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Weak Bruhat order on the set of faces of the permutohedron and the associahedron

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Abstract

We define a partial order on the set of faces of the permutohedron, which extends the weak Bruhat order of the symmetric group. The restriction of this order gives a partially ordered set structure to the set of faces of the associahedron, which extends the Tamari order defined on the set of planar binary rooted trees. These orders are used to describe associative algebra structures on the vector spaces spanned by the set of faces of the permutohedra on one hand, and on the vector space spanned by the set of planar rooted trees on the other hand. Our results extend the description of the dendriform algebra structure on the spaces spanned by permutations and by planar binary trees in terms of the weak Bruhat order and the Tamari order, respectively, obtained by J.-L. Loday and the second author in [J.-L. Loday, M. Ronco, Order structure and the algebra of permutations and of planar binary trees, J. Algebraic Combin. 15 (2002) 253–270].

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0. Introduction

In this paper, we consider the graded vector spaces spanned by the faces of two types of convex polytopes: the permutohedron and the associahedron.

Let S_n be the group of permutations of n elements. The Coxeter poset of S_n is the set of right coclasses

$$\mathcal{P}_n := \{S_{n_1} \times \cdots \times S_{n_r} \circ \sigma / (n_1, \ldots, n_r) \text{ is a composition of } n, \text{ and } \sigma \in S_n\},$$

equipped with the set inclusion order denoted $\subseteq$.

The geometric realization of the poset $(\mathcal{P}_n, \subseteq)$, is a convex polytope of dimension $n-1$ called the permutohedron (cf. [19]). Its vertices are in one-to-one correspondence with the elements of S_n.

For $n \geqslant 1$, we denote by $(\mathcal{T}_n, \subseteq)$ the set of planar rooted trees with $n+1$ leaves, partially ordered by the relation:

$$t \subseteq w \text{ if } w \text{ is obtained from } t \text{ by contracting internal edges, for trees } t \text{ and } w \text{ in } \mathcal{T}_n.$$

The geometric realization of $(\mathcal{T}_n, \subseteq)$ is also a convex polytope of dimension $n-1$ called the associahedron (cf. [10,18,19]). The faces of dimension 0 of the associahedron are in one-to-one correspondence with the elements of the set of planar binary rooted trees with $n+1$ leaves, denoted by $\mathcal{Y}_n$.

Moreover, there exists an epimorphism of posets $\Gamma_n : (\mathcal{P}_n, \subseteq) \to (\mathcal{T}_n, \subseteq)$, which induces the surjective map $\psi_n : S_n \to \mathcal{Y}_n$ described in [11].

The weak Bruhat order of the group S_n induces a partial order in $\mathcal{Y}_n$, also called the Tamari order. We denote both orders by $\leqslant_B$.

The main goal of our work is to define an order on the set $\mathcal{P}_n$, which extends the weak Bruhat order of S_n, and to show that this order induces a partially ordered set structure on $\mathcal{T}_n$ in such a way that the diagram:

$$\begin{array}{ccc} S_n & \longrightarrow & \mathcal{Y}_n \\ \downarrow & & \downarrow \\ \mathcal{P}_n & \longrightarrow & \mathcal{T}_n \end{array} \qquad (*)$$

of partially ordered sets, commutes for $n \geqslant 1$.

C. Malvenuto and C. Reutenauer defined in [15] a Hopf algebra structure on the graded vector space spanned by the set of all permutations $\bigcup_{n\geqslant 1} S_n$. This Hopf algebra is denoted $\mathcal{Q}Sym$, and is widely studied in [1,5–7].

In [11], the authors show that the vector space $k[\mathcal{Y}_\infty]$, spanned by the set $\bigcup_{n\geqslant 1} \mathcal{Y}_n$ of binary rooted trees, has a natural structure of Hopf subalgebra of $\mathcal{Q}Sym$. In a second paper (cf. [12]), they prove that the products of the algebras $\mathcal{Q}Sym$ and $k[\mathcal{Y}_\infty]$ may be described in terms of the weak Bruhat order.

