mathfrak{H}^*}) = c_{H^*}$.*
(3) $F(\mathsf{ev}_M) = \mathsf{ev}_{F(M)}$ *and* $F(\mathsf{coev}_M) = \mathsf{coev}_{F(M)}$ *for each rigid object $M \in \mathcal{YD}_L^L$, where $\mathsf{ev}_M : M^* \otimes M \to k$ and $\mathsf{coev}_M : k \to M \otimes M^*$ are the evaluation and coevaluation maps of M.*

Furthermore, if H is a braided Hopf algebra, we can take $\mathfrak{H}$ as a Hopf algebra in $\mathcal{YD}_L^L$. In this case, we have $F(S_{\mathfrak{H}}) = S_H$.

As was pointed in [T2, Section 1], each map $f : V \otimes W \to \mathbf{1}$, in a braided tensor category $\mathcal{C}$ with initial object $\mathbf{1}$, satisfies

$$(U \otimes f)\circ(c_{VU} \otimes W) = (f \otimes U)\circ\big(V \otimes c_{WU}^{-1}\big) \quad \text{and}$$
$$(U \otimes f)\circ\big(c_{UV}^{-1} \otimes W\big) = (f \otimes U)\circ(V \otimes c_{UW}),$$

for all object $U \in \mathcal{C}$. Similarly, each map $g : \mathbf{1} \to V \otimes W$ satisfies

$$(c_{UV} \otimes W)\circ(U \otimes g) = \big(V \otimes c_{UW}^{-1}\big)\circ(g \otimes U) \quad \text{and}$$
$$\big(c_{VU}^{-1} \otimes W\big)\circ(U \otimes g) = (V \otimes c_{WU})\circ(g \otimes U).$$

Let H be a rigid braided bialgebra. Thanks to Theorem 1.7 this remark applies to the maps ev_H and coev_H. More importantly, as was noted in [T1, Section 6], this theorem allows to reformulate all the known results about rigid bialgebras in a Yetter–Drinfeld category as results about rigid braided bialgebras. Next, we recall those ones that we will need later.

Theorem 1.8. [T2, Theorem 4.1] *The antipode of a rigid braided Hopf algebra is bijective.*

Definition 1.9. Let H be a rigid braided Hopf algebra. An element $t \in H$ is a *left integral* if $ht = \epsilon(h)t$, for all $h \in H$, and it is a *right integral* if $th = \epsilon(h)t$, for all $h \in H$. We let $\int_H^l$ denote the set of left integrals and we let $\int_H^r$ denote the set of right integrals.

Theorem 1.10. ([L2, Theorem 1.6], [F-M-S, Corollary 5.8], [T2, Theorem 4.6], [D, Theorem 3]) *The sets $\int_H^l$ and $\int_H^r$ are one-dimensional vector subspaces of H.*

Theorem 1.11. [T1, Section 7] *The sets $\int_H^l$ and $\int_H^r$ satisfy $c(\int_H^l \otimes H) = H \otimes \int_H^l$, $c(H \otimes \int_H^l) = \int_H^l \otimes H$, $c(\int_H^r \otimes H) = H \otimes \int_H^r$ and $c(H \otimes \int_H^r) = \int_H^r \otimes H$.*

Corollary 1.12. *Let H be a rigid braided Hopf algebra. There exist unique isomorphisms of braided Hopf algebras*

$$f_H^l : H \to H \quad \textit{and} \quad f_H^r : H \to H,$$

such that $c(h \otimes t) = t \otimes f_H^l(h)$ and $c(u \otimes h) = f_H^r(h) \otimes u$, for all $t \in \int_H^l \setminus \{0\}$, $u \in \int_H^r \setminus \{0\}$ and $h \in H$.

Corollary 1.13. *Let $t \in \int_H^l$. Since $S(t)S(f_H^l(h)) = \mu\circ(S \otimes S)\circ c(h \otimes t) = S(ht) = \epsilon(h)S(t) = \epsilon(S(f_H^l(h)))S(t)$, we have $S(\int_H^l) = \int_H^r$. In a similar way it can be proven that $S(\int_H^r) = \int_H^l$, $\int_H^l = \int_{H_c^{\mathsf{op}}}^r$ and $\int_H^r = \int_{H_c^{\mathsf{op}}}^l$.*

Remark 1.14. Let $t \in \int_H^l \setminus\{0\}$, $u \in \int_H^r \setminus\{0\}$ and $h \in H$. From the fact that c and S are compatible, $S(\int_H^l) = \int_H^r$ and $S(\int_H^r) = \int_H^l$, it follows that $c(t \otimes h) = f_H^r(h) \otimes t$ and $c(h \otimes u) = u \otimes f_H^l(h)$. Using this it is easy to see that there exists $q \in k \setminus \{0\}$ such that $c(t \otimes t) = qt \otimes t$ and $c(u \otimes u) = qu \otimes u$.

Let H be a rigid braided Hopf algebra and let t be a nonzero left integral of H. There is an algebra map $\alpha : H \to k$ such that $th = \alpha(h)t$, for all $h \in H$. This map is called the *modular function* of H. From Corollary 1.13 and Remark 1.14 it follows that if u is a nonzero right integral, then $hu = \alpha(S(f_H^l(h)))u$.

Theorem 1.15. [T1, Section 7] *We have* $(\alpha \otimes H)\circ c = H \otimes \alpha$ *and* $(H \otimes \alpha)\circ c = \alpha \otimes H$.

Theorem 1.16 *(Maschke's Theorem). A rigid braided Hopf algebra H is semisimple iff there exists $t \in \int_H^l$, such that $\epsilon(t) \neq 0$.*

Using this theorem it is easy to see that if H is semisimple, then $\int_H^l = \int_H^r$ and the maps f_H^r and f_H^l of Corollary 1.12 are the identity maps.

Theorem 1.17. [T2, Theorem 2.16] *If H is a rigid braided bialgebra, then H^* is also a rigid braided bialgebra, with braid $c_{H^*} = (c_H)^{\flat\flat}$ and multiplication, unit, comultiplication and counit given by,*

$$\mu_{H^*} := (\mathsf{ev}_H \otimes H^*)\circ(H^* \otimes \mathsf{ev}_H \otimes H \otimes H^*)\circ(H^{*\otimes 2} \otimes \Delta_H \otimes H^*)\circ(c_{H^*} \otimes \mathsf{coev}_H),$$

$$\eta_{H^*}(\lambda) := \lambda \cdot \epsilon_H \quad \text{for all } \lambda \in k,$$

$$\Delta_{H^*} := (\mathsf{ev}_H \otimes c_{H^*}^{-1})\circ(H^* \otimes \mu_H \otimes H^{*\otimes 2})\circ(H^* \otimes H \otimes \mathsf{coev}_H \otimes H^*)\circ(H^* \otimes \mathsf{coev}_H),$$

$$\epsilon_{H^*}(\varphi) := \varphi(1) \quad \text{for all } \varphi \in H^*.$$

Furthermore, if H is a braided Hopf algebra, then so is H^*, with antipode $S_{H^*}(\varphi) := \varphi\circ S_H$. Finally, the correspondence $H \mapsto H^*$ is functorial in an evidence sense.

Remark 1.18. For each rigid braided bialgebra H, the canonical bijection $H \to H^{**}$ is a bialgebra isomorphism (in the sense of Definition 1.3, but not as bialgebras in a Yetter–Drinfeld category $\mathcal{YD}_L^L$, since i is not compatible with the actions of L on H and H^{**}).

Notation 1.19. For each rigid braided bialgebra H, we write $H^\dagger := (H^*)^{\mathsf{op\,cop\,op\,cop}}$. Note that the multiplication and the comultiplication of $H^\dagger$ are described by interchanging c and c^{-1} in Theorem 1.17. We will write $c_{H^\dagger H} := c_{H^*H}$, $c_{HH^\dagger} := c_{HH^*}$ and $c_{H^\dagger} := c_{H^*}$. It is immediate that $H^{\dagger\dagger} = H^{**}$ and that, if c is involutive, then $H^\dagger = H^*$. Finally, when H is a braided Hopf algebra, then $S^2 : H^\dagger \to H^*$ is an isomorphism of braided Hopf algebras and, by Corollary 1.13, $\int_{H^\dagger}^l = \int_{H^*}^l$ and $\int_{H^\dagger}^r = \int_{H^*}^r$.

Throughout this work $\varphi \leftharpoonup h$ and $h \rightharpoonup \varphi$ denote the right and left standard actions of H on $H^\dagger$, given by $(\varphi \leftharpoonup h)(l) = \varphi(hl)$ and $(h \rightharpoonup \varphi)(l) = \varphi(lh)$, respectively.

Proposition 1.20. [D, Theorem 3b] *Let H be a rigid braided Hopf algebra and let ϕ be a nonzero left or right integral of $H^\dagger$. The map $\vartheta : H \to H^\dagger$, defined by $\vartheta(h) := \phi \leftharpoonup h$, is a right H-module isomorphism. Similarly the map $\vartheta' : H \to H^\dagger$, defined by $\vartheta'(h) := h \rightharpoonup \phi$, is a left H-module isomorphism.*

Corollary 1.21. *Given a left or right nonzero integral ϕ of $H^\dagger$ there exist $t \in \int_H^l$ and $u \in \int_H^r$ such that $t \rightharpoonup \phi = \epsilon$ and $\phi \leftharpoonup u = \epsilon$. Note that this is equivalent to say that $\phi(t) = \phi(u) = 1$.*

Remark 1.22. Let H be a rigid braided bialgebra. Then $H^\dagger$ acts on the left on H via $\varphi \rightharpoonup h := \sum_i h_{(1)_i}\varphi(h_{(2)_i})$, where $\sum_i h_{(2)_i} \otimes h_{(1)_i} = c^{-1}(h_{(1)} \otimes h_{(2)})$. Similarly $H^\dagger$ acts on the right on H via $h \leftharpoonup \varphi := \sum_i \varphi(h_{(1)_i})h_{(2)_i}$. Composing these actions with the canonical bijection $H \to H^{\dagger\dagger}$ we recover the standard left and right actions of $H^\dagger$ on $H^{\dagger\dagger}$. Assume that H is a rigid braided Hopf algebra. Hence, by Corollary 1.21, for each left or right nonzero integral l of H there exist $T \in \int_{H^\dagger}^l$ and $U \in \int_{H^\dagger}^r$ such that $T \rightharpoonup l = 1$ and $l \leftharpoonup U = 1$. Note that this is equivalent to say that $T(l) = U(l) = 1$. Since $(H_c^{\mathrm{cop}})^\dagger = (H^*)^{\mathrm{op}}$, applying this result to H_c^{cop} and taking into account Corollary 1.21 and Notation 1.19, we obtain that there exist $T' \in \int_{H^\dagger}^l$ and $U' \in \int_{H^\dagger}^r$ such that $l_{(1)}T'(l_{(2)}) = 1$ and $U'(l_{(1)})l_{(2)} = 1$. Since $T'(l) = 1 = T(l)$ and $U'(l) = 1 = U(l)$, we have that $T' = T$ and $U' = U$.

Remark 1.23. Let H be a rigid braided Hopf algebra and let α be the modular function. For $h \in H$ we write $h^\alpha := \alpha \rightharpoonup h$. So, $h^\alpha = \sum_i h_{(1)_i}\alpha(h_{(2)_i})$, where $\sum_i h_{(2)_i} \otimes h_{(1)_i} = c^{-1}(h_{(1)} \otimes h_{(2)})$. From Theorem 1.15 it follows easily that $h^\alpha = h_{(1)}\alpha(h_{(2)})$ and that the map $h \mapsto h^\alpha$ is an algebra automorphism.

Let $t \in \int_H^l$ be a nonzero left integral, α the modular function and $q \in k$ such that $c(t \otimes t) = qt \otimes t$. By [G-G, Lemma 8.3] we know that $S(t) = qt_{(1)}\alpha(t_{(2)})$. Applying this result to H_c^{cop} we obtain $S^{-1}(t) = q^{-1}\sum_i \alpha(t_{(1)_i})t_{(2)_i}$, where $\sum_i t_{(2)_i} \otimes t_{(1)_i} = c^{-1}(t_{(1)} \otimes t_{(2)})$. We will use this formula in the proof of the following proposition.

Proposition 1.24. *Let H be a rigid braided Hopf algebra. If $T \in \int_{H^\dagger}^l$ and $t \in \int_H^l$ satisfy $t \rightharpoonup T = \epsilon$, then $T \leftharpoonup t = \alpha$, $T \leftharpoonup S^{-1}(t) = q^{-1}\epsilon$, $T \rightharpoonup t = 1$ and $T \rightharpoonup S^{-1}(t) = q^{-1}1$.*

Proof. For each $h \in H$ we have $(T \leftharpoonup t)(h) = T(th) = \alpha(h)T(t) = \alpha(h)$. So, the first formula holds. Let us prove the third one. For $h \in H$, let $c^{-1}(\Delta(h)) = \sum_i h_{(2)_i} \otimes h_{(1)_i}$. Since $\sum_i \varphi(h_{(1)_i})T(h_{(2)_i}) = (\varphi T)(h) = \varphi(1)T(h)$, where φT is the product in $H^\dagger$, we have $\sum_i h_{(1)_i}T(h_{(2)_i}) = T(h)1$. Thus $T \rightharpoonup t = T(t)1 = 1$. Next we prove the second and fourth equalities. By the discussion preceding this proposition, $T(S^{-1}(t)) = q^{-1}\sum_i \alpha(t_{(1)_i})T(t_{(2)_i}) = q^{-1}(\alpha T)(t) = T(S^{-1}(t)) = q^{-1}\alpha(1)T(t) = q^{-1}$. Hence,

$$\big(T \leftharpoonup S^{-1}(t)\big)(h) = T\big(S^{-1}(t)h\big) = T\big(S^{-1}(t)\big)\epsilon(h) = q^{-1}\epsilon(h)$$

for all $h \in H$, and

$$\begin{aligned}\varphi\big(T \rightharpoonup S^{-1}(t)\big) &= \sum_i \varphi\big(S^{-1}(t)_{(1)_i}\big)T\big(S^{-1}(t)_{(2)_i}\big)\\ &= (\varphi T)\big(S^{-1}(t)\big) = \varphi(1)T\big(S^{-1}(t)\big) = q^{-1}\varphi(1)\end{aligned}$$

for all $\varphi \in H^\dagger$. From these facts it follows immediately that $T \leftharpoonup S^{-1}(t) = q^{-1}\epsilon$ and $T \rightharpoonup S^{-1}(t) = q^{-1}1$. $\square$

Let H be a rigid braided Hopf algebra and ϕ a nonzero left or right integral of $H^\dagger$. Given a map $\phi: H \to H$ we let ϕ^* denote the transpose of ϕ. Let $\widetilde{f}^l_H: H \to H$ and $\widetilde{f}^r_H: H \to H$ be the automorphisms of H defined by $f^l_{H^\dagger} := (\widetilde{f}^l_H)^*$ and $f^r_{H^\dagger} := (\widetilde{f}^r_H)^*$. Using that

$$c^{-1}_{HH^\dagger} = \big(H \otimes H^\dagger \otimes \mathsf{ev}_H\big)\circ(H \otimes c_{H^\dagger} \otimes H)\circ\big(\mathsf{coev}_H \otimes H^\dagger \otimes H\big),$$
$$c_{H^\dagger H} = \big(H \otimes H^\dagger \otimes \mathsf{ev}_H\big)\circ\big(H \otimes c^{-1}_{H^\dagger} \otimes H\big)\circ\big(\mathsf{coev}_H \otimes H^\dagger \otimes H\big),$$

we easily obtain that $c_{HH^\dagger}(h \otimes \phi) = \phi \otimes (\widetilde{f}^l_H)^{-1}(h)$ and $c_{H^\dagger H}(\phi \otimes h) = (\widetilde{f}^r_H)^{-1}(h) \otimes \phi$.

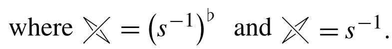
2. Transpositions

Let H be a braided bialgebra. We recall from [G-G] that a *left H-braided space* (V, s) is a vector space V endowed with a bijective map $s: H \otimes V \to V \otimes H$, which is compatible with the bialgebra structure of H and satisfies $(s \otimes H)\circ(H \otimes s)\circ(c \otimes V) = (V \otimes c)\circ(s \otimes H)\circ(H \otimes s)$ (compatibility of s with the braid). Actually in the definition given in [G-G] is not required that s be bijective, but here we add this condition, since it is necessary to prove most of the properties. When H is a braided Hopf algebra it is also true that $s\circ(S \otimes V) = (V \otimes S)\circ s$, as was shown in [D-G-G, Section 1]. It is easy to check that (V, s) is a left H-braided space iff (V, s) is a left H_c^{op}-braided space and that this happens iff (V, s) is a left H_c^{cop}-braided space. A map $g: V \to V'$ is said to be an homomorphism of left H-braided spaces, from (V, s) to (V', s'), if $(g \otimes H)\circ s = s'\circ(H \otimes g)$.