Let $k[\mathcal{P}_\infty]$ be the vector space spanned by the set $\bigcup_{n\geqslant 1} \mathcal{P}_n$ of all the faces of the permutohedra, and $k[\mathcal{T}_\infty]$ be the vector space spanned by the set $\bigcup_{n\geqslant 1} \mathcal{T}_n$ of all planar rooted trees. We use the new orders defined on $\mathcal{P}_n$ and $\mathcal{T}_n$ to construct associative products on the spaces $k[\mathcal{P}_\infty]$ and $k[\mathcal{T}_\infty]$, in such a way that the diagram $(*)$ induces the following commutative diagram of associative algebras:

$$\begin{array}{ccc} k[\mathcal{Y}_\infty] & \hookrightarrow & \mathcal{Q}Sym \\ \downarrow & & \downarrow \\ k[\mathcal{T}_\infty] & \hookrightarrow & k[\mathcal{P}_\infty] \end{array}$$

Finally, we prove that the products of the algebras $k[\mathcal{P}_\infty]$ and $k[\mathcal{T}_\infty]$ coincide with the ones defined in [14], and that the dendriform trialgebra structure of both algebras may also be described in terms of the weak Bruhat order.

The paper is organized as follows:

In Section 1, the definition of weak Bruhat order $\leqslant_B$ on the set $\mathcal{P}^{(W,S)} := \{W_J \circ w$: for W_J a standard parabolic subgroup of $W\}$ is introduced, for any finite Coxeter system (W, S).

In Section 2, we construct subsets $SH(n_0, \ldots, n_r)$ of $\mathcal{P}_n$, which generalize the well-known notion of shuffles in S_n. We study the order $\leqslant_B$ in this context, as well as the order induced by $\leqslant_B$ on the set of rooted planar trees $\mathcal{T}_n$. We show that the epimorphism $\Gamma : (\mathcal{P}_n, \leqslant_B) \to (\mathcal{T}_n, \leqslant_B)$ is a homomorphism of posets.

Section 3 is devoted to describe the associative algebra structure of the spaces $k[\mathcal{P}_\infty]$ and $k[\mathcal{T}_\infty]$, and to show that their structure of dendriform trialgebras may be obtained in terms of the generalized weak Bruhat orders.

1. Weak Bruhat order

Definition 1. A finite *Coxeter system* is a pair (W, S), where W is a finite group generated by the set S, with relations of the form

$$(s \cdot s')^{m(s,s')} = 1 \quad \text{for } s, s' \in S,$$

for certain positive integers $m(s, s')$, with $m(s, s) = 1$ for all $s \in S$.

For any Coxeter system (W, S), we call W a *Coxeter group*.

Let J be a subset of S, W_J denotes the subgroup of W generated by the elements of J. Such subgroups are called *standard parabolic subgroups*.

In this section we define an order on the set of right coclasses

$$\mathcal{P}^{(W,S)} := \{W_J \circ w\colon\ w \in W \text{ and } W_J \text{ is a standard parabolic subgroup of } W\},$$

which differs from the canonical inclusion order and extends the weak Bruhat order of the group W. For the elementary definitions and results about Coxeter groups, Solomon algebra and weak Bruhat order we refer to [2,3,8,17].

Notation 2.

(1) The length of an element $w \in W$ as a word in the elements of S is denoted $l(w)$. The element ω_S is the unique element of maximal length in W.
(2) Given a subset $J \subseteq S$, the pair (W_J, J) is also a Coxeter system.
(3) For a subset J of S, X_J denotes the Solomon subset of elements which have no descent in J, that is, $X_J := \{x \in W \mid l(x \circ s) > l(x),\ \forall s \in J\}$, where $\circ$ denotes the product of the group W.

Recall, from [3], that $l(x \circ w) = l(x) + l(w)$, for all $x \in X_J$, $w \in W_J$.

Definition 3. The *weak Bruhat order* on W is defined by $x \leqslant_B x'$ if $x' = y \circ x$, with $l(x') = l(y) + l(x)$.

The group W equipped with the weak Bruhat order $\leqslant_B$ is a partially ordered set. The minimal element of $(W, \leqslant_B)$ is the identity element 1_W of the group, and its maximal element is ω_S.

Given a subset $J \subseteq S$, there exist unique elements $\xi_J \in X_J$ and $\omega_J \in W_J$ such that $\omega_S = \xi_J \circ \omega_J$. It is easy to check that ω_J is the maximal element of (W_J, J), and that ξ_J is the longest element of X_J. Note that $\omega_J^{-1} = \omega_J$ since $\omega_J^{-1} \in W_J$ and $l(\omega_J^{-1}) = l(\omega_J)$.

The following result is proved in [12].

Lemma 4. *Let* (W, S) *be a Coxeter system and let* $J \subseteq S$:

(1) *The subset* X_J *of* W *verifies that* $X_J = \{x \in W\colon\ x \leqslant_B \xi_J\}$.
(2) *The subgroup* W_J *of* W *satisfies* $W_J = \{x \in W\colon\ x \leqslant \omega_J\}$.

Definition 5. The *Coxeter poset* of (W, S) is the set $\mathcal{P}^{(W,S)}$ of right coclasses modulo the parabolic standard subgroups, ordered by the inclusion $\subseteq$.