The notion of right H-braided space can be introduced in a similar way. We leave the details to the reader.

Let H be a braided bialgebra, V a vector space and $s: H \otimes V \to V \otimes H$ a bijective map. It is easy to check that (V, s) is a left H-braided space iff (V, s^{-1}) is a right H-braided space.

Let H be a rigid braided bialgebra and let (V, s) be a left H-braided space. From the definition of $(s^{-1})^\flat$ it follows immediately that

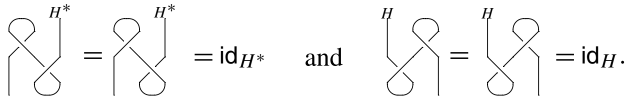


For each result about left H-braided spaces there is an analogous result about right H-braided spaces. The same is valid for the notions of transposition, H-module, H-module algebra, H-comodule and H-comodule algebra that we will consider later. In general we will announce the left version for H-braided spaces, transpositions and modules and the right version for comodules, and we leave the other ones to the reader.

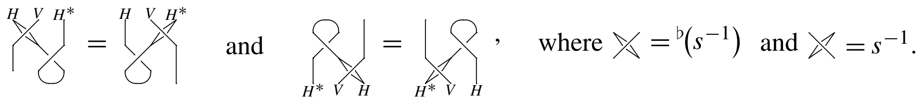
Let H be a rigid braided bialgebra, V a vector space and $s: H^* \otimes V \to V \otimes H^*$ a bijective map. We define the map ${}^\flat(s^{-1}): H \otimes V \to V \otimes H$, by

$${}^\flat\big(s^{-1}\big) := (\mathsf{ev}_H\circ c_{HH^*} \otimes V \otimes H)\circ\big(H \otimes s^{-1} \otimes H\big)\circ\big(H \otimes V \otimes c^{-1}_{H^*H}\circ\mathsf{coev}_H\big).$$

A direct computation shows that



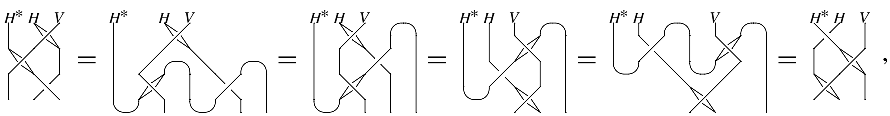
Using this and the definition of ${}^{\flat}(s^{-1})$ it is easy to check that



Lemma 2.1. *Let H be a rigid braided bialgebra. For each left H-braided space (V,s), we have*

$$\big(V\otimes c_{HH^*}^{-1}\big)\circ\big(\big(s^{-1}\big)^{\flat}\otimes H\big)\circ\big(H^*\otimes s\big)=\big(s\otimes H^*\big)\circ\big(H\otimes \big(s^{-1}\big)^{\flat}\big)\circ\big(c_{HH^*}^{-1}\otimes V\big),$$
$$(V\otimes c_{H^*H})\circ\big(\big(s^{-1}\big)^{\flat}\otimes H\big)\circ\big(H^*\otimes s\big)=\big(s\otimes H^*\big)\circ\big(H\otimes \big(s^{-1}\big)^{\flat}\big)\circ(c_{H^*H}\otimes V).$$

Proof. By basic properties of the evaluation and coevaluation maps and the fact that (V,s) is a left H-braided space, we have

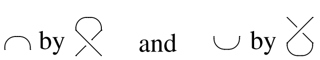


which proves the first equality. The second one can be checked in a similar way. □

Lemma 2.2. *Let H be a rigid braided bialgebra. For each left H^*-braided space (V,s), we have*

$$\big(V\otimes c_{H^*H}^{-1}\big)\circ\big({}^{\flat}\big(s^{-1}\big)\otimes H^*\big)\circ(H\otimes s)=(s\otimes H)\circ\big(H^*\otimes {}^{\flat}\big(s^{-1}\big)\big)\circ\big(c_{H^*H}^{-1}\otimes V\big),$$
$$(V\otimes c_{HH^*})\circ\big({}^{\flat}\big(s^{-1}\big)\otimes H^*\big)\circ(H\otimes s)=(s\otimes H)\circ\big(H^*\otimes {}^{\flat}\big(s^{-1}\big)\big)\circ(c_{HH^*}\otimes V).$$

Proof. The first equality can be proven by replacing H^* by H, H by H^*,



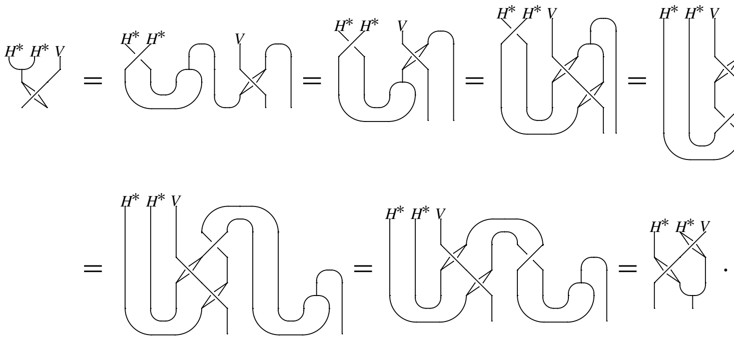
in the diagrams used in the proof of Lemma 2.1. The second one is similar. □

Proposition 2.3. *Let H be a rigid braided bialgebra. The following assertions hold*:

(1) *If (V,s) is a left H-braided space, then $(V,(s^{-1})^{\flat})$ is a left H^*-braided space.*
(2) *If (V,s) is a left H^*-braided space, then $(V,{}^{\flat}(s^{-1}))$ is a left H-braided space.*
(3) *The above constructions are inverse one of each other.*

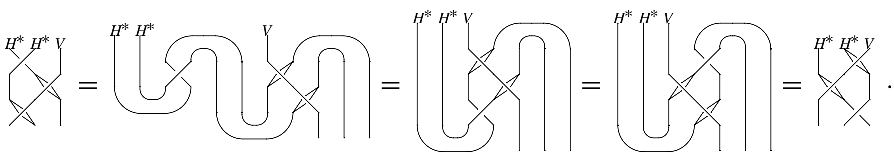
Proof. (1): Using the definition of η_{H^*} and the fact that s^{-1} is compatible with ϵ_H it is easy to check diagrammatically that $(s^{-1})^\flat$ is compatible with η_{H^*}. We left the details to the reader. Similarly, $(s^{-1})^\flat$ is compatible with ϵ_{H^*}. Next, we check the other compatibilities and that $(s^{-1})^\flat$ is bijective.

Compatibility of $(s^{-1})^\flat$ with μ_{H^*}: Since $(H \otimes s^{-1})\circ(s^{-1} \otimes H)\circ(V \otimes \Delta) = (\Delta \otimes V)\circ s^{-1}$ and $(c \otimes V)\circ(H \otimes s^{-1})\circ(s^{-1} \otimes H) = (H \otimes s^{-1})\circ(s^{-1} \otimes H)\circ(V \otimes c)$, we have

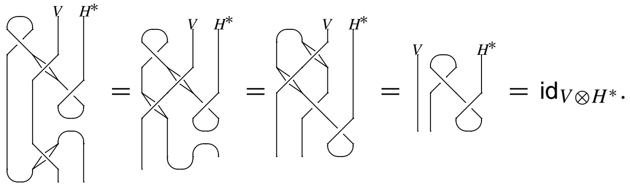


Compatibility of $(s^{-1})^\flat$ with Δ_{H^*}: This can be checked dualizing the proof that $(s^{-1})^\flat$ es compatible con μ_{H^*}.

Compatibility of $(s^{-1})^\flat$ with c_{H*}: We have



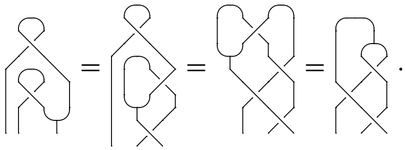
$(s^{-1})^\flat$ is bijective: By the discussion at the beginning of this section, basic properties of the evaluation and coevaluation maps and Lemma 2.1,



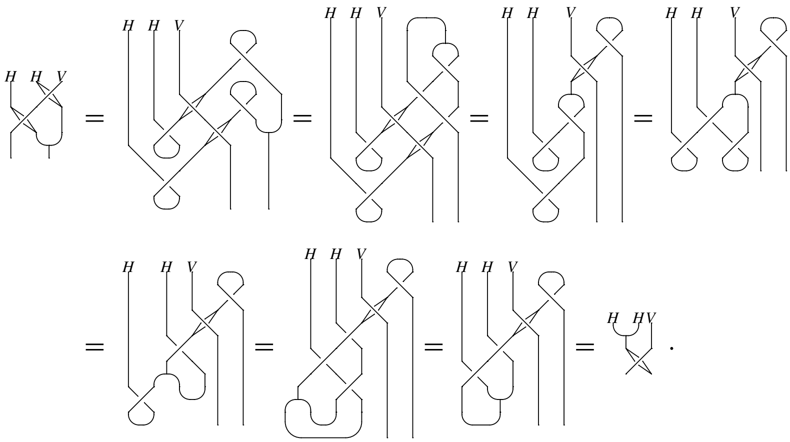
A similar argument shows that $(s^{-1})^\flat$ is left invertible.

(2): Using the definition of ϵ_{H^*}, that η is compatible with c_{HH^*}, and that ϵ_{H^*} is compatible with s^{-1} and $c^{-1}_{H^*H}$, it is easy to check diagrammatically that ${}^\flat(s^{-1})$ is compatible with η_H. We leave the details to the reader. Similarly, ${}^\flat(s^{-1})$ is compatible with ϵ_H. Next, we check the other compatibilities and that ${}^\flat(s^{-1})$ is bijective.

Compatibility of ${}^{\flat}(s^{-1})$ with μ_H: By the compatibility of $c^{-1}_{HH^*}$ with μ_H and the definition of Δ_{H^*},

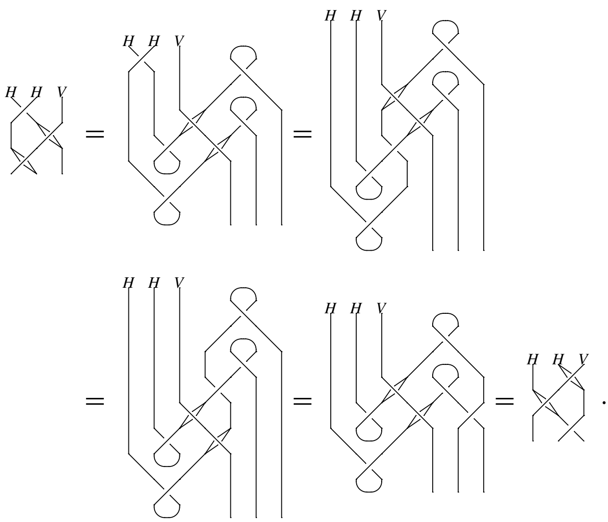


Using this fact, the definitions of ${}^{\flat}(s^{-1})$ and Δ_{H^*}, and the compatibility of $c^{-1}_{H^*}\circ\Delta_{H^*}$ with $c^{-1}_{HH^*}$ and s^{-1}, we obtain

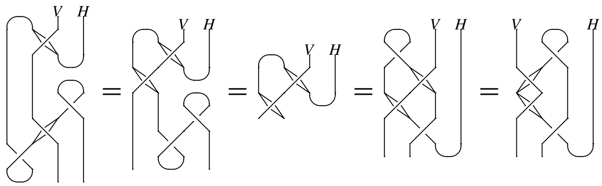


Compatibility of ${}^{\flat}(s^{-1})$ with Δ_H: This can be checked dualizing the proof that ${}^{\flat}(s^{-1})$ is compatible con μ_H.

Compatibility of ${}^{\flat}(s^{-1})$ with c_H: By the compatibility of s^{-1} with c_{H^*}, we have



${}^{\flat}(s^{-1})$ is bijective: By the discussion before Lemma 2.1, basic properties of the evaluation and coevaluation maps and Lemma 2.2,



$= \mathrm{id}_{V\otimes H}.$

A similar argument shows that ${}^{\flat}(s^{-1})$ is left invertible.

(3): Left to the reader (use the formulas for the inverse of $(s^{-1})^{\flat}$ and ${}^{\flat}(s^{-1})$ obtained in the proofs of items (1) and (2), respectively). □

Remark 2.4. Let H be a rigid braided bialgebra and (V, s) a left H-braided space. In the proof of item (1) of Proposition 2.3, it was shown that

$$\big(H^* \otimes V \otimes \mathrm{ev}_H \circ c_{HH^*}\big)\circ\big(H^* \otimes s \otimes H^*\big)\circ\big(c^{-1}_{H^*H} \circ \mathrm{coev}_H \otimes V \otimes H^*\big)$$

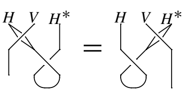
is the compositional inverse of $(s^{-1})^{\flat}$. Applying this result with H replaced by H^{op}, we obtain that

$$\big(H^* \otimes V \otimes \mathrm{ev}_H \circ c^{-1}_{H^*H}\big)\circ\big(H^* \otimes s \otimes H^*\big)\circ\big(c_{HH^*} \circ \mathrm{coev}_H \otimes V \otimes H^*\big)$$

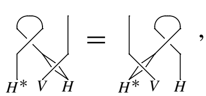
is also the inverse of $(s^{-1})^{\flat}$. So, both maps coincide. In a similar way, we can check that if (V, s) is a left H^*-braided space, then

$${}^{\flat}\big(s^{-1}\big) = \big(\mathrm{ev}_H \circ c^{-1}_{H^*H} \otimes V \otimes H\big)\circ\big(H \otimes s^{-1} \otimes H\big)\circ(H \otimes V \otimes c_{HH^*} \circ \mathrm{coev}_H).$$

Using the last equality of the Remark 2.4, and arguing as in the discussion above Lemma 2.1, it is easy to check that



and



, where  $= {}^{\flat}\big(s^{-1}\big)$ and  $= s^{-1}$.

Definition 2.5. [G-G] Let H be a braided bialgebra and A an algebra. A *left transposition* of H on A is a map $s : H \otimes A \to A \otimes H$, satisfying

(1) (A, s) is a left H-braided space,
(2) s is compatible with the algebra structure of A.

Note that by condition (1) s is bijective, which was not required in the definition given in [G-G]. It is immediate that s is a left transposition of H on A iff it is a left transposition of H_c^{op} on A and that this happens iff s is a left transposition of H_c^{cop} on A.

Definition 2.6. Let H be a braided bialgebra and A an algebra. A *right transposition* of H on A is a map $s : A \otimes H \to H \otimes A$, satisfying

(1) (A, s) is a right H-braided space,
(2) s is compatible with the algebra structure of A.

As for left transpositions, to have a right transposition of H on A is the same that to have a right transposition of H_c^{op} on A and this is equivalent to have a right transposition of H_c^{cop} on A.

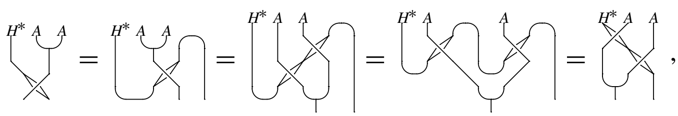
Let H be a braided bialgebra, A an algebra and $s: H \otimes A \to A \otimes H$ a bijective map. It is immediate that s is an left transposition of H on A iff s^{-1} is a right transposition of H on A.

A pair (A, s) consisting of an algebra A and a left transposition of H on A will be called a *left H-braided algebra*. Similarly, if s is a right transposition, then (A, s) will be called a *right H-braided algebra*.

Proposition 2.7. *Let H be a rigid braided bialgebra and let A be an algebra. The following facts hold*:

(1) *If $s: H \otimes A \to A \otimes H$ is a left transposition, then $(s^{-1})^{\flat}: H^* \otimes A \to A \otimes H^*$ is so too.*
(2) *If $s: H^* \otimes A \to A \otimes H^*$ is a left transposition, then ${}^{\flat}(s^{-1}): H \otimes A \to A \otimes H$ is so too.*

Proof. We prove the first assertion and we leave the second one to the reader. By Proposition 2.3 we only must check that $(s^{-1})^{\flat}$ is compatible with the algebra structure of A. It is easy to check that $(s^{-1})^{\flat}$ is compatible with η_A. Let us see that it is compatible with μ_A. Since $(H \otimes \mu)\circ(s^{-1} \otimes A)\circ(A \otimes s^{-1}) = s^{-1}\circ(\mu \otimes H)$,



as desired. □

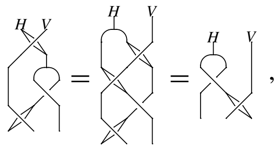
Theorem 2.8. [G-G, Theorem 4.3.1] *Let H be a rigid braided Hopf algebra, A an algebra and s a left transposition of H on A. There is a unique automorphism of algebras $g_s: A \to A$, such that $s(t \otimes a) = g_s(a) \otimes t$ for all left or right integral $t \in H$. Furthermore, we have $s\circ(f_H^r \otimes g_s) = (g_s \otimes f_H^r)\circ s$ and $s\circ(f_H^l \otimes g_s^{-1}) = (g_s^{-1} \otimes f_H^l)\circ s$, where f_H^r and f_H^l are the maps introduced in Corollary* 1.12.

Proposition 2.9. [G-G, Proposition 4.17] *Let H be a rigid braided Hopf algebra, $\alpha: H \to k$ the modular function, A an algebra and $s: H \otimes A \to A \otimes H$ a bijective transposition. Then, $(A \otimes \alpha)\circ s = \alpha \otimes A$.*

Lemma 2.10. *Let H be a braided bialgebra and (V, s) a left H-braided space. The following equality holds*:

$$\big(s^{-1} \otimes H\big)\circ\big(V \otimes c^{-1}\big)\circ(V \otimes \Delta_H)\circ s = (H \otimes s)\circ\big(c^{-1} \otimes V\big)\circ(\Delta_H \otimes V).$$

Proof. Since $(V \otimes c^{-1})\circ(s \otimes H)\circ(H \otimes s) = (s \otimes H)\circ(H \otimes s)\circ(c^{-1} \otimes V)$, we have

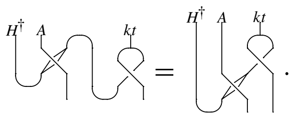


as desired. □

Proposition 2.11. *Let H be a rigid braided Hopf algebra, $s : H \otimes A \to A \otimes H$ a left transposition and $\theta : H^\dagger \to H$ the map defined by $\theta(\varphi) = \varphi \rightharpoonup t$, where t is a nonzero left integral of H. Then, $(A \otimes \theta)\circ(s^{-1})^\flat = s\circ(\theta \otimes g_s^{-1})$.*

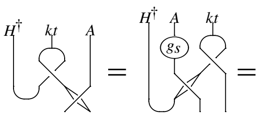
Proof. By definition

$$(A \otimes \theta) \circ \left(s^{-1}\right)^\flat =$$



Using this fact, Lemma 2.10 and Theorem 2.8, we obtain

$$s \circ (\theta \otimes A) =$$



$$= (A \otimes \theta) \circ \left(s^{-1}\right)^\flat \circ \left(H^\dagger \otimes g_s\right),$$

as we want. □

Corollary 2.12. *Let H be a rigid braided Hopf algebra and $s : H \otimes A \to A \otimes H$ a left transposition. Then $g_{(s^{-1})^\flat} = g_s^{-1}$.*

Proof. Let $T \in \int_{H^\dagger}^l$ such that $\theta(T) = 1$. By Proposition 2.11,

$$g_{(s^{-1})^\flat}(a) \otimes 1 = (A \otimes \theta)\circ\left(s^{-1}\right)^\flat(T \otimes a) = s\circ\left(\theta \otimes g_s^{-1}\right)(T \otimes a) = g_s^{-1}(a) \otimes 1,$$

as we want. □

Remark 2.13. Let $\widetilde{f}_H^r$ be as in the discussion at the end of Section 1. By the comments at the end of Section 1, we know that $\widetilde{f}_H^r = f_H^r$. Consequently, if $f_H^r = \mathrm{id}$, then $f_{H^\dagger}^r = \mathrm{id}$. Similarly, $\widetilde{f}_H^l = f_H^l$ and if $f_H^l = \mathrm{id}$, then $f_{H^\dagger}^l = \mathrm{id}$.

3. Modules and comodules

Let H be a braided bialgebra. In this section we recall from [G-G] the notions of left H-braided module and right H-braided comodule and we establish a relation between these concepts.

Definition 3.1. [G-G] Let (V, s) be a left H-braided space.

(1) We say that (V, s) is a *left H-module* or V is a left *H-braided module* if V is a left H-module in the standard sense and the action $\rho : H \otimes V \to V$ satisfies $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes V)$.
(2) We say that (V, s) is a *right H-comodule* or V is a right *H-braided comodule* if V is a right H-comodule in the standard sense and the coaction $\nu : V \to V \otimes H$ satisfies $(\nu \otimes H) \circ s = (V \otimes c) \circ (s \otimes H) \circ (H \otimes \nu)$.

A map $f : (V, s) \to (V', s')$ is a *morphism of left H-modules* if it is morphism of left H-braided spaces and $f \circ \rho = \rho' \circ (H \otimes f)$, where ρ and ρ' are the actions of H on (V, s) and (V', s'), respectively. The definition of morphism of right H-comodules is similar.

Let (V, s) be a right H-braided space. The concepts of right action of H on (V, s) and left coaction of H on (V, s) can be introduced in a similar way. We leave the details to the reader. Next, we establish a relation between these last notions and the ones introduced in Definition 3.1.

Proposition 3.2. *Let H be a braided bialgebra and (V, s) a left H-braided space. Then (V, s) is a left H-module via ρ iff (V, s^{-1}) is a right H_c^{op}-module via $\rho \circ s^{-1}$. Similarly, (V, s) is a right H-comodule via ν iff (V, s^{-1}) is a left H_c^{cop}-comodule via $s^{-1} \circ \nu$.*

Proof. Left to the reader. □

Let H be a rigid braided bialgebra and let (V, s) be a left H-braided space. Given a map $\nu : V \to V \otimes H$, we define a map $\rho_\nu : H^\dagger \otimes V \to V$ by

$$\rho_\nu := (V \otimes \mathrm{ev}_H) \circ \big((s^{-1})^\flat \otimes H \big) \circ \big(H^\dagger \otimes \nu \big).$$

Conversely, given a map $\rho : H^\dagger \otimes V \to V$, we define $\nu_\rho : V \to V \otimes H$ by

$$\nu_\rho := s \circ (H \otimes \rho) \circ (\mathrm{coev}_H \otimes V).$$

It is easy to check that these constructions are inverse one of each other.

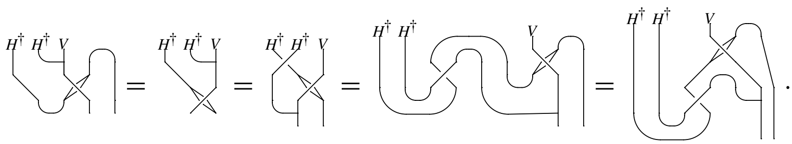
Let H be a rigid braided bialgebra and let $H^\dagger$ be as in Notation 1.19. By Proposition 2.3 and the discussion at the beginning of Section 2, we know that (V, s) is a left H-braided space iff $(V, (s^{-1})^\flat)$ is a left $H^\dagger$-braided space.

Lemma 3.3. *Let H be a rigid braided bialgebra and let (V, s) be a left H-braided space. Then, each left $H^\dagger$-module structure on $(V, (s^{-1})^\flat)$, satisfies*

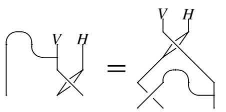
$$(H \otimes \rho \otimes H) \circ (\mathrm{coev}_H \otimes s) = (H \otimes s) \circ (c \otimes \rho) \circ (H \otimes \mathrm{coev}_H \otimes V),$$

where $\rho : H^\dagger \otimes V \to V$ is the action of $H^\dagger$ on V.

Proof. By the definitions of $(s^{-1})^\flat$ and $c_{H^\dagger}$, the fact that $(V, (s^{-1})^\flat)$ is a left $H^\dagger$-module and basic properties of the evaluation and coevaluation maps, we have



From this it follows that



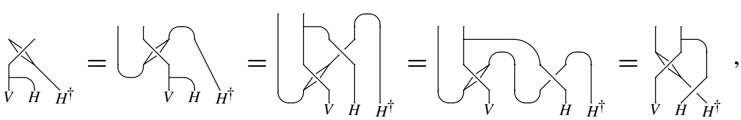
which clearly implies our assertion. □

Lemma 3.4. *Let H be a rigid braided bialgebra. If (V, s) is a right H-comodule, then*

$$\big(\nu \otimes H^\dagger\big)\circ\big(s^{-1}\big)^\flat = (V \otimes c_{H^\dagger H})\circ\big(\big(s^{-1}\big)^\flat \otimes H\big)\circ\big(H^\dagger \otimes \nu\big),$$

where ν is the coaction of (V, s).

Proof. Since ν is a map of H-braided spaces, we have



where the first equality follows from the definition of $(s^{-1})^\flat$. □

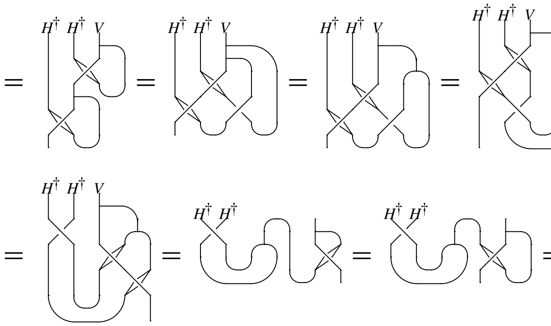
Theorem 3.5. *Let H be a rigid braided bialgebra and (V, s) a left H-braided space. Then (V, s) is a right H-comodule via ν iff $(V, (s^{-1})^\flat)$ is a left $H^\dagger$-module via ρ_ν.*

Proof. $(\Rightarrow)$: Since $(V, (s^{-1})^\flat)$ is a left $H^\dagger$-braided space and ν is a coaction, we have

$$\rho_\nu(\epsilon \otimes v) = (V \otimes \mathsf{ev}_H)\circ\big(\big(s^{-1}\big)^\flat \otimes H\big)\circ\big(H^\dagger \otimes \nu\big)(\epsilon \otimes v) = \epsilon(v_{(1)})v_{(0)} = v,$$

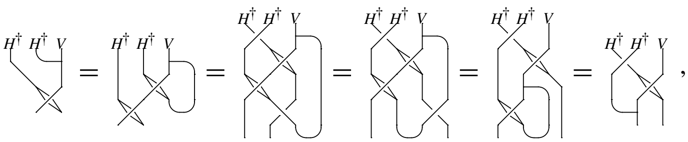
for all $v \in V$. Hence, ρ_ν is unitary. By Lemma 3.4, the fact that (V, ν) is a right H-comodule, the discussion following Theorem 1.7, the compatibility of s with c, the discussion at the beginning of Section 2, and the compatibility of s with Δ, we have

$$\rho_\nu \circ \left(H^\dagger \otimes \rho_\nu\right) =$$



$$= \rho_\nu \circ (\mu_{H^\dagger} \otimes V),$$

which shows that ρ_ν is associative. It remains to check that $(s^{-1})^\flat \circ (H^\dagger \otimes \rho_V) = (\rho_V \otimes H^\dagger) \circ (H^\dagger \otimes (s^{-1})^\flat) \circ (c_{H^\dagger} \otimes V)$. But, by the compatibility of $(s^{-1})^\flat$ with $c_{H^\dagger}$, the discussion following Theorem 1.7 and Lemma 3.4, we have



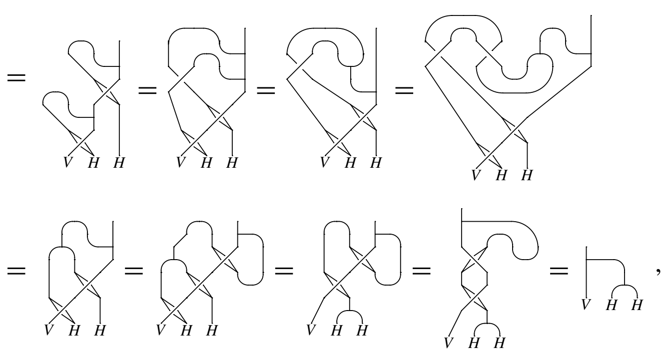
as desired.

($\Leftarrow$): Let $v \in V$. Since s is compatible with ϵ and ρ_ν is unitary, we have

$$\begin{aligned}
(V \otimes \epsilon) \circ \nu(v) &= (V \otimes \epsilon) \circ s \circ (H \otimes \rho_\nu) \circ (\mathsf{coev}_H \otimes V)(v) \\
&= (\epsilon \otimes V) \circ (H \otimes \rho_\nu) \circ (\mathsf{coev}_H \otimes V)(v) \\
&= \rho_\nu(\epsilon \otimes v) \\
&= v,
\end{aligned}$$

where the first equality follows from the discussion following Proposition 3.2. Hence, ν is counitary. By Lemma 3.3, the fact that (V, s) is a left $H^\dagger$-module, the definition of the multiplication in $H^\dagger$, basic properties of the evaluation and coevaluation maps, the relation between ρ_ν and ν, the compatibility of s with Δ and the discussion at the beginning of Section 2, we have

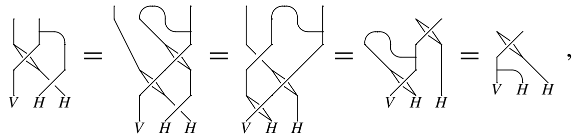
$$\rho_\nu \circ \left(H^\dagger \otimes \rho_\nu\right) =$$



which shows that ν is coassociative. It remains to check that

$$(\nu \otimes H)\circ s = (V \otimes c)\circ(s \otimes H)\circ(H \otimes \nu).$$

But, by the relation between ρ_ν and ν, the compatibility of s with c and Lemma 3.3, we have



as we want. □

4. Module algebras and comodule algebras

Let H be a braided bialgebra. In this section we introduce the notions of left H-module algebra and right H-comodule algebra and we study the relation between these concepts.

Definition 4.1. [G-G] Let (A, s) be a left H-braided algebra.

(1) We say that (A, s) is a *left H-module algebra* if (A, s) is a left H-module and the action $\rho : H \otimes A \to A$ satisfies:
  (a) $\rho(h \otimes 1) = \epsilon(h)1$,
  (b) $\rho\circ(H \otimes \mu) = \mu\circ(\rho \otimes \rho)\circ(H \otimes s \otimes A)\circ(\Delta \otimes A \otimes A)$.
(2) We say that (A, s) is a *right H-comodule algebra* if (A, s) is a right H-comodule and the coaction $\nu : A \to A \otimes H$ satisfies:
  (a) $\nu(1) = 1 \otimes 1$,
  (b) $\nu\circ\mu = (\mu \otimes \mu)\circ(A \otimes s \otimes A)\circ(\nu \otimes \nu)$.

Items (1) and (2) of the above definition can be expressed saying that ρ and ν are compatible with the algebra structure of A.

A map $f : (A, s) \to (A', s')$ is a *morphism* of left H-module algebras if it is morphism of left H-modules and a morphism of algebras. The definition of morphism of right H-comodule algebras is similar.

Let A be a k-algebra and $s : A \otimes H \to H \otimes A$ a right transposition. The notion of right H-module algebra structure and left H-comodule algebra structure on (A, s) can be introduced in a similar way. We leave the details to the reader. Next, we establish a relation between these notions and the ones introduced in Definition 4.1.

Proposition 4.2. *Let H be a braided bialgebra and let (A, s) be a left H-braided algebra. Then (A, s) is a left H-module algebra via $\rho : H \otimes A \to A$ iff (A, s^{-1}) is a right H_c^{op}-module algebra via $\rho\circ s^{-1}$. Similarly, (A, s) is a right H-comodule algebra via ν iff (A, s^{-1}) is a right H_c^{cop}-comodule algebra via $s^{-1}\circ\nu$.*

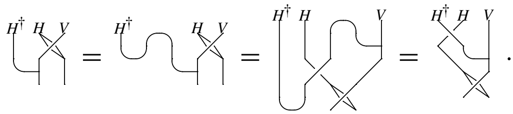
Proof. From Proposition 3.2 and the discussion following Definition 2.6 it follows immediately that in order to check the first assertion it suffices to show that ρ satisfies conditions (a) and (b) of item (1) of Definition 4.1 iff $\rho\circ s^{-1}$ satisfies the analogous conditions. We leave this task to the reader. The second assertion can be checked similarly. □

Lemma 4.3. *Let H be a rigid braided bialgebra and let (V, s) be a left H-braided space. If $(V, (s^{-1})^\flat)$ is a left $H^\dagger$-module, then*

$$s^{-1}\circ(\rho\otimes H) = (H\otimes\rho)\circ\big(c_{HH^\dagger}^{-1}\otimes V\big)\circ\big(H^\dagger\otimes s^{-1}\big),$$

where ρ denotes the action of $H^\dagger$ on V.