The result below follows immediately from [3, Chapitre IV, p. 37, ex. 3].

Lemma 6. *The set* $\mathcal{P}^{(W,S)}$ *is described as*

$$\mathcal{P}^{(W,S)} := \left\{W_J \circ w,\ J \subseteq S \text{ and } w \in X_J^{-1}\right\}.$$

For $J = \emptyset$ one obtains the equality $X_\emptyset = W$. So, the group W is embedded in $\mathcal{P}^{(W,S)}$, via the map $w \mapsto W_\emptyset \circ w$.

The family of subsets $\mathcal{P}_r^{(W,S)} := \{W_J \circ w\colon\ |J| = r\}$, for $0 \leqslant r \leqslant |S|$, defines a graduation on the set $\mathcal{P}^{(W,S)}$.

Let us define an order, different from $\subseteq$, on the set $\mathcal{P}^{(W,S)}$.

In [2] the authors prove that for any subset $J \subseteq S$ and $s_0 \in S \setminus J$, the equality $X_{J\cup\{s_0\}} \circ X_J^{W_{J\cup\{s_0\}}} = X_J$ holds, where $X_J^{W_{J\cup\{s_0\}}} = \{x \in W_{J\cup\{s_0\}}$ which have no descent at $J\}$.

Notation 7. Let $J \subseteq S$. For any $s \in S \setminus J$, the result above implies that there exists an element $\alpha_{J,s} \in X_J^{W_{J\cup\{s\}}} = X_J \cap W_{J\cup\{s\}}$ such that $\xi_J = \xi_{J\cup\{s\}} \circ \alpha_{J,s}$.

Definition 8 *(weak Bruhat order on $\mathcal{P}^{(W,S)}$).* The *weak Bruhat order* $\leqslant_B$ on the set $\mathcal{P}^{(W,S)}$ is the transitive relation generated by the following conditions:

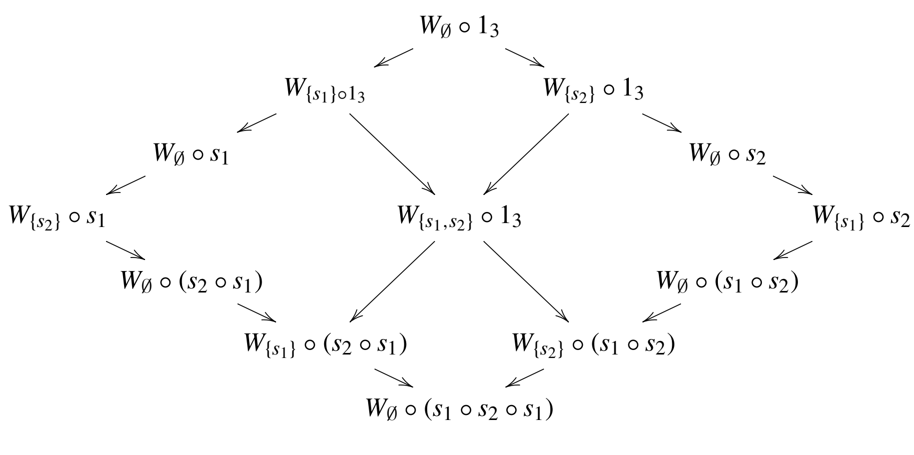
(1) For $J \subseteq S$, $s \in S \setminus J$ and $w \in X_{J\cup\{s\}}^{-1}$, the inequality $W_J \circ w <_B W_{J\cup\{s\}} \circ w$ holds.
(2) For $J \subseteq S$, $s \in S \setminus J$ and $w \in X_{J\cup\{s\}}^{-1}$, the following inequality holds

$$W_{J\cup\{s\}} \circ w <_B W_J \circ \left(\alpha_{J,s}^{-1} \circ w\right).$$

Note that, if $W_J \circ w <_B W_K \circ z$, then $w <_B z$.

To check that (1) and (2) give a partial order relation it suffices to see that, given a coclass $W_J \circ w$ it is impossible to get, applying several times the relations (1) and (2), a sequence of coclasses that ends at $W_J \circ w$. By the remark above, $W_J \circ w <_B W_K \circ z$ only for $w <_B z$. So, we restrict ourselves to consider a sequence of coclasses obtained by applying only the first relation. But in this case, the inequality $W_J \circ w <_B W_K \circ z$ implies that J is strictly contained in K.

For instance, let W be the symmetric group S_3 and let S be the set of transpositions $\{s_1, s_2\}$, where s_i denotes the permutation with exchanges i and $i+1$. The Hasse diagram for this group is:



where 1_3 denotes the identity of the group.

The proof of the following lemma is straightforward.

Lemma 9. *Let $x, z \in W$. Then one has $x \leqslant_B z$ in W if, and only if $W_\emptyset \circ x \leqslant_B W_\emptyset \circ z$ in $\mathcal{P}^{(W,S)}$.*

Let us point out that the result above is false if we replace $\emptyset$ by any $J \subseteq S$. As an example, consider the symmetric group S_4 with the set of generators $\{s_1, s_2, s_3\}$, where s_i is the permutation that exchanges i and $i+1$. The coclasses $W_{\{s_2,s_3\}} \circ 1_{S_4}$ and $W_{\{s_2,s_3\}} \circ s_1$ are not comparable for the weak Bruhat order.

Lemma 10. *Let $K \subseteq S$. If x and z are two elements of W such that $x \in X_K^{-1}$ and $\omega_K \circ x \leqslant_B z$, then $W_K \circ x \leqslant_B W_J \circ z$, for any J such that $z \in X_J^{-1}$.*

Proof. We prove the lemma by induction on the number of elements of K.

Suppose $|K| = 0$. The inequality $x = \omega_K \circ x \leqslant_B z$ holds, and Lemma 9 shows that $W_\emptyset \circ x \leqslant_B W_\emptyset \circ z$. Applying several times the relation (1) of Definition 8, we obtain the inequality $W_\emptyset \circ z \leqslant_B W_J \circ z$, which implies the result.

For $|K| \geqslant 1$, one has that $K = K' \cup \{s\}$, with $|K'| = |K| - 1$. The relation (2) of Definition 8 states that $W_K \circ x <_B W_{K'} \circ (\alpha_{K',s}^{-1} \circ x)$.

The equalities $\omega_S = \xi_{K'} \circ \omega_{K'} = \xi_K \circ \omega_K$ and $\xi_{K'} = \xi_K \circ \alpha_{K',s}$, imply that $\omega_K = \alpha_{K',s} \circ \omega_{K'}$. Since $\omega_{K'} \circ \alpha_{K',s}^{-1} = \omega_{K'}^{-1} \circ \alpha_{K',s}^{-1} = \omega_K^{-1} = \omega_K$, we obtain that:

$$\omega_{K'} \circ \left(\alpha_{K',s}^{-1} \circ x\right) = \omega_K \circ x < z.$$

Applying a recursive argument, it follows that $W_K \circ x <_B W_{K'} \circ (\alpha_{K',s}^{-1} \circ x) <_B W_J \circ z$. $\square$

2. Weak Bruhat order on the faces of permutohedron and on planar rooted trees

2.1. Permutohedron

2.1.1. Poset associated to the symmetric group

Let $(S_n, \circ)$ be the group of permutations of n elements. The group S_n is a Coxeter group generated by $n-1$ transpositions $s_1, \ldots, s_{n-1}$, where s_i is the permutation that exchanges i and $i+1$. We denote the set $\{s_1, \ldots, s_{n-1}\}$ by $\mathcal{S}_n$. Any element $\sigma \in S_n$ is identified with its image $(\sigma(1), \ldots, \sigma(n))$. The identity element of S_n is denoted by 1_n, and the longest one is denoted ω_n.

Definition 11. A *composition* $(n_1, \ldots, n_r)$ of n is a sequence of nonnegative integers $n_1, \ldots, n_r$ such that $\sum_{i=1}^r n_i = n$.

Given elements $\sigma_i \in S_{n_i}$, for $1 \leqslant i \leqslant r$, the permutation $\sigma_1 \times \cdots \times \sigma_r$ in S_n is defined as:

$$\sigma_1 \times \cdots \times \sigma_r := \left(\sigma_1(1), \ldots, \sigma_1(n_1), \sigma_2(1) + n_1, \ldots, \sigma_r(n_r) + n_1 + \cdots + n_{r-1}\right).$$

We denote by $S_{n_1,\ldots,n_r}$ the subgroup of S_n which is the image of the embedding $S_{n_1} \times \cdots \times S_{n_r} \hookrightarrow S_n$.

Let J be the set $\mathcal{S}_n \setminus \{s_{n_1}, s_{n_1+n_2}, \ldots, s_{n_1+\cdots+n_{r-1}}\}$, where $(n_1, \ldots, n_r)$ is a composition of n. It is immediate to check that the standard parabolic subgroup W_J is the subgroup $S_{n_1,\ldots,n_r}$ of S_n. Moreover, the longest element of $S_{n_1,\ldots,n_r}$ is $\omega_{n_1,\ldots,n_r} := \omega_{n_1} \times \cdots \times \omega_{n_r}$.