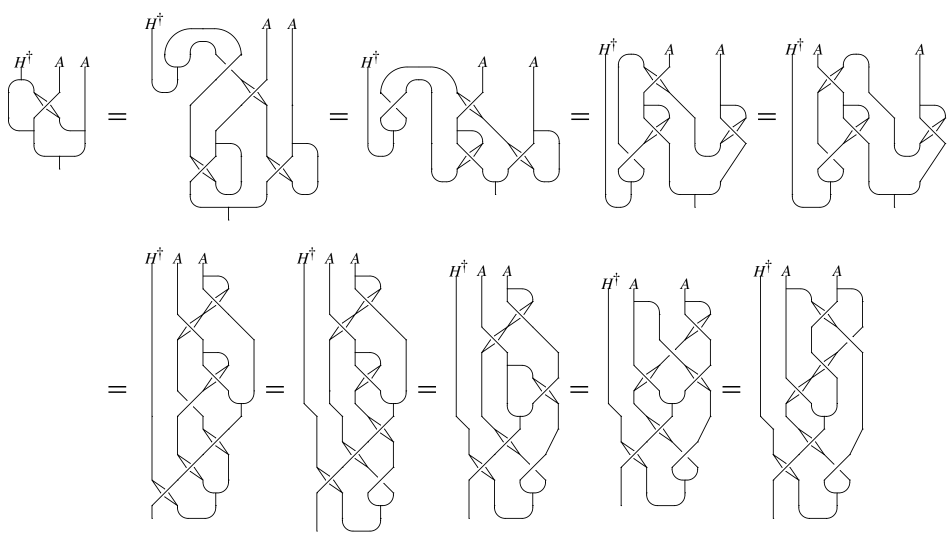
Proof. By Lemma 3.3, the discussion following Theorem 1.7, and basic properties of the evaluation and coevaluation maps

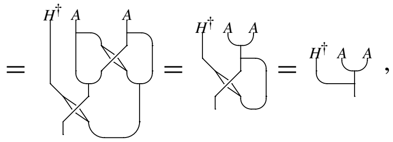


The assertion follows immediately from this equality. □

Theorem 4.4. *Let H be a rigid braided bialgebra and let (A, s) be a left H-braided algebra. Then (A, s) is a right H-comodule algebra via ν iff $(A, (s^{-1})^\flat)$ is a left $H^\dagger$-module algebra via ρ_ν.*

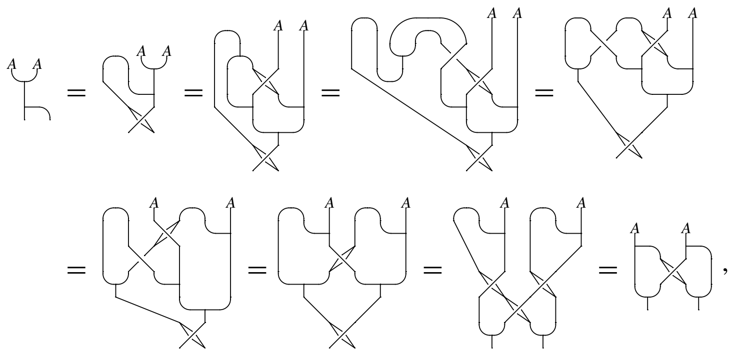
Proof. (⇒): By Proposition 2.7 and Theorem 3.5 it suffices to check that $\rho_\nu(\varphi\otimes 1) = \varphi(1)1$ for all $\varphi\in H^\dagger$ and $\rho_\nu\circ(H^\dagger\otimes\mu) = \mu\circ(\rho_\nu\otimes\rho_\nu)\circ(H^\dagger\otimes(s^{-1})^\flat\otimes A)\circ(\Delta_{H^\dagger}\otimes A\otimes A)$. The first assertion is immediate. Let us consider the second one. By the definitions of $\Delta_{H^\dagger}$ and ρ_ν, the discussion following Theorem 1.7, the discussion at the beginning of Section 2, the compatibility of s with μ_H, μ_A and c, the fact that (V, s) is a right H-comodule via ν and $\nu\circ\mu = (\mu\otimes\mu)\circ(A\otimes s\otimes A)\circ(\nu\otimes\nu)$, we have





as desired.

(⇐): We must check that $\nu(1) = 1 \otimes 1$ and $\nu \circ \mu = (\mu \otimes \mu) \circ (A \otimes s \otimes H) \circ (\nu \otimes \nu)$. The first assertion is immediate. Let us consider the second one. By the relation between ν and ρ_ν, the facts that $(A, (s^{-1})^\flat)$ is a left $H^\dagger$-module algebra and s is a left transposition, the definition of $\Delta_{H^\dagger}$, the discussion following Theorem 1.7, the discussion at the beginning of Section 2, and Lemma 4.3, we have



as we want. □

5. *H*-invariants

Let H be a braided bialgebra and let V be a standard left (right) H-module. Recall that an element v of V is *H-invariant* if $h \cdot v = \epsilon(h)v$ ($v \cdot h = \epsilon(h)v$) for all $h \in H$. We let ${}^H V$ (V^H) denote the set of H-invariants of V. Note that this is not the notation used in [G-G], where the set of invariants of a left action is denoted V^H.

Proposition 5.1. *Let (V, s) be a left H-module and let $\chi : H \otimes V \to V \otimes H$ be the map $\chi := (\rho \otimes H) \circ (H \otimes s) \circ (\Delta \otimes V)$. Then, an element $v \in V$ is H-invariant iff $\chi(h \otimes v) = s(h \otimes v)$ for all $h \in H$.*

Proof. For (V, s) a left H-module algebra this is [G-G, Proposition 7.2]. The same proof works for left H-modules. □

Proposition 5.2. *$s(H \otimes {}^H V) \subseteq {}^H V \otimes H$ for each left H-module (V, s).*

Proof. For (V, s) a left H-module algebra this is [G-G, Proposition 7.4]. Se same proof works for left H-modules. □

Let V be a standard right (left) H-comodule with coaction ν. Recall that an element v of V is H-coinvariant if $\nu(v) = v \otimes 1$ ($\nu(v) = 1 \otimes v$). We let V^{coH} (${}^{\mathrm{coH}}V$) denote the set of H-coinvariant elements of V under a right (left) action. In [G-G, Remark 5.1] it was note that if (V,s) is a right H-comodule, then V^{coH} is stable under s (that is, $s(H\otimes V^{\mathsf{coH}})\subseteq V^{\mathsf{coH}}\otimes H$). Assume that H is a rigid braided bialgebra. By Theorem 3.5, we know that if (V,s) is a right H-comodule, then $(V,(s^{-1})^{\flat})$ is a left $H^{\dagger}$-module.

Proposition 5.3. *It is true that* $V^{\mathsf{coH}} = {}^{H^{\dagger}}V$.

Proof. Let $v\in V^{\mathsf{coH}}$. Since $(s^{-1})^{\flat}$ is compatible with $\epsilon_{H^{\dagger}}$,

$$\varphi\cdot v=(V\otimes \mathsf{ev}_H)\circ\big(\big(s^{-1}\big)^{\flat}\otimes H\big)(\varphi\otimes v\otimes 1)=(V\otimes\epsilon_{H^{\dagger}})\big(\big(s^{-1}\big)^{\flat}(\varphi\otimes v)\big)=\varphi(1)v,$$

for all $\varphi\in H^{\dagger}$. So, $v\in {}^{H^{\dagger}}V$. Conversely, if $v\in {}^{H^{\dagger}}V$, then by the discussion following Proposition 3.2 and the compatibility of s with 1, we have

$$\nu(v)=s\left(\sum_{i=1}^{n}h_i\otimes h_i^{*}\cdot v\right)=s\left(\sum_{i=1}^{n}h_ih_i^{*}(1)\otimes v\right)=s(1\otimes v)=v\otimes 1,$$

where ν is the coaction of V and $(h_i,h_i^{*})_{1\leqslant i\leqslant n}$ are dual basis of H. □

The following theorem was communicated to us by the referee of [G-G].

Theorem 5.4. *Let* (V,s) *be a left* H*-module. If* H *be a rigid braided bialgebra, then* $s(H\otimes {}^{H}V)={}^{H}V\otimes H$.

Proof. Consider s as a map from $H^{\dagger\dagger}\otimes V$ to $V\otimes H^{\dagger\dagger}$ and (V,s) as a left $H^{\dagger\dagger}$-module. By Theorem 3.5 $(V,{}^{\flat}(s^{-1}))$ is a right $H^{\dagger}$-comodule. Let $\nu: V\to V\otimes H^{\dagger}$ denote the corresponding coaction. By Proposition 5.3, ${}^{H}V=V^{\mathsf{coH}^{\dagger}}$. Let $\zeta: V\to V\otimes H^{\dagger}$ be the map defined by $\zeta(v)=\nu(v)-v\otimes\epsilon$. Let $s_{|}: H\otimes {}^{H}V\to {}^{H}V\otimes H$ be the map induced by s and let $s': H\otimes(V\otimes H^{\dagger})\to(V\otimes H^{\dagger})\otimes H$ be the map $s'=(V\otimes c_{HH^{\dagger}})\circ(s\otimes H^{\dagger})$. Since the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H\otimes {}^{H}V & \longrightarrow & H\otimes V & \xrightarrow{H\otimes\zeta} & H\otimes(V\otimes H^{\dagger})\\
 & & \big\downarrow{\scriptstyle s_{|}} & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s'}\\
0 & \longrightarrow & {}^{H}V\otimes H & \longrightarrow & H\otimes V & \xrightarrow{\zeta\otimes H} & (V\otimes H^{\dagger})\otimes H
\end{array}$$

has exact rows and s and s' are isomorphisms, $s_{|}$ is also. □

By this theorem, $({}^{H}V,s_{|})$ is an H-braided space. Furthermore, by [G-G, Proposition 7.3], if (A,s) is a left H-module algebra, then $s_{|}: H\otimes {}^{H}A\to {}^{H}A\otimes H$ is a transposition.

Recall that, by Proposition 3.2, if (V,s) is a left H-module, then (V,s^{-1}) is a right H_c^{op}-module. The following result is used without mention in the proof of [G-G, Proposition 8.1].

Proposition 5.5. *Suppose that* (V,s) *is a left* H*-module. Then* ${}^{H}V=V^{H_c^{\mathsf{op}}}$.

Proof. Left to the reader. □

Proposition 5.6. *Let H be a rigid Hopf algebra and (A, s) a left H-module algebra. The automorphism $g_s : A \to A$ of Theorem* 2.8 *satisfies $g_s({}^H\!A) = {}^H\!A$.*

Proof. By Theorem 2.8 and the discussion preceding Proposition 2.7, we know that the map $s : kt \otimes A \to A \otimes kt$ is bijective. Hence, by Theorem 5.4, the map $s : kt \otimes {}^H\!A \to {}^H\!A \otimes kt$ is also. The proposition follows immediately from this fact. □

6. Smash products

Let H be a braided bialgebra. By [G-G, Theorem 6.3 and Proposition 6.5], we know that if (A, s) is a left H-module algebra, then the map $\chi : H \otimes A \to A \otimes H$, defined by $\chi := (\rho \otimes H)\circ(H \otimes s)\circ(\Delta \otimes A)$, is compatible with the algebra structures of A and H. Hence, as was shown in [C-S-V], the tensor product $A \otimes H$ is an algebra $A \otimes_\chi H$, with multiplication $\mu_{A\otimes_\chi H} := (\mu_A \otimes \mu_H)\circ(A \otimes \chi \otimes H)$. This algebra is called the *smash product* of A with H associated with (s, ρ), and it is also denoted $A \# H$. We frequently identify A and H with the subsets $A \otimes 1$ and $1 \otimes H$ of $A \# H$, respectively. Consequently, we sometimes write ah instead of $a \# h$.

It is easy to check that A is an $(A \# H, {}^H\!A)$-bimodule via the regular right action and the left action

$$(a \# h) \cdot b = a(h \cdot b). \tag{1}$$

Furthermore, arguing as in the proof of [G-G, Proposition 8.1] it can be shown that if H is a braided Hopf algebra with bijective antipode S, then A is an $(A^H, A \# H)$-bimodule via the regular left action and the right action

$$b \cdot (a \# h) = \sum_i S^{-1}(h_i) \cdot (ba)_i, \tag{2}$$

where $\sum_i h_i \otimes (ba)_i = s^{-1}(ba \otimes h)$ and A^H is the set of invariants of A under the right action of H obtained by restriction of (2).

Let H be a braided Hopf algebra with bijective antipode S. We just note that if (A, s) is left H-module algebra via $\rho : H \otimes A \to A$, then (A, s^{-1}) is a right H^{cop}-module algebra via $\rho\circ s^{-1}\circ(A \otimes S^{-1})$. By Proposition 5.5 we know that $A^H = {}^H\!A$. To unify expressions, from now on we always will write ${}^H\!A$ to denote this set of invariants.

The next results generalize Lemmas 0.3 and 0.6 of [C-F-M], and their proof are closed to the ones given in that paper.

Proposition 6.1. *Let H be a braided bialgebra and let (A, s) be a left H-module algebra. The following assertions hold*:

(1) ${}^H\!A \simeq \mathrm{End}({}_{A\#H}A)^{\mathrm{op}}$, *where we consider A as a left $A \# H$-module as in* (1).
(2) *If H is a braided Hopf algebra with bijective antipode, then ${}^H\!A \simeq \mathrm{End}(A_{A\#H})$, where A is considered as a right $A \# H$-module as in* (2).

Proof. We prove the second assertion because the first one is easier. Let $L_a : A \to A$ denote the left multiplication by a. Is is clear that the map $a \mapsto L_a$, from ${}^H\!A$ to $\mathrm{End}(A_{A\#H})$, is well defined and injective. We claim that it is surjective. Let $f \in \mathrm{End}(A_{A\#H})$. Since, $f(a) = f(1 \cdot a) =$

$f(1)\cdot a$, for each $a\in A$, in order to check the claim it suffices to show that $f(1)\in {}^H\!A$. This follows from the above discussion and the fact that $f(1)\cdot h = f(1\cdot h) = f(S^{-1}(h)\cdot 1) = \epsilon(h)f(1)$, for all $h\in H$. □

Proposition 6.2. *Let H be braided bialgebra and let (A,s) be a left H-module algebra. Consider A as a left $A\#H$-module via the action (a). For each left $A\#H$-module M and each $m_0\in {}^H\!M$, the map $\Psi: A\to M$, defined by $\Psi(a) := a\cdot m_0$, is a left $A\#H$-homomorphism. Furthermore, if $a\in {}^H\!A$, then $a\cdot m_0\in {}^H\!M$.*

Proof. Let $a,b\in A$ and $h\in H$. Since m_0 is invariant and s is compatible with ϵ,

$$\begin{aligned}
(b\#h)\cdot\Psi(a) &= (b\#h)\cdot(a\cdot m_0)\\
&= \big((b\#h)(a\#1)\big)\cdot m_0\\
&= \sum_i b(h_{(1)}\cdot a_i\# h_{(2)_i})\cdot m_0\\
&= \sum_i b(h_{(1)}\cdot a_i)\epsilon(h_{(2)_i})\cdot m_0\\
&= \sum_i b(h\cdot a)\cdot m_0\\
&= \Psi\big((b\#h)\cdot a\big),
\end{aligned}$$

where $\sum_i h_{(1)}\otimes a_i\otimes h_{(2)_i} = (H\otimes s)\circ(\Delta\otimes A)(h\otimes a)$. The last assertion can be easily checked. □

Proposition 6.3. *Let H be braided Hopf algebra with bijective antipode S and (A,s) a left H-module algebra. Consider A as a right $A\#H$-module via the action (b). For each right $A\#H$-module M and each $m_0\in M^H$, the map $\Psi': A\to M$, defined by $\Psi'(a) := m_0\cdot a$, is a right $A\#H$-homomorphism. Furthermore, if $a\in {}^H\!A$, then $m_0\cdot a\in M^H$.*

Proof. Let $a,b\in A$ and $h\in H$. By the proof of [G-G, Proposition 6.9],

$$ab\#h = \sum_{ij}(1\# h_{i(2)_j})\big(S^{-1}(h_{i(1)_j})\cdot(ab)_i\#1\big),$$

where $\sum_{ij} h_{i(2)_j}\otimes S^{-1}(h_{i(1)_j})\cdot(ab)_i = (H\otimes\rho\circ(S^{-1}\otimes A))\circ(c^{-1}\circ\Delta\otimes A)\circ s^{-1}(ab\otimes h)$. Using this, the fact that $m_0\in M^H$ and the compatibility of c^{-1} with ϵ, we obtain

$$\begin{aligned}
\Psi'(a)\cdot(b\#h) &= (m_0\cdot a)\cdot(b\#h)\\
&= m_0\cdot(ab\#h)\\
&= m_0\cdot\left(\sum_{ij}(1\# h_{i(2)_j})\big(S^{-1}(h_{i(1)_j})\cdot(ab)_i\#1\big)\right)
\end{aligned}$$

$$= m_0 \cdot \left(\sum_{ij} \epsilon(h_{i\,(2)_j}) \big(S^{-1}(h_{i\,(1)_j}) \cdot (ab)_i \,\#\, 1\big) \right)$$
$$= m_0 \cdot \big(S^{-1}(h_i) \cdot (ab)_i \,\#\, 1\big)$$
$$= \Psi'\big(a \cdot (b \,\#\, h)\big),$$

as we want. The last assertion can be easily checked. □

Proposition 6.4. *Let H be a rigid braided Hopf algebra and (A, s) a left H-module algebra. Let t be a nonzero left or right integral of H. Then AtA is an ideal of $A \# H$.*

Proof. For left integrals t this result is [G-G, Proposition 7.8]. For right integrals t the assertion can be check in a similar way. We leave the details to the reader. □

Let H be a rigid braided Hopf algebra, (A, s) a left H-module algebra and $A \# H$ the smash product constructed from these data. Consider $H \otimes A$ and $A \# H$ as left H-modules via the actions $l \cdot (h \otimes a) := lh \otimes a$ and $l \cdot (ah) := lah$, respectively. Let t be a nonzero left integral of H. We assert that ${}^H(A \# H) = tA$. In fact, in [G-G, Proposition 6.9] it was proved that the H-linear map $\theta : H \otimes A \to A \# H$ given by $\theta(h \otimes a) = ha$ is bijective. So, ${}^H(A \# H) = \theta({}^H(H \otimes A)) = \theta(t \otimes A) = tA$.