Given a composition $n = (n_1, \ldots, n_r)$, a $(n_1, \ldots, n_r)$-*shuffle* is a permutation $\sigma \in S_n$ verifying that:

$$\sigma(1) < \cdots < \sigma(n_1),$$
$$\vdots$$
$$\sigma(n_1 + \cdots + n_{r-1} + 1) < \cdots < \sigma(n).$$

The set of all $(n_1, \ldots, n_r)$-shuffles in S_n is denoted by $Sh(n_1, \ldots, n_r)$.

Remark 12.

(1) Let σ be an element of S_n and s_i be a transposition of $\mathcal{S}_n$. It is easily verified that $l(s_i \circ \sigma) = l(\sigma) + 1$ if, and only if, $\sigma^{-1}(i) < \sigma^{-1}(i + 1)$.
(2) Let $(n_1, \ldots, n_r)$ be a composition of n. The assertion above implies that the set $Sh(n_1, \ldots, n_r)$ coincides with the set X_J described in the first section, for $J = \mathcal{S}_n \setminus \{s_{n_1}, s_{n_1+n_2}, \ldots, s_{n_1+\cdots+n_{r-1}}\}$.

The longest element of $Sh(n_1, \ldots, n_r)$ (cf. [12]) is the permutation $\xi_{n_1,\ldots,n_r}$ defined as:

$$\xi_{n_1,\ldots,n_r}(h) := h - \sum_{i=1}^{j-1} n_i + \sum_{i=j+1}^{r} n_i, \quad \text{for } \sum_{i=1}^{j-1} n_i < h \leqslant \sum_{i=1}^{j} n_i, \text{ for } 1 \leqslant j \leqslant r.$$

For example, the longest element of $Sh(2, 1, 3, 2)$ is $\xi_{2,1,3,2} = (7, 8, 6, 3, 4, 5, 1, 2)$.

Notation 13. In order to simplify notation, we denote by $\mathcal{P}_n$ the set $\mathcal{P}^{(S_n, \mathcal{S}_n)}$.

The description of the set $\mathcal{P}^{(W,S)}$ given in Section 1 and the results above imply that for any right coclass $W_J \circ \tau \in \mathcal{P}_n$, there exist a unique composition $(n_1, \ldots, n_r)$ of n and a unique element $\sigma \in Sh(n_1, \ldots, n_r)^{-1}$ such that $W_J \circ \tau = S_{n_1,\ldots,n_r} \circ \sigma$.

The coclasses $S_{n_1,\ldots,n_r} \circ \sigma \in \mathcal{P}_n$ are in one-to-one correspondence with the faces of dimension $n - r$ of the permutohedron of dimension $n - 1$.

For any composition $(n_1, \ldots, n_r)$ of n and any $1 \leqslant k \leqslant r - 1$, we denote by $\alpha^k_{n_1,\ldots,n_r}$ the permutation such that $\xi_{n_1,\ldots,n_r} = \xi_{n_1,\ldots,n_k+n_{k+1},\ldots,n_r} \circ \alpha^k_{n_1,\ldots,n_r}$.

Recall that an element σ of S_n is identified with the coclass $S_{1,\ldots,1} \circ \sigma$, and that this identification defines an embedding of partially ordered sets $(S_n, \leqslant_B) \hookrightarrow (\mathcal{P}_n, \leqslant_B)$, by Lemma 9.

Let us give another description of the sets $\mathcal{P}_n$, which makes it easier to deal with them.

Lemma 14. *There exists a bijection between the set $\mathcal{P}_n$ and the set of all surjective maps $\gamma : \{1, \ldots, n\} \to \{1, \ldots, r\}$, with $r \geqslant 1$.*

Proof. Let $x = S_{n_1,\ldots,n_r} \circ \sigma$ be an element of $\mathcal{P}_n$.

Consider the map $\Psi(x):\{1,\dots,n\}\to\{1,\dots,r\}$ defined by:

$$\Psi(x)(i) := j, \quad \text{if } n_1+\cdots+n_{j-1} < \sigma(i) \leqslant n_1+\cdots+n_j.$$

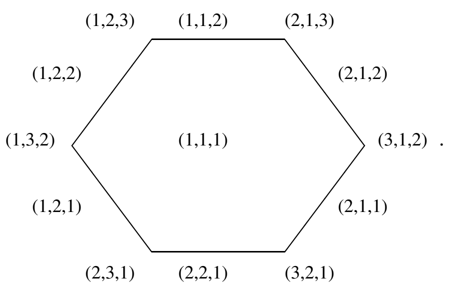
Given $\gamma:\{1,\dots,n\}\to\{1,\dots,r\}$, let n_i is the number of elements in $\gamma^{-1}(i)$. Define a permutation σ by $\sigma(j) := n_1+\cdots+n_{\gamma(j)-1}+k$, for $\gamma^{-1}(\gamma(j)) = \{i_1<\cdots<i_{n_j}\}$ and $i_k = j$. The map ψ has an inverse map given by $\Psi^{-1}(\gamma) = S_{n_1,\dots,n_r}\circ\sigma$. □

Example 15.

(1) The image of the element $S_{1,1,\dots,1}\circ\sigma$ under Ψ is the permutation σ.
(2) Consider the element $x = S_{2,1,3,4}\circ(3,4,1,5,7,8,6,2,9,10)\in\mathcal{P}_{10}$, the image of the map $\Psi(x):\{1,\dots,10\}\to\{1,\dots,4\}$ is $(2,3,1,3,4,4,3,1,4,4)$.

In what follows, we identify the elements of $\mathcal{P}_n$ with surjective maps $\gamma:\{1,\dots,n\}\to\{1,\dots,r\}$. We write (0) for the unique element of $\mathcal{P}_0$.

For $n=3$, the correspondence between the elements of $\mathcal{P}_3$ and the faces of the permutohedron of dimension 2 is given by the following picture:



Let $\mathcal{P}_\infty$ denote the disjoint union of all the sets $\mathcal{P}_n$, for $n\geqslant 0$. The usual composition of maps defines a multiplication on $\mathcal{P}_\infty$:

$$\gamma\circ\delta := \begin{cases} \gamma\circ\delta & \text{if } \gamma\in\mathcal{P}_r \text{ and } \mathrm{Im}(\delta)=\{1,\dots,r\},\\ (0) & \text{otherwise.}\end{cases}$$

From now on, we denote an element $\gamma\in\mathcal{P}_n$ by its image $\gamma = (\gamma(1),\dots,\gamma(n))$. If γ and δ are permutations, then their composite coincides with their product in the symmetric group S_n.

Remark 16. Note that any element $\gamma\in\mathcal{P}_n$, with $n\geqslant 1$, is a composition $\gamma=\gamma'\circ\sigma$ with $\sigma\in S_n$ and γ' a nondecreasing map. The element γ' is unique, while there exist many permutations σ verifying this equality. However there exists a unique σ of minimal length, which is precisely the permutation defined in the proof of Lemma 14 verifying $\Psi^{-1}(\gamma) = S_{n_1,\dots,n_r}\circ\sigma$.

For $1 \leqslant r \leqslant n$, the subset $\mathcal{P}_{n,r} := \{x \in \mathcal{P}_n\colon \operatorname{Im}(x) = \{1, \dots, r\}\}$ corresponds to the subset $\mathcal{P}_{n-r}^{(S_n, \mathcal{S}_n)}$ defined in Section 1.

The embedding $\mathcal{P}_n \times \mathcal{P}_m \hookrightarrow \mathcal{P}_{n+m}$ is given by:

$$(\gamma, \delta) \mapsto \big(\gamma(1), \dots, \gamma(n), \delta(1) + r, \dots, \delta(m) + r\big),$$

where $\gamma \in \mathcal{P}_{n,r}$. The image of (γ, δ) under this embedding is denoted by $\gamma \times \delta$. Note that $\gamma \times (0) = \gamma = (0) \times \gamma$, for all $\gamma \in \mathcal{P}_\infty$.

For $n \geqslant 1$, let us denote by t_i the element of $\mathcal{P}_{n,n-1}$ defined by:

$$t_i(j) := \begin{cases} j & \text{for } 1 \leqslant j \leqslant i, \\ j - 1 & \text{for } i < j \leqslant n. \end{cases}$$

Translating the definition of inclusion order and weak Bruhat order to the set of surjective maps $\gamma : \{1, \dots, n\} \to \{1, \dots, r\}$, we get the following result.

Lemma 17. *The weak Bruhat order $\leqslant_B$ on the set $\mathcal{P}_n$ is the transitive relation spanned by the conditions*:

(1) *if $\gamma^{-1}(i) < \gamma^{-1}(i+1)$ for some $1 \leqslant i \leqslant r$ then $\gamma <_B t_i \circ \gamma$,*
(2) *if $\gamma^{-1}(i) > \gamma^{-1}(i+1)$ for some $1 \leqslant i \leqslant r$ then $t_i \circ \gamma <_B \gamma$,*

for $\gamma \in \mathcal{P}_{n,r}$, where $\gamma^{-1}(i) < \gamma^{-1}(i+1)$ means that $j < k$ for all $j \in \gamma^{-1}(i)$ and $k \in \gamma^{-1}(i+1)$.