Proposition 6.5. *Let H be a rigid braided Hopf algebra, α the modular function of H and (A, s) a left H-module algebra. The map $(-)^\alpha : A \# H \to A \# H$, defined by $(a \# h)^\alpha = a \# h^\alpha$, where h^α is the map introduced in Remark* 1.23, *is an automorphism of algebras.*

Proof. Clearly $(-)^\alpha$ is bijective and $(1 \# 1)^\alpha = 1 \# 1$. For $h \in H$ and $b \in A$ let $s(h \otimes b) = \sum_i b_i \otimes h_i$. Then

$$\begin{aligned}
\big((a \# h)(b \# l)\big)^\alpha &= \sum_i \big(a(h_{(1)} \cdot b_i) \# h_{(2)_i} l\big)^\alpha \\
&= \sum_i a(h_{(1)} \cdot b_i) \# (h_{(2)_i} l)^\alpha \\
&= \sum_i a(h_{(1)} \cdot b_i) \# (h_{(2)_i})^\alpha l^\alpha \\
&= \sum_i a(h_{(1)} \cdot b_i) \# h_{(2)_{i\,(1)}} \alpha(h_{(2)_{i\,(2)}}) l^\alpha \\
&= \sum_i a(h_{(1)} \cdot b_i) \# h_{(2)_i} \alpha(h_{(3)}) l^\alpha \\
&= (a \# h)^\alpha (b \# l)^\alpha,
\end{aligned}$$

where the third and fourth equality follow from Remark 1.23 and the last one from the compatibility of s with the comultiplication and Proposition 2.9. □

7. Galois extensions

Let H be a braided bialgebra and let (A, s) be a right H-comodule algebra. Let ν denote the coaction of A. Recall from [G-G, Section 10] that $(A^{\mathrm{coH}} \hookrightarrow A, s)$ is called a right H-extension of A^{coH}, and that such an H-extension is said to be H-Galois if the map $\beta_A : A \otimes_{A^{\mathrm{coH}}} A \to A \otimes H$, defined by $\beta_A(a \otimes b) = (a \otimes 1)\nu(b)$, is bijective. Furthermore, by Theorem 4.4 and Proposition 5.3, if H is rigid, then $(A, (s^{-1})^\flat)$ is a left $H^\dagger$-module algebra and ${}^{H^\dagger}\!A = A^{\mathrm{coH}}$.

Remark 7.1 and Theorem 7.2 below generalize results of [K-T]. In the proof of the second one we follow closely an argument of Schneider [Sch].

Remark 7.1. Let H be a braided Hopf algebra with bijective antipode S and let (A, s) be a right H-comodule algebra. Let

$$\beta'_A : A \otimes_{A^{\mathrm{coH}}} A \to A \otimes H \quad \text{and} \quad \Phi : A \otimes H \to A \otimes H$$

be the maps defined by $\beta'_A := (\mu_A \otimes H)\circ(A \otimes s)\circ(\nu_A \otimes A)$ and $\Phi := (A \otimes \mu_H)\circ(\nu_A \otimes S)$. Then, the following facts hold:

(1) Φ is bijective, with inverse $\Phi^{-1} = (A \otimes \mu_H)\circ(A \otimes c^{-1})\circ(\nu_A \otimes S^{-1})$.
(2) $\Phi\circ\beta_A = \beta'_A$. Consequently, β_A is injective (surjective) if and only if β'_A is.

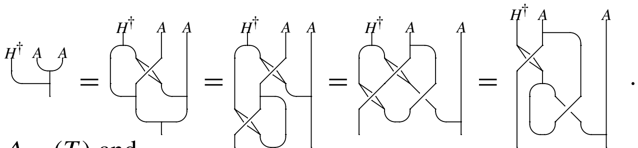
Theorem 7.2. *Let H be a rigid braided Hopf algebra and (A, s) a right H-comodule algebra such that the Galois map β_A is surjective. Let $T \in H^\dagger$ be a nonzero left integral. Then*:

(1) *There exist elements $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $(x \mapsto T \cdot (b_i x), a_i)$ is a projective basis of A as a right ${}^{H^\dagger}\!A$-module.*
(2) *β_A is injective, and so bijective.*

Proof. (1): By Corollary 1.21 there exists a right integral $u \in H$, such that $T(uh) = \epsilon(h)$, for all $h \in H$. Since $\beta_A : A \otimes_{{}^{H^\dagger}\!A} A \to A \otimes H$ is surjective and $g_{(s^{-1})^\flat} : A \to A$ is an algebra automorphism, there exists $\sum_i a_i \otimes b_i \in A \otimes_{{}^{H^\dagger}\!A} A$, such that

$$\sum_i a_i g_{(s^{-1})^\flat}(b_{i\,(0)}) \otimes b_{i\,(1)} = 1 \otimes u. \tag{3}$$

By Theorem 4.4, the definition of ρ_ν, Lemma 3.4 and the compatibility of $\Delta_{H^\dagger}$ with $(s^{-1})^\flat$,



(4)

Let $T_{(1)} \otimes T_{(2)} = \Delta_{H^\dagger}(T)$ and

$$\sum_j b_{i\,(0)} \otimes T_{(1)} \otimes b_{i\,(1)_j} \otimes T_{(2)_j} = b_{i\,(0)} \otimes T_{(1)} \otimes c_{H^\dagger H}(T_{(2)} \otimes b_{i\,(1)}).$$

Let $\phi_i : A \to {}^{H^\dagger}A$ denote the right ${}^{H^\dagger}A$-linear map defined by $\phi_i(a) = T \cdot (b_i a)$. Using (4) and that $(s^{-1})^\flat(T \otimes b) = g_{(s^{-1})^\flat}(b) \otimes T$, we obtain

$$\sum_i a_i \phi_i(a) = \sum_i a_i T \cdot (b_i a) = \sum_i a_i g_{(s^{-1})^\flat}(b_{i\,(0)}) \langle T_{(1)}, b_{i\,(1)_j} \rangle (T_{(2)_j} \cdot a). \tag{5}$$

Now, from Remark 1.14 it follows easily that $c_{H^\dagger H}(\varphi \otimes u) = u \otimes \varphi \circ (f_H^l)^{-1}$ for all $\varphi \in H^\dagger$. Using this fact, (3) and (5) we obtain

$$\sum_i a_i \phi_i(a) = \langle T_{(1)}, u \rangle \big(T_{(2)} \circ (f_H^l)^{-1}\big) \cdot a.$$

Since, by the definition of $\Delta_{H^\dagger}$ and Remark 1.14,

$$\langle T_{(1)}, u \rangle \big\langle T_{(2)}, \big(f_H^l\big)^{-1}(h)\big\rangle = \big\langle T, \mu \circ c\big(\big(f_H^l\big)^{-1}(h) \otimes u\big)\big\rangle = \langle T, uh \rangle = \epsilon(h),$$

for all $h \in H$, we have $\sum_i a_i \phi_i(a) = \epsilon \cdot a = a$, as we want.

(2): By Remark 7.1 it suffices to show that β'_A is injective. To simplify notations we set $B = A^{\mathrm{coH}}$. Let

$$\sum_j u_j \otimes_B v_j \in \ker\big(\beta'_A\big), \quad \text{so that } \sum_j u_{j\,(0)} v_{j_i} \otimes u_{j\,(1)_i} = 0,$$

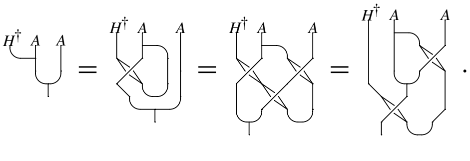
where $\sum_j u_{j\,(0)} \otimes v_{j_i} \otimes u_{j\,(1)_i} = \sum_j u_{j\,(0)} \otimes s(u_{j\,(1)} \otimes v_j)$. Let

$$\chi = \big(\rho_\nu \otimes H^\dagger\big) \circ \big(H^\dagger \otimes \big(s^{-1}\big)^\flat\big) \circ (\Delta_{H^\dagger} \otimes A) \quad \text{and} \quad x = \sum_{ij} a_i \otimes T \otimes b_i \otimes u_j \otimes_B v_j.$$

We have

$$\begin{aligned}
\sum u_j \otimes_B v_j &= (\mu_A \otimes A) \circ (A \otimes \rho_\nu \otimes_B A) \circ \big(A \otimes H^\dagger \otimes \mu_A \otimes_B A\big)(x) \\
&= (A \otimes_B \mu_A) \circ (A \otimes \rho_\nu \otimes_B A) \circ \big(A \otimes H^\dagger \otimes \mu_A \otimes_B A\big)(x) \\
&= (A \otimes_B \mu_A) \circ (A \otimes A \otimes \mu_A) \circ (A \otimes A \otimes \rho_\nu \otimes A) \circ (A \otimes \chi \otimes A \otimes A)(x) \\
&= (A \otimes_B \mu_A \otimes \mathrm{ev}_H) \circ \big(A \otimes A \otimes \big(s^{-1}\big)^\flat \otimes H\big) \circ \big(A \otimes \chi \otimes \beta'_A\big)(x) = 0,
\end{aligned}$$

where the first equality follows from item (1), the second one from Proposition 5.3, the third one from the fact that A is a left $H^\dagger$-module algebra and the last one from the fact that

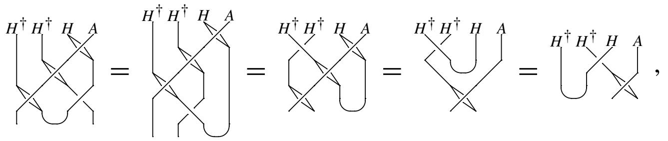


Thus β'_A is injective. $\square$

Lemma 7.3. *Let H be a rigid braided bialgebra and $s : H \otimes A \to A \otimes H$ a left transposition. Then,*

$$\big(A \otimes \mathsf{ev}_H \otimes H^\dagger\big)\circ\big(\big(s^{-1}\big)^\flat \otimes c_{H^\dagger H}\big)\circ\big(H^\dagger \otimes \big(s^{-1}\big)^\flat \otimes H\big)\circ\big(\big(H^\dagger\big)^{\otimes 2} \otimes s\big)$$
$$= \big(\mathsf{ev}_H \otimes \big(s^{-1}\big)^\flat\big)\circ\big(H^\dagger \otimes c_{H^\dagger H} \otimes A\big).$$

Proof. By the discussion following Theorem 1.7, the discussion at the beginning of Section 2 and the compatibility of $(s^{-1})^\flat$ with $c_{H^\dagger}$,



as we assert. □

Theorems 7.4 and 7.5 below generalize Theorems 1.2 and 1.2′ of [C-F-M] and their proofs are very close to the ones given in that paper.

Theorem 7.4. *Let H be a rigid braided Hopf algebra and let (A, s) be a right H-comodule algebra. Consider A as an $(A \# H^\dagger, {}^{H^\dagger}\!A)$-bimodule via the actions defined at the beginning of Section 6. Let $g_s : A \to A$ be the automorphism of algebras introduced in Theorem 2.8. Let us ${}^g\!A$ denote A endowed with the left A-module structure given by $a \cdot b = g_s(a)b$. The following assertions are equivalent*:

(1) *$({}^{H^\dagger}\!A \hookrightarrow A, s)$ is H-Galois.*
(2) (a) *The map $\pi : A \# H^\dagger \to \mathsf{End}(A_{{}^{H^\dagger}\!A})$, defined by $\pi(a \# \varphi)(b) := (a \# \varphi) \cdot b$, is an algebra isomorphism.*
    (b) *A is a finitely generated projective right ${}^{H^\dagger}\!A$-module.*
(3) *A is a left $A \# H^\dagger$-generator.*
(4) *If $0 \neq T \in \int^l_{H^\dagger}$, then the map $[\,,\,] : A \otimes_{{}^{H^\dagger}\!A} {}^g\!A \to A \# H^\dagger$ given by $[a, b] = aTb$ is surjective.*
(5) *For any left $A \# H^\dagger$-module M, the map $F_M : A \otimes_{{}^{H^\dagger}\!A} {}^{H^\dagger}\!M \to M$ defined by $F_M(a \otimes m) = a \cdot m$ is a left $A \# H^\dagger$-module isomorphism, where $A \otimes_{{}^{H^\dagger}\!A} {}^{H^\dagger}\!M$ has the $A \# H^\dagger$-module structure*

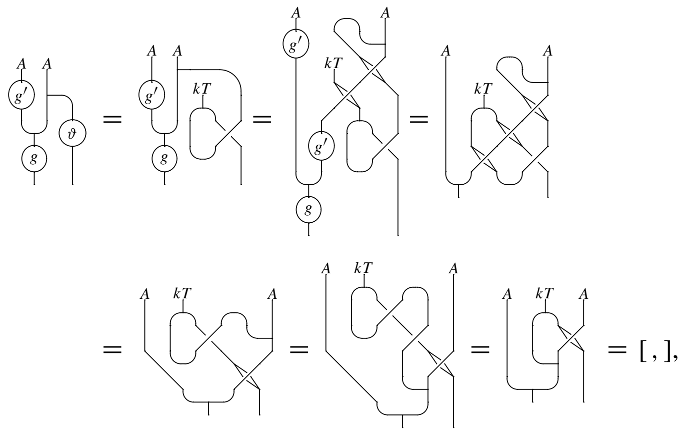
$$(a \# \varphi) \cdot (b \otimes m) = \pi(a \# \varphi)(b) \otimes m.$$

Proof. (2) ⇔ (3): As in the classical setting it follows from a well-known theorem of Morita [Fa, 4.1.3].

(1) ⇔ (4): Let $\vartheta : H \to H^\dagger$ be the map defined by

$$\vartheta(h) = \big(\mathsf{ev}_H \otimes H^\dagger\big)\circ\big(H^\dagger \otimes c_{H^\dagger H}\big)\circ\big(\Delta_{H^\dagger}(T) \otimes h\big).$$

A direct computation shows that $\vartheta(h)(l) = T(hl)$ for all $h, l \in H$. Hence, by [D, Theorem 3.b], we know that ϑ is a bijective map. Since $g_{(s^{-1})^\flat}$ is also bijective, to prove that (1) ⇔ (4) it suffices to show that $(g_{(s^{-1})^\flat} \otimes \vartheta)\circ\beta_A = [\,,\,]\circ(g_{(s^{-1})^\flat} \otimes_{H^\dagger A} A)$. Let $g = g_{(s^{-1})^\flat}$ and $g' = g_s$. By the definition of ϑ, the discussion following Proposition 3.2, the definition of g, the fact that g is an algebra map, the compatibility of $(s^{-1})^\flat$ with $\Delta_{H^\dagger}$, Corollary 2.12, Lemma 7.3, the fact that $(A, (s^{-1})^\flat)$ is a left $H^\dagger$-module and basic properties of the evaluation and coevaluation maps, we have



as desired.

(4) ⇒ (3): By hypothesis there exist $x_1, \dots, x_s, y_1, \dots, y_s \in A$ such that $1 \# \varepsilon = \sum_i x_i T y_i$. By this fact, Proposition 6.2 and the discussion following Proposition 6.4, the map $f : A^{(s)} \to A \# H^\dagger$ given by $f(a_1, \dots, a_s) = \sum_i a_i T y_i$ is $A \# H^\dagger$-linear and surjective. Hence A is a left $A \# H^\dagger$-generator.

(5) ⇒ (4): By the discussion following Proposition 6.4, ${}^{H^\dagger}(A \# H^\dagger) = TA$. The assertion follows immediately from this fact.

(2) ⇒ (4): By Proposition 6.4 it suffices to prove that $1 \# \epsilon \in ATA$. This can be checked as in [C-F-M].