Proof. It is easily seen that the weak Bruhat order $\leqslant_B$ defined on the elements of $\mathcal{P}_n$, seen as right coclasses, is the transitive relation spanned by the following conditions:

(1) $S_{n_1,\dots,n_r} \circ \sigma <_B S_{n_1,\dots,n_k+n_{k+1},\dots,n_r} \circ \sigma$,
(2) $S_{n_1,\dots,n_k+n_{k+1},\dots,n_r} \circ \sigma <_B S_{n_1,\dots,n_r} \circ (\alpha^k_{n_1,\dots,n_r})^{-1} \circ \sigma$,

for any composition $(n_1, \dots, n_r)$ of n, any $1 \leqslant k \leqslant r - 1$ and any $\sigma \in Sh(n_1, \dots, n_k + n_{k+1}, \dots, n_r)$. Translating these conditions from the elements of type $S_{n_1,\dots,n_r} \circ \sigma$ to the surjective maps $\{1, \dots, n\} \to \{1, \dots, r\}$ the assertion follows. □

Remark 18.

(1) Given $\gamma \in \mathcal{P}_n$, the inequality $1_n \leqslant_B \gamma \leqslant_B \omega_n$ holds, where 1_n is the identity element of S_n and $\omega_n = (n, n-1, \dots, 1)$.
(2) Suppose $\gamma \leqslant_B \gamma'$ in $\mathcal{P}_n$. If $\gamma'(j) < \gamma'(k)$ for some $1 \leqslant j < k \leqslant n$, then $\gamma(j) < \gamma(k)$. Conversely, if $\gamma(j) > \gamma(k)$ for some $1 \leqslant j < k \leqslant n$, then $\gamma'(j) > \gamma'(k)$.

Consider the involution $\mathcal{P}_n \to \mathcal{P}_n$, $\gamma \mapsto \bar{\gamma}$, where $\bar{\gamma}$ is defined by $\bar{\gamma}(i) := \gamma(n - i + 1)$, for $\gamma \in \mathcal{P}_n$ and $1 \leqslant i \leqslant n$. Note that $\bar{\gamma} = (\gamma(n), \dots, \gamma(1))$, and that the following inequality is verified.

Let $1 \leqslant i \leqslant r$, $\gamma^{-1}(i) < \gamma^{-1}(i+1)$ if, and only if, $\bar{\gamma}^{-1}(i) > \bar{\gamma}^{-1}(i+1)$.

Remark 19. Lemma 17 and the argument above imply that $\gamma \mapsto \bar{\gamma}$ is an anti-isomorphism of ordered sets. Moreover, the order $\leqslant_B$ is the transitive relation spanned by the conditions:

(1) if $\gamma^{-1}(i) < \gamma^{-1}(i+1)$ for some $1 \leqslant i \leqslant r$ then $\gamma <_B t_i \circ \gamma$,
(2) $\gamma \mapsto \bar{\gamma}$ is an anti-isomorphism of ordered sets.

Proposition 20. *Let $\gamma_0, \ldots, \gamma_k$ and $\delta_0, \ldots, \delta_k$ be families of elements in $\mathcal{P}_\infty$ verifying that $\gamma_i \leqslant_B \delta_i$ in $\mathcal{P}_{n_i}$. The following inequality holds*:

$$\gamma_0 \times \cdots \times \gamma_k \leqslant_B \delta_0 \times \cdots \times \delta_k.$$

Proof. Clearly, it suffices to show that

$$\gamma \times \alpha \leqslant_B \gamma \times \beta \quad \text{and} \quad \alpha \times \gamma \leqslant_B \beta \times \gamma,$$

whenever $\alpha, \beta, \gamma \in \mathcal{P}_\infty$ are such that $\alpha \leqslant_B \beta$.

We prove the first inequality, the second one is proved in a similar way. Suppose that $\gamma \in \mathcal{P}_{n,r}$, $\alpha \in \mathcal{P}_{m,s}$ and $\beta \in \mathcal{P}_{m,p}$. We may restrict ourselves to the following two cases:

(1) When $\alpha^{-1}(j) < \alpha^{-1}(j+1)$, for some $1 \leqslant j \leqslant s$, and $\beta = t_j \circ \alpha$.
(2) When $\beta^{-1}(j) > \beta^{-1}(j+1)$, for some $1 \leqslant j \leqslant p$, and $\alpha = t_j \circ \beta$.