(4) ⇒ (5): The proof given in [C-F-M] works in our setting. □

Theorem 7.5. *Let H be a rigid braided Hopf algebra and let (A, s) be a right H-comodule algebra. Consider A as an $({}^{H^\dagger}A, A \# H^\dagger)$-bimodule via the actions defined at the beginning of Section* 6. *Let ${}^gA$ be as in Theorem* 7.4. *The following assertions are equivalent*:

(1) *$({}^{H^\dagger}A \hookrightarrow A, s)$ is H-Galois.*
(2) (a) *The map $\pi' : A \# H^\dagger \to \mathsf{End}({}_{H^\dagger A}A)^{\mathsf{op}}$, defined by $\pi'(a \# \varphi)(b) := b \cdot (a \# \varphi)$, is an algebra isomorphism.*
  (b) *A is a finitely generated projective left ${}^{H^\dagger}A$-module.*
(3) *A is a right $A \# H^\dagger$-generator.*
(4) *If $0 \neq U \in \int^r_{H^\dagger}$, then the map $[\,,\,]' : A \otimes_{H^\dagger A} {}^gA \to A \# H^\dagger$ given by $[a, b]' = aUb$ is surjective.*
(5) *For any right $A \# H^\dagger$-module M, the map $G_M : M^{H^\dagger} \otimes_{H^\dagger A} A \to M$, defined by $G_M(m \otimes a) = m \cdot a$, is a right $A \# H^\dagger$-module isomorphism, where $M^{H^\dagger} \otimes_{H^\dagger A} A$ has the $A \# H^\dagger$-module structure*

$$(m \otimes b) \cdot (a \# \varphi) = m \otimes \pi'(a \# \varphi)(b).$$

Proof. (2) $\Leftrightarrow$ (3): Is the same as Theorem 7.4.

(1) $\Leftrightarrow$ (4): Mimic the argument given in the proof of Theorem 7.4, replacing T by U.

(4) $\Rightarrow$ (3): By hypothesis there exist $x_1, \dots, x_s, y_1, \dots, y_s \in A$ such that $1 \# \epsilon = \sum_i x_i U y_i$. By this, the fact that $(A \# H^\dagger)^{H^\dagger} = AU$ and Proposition 6.3, the map $f : A^{(s)} \to A \# H^\dagger$ given by $f(a_1, \dots, a_s) = \sum_i a_i U y_i$ is $A \# H^\dagger$-linear and surjective. Hence A is a right $A \# H^\dagger$-generator.

(5) $\Rightarrow$ (4): This follows immediately from the fact that $(A \# H^\dagger)^{H^\dagger} = AU$.

(2) $\Rightarrow$ (4): By Proposition 6.4 it suffices to prove that $1 \# \epsilon \in AUA$. This can be checked as in [C-F-M].

(4) $\Rightarrow$ (5): The proof given in [C-F-M] works in our setting. □

Corollary 7.6. *Let H be a rigid braided Hopf algebra, (A, s) a right H-comodule algebra and $T \in \int_{H^\dagger}^{l} \setminus \{0\}$. Let ${}^{g}A$ be as in Theorem 7.4. If $({}^{H^\dagger}A \hookrightarrow A, s)$ is H-Galois, then the map $\overline{\pi} : {}^{g}A \to \mathsf{Hom}(A_{{}^{H^\dagger}A}, {}^{H^\dagger}A_{{}^{H^\dagger}A})$, defined by $\overline{\pi}(a)(b) = T \cdot (ab)$, is an isomorphism of $({}^{H^\dagger}A, A)$-bimodules.*

Proof. It is clear that $\overline{\pi}$ is right A-linear. Let $c \in {}^{H^\dagger}A$. We have

$$\overline{\pi}(c \cdot a)(b) = \overline{\pi}\big(g_s(c)a\big)(b) = T \cdot \big(g_s(c)ab\big) = cT \cdot (ab),$$

where the last equality follows from Corollary 2.12 and the fact that, by Proposition 5.6, we know that $g_s(c) \in {}^{H^\dagger}A$. This shows that $\overline{\pi}$ is left ${}^{H^\dagger}A$-linear. Let us prove that it is bijective. Consider $A \# H^\dagger$ and $\mathsf{End}(A_{{}^{H^\dagger}A})$ as left $H^\dagger$-modules via $\varphi \cdot (a\psi) = \varphi a \psi$ and $(\varphi \cdot f)(a) = \varphi \cdot f(a)$. The map $\pi : A \# H^\dagger \to \mathsf{End}(A_{{}^{H^\dagger}A})$, introduced in Theorem 7.4, is $H^\dagger$-linear. It is immediate that $\mathsf{Hom}(A_{{}^{H^\dagger}A}, {}^{H^\dagger}A_{{}^{H^\dagger}A})$ is the set of invariants of $\mathsf{End}(A_{{}^{H^\dagger}A})$. Hence, by the discussion following Proposition 6.4, the map π restricts to a bijective map from TA to $\mathsf{Hom}(A_{{}^{H^\dagger}A}, {}^{H^\dagger}A_{{}^{H^\dagger}A})$. The assertion follows immediately from this fact. □

Corollary 7.7. *Let H be a rigid braided Hopf algebra, (A, s) a right H-comodule algebra and $U \in \int_{H^\dagger}^{r} \setminus \{0\}$. Consider A as a right $A \# H^\dagger$-module as in Theorem 7.5. Let A^g denote A endowed with the right A-module structure given by $b \cdot a = b g_{(s^{-1})^\flat}(a)$, where $g_{(s^{-1})^\flat} : A \to A$ is the automorphism of algebras introduced in Theorem 2.8. If $({}^{H^\dagger}A \hookrightarrow A, s)$ is H-Galois, then the map $\overline{\pi'} : A^g \to \mathsf{Hom}({}_{{}^{H^\dagger}A}A, {}_{{}^{H^\dagger}A}{}^{H^\dagger}A)$, defined by $\overline{\pi'}(a)(b) = (ba) \cdot U$, is an isomorphism of $(A, {}^{H^\dagger}A)$-bimodules.*

Proof. It is clear that $\overline{\pi'}$ is left A-linear and we leave to the reader the task to check that it is right ${}^{H^\dagger}A$-linear. Let us prove that it is bijective. Consider $A \# H^\dagger$ and $\mathsf{End}({}_{{}^{H^\dagger}A}A)^{\mathsf{op}}$ as right $H^\dagger$-modules via $(a\psi) \cdot \varphi = a\psi\varphi$ and $(f \cdot \varphi)(a) = f(a) \cdot \varphi$. The map $\pi' : A \# H^\dagger \to \mathsf{End}({}_{{}^{H^\dagger}A}A)^{\mathsf{op}}$, introduced in Theorem 7.5, is $H^\dagger$-linear. It is immediate that AU and $\mathsf{Hom}({}_{{}^{H^\dagger}A}A, {}_{{}^{H^\dagger}A}{}^{H^\dagger}A)$ are the sets of invariants of $A \# H^\dagger$ and $\mathsf{End}({}_{{}^{H^\dagger}A}A)^{\mathsf{op}}$, respectively. Hence, the map π' restricts to a bijective map from AU to $\mathsf{Hom}({}_{{}^{H^\dagger}A}A, {}_{{}^{H^\dagger}A}{}^{H^\dagger}A)$. This implies that $\overline{\pi'}$ is also bijective. □

8. Existence of elements of trace 1

Let H be a rigid braided Hopf algebra and (A, s) a right H-comodule algebra. Recall from Corollary 2.12 that $g^{-1}_{(s^{-1})^\flat} = g_s$. In [G-G, Section 8] it was shown that A is a left ${}^{H^\dagger}\!A$-module via $a \triangleright b := g_s(a)b$ and a right $A \# H^\dagger$-module via $b \leftarrow (a \# \varphi) = \sum_i S^{-1}((\varphi^\alpha)_i) \cdot (ba)_i$, where $\alpha : H^\dagger \to k$ is the modular function and $\sum_i (\varphi^\alpha)_i \otimes (ba)_i = ((s^{-1})^\flat)^{-1}(ba \otimes \varphi^\alpha)$. Furthermore, A is an $({}^{H^\dagger}\!A, A \# H^\dagger)$-bimodule. Consider the bimodules

$$M = {}_{{}^{H^\dagger}\!A}A_{A\#H^\dagger} \quad \text{and} \quad N = {}_{A\#H^\dagger}A_{{}^{H^\dagger}\!A},$$

where the actions on N are the ones introduced in Section 6. In [G-G, Theorem 8.4] it was proved that the maps

$$[\,,\,] : N \otimes_{{}^{H^\dagger}\!A} M \to A \# H^\dagger, \quad \text{given by } [a, b] = aTb,$$
$$(\,,\,) : M \otimes_{A\#H^\dagger} N \to {}^{H^\dagger}\!A, \quad \text{given by } (a, b) = T \cdot (ab),$$

in which $T \neq 0$ is a left integral of $H^\dagger$, give a Morita context for ${}^{H^\dagger}\!A$ and $A \# H^\dagger$. The purpose of this section is to study the implications of the surjectivity of the map $(\,,\,)$. Proposition 8.1 generalizes item (1) of Proposition 2.5 of [C-F-M] and Propositions 8.2, 8.4 and 8.9 generalize Propositions 1.4, 1.5 and 1.7 of [C-F], respectively. We prove neither the first one, the second one nor the fourth one, because the proofs given there work in our context. Before beginning we recall that an element c of A is called a trace 1 element of A if $T \cdot c = 1$. It is clear that the existence of a trace 1 element is equivalent to the surjectivity of $(\,,\,)$.

Proposition 8.1. *The map $(\,,\,)$ is surjective if and only if there exists $x \in A \# H^\dagger$ such that $TxT = T$.*

Proposition 8.2. *Let V be a left ${}^{H^\dagger}\!A$-module. Consider $A \otimes_{{}^{H^\dagger}\!A} V$ as a left $H^\dagger$-module via $\varphi \cdot (a \otimes v) = \varphi \cdot a \otimes v$. If $(\,,\,)$ is surjective, then the map $i_V : V \to A \otimes_{{}^{H^\dagger}\!A} V$, defined by $i_V(v) = 1 \otimes v$, is injective and its image is ${}^{H^\dagger}(A \otimes_{{}^{H^\dagger}\!A} V)$.*

Definition 8.3. A *total integral* is a right H-colinear map $g : H \to A$ such that $g(1) = 1$.

Proposition 8.4. *The following facts hold*:

(1) *The map $(\,,\,)$ is surjective if and only if there is a total integral $g : H \to A$.*
(2) *There exists $c \in A$ such that $T \cdot c = 1$ and $ca = g_s(a)c$, for all $a \in {}^{H^\dagger}\!A$ if and only if there exists a total integral $g : H \to A$ satisfying $\mu_A \circ (g \otimes {}^{H^\dagger}\!A) = \mu_A \circ (A \otimes g) \circ s_{|H \otimes {}^{H^\dagger}\!A}$.*

Proof. (1): If $(\,,\,)$ is surjective, then there exists $c \in A$ such that $T \cdot c = 1$. Define $g(h) = \theta^{-1}(h) \cdot c$, where $\theta : H^\dagger \to H$ is the bijection given by $\theta(\varphi) = \varphi \rightharpoonup t$. It is immediate that g is $H^\dagger$-linear and that $g(1) = \theta^{-1}(1) \cdot c = T \cdot c = 1$. Conversely, if $g : H \to A$ is a total integral, then, by Proposition 1.24, $T \cdot g(t) = g(T \rightharpoonup t) = g(1) = 1$.

(2): Let $h \in H$ and $a \in {}^{H^\dagger}\!A$. If $c \in A$ satisfies the hypothesis of item (2), then

$$g(h)a = \big(\theta^{-1}(h) \cdot c\big)a = \theta^{-1}(h) \cdot (ca) = \theta^{-1}(h) \cdot \big(g_s(a)c\big) = \sum_i g_s(a)_i \big(\theta^{-1}(h)_i \cdot c\big),$$

where $\sum_i g_s(a)_i \otimes \theta^{-1}(h)_i = (s^{-1})^\flat(\theta^{-1}(h) \otimes g_s(a))$ and the last equality follows from Proposition 5.6. So, by Proposition 2.11,

$$g(h)a = \mu_A \circ (A \otimes g) \circ s(h \otimes a). \tag{6}$$

Conversely, if g satisfies (6), then $c = g(t)$ satisfies $ca = \mu_A \circ (A \otimes g) \circ s(t \otimes a) = g_s(a)g(t) = g_s(a)c$, for all $a \in {}^{H^\dagger}\!A$. $\square$

Now we recall the notion of trace ideal. For any ring R and any right R-module M we let $\mathcal{T}(M)$ denote the image of the evaluation map $\mathsf{Hom}(M_R, R_R) \otimes M \to R$. It is easy to see that $\mathcal{T}(M)$ is a two sided ideal of R. It is well known that $\mathcal{T}(M) = R$ if and only if M is a generator of the category of right R-modules. Also, if R is a subring of a ring S, then $\mathcal{T}(S) = R$ if and only if R is a right R-summand of S [Fa, 3.26 and 3.27]. Of course, similar results are valid for left R-modules.

The following result generalizes [K-T, Proposition 1.9] and [C-F-M, Theorem 2.2]. Our proof follows closely the ones given in those papers.

Proposition 8.5. *Let $\widehat{T} : A \to {}^{H^\dagger}\!A$ and $\widehat{U} : A \to {}^{H^\dagger}\!A$ denote the trace maps defined by $\widehat{T}(a) = T \cdot a$ and $\widehat{U}(a) = a \cdot U$, where $U = S(T)$. Assume that $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is right H-Galois and consider A as a $(A \# H^\dagger, {}^{H^\dagger}\!A)$-bimodule and a $({}^{H^\dagger}\!A, A \# H^\dagger)$-bimodule via the actions given in Section 6. The following facts are equivalent:*

1. *$\widehat{T}$ is surjective.*
2. *$\mathcal{T}(A_{{}^{H^\dagger}\!A}) = {}^{H^\dagger}\!A$.*
3. *${}^{H^\dagger}\!A$ is a right direct ${}^{H^\dagger}\!A$-summand of A.*
4. *A is a generator of the category of right ${}^{H^\dagger}\!A$-modules.*
5. *A is a finitely generated projective left $A \# H^\dagger$-module.*
6. *$\widehat{U}$ is surjective.*
7. *$\mathcal{T}({}_{{}^{H^\dagger}\!A}A) = {}^{H^\dagger}\!A$.*
8. *${}^{H^\dagger}\!A$ is a left direct ${}^{H^\dagger}\!A$-summand of A.*
9. *A is a generator of the category of left ${}^{H^\dagger}\!A$-modules.*
10. *A is a finitely generated projective right $A \# H^\dagger$-module.*

Furthermore any of these conditions implies that $A \# H^\dagger$ and ${}^{H^\dagger}\!A$ are Morita equivalent.

Proof. From Corollary 7.6, it follows easily that $\mathcal{T}(A_{{}^{H^\dagger}\!A}) = T \cdot A$. Hence, (1) $\Leftrightarrow$ (2). That (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from the discussion above this proposition. Now (4) $\Leftrightarrow$ (5) follows from Morita's Theorem [Fa, 4.1.3], since ${}^{H^\dagger}\!A \simeq \mathsf{End}({}_{A\#H^\dagger}A)^{\mathsf{op}}$ by Proposition 6.1 and $A \# H^\dagger \simeq \mathsf{End}(A_{{}^{H^\dagger}\!A})$ by Theorem 7.4(2). Finally, (4) and Theorem 7.4(2) say that A is a right ${}^{H^\dagger}\!A$-progenerator and $A \# H^\dagger \simeq \mathsf{End}(A_{{}^{H^\dagger}\!A})$, which imply that $A \# H^\dagger$ and ${}^{H^\dagger}\!A$ are Morita equivalent

[Fa, 4.29]. Alternatively we can use Corollary 8.5 of [G-G]. The equivalence of (6), (7), (8), (9) and (10) can be proven in a similar way. We end the proof by noting that (1) ⇔ (6), since $\widehat{U}(a) = a \cdot U = T \cdot g_s(a) = \widehat{T}(g_s(a))$. □

Remark 8.6. Note that the hypothesis that $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is right H-Galois is not necessary to prove (1) ⇒ (2) ⇔ (3) ⇔ (4) ⇔ (5), (6) ⇒ (7) ⇔ (8) ⇔ (9) ⇔ (10) and (1) ⇔ (6).

Remark 8.7. Let $\pi : A \# H^\dagger \to \mathrm{End}(A_{{}^{H^\dagger}\!A})$ be the morphism introduced in Theorem 7.4. Arguing as in [C-F, Proposition 1.6] it can be proven that:

(1) if $T \cdot c = 1$, then $\pi(Tc)$ is a right ${}^{H^\dagger}\!A$-linear projection of A on ${}^{H^\dagger}\!A$.
(2) Assume that $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is right H-Galois. By Theorem 7.4, the map π is bijective and $[\,,\,]$ is surjective. If $x_1, \dots, x_r, y_1, \dots, y_r \in A$ satisfy $\sum_{i=1}^r [x_i, y_i] = 1$ and $p \in \mathrm{End}(A_{{}^{H^\dagger}\!A})$ is a projection of A onto ${}^{H^\dagger}\!A$, then $c = \sum_{i=1}^r g_s(e \cdot x_i) y_i$, where $e = \pi^{-1}(p)$, satisfy $T \cdot c = 1$.
(3) If $T \cdot c = 1$ and $ca = g_s(a)c$ for all $a \in {}^{H^\dagger}\!A$, then ${}^{H^\dagger}\!A$ is a direct ${}^{H^\dagger}\!A$-bimodule summand of A.
(4) If $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is right H-Galois and ${}^{H^\dagger}\!A$ is a direct ${}^{H^\dagger}\!A$-bimodule summand of A, then A has a trace 1 element c, such that $ca = g_s(a)c$ for all $a \in {}^{H^\dagger}\!A$.

Corollary 8.8. *Let H be a rigid braided Hopf algebra and let (A, s) be a right comodule algebra. If $H^\dagger$ is semisimple, then ${}^{H^\dagger}\!A$ is an ${}^{H^\dagger}\!A$-bimodule direct summand of A.*

Proof. Let $T \in \int_{H^\dagger}^l$ such that $\epsilon(T) = 1$. By Corollary 2.12 and [G-G, Remark 4.16], we know that $g_s = \mathrm{id}$ and so $c = 1$ satisfies the condition of item (3) of Remark 8.7. □

The following result generalizes items (1) and (3) of Remark 8.7.

Proposition 8.9. *Let B be a k-algebra and let M be a $(A \# H^\dagger, B)$-bimodule.*

(1) *If there exists an element c of A of trace 1, then ${}^{H^\dagger}\!M$ is a right B-direct summand of M.*
(2) *If there exists an element c of A of trace 1, such that $ca = g_s(a)c$ for all $a \in {}^{H^\dagger}\!A$, then ${}^{H^\dagger}\!M$ is a $({}^{H^\dagger}\!A, B)$-direct summand of M.*

Recall that a ring A has invariant basis number if in each free A-module M each pair of basis of M has equal cardinality. Given such a ring and a free left A-module M, we let $[M : A]_l$ denote the dimension of M as a left A-module. Similarly if M is a free right A-module, then $[M : A]_r$ denotes the dimension of M as a right A-module. The following result generalizes Corollary 8.3.5 of [M] and its proof is similar to the one given there.

Corollary 8.10. *Let H be a rigid braided Hopf algebra and (A, s) a right H-comodule algebra. The following assertions hold*:

(1) *If $A \# H^\dagger$ is simple, then $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is H-Galois.*
(2) *$A \# H^\dagger$ is simple and any of the conditions in Proposition 8.5 is valid if and only if $({}^{H^\dagger}\!A \hookrightarrow A, s)$ is H-Galois and ${}^{H^\dagger}\!A$ is simple.*

(3) *$A \# H^{\dagger}$ is simple Artinian if and only if ${}^{H^{\dagger}}A$ is, A is a free left and right ${}^{H^{\dagger}}A$-module of rank $n = \dim H$ and $A \# H^{\dagger} \simeq M_n({}^{H^{\dagger}}A)$.*

Proof. (1): From Proposition 6.4 we know that $[A, A] = ATA$ is a two sided ideal of $A \# H^{\dagger}$. So, if $A \# H^{\dagger}$ is simple, then $A \# H^{\dagger} = [A, A]$ and by Theorem 7.4, $({}^{H^{\dagger}}A \hookrightarrow A, s)$ is H-Galois.

(2): Assume that $A \# H^{\dagger}$ is simple and any of the conditions in Proposition 8.5 is valid. Then, by item (1), the extension $({}^{H^{\dagger}}A \hookrightarrow A, s)$ is H-Galois. Hence Proposition 8.5 implies that $A \# H^{\dagger}$ and ${}^{H^{\dagger}}A$ are Morita equivalent and so ${}^{H^{\dagger}}A$ is simple. Conversely, assume that $({}^{H^{\dagger}}A \hookrightarrow A, s)$ is H-Galois and ${}^{H^{\dagger}}A$ is simple. By the equivalence between items (1) and (2) of Theorem 7.4, the map $\widehat{T} : A \to {}^{H^{\dagger}}A$ is nonzero, and since its image is a two sided ideal of ${}^{H^{\dagger}}A$, it is surjective. Hence, again by Proposition 8.5, $A \# H^{\dagger}$ and ${}^{H^{\dagger}}A$ are Morita equivalent and so $A \# H^{\dagger}$ is simple.

(3): Assume that $A \# H^{\dagger}$ is simple Artinian. Then, it is von Neumann regular, and thus there exists $x \in A \# H^{\dagger}$ such that $T = TxT$. By Proposition 8.2 the map $\widehat{T} : A \to {}^{H^{\dagger}}A$ is surjective. Then, by Proposition 8.5, $A \# H^{\dagger}$ and ${}^{H^{\dagger}}A$ are Morita equivalent, and so, ${}^{H^{\dagger}}A$ is Artinian semisimple. It then follows by the lemma of Artin–Whaples that A is a free left ${}^{H^{\dagger}}A$-module, say of rank m. Now, by Theorem 7.4, $A \# H^{\dagger} \simeq \mathsf{End}(A_{{}^{H^{\dagger}}A}) \simeq M_m({}^{H^{\dagger}}A)$. But then

$$m^2 = \left[A \# H^{\dagger} : {}^{H^{\dagger}}A\right]_l = \left[A \# H^{\dagger} : A\right]_l \left[A : {}^{H^{\dagger}}A\right]_l = nm.$$

Thus $m = n$. Similarly A is a free right ${}^{H^{\dagger}}A$-module of rank n. The converse is trivial. □

Theorem 8.11. *Let H be a rigid braided Hopf algebra and let (D, s) be a right H-comodule algebra, where D is a division ring. Then the following facts are equivalent*:

(1) *$({}^{H^{\dagger}}D \hookrightarrow D, s)$ is H-Galois.*
(2) $[D : {}^{H^{\dagger}}D]_r = \dim H$.
(3) $[D : {}^{H^{\dagger}}D]_l = \dim H$.
(4) *$D \# H^{\dagger}$ is simple.*
(5) *D is a faithful right $D \# H^{\dagger}$-module.*
(6) *D is a faithful left $D \# H^{\dagger}$-module.*

Proof. Arguing as in [M, Theorem 8.3.7] we can see that (1) ⇒ (5), (1) ⇒ (6), (5) ⇒ (4), (6) ⇒ (4), (2) ⇒ (1) and (3) ⇒ (1). Corollary 8.10 implies that (4) ⇒ (1), (1) ⇒ (2) and (1) ⇒ (3). □

Example 8.12. Let H be the algebra $k[X]/\langle X^2\rangle$, endowed with the braided Hopf algebra structure given by $\Delta(X) = 1 \otimes X + X \otimes 1$ and $c(X \otimes X) = -X \otimes X$. It is easy to check that $H^{\dagger} \simeq H$. In fact if ϵ, ξ_X is the dual basis of $1, X$, then the map $1 \mapsto \epsilon$, $X \mapsto \xi_X$ is an isomorphism. In [D-G-G, Example 2.4] it was shown that if

(1) $\alpha : A \to A$ is an automorphism,
(2) $\delta : A \to A$ is an α-derivation,
(3) $\delta \circ \alpha + \alpha \circ \delta = 0$,
(4) $\delta^2 = 0$,

then the formulas $s(\xi_X \otimes a) = \alpha(a) \otimes \xi_X$ and $\rho(\xi_X \otimes a) = \delta(a)$, define a transposition $s : H^\dagger \otimes A \to A \otimes H^\dagger$ and a s-action $\rho : H^\dagger \otimes A \to A$. By Proposition 2.7 the map ${}^\flat(s^{-1}) : H \otimes A \to A \otimes H$, defined at the beginning of Section 2, is a transposition, and, by Theorem 4.4, we know that $(A, {}^\flat(s^{-1}))$ is a right H-comodule algebra via the map $\nu_\rho : A \to A \otimes H$, introduced below Proposition 3.2. A direct computation shows that ${}^\flat(s^{-1})(X \otimes a) = \alpha^{-1}(a) \otimes X$ and $\nu_\rho(a) = a \otimes 1 + \alpha^{-1} \circ \delta(a) \otimes X$. The following concrete examples satisfy the hypothesis of Theorem 8.11.

(1) $k = \mathbb{Q}$, $A = \mathbb{Q}(\sqrt{2})$, $\alpha(a + b\sqrt{2}) = a - b\sqrt{2}$ and $\delta(a + b\sqrt{2}) = b$.
(2) A the field of Laurent series $\sum a_i Y^i$ with coefficients in k, $\alpha(\sum a_i Y^i) = \sum (-1)^i a_i Y^i$ and $\delta(\sum a_i Y^i) = \sum a_{2i+1} Y^{2i}$.

In fact, in the former ${}^{H^\dagger}\!A = \mathbb{Q}$ and then $[A : {}^{H^\dagger}\!A]_r = 2 = \dim H$, and, in the second one ${}^{H^\dagger}\!A = \{\sum_i a_{2i} Y^{2i}\}$ and then $[A : {}^{H^\dagger}\!A]_r = 2 = \dim H$.

From now on, given a left braided space (V, s), we let $\widetilde{s}$ denote $(((s^{-1})^\flat)^{-1})^\flat$.

Let H be a rigid braided bialgebra and let (A, s) be a right H-comodule algebra. Then, $(A, (s^{-1})^\flat)$ is a left $H^\dagger$-module algebra. Let us consider the smash product $A \# H^\dagger$ and let $\breve{s} = (A \otimes c_{H^\dagger}) \circ ((s^{-1})^\flat \otimes H^\dagger)$. Note that with the notations of [G-G, Proposition 10.3], $\breve{s} = \widehat{(s^{-1})^\flat}$. By [G-G, Proposition 10.4] we know that $(A \# H^\dagger, \breve{s})$ is a right $H^\dagger$-comodule algebra via $\nu_\Delta := A \otimes \Delta_{H^\dagger}$. Then, by Theorem 4.4, $(A \# H^\dagger, (\breve{s}^{-1})^\flat)$ is a left $H^{\dagger\dagger}$-module algebra via ρ_{ν_Δ}. We let $\Psi * (a \# \varphi)$ denote $\rho_{\nu_\Delta}(\Psi \otimes a \# \varphi)$.

Let H be a rigid braided Hopf algebra. Given a nonzero left integral $T \in H^\dagger$ we let $\mathcal{T}$ denote from now on the unique left integral of $H^{\dagger\dagger}$ such that $\mathcal{T}(T) = 1$. Note that $\mathcal{T} = t^{**}$, where t is the left integral of H satisfying $T(t) = 1$.

Proposition 8.13. *The following assertions hold*:

(1) $(\breve{s}^{-1})^\flat = (A \otimes c_{H^{\dagger\dagger} H^\dagger}) \circ (\widetilde{s} \otimes H^\dagger)$.
(2) $\mathcal{T} * T = \epsilon$.

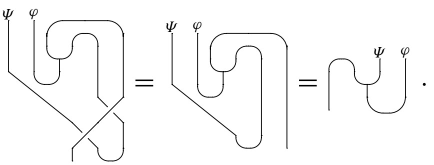
Proof. (1): A direct computation shows that if (V, s_V) and (W, s_W) are left $H^\dagger$-braided spaces, then

$$\left(s_{V \otimes W}^{-1}\right)^\flat = \left(V \otimes \left(s_W^{-1}\right)^\flat\right) \circ \left(\left(s_V^{-1}\right)^\flat \otimes W\right).$$

Using this we immediately see that

$$\left(\breve{s}^{-1}\right)^\flat = \left(A \otimes \left(c_{H^\dagger}^{-1}\right)^\flat\right) \circ \left(\widetilde{s} \otimes H^\dagger\right) = (A \otimes c_{H^{\dagger\dagger} H^\dagger}) \circ \left(\widetilde{s} \otimes H^\dagger\right).$$

(2): Let $\Psi \in H^{\dagger\dagger}$ and $\varphi \in H^\dagger$. We have

$\Psi * (1 \# \varphi) =$ 

Using this and that T and $\mathcal{T}$ are left integrals satisfying $\mathcal{T}(T) = 1$, we obtain $\mathcal{T} * T = \epsilon$. □

The next remark is an adaptation of [C-F, Remark 0.13].

Remark 8.14. Let $({}^{H^\dagger}A \hookrightarrow A, s)$ be a right H-Galois extension. By Theorem 7.4 there exists $\sum_i x_i \otimes y_i \in A \otimes_{{}^{H^\dagger}A} {}^{g}A$ such that $\sum_i [x_i, y_i] = 1$. Then, for all $a \in A$,

$$a = \sum_i [x_i, y_i] \cdot a = \sum_i x_i (y_i, a) = \sum_i x_i T \cdot (y_i a)$$

and

$$a = a \leftarrow \sum_i [x_i, y_i] = \sum_i (a, x_i) \triangleright y_i = \sum_i g_s\bigl(T \cdot (a x_i)\bigr) y_i = \sum_i q^{-1}\bigl(T \cdot g_s(a x_i)\bigr) y_i,$$

where $q \in k$ is such that $c_{H^\dagger}(T \otimes T) = qT \otimes T$ and the last equality can be easily checked using Corollary 2.12 and Lemma 8.2 of [G-G]. In particular, $1 = \sum_i x_i (T \cdot y_i) = \sum_i q^{-1}(T \cdot g_s(x_i)) y_i$. Furthermore, surjectivity of $[\,,\,]$ implies its injectivity by the Morita theorems. Thus,

(1) If $\sum_i u_i \otimes v_i \in A \otimes_{{}^{H^\dagger}A} {}^{g}A$ satisfies $\sum_i [u_i, v_i] a = g_s(a) \sum_i [u_i, v_i]$ for all $a \in A$, then

$$\sum_i g_s(a) u_i \otimes v_i = \sum_i u_i \otimes v_i a \quad \text{for all } a \in A.$$

(2) If $\sum_i u_i \otimes v_i \in A \otimes_{{}^{H^\dagger}A} {}^{g}A$ is such that $\sum_i [u_i, v_i] \in C_{A\#H^\dagger}(A)$ (the centralizer of A in $A \# H^\dagger$), then

$$\sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a \quad \text{for all } a \in A.$$

The following result generalizes Theorem 1.8 of [C-F] and its proof follows closely the one given there.

Theorem 8.15. *Let H be a rigid Hopf algebra and let (A, s) be a right H-comodule algebra. Assume that $({}^{H^\dagger}A \hookrightarrow A, s)$ is a Galois extension. The following conditions are equivalent*:

(1) *$A/{}^{H^\dagger}A$ is separable.*
(2) *There exists $w \in A \# H^\dagger$ such that $\mathcal{T} * w = 1$ and $wa = g_s^{-1}(a) w$ for all $a \in A$.*
(3) *A is a direct summand of $A \# H^\dagger$ as an A-bimodule.*
(4) *Let $\sum_{i \in I} x_i \otimes y_i \in A \otimes_{{}^{H^\dagger}A} {}^{g}A$ such that $\sum_{i \in I} [x_i, y_i] = 1$. There exists $c \in A$ such that $\sum_{i \in I} x_i c y_i = 1$ and $ac = c g_s(a)$ for all $a \in {}^{H^\dagger}A$.*

Proof. (1) $\Rightarrow$ (2): Let $\sum_{j \in J} a_j \otimes b_j$ be an idempotent of separability of $A/{}^{H^\dagger}A$. Let $w = \sum_{j \in J} g_s^{-1}(a_j) T b_j \in A \# H^\dagger$, where g_s is as in Theorem 2.8. Corollary 2.12 implies that $g_s = g_{\tilde{s}}$. Using this fact, that $A = {}^{H^{\dagger\dagger}}(A \# H^\dagger)$ and Proposition 8.13, we obtain

$$\mathcal{T} * w = \sum_{j \in J} a_j (\mathcal{T} * T) b_j = \sum_{j \in J} a_j b_j = 1.$$

Furthermore,

$$wa = \sum_{j\in J} g_s^{-1}(a_j)Tb_ja = \sum_{j\in J} g_s^{-1}(aa_j)Tb_j = g_s^{-1}(a)w,$$

for all $a \in A$, since $\sum_{j\in J} a_j \otimes b_ja = \sum_{j\in J} aa_j \otimes b_j$.

(2) $\Rightarrow$ (1): Since g_s and $[\,,\,]$ are bijective maps, there exists $\sum_{i\in I} u_i \otimes v_i \in A \otimes_{H^\dagger A} {}^gA$, such that $\sum_{i\in I}[g_s^{-1}(u_i), v_i] = w$. Furthermore, since

$$\sum_{i\in I}\big[g_s^{-1}(au_i), v_i\big] = g_s^{-1}(a)w = wa = \sum_{i\in I}\big[g_s^{-1}(u_i), v_ia\big],$$

we have

$$\sum_{i\in I} au_i \otimes v_i = \sum_{i\in I} u_i \otimes v_ia.$$

Finally, $\sum_{i\in I} u_iv_i = \sum_{i\in I} u_i\mathcal{T} * Tv_i = \mathcal{T} * (\sum_{i\in I} g_s^{-1}(u_i)Tv_i) = \mathcal{T} * w$.

(2) $\Leftrightarrow$ (3): This follows immediately from Corollary 2.12 and item (3) of Remark 8.7 applied to the left $H^{\dagger\dagger}$-module algebra $A \# H^\dagger$.

(1) $\Rightarrow$ (4): Let $\sum a_i \otimes b_i \in A \otimes_{H^\dagger A} A$ be a separability idempotent for $A/{}^{H^\dagger}A$, and set $c = \sum_j q^{-1}a_jT \cdot g_s(b_j)$, where $q \in k$ is such that $c_{H^\dagger}(T \otimes T) = qT \otimes T$. By Remark 8.14,

$$\sum_i x_icy_i = \sum_{ij} q^{-1}x_ia_jT \cdot g_s(b_j)y_i = \sum_{ij} q^{-1}a_jT \cdot g_s(b_jx_i)y_i = \sum_j a_jb_j = 1,$$

where the second equality follows from the fact that $\sum_j x_ia_j \otimes b_j = \sum_j a_j \otimes b_jx_i$. Furthermore, using that $\sum a_i \otimes b_i$ is a separability idempotent and Proposition 5.6, we see that

$$ac = \sum_j q^{-1}aa_jT \cdot g_s(b_j) = \sum_j q^{-1}a_jT \cdot g_s(b_ja) \sum_j q^{-1}a_jT \cdot g_s(b_j)g_s(a) = cg_s(a)$$

for all $a \in {}^{H^\dagger}A$.

(4) $\Rightarrow$ (1): Since $ac = cg_s(a)$ for all $a \in {}^{H^\dagger}A$, the map $\Upsilon : A \otimes_{H^\dagger A} {}^gA \to A \otimes_{H^\dagger A} A$, given by $\Upsilon(x \otimes y) = xc \otimes y$, is well defined. Let $\sum_i a_i \otimes b_i = \sum_i x_ic \otimes y_i = \Upsilon(\sum_i x_i \otimes y_i)$. By item (2) of Remark 8.14, the equality $\sum_i aa_i \otimes b_i = \sum_i a_i \otimes b_ia$ is satisfied for all $a \in A$, and by hypothesis, $\sum a_ib_i = \sum_i x_icy_i = 1$. □

Corollary 8.16. *Let H be a semisimple braided Hopf algebra. Let t be a left integral of H such that $\epsilon(t) = 1$. Let $A \#_f H$ be a braided Hopf crossed product in the sense of* [G-G, Definition 9.4]. *If f is invertible, then $A \#_f H$ is a separable extension of A.*

Proof. Let s be the transposition of H on A. By [G-G, Theorem 10.6] we know that $(A \hookrightarrow A \#_f H, \hat{s})$ is a right Galois extension. Then, by Theorem 8.15, the comment following Theorem 1.16, and [G-G, Remark 4.16] we must pick a $w \in (A \#_f H) \# H^\dagger$ such that $bw = wb$ for all $b \in A \#_f H$ and $\mathcal{T} \star w = 1$. But an easy computation shows that $w = 1 \# 1 \# \epsilon$ satisfies these conditions. □

The following result generalizes a theorem due to Doi.

Corollary 8.17. *Let H be a semisimple braided Hopf algebra and let (A, s) be a right H-comodule algebra. Let t be a left integral of H such that $\epsilon(t) = 1$. If $({}^{H^{\dagger}}\!A \hookrightarrow A, s)$ is a Galois extension, then $A/{}^{H^{\dagger}}\!A$ is separable.*

Proof. Letting $w = 1 \# \epsilon$, condition (2) of Theorem 8.15 is satisfied. □

Note that Corollary 8.16 is a particular case of Corollary 8.17.

Recall that a braided bialgebra H is cocommutative if c_H is involutive and $c_H \circ \Delta_H = c_H$. The following result generalizes Theorem 1.11 of [C-F].

Theorem 8.18. *Let H be a cocommutative rigid braided Hopf algebra with braid c and let $s : H \otimes A \to A \otimes H$ be a transposition. Let $A \#_f H$ be a braided Hopf crossed product in the sense of* [G-G, Definition 9.4]. *If there exists a left integral t of H and an element c of A such that*:

(1) $t \cdot c = 1$,
(2) $c(t \otimes h) = h \otimes t$ *for all* $h \in H$,
(3) $s(h \otimes c) = c \otimes h$ *for all* $h \in H$,
(4) $s(t \otimes a) = a \otimes t$ *for all* $a \in A$,

then, $A \#_f H$ is a separable extension of A.

Proof. By [G-G, Theorem 10.6] we know that $(A \hookrightarrow A \#_f H, \widehat{s})$ is a right H-Galois extension with transposition $\widehat{s} = (A \otimes c) \circ (s \otimes H)$ and coaction $\nu = A \otimes \Delta$. Let $\gamma : H \to A \#_f H$ be the map defined by $\gamma(h) = 1 \#_f h$ and let γ^{-1} be the convolution inverse of γ. Let

$$w = \big(\gamma^{-1}(u_{(1)}) \# T\big)(c \#_f u_{(2)} \# \epsilon),$$

where $u = S(t)$ and T is a left integral of $H^{\dagger}$ satisfying $T(t) = 1$. By Theorem 8.15 (applied to $(A \hookrightarrow A \#_f H)$) we must see that

$$\mathcal{T} * w = 1 \#_f 1 \# \epsilon \quad \text{and} \quad w(a \#_f l) = (a \#_f l)w,$$

for all $a \in A$ and $l \in H$. Let $\widetilde{\widehat{s}} : H^{\dagger\dagger} \otimes (A \#_f H) \to (A \#_f H) \otimes H^{\dagger\dagger}$ be as in the discussion preceding Proposition 8.13. By Corollary 2.12, we know that $g_{\widetilde{\widehat{s}}} = g_{\widehat{s}} = \mathrm{id}$. So,

$$\begin{aligned}
\mathcal{T} * w &= \big(\gamma^{-1}(u_{(1)}) \# \epsilon\big)\big(\mathcal{T} * (1 \#_f 1 \# T)\big)(c \#_f u_{(2)} \# \epsilon) \\
&= \big(\gamma^{-1}(u_{(1)}) \# \epsilon\big)(c \#_f u_{(2)} \# \epsilon) \\
&= \big(\gamma^{-1}(u_{(1)}) \# \epsilon\big)(1 \#_f u_{(2)} \# \epsilon)\big(S(u_{(3)}) \cdot c \#_f 1 \# \epsilon\big) \\
&= S(u) \cdot c \#_f 1 \# \epsilon \\
&= 1 \#_f 1 \# \epsilon,
\end{aligned}$$

where the second equality follows from item (2) of Proposition 8.13 and the third one from the hypothesis and the fact that, since H is cocommutative,

$$(H \otimes c) \circ (\Delta \otimes S) \circ \Delta = (H \otimes S \otimes H) \circ (H \otimes c) \circ (H \otimes \Delta) \circ \Delta = (H \otimes S \otimes H) \circ (H \otimes \Delta) \circ \Delta.$$

By the bijectivity of $[\,,\,] : A \#_f H \otimes_A A \#_f H \to A \#_f H \# H^\dagger$, in order to check that $w(a \#_f l) = (a \#_f l)w$, it suffices to show that for each $z \in A \# H$,

$$z\big(\gamma^{-1}(u_{(1)})\big) \otimes_A c\gamma(u_{(2)}) = \gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)})z.$$

First let $z = a \in A$. For $h \in H$ and $b \in A$, write $\sum_i b_i \otimes h_i = s(h \otimes b)$. Since $h \cdot b = \sum_i \gamma(h_{(1)})(b_i \#_f 1)\gamma^{-1}(h_{(2)_i})$,

$$\begin{aligned} \gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)})a &= \sum_{ij} \gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)})a_{ij}\gamma^{-1}(u_{(3)_j})\gamma(u_{(4)_i}) \\ &= \sum_{ij} \gamma^{-1}(u_{(1)})\gamma(u_{(2)})a_{ij}\gamma^{-1}(u_{(3)_j}) \otimes_A c\gamma(u_{(4)_i}) \\ &= \sum_{ij} a_{ij}\gamma^{-1}(u_{(1)_j}) \otimes_A c\gamma(u_{(2)_i}) \\ &= a\gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)}). \end{aligned}$$

Now, let $z = 1 \# l$. For $h, h' \in H$, write $\sum_i h'_i \otimes h_i = c(h \otimes h')$. Since $f(h, h') = \sum_i \gamma(h_{(1)}) \times \gamma(h'_{(1)_i})\gamma^{-1}(h_{(2)_i}h'_{(2)})$,

$$\begin{aligned} \gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)})z &= \sum_i \gamma^{-1}(u_{(1)}) \otimes_A cf(u_{(2)}, l_{(1)_i})\gamma(u_{(3)_i}l_{(2)}) \\ &= \sum_i \gamma^{-1}(u_{(1)})f(u_{(2)}, l_{(1)_i}) \otimes_A c\gamma(u_{(3)_i}l_{(2)}) \\ &= \sum_{ij} \gamma(l_{(1)_{ij}})\gamma^{-1}(u_{(1)_j}l_{(2)}) \otimes_A c\gamma(u_{(2)_i}l_{(2)}) \\ &= z\gamma^{-1}(u_{(1)}) \otimes_A c\gamma(u_{(2)}). \end{aligned}$$

This finishes the proof. □

Corollary 8.19. *Under the hypothesis of Theorem* 8.18*, if A is semisimple Artinian, so is $A \#_f H$.*

Lemma 8.20. *Let H be a rigid braided Hopf algebra and let H' a be a braided Hopf subalgebra of H. Let $t' \in H' \setminus \{0\}$ be a left integral. If $H^\dagger$ is semisimple, then there exist $T \in \int_l^{H^\dagger}$ such that $T \rightharpoonup t' = 1$.*

Proof. The proof of [C-F, Proposition 1.14] works in our setting. □

Theorem 8.21. *Let H be a semisimple braided Hopf algebra and let $H^{\dagger'} \subseteq H^\dagger$ be a braided Hopf subalgebra of $H^\dagger$. Let $T' \in \int_l^{H^{\dagger'}} \setminus \{0\}$ and $T \in \int_l^{H^\dagger} \setminus \{0\}$. Assume that there exist $\varphi, \psi \in H^\dagger$ such that $T = T'\varphi = \psi T'$. If (A, s) is a Galois right H-comodule algebra with a trace 1 element and $(s^{-1})^\flat(H^{\dagger'} \otimes A) \subseteq A \otimes H^{\dagger'}$, then ${}^{H^{\dagger'}}A/{}^{H^\dagger}A$ is separable.*

Proof. We follow closely the proof of [C-F, Theorem 1.15]. Let $\sum_i x_i \otimes y_i \in A \otimes_{{}^{H^\dagger}A} A$ such that $\sum_i [x_i, y_i] = 1$ and let $c \in A$ be an element of trace one. We claim that $\sum_i T' \cdot x_i \otimes_{{}^{H^\dagger}A} T' \cdot (y_i(\varphi \cdot c))$ is a separability idempotent for ${}^{H'}A/{}^{H^\dagger}A$. By Lemma 8.20 we know that there exists a left integral $\mathcal{T}$ of $H^{\dagger\dagger}$ such that $\mathcal{T} * T' = \epsilon$. We have

$$\sum_i (T' \cdot x_i) y_i = \mathcal{T} * \sum_i (T' \cdot x_i) T y_i = \mathcal{T} * \sum_i T' x_i T y_i = \mathcal{T} * T' = \epsilon,$$

and so,

$$\sum_i (T' \cdot x_i)\big(T' \cdot \big(y_i(\varphi \cdot c)\big)\big) = T' \cdot \sum_i (T' \cdot x_i)\big(y_i(\varphi \cdot c)\big) = T' \cdot (\varphi \cdot c) = (T'\varphi) \cdot c = T \cdot c = 1.$$

To finish the proof it remains to check that

$$\sum_i wT' \cdot x_i \otimes_{{}^{H^\dagger}A} T' \cdot \big(y_i(\varphi \cdot c)\big) = \sum_i T' \cdot x_i \otimes_{{}^{H^\dagger}A} T' \cdot \big(y_i(\varphi \cdot c)\big) w \tag{7}$$

for all $w \in {}^{H'}A$. To prove (7) we will use that, for all $w \in {}^{H^{\dagger'}}A$,

(1) $\sum_i (T' \cdot x_i) T(y_i(\varphi \cdot c) w) = w$,
(2) $\sum_i (T \cdot (w x_i)) T'(y_i(\varphi \cdot c)) = w$,
(3) $T \cdot (y_j(\varphi \cdot c) T' \cdot (w x_i)) = T \cdot (T' \cdot (y_j(\varphi \cdot c) w) x_i)$.

The proof of (1) is similar to the proof of (2) but easier. Let us see (2). First note that there exists a quotient braided Hopf algebra H' of H such that $H^{\dagger'} \simeq H'^\dagger$. Since H is semisimple, the map g_s introduced in Theorem 2.8 is the identity map. By Corollary 2.12 we also have $g_{(s^{-1})^\flat} = \mathrm{id}$ and so $(s^{-1})^\flat(T \otimes a) = a \otimes T$ for all $a \in A$. Similarly, since H' is semisimple, $(s^{-1})^\flat(T' \otimes a) = a \otimes T'$ for all $a \in A$. Furthermore, from Remark 2.13 it follows that $c_{H^\dagger}(T \otimes T) = T \otimes T$ and so, Remark 8.14 implies that $a = \sum_i x_i T \cdot (y_i a) = \sum_i (T \cdot (a x_i)) y_i$, for all $a \in A$. Using all these facts we obtain that

$$\begin{aligned}\sum_i \big(T \cdot (w x_i)\big) T'\big(y_i(\varphi \cdot c)\big) &= T' \cdot \sum_i \big(T \cdot (w x_i)\big) y_i(\varphi \cdot c) = T' \cdot \big(w(\varphi \cdot c)\big) \\ &= wT' \cdot (\varphi \cdot c) = w(T'\varphi) \cdot c = wT \cdot c = w.\end{aligned}$$

Let us see (3): We have

$$\begin{aligned}T \cdot \big(y_j(\varphi \cdot c) T' \cdot (w x_i)\big) &= T \cdot \big(y_j(\varphi \cdot c) w (T' \cdot x_i)\big) \\ &= \psi \cdot \big(T' \cdot \big(y_j(\varphi \cdot c) w\big)(T' \cdot x_i)\big)\end{aligned}$$

$$= T \cdot \big(T' \cdot \big(y_j(\varphi \cdot c)w\big)x_i\big).$$

The proof of (7) now can be finished as in [C-F, Theorem 1.15]. □

Let G be a finite group. The $k[G]$-module algebra structures were described in [G-G, Theorem 4.14 and Example 9.8]. The following result generalizes [H-S, Proposition 3.4].

Corollary 8.22. *Let (A, s) be a $k[G]$-module algebra. If A has an element of trace 1, then, for each subgroup $G' < G$, the extension ${}^{G'}A/{}^{G}A$ is separable.*

Proof. The proof given in [C-F, Corollary 1.18] works in our setting. □

Corollary 8.23. *Let H be a semisimple braided Hopf algebra and let $H^{\dagger'} \subseteq H^{\dagger}$ be a braided Hopf subalgebra of $H^{\dagger}$. Let $T' \in \int_l^{H^{\dagger'}} \setminus \{0\}$ and $T \in \int_l^{H^{\dagger}} \setminus \{0\}$. Assume that $H^{\dagger}$ is also semisimple, that $T(1) = 1$ and that there exists $\varphi \in H^{\dagger}$ such that $T = T'\varphi$. If (A, s) is a Galois right H-comodule algebra and $(s^{-1})^{\flat}(H^{\dagger'} \otimes A) \subseteq A \otimes H^{\dagger'}$, then ${}^{H^{\dagger'}}A/{}^{H^{\dagger}}A$ is separable.*

Proof. The element $c = 1$ is a trace 1 element since $\epsilon(T) = T(1) = 1$ and $\epsilon(T') \neq 0$, since $1 = \epsilon(T) = \epsilon(T')\epsilon(\varphi)$. Since $T = S(T) = S(\varphi)S(T') = S(\varphi)T'$ we are in the hypothesis of Theorem 8.21 and so the result follows from that theorem. □

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