(1) In this case, one has that

$$(\gamma \times \alpha)^{-1}(r+j) < (\gamma \times \alpha)^{-1}(r+j+1), \quad \text{and} \quad \gamma \times \beta = t_{r+j} \circ (\gamma \times \alpha).$$

So, $\gamma \times \alpha <_B \gamma \times \beta$.

(2) It is easy to check that

$$(\gamma \times \beta)^{-1}(r+j) > (\gamma \times \beta)^{-1}(r+j+1), \quad \text{and} \quad \gamma \times \alpha = t_{r+j} \circ (\gamma \times \beta).$$

So, $\gamma \times \alpha <_B \gamma \times \beta$. $\square$

2.1.2. Shuffles in $\mathcal{P}_n$

We proceed to introduce first the notion of shuffle in $\mathcal{P}_n$ which extends the definition of shuffle in S_n.

Definition 21. Let $\gamma \in \mathcal{P}_n$ and let $n = (n_1, \ldots, n_r)$ be a composition of n, the element γ is a $(n_1, \ldots, n_r)$-*shuffle* in $\mathcal{P}_n$ if

$$\gamma(1) < \cdots < \gamma(n_1),$$
$$\vdots$$
$$\gamma(n_1 + \cdots + n_{r-1} + 1) < \cdots < \gamma(n).$$

We denote by $SH(n_1,\dots,n_r)$ the subset of $(n_1,\dots,n_r)$-shuffles in $\mathcal{P}_n$. The following result is a generalized version of [3, Chapitre IV, p. 37, ex. 3], to the set $\mathcal{P}_n$.

Lemma 22. *For any element $\gamma\in\mathcal{P}_n$, and any composition $(n_1,\dots,n_r)$ of n, there exist unique elements $\omega\in\bigcup_{1\leqslant i_j\leqslant n_j,\,1\leqslant j\leqslant r} SH(i_1,\dots,i_r)$ and $\gamma_j\in\mathcal{P}_{n_j,i_j}$, for $1\leqslant j\leqslant r$, such that $\gamma=\omega\circ(\gamma_1\times\cdots\times\gamma_r)$.*

Proof. Let γ be an element of $\mathcal{P}_{n,r}$. For $1\leqslant j\leqslant r$, there exist unique integers $1\leqslant i_j\leqslant n_j$ and unique bijective order-preserving morphisms

$$\phi_j:\gamma\bigl(\{n_1+\cdots+n_{j-1}+1,\dots,n_1+\cdots+n_j\}\bigr)\to\{1,\dots,i_j\}.$$

Define $\gamma_j:=(\phi_j(\gamma(n_1+\cdots+n_{j-1}+1)),\dots,\phi_j(\gamma(n_1+\cdots+n_j)))$, and $\omega:=(\phi_1^{-1}(1),\dots,\phi_1^{-1}(i_1),\dots,\phi_r^{-1}(1),\dots,\phi_r^{-1}(i_r))$. It is easy to check that $\omega\in SH(i_1,\dots,i_r)$ and that the elements are unique. □

Remark 23. Suppose $\gamma=\tau\circ\sigma$, with τ a nondecreasing map and $\sigma\in S_n$. If $\gamma\in SH(n_1,\dots,n_r)$, then $\sigma\in Sh(n_1,\dots,n_r)$.

Proposition 24. *The following equality holds*:

$$SH(n_1,\dots,n_r)=\{x\in\mathcal{P}_n\colon\ x\leqslant_B\xi_{n_1,\dots,n_r}\}.$$

Proof. Suppose $\omega\in\mathcal{P}_n$ is such that $\omega\leqslant_B\xi_{n_1,\dots,n_r}$. Remark 18 implies that $\omega\in SH(n_1,\dots,n_r)$.

Let $\omega\in SH(n_1,\dots,n_r)\cap\mathcal{P}_{n,r}$. We proceed by induction on $n-r$. If $n-r=0$, then $\omega\in Sh(n_1,\dots,n_r)$ and the result follows from Lemma 4 and Remark 12.

If $n-r>0$, there exists $i\leqslant r$ such that $\omega^{-1}(i)=\{l_1<\cdots<l_k\}$, with $k\geqslant 2$. Define $\tilde{\omega}\in\mathcal{P}_{n,r+1}$ as follows:

$$\tilde{\omega}(j):=\begin{cases}\omega(j) & \text{if } \omega(j)<i \text{ or } j=l_k,\\ \omega(j)+1 & \text{otherwise.}\end{cases}$$

It is easy to see that $\tilde{\omega}\in SH(n_1,\dots,n_r)$, and that $\tilde{\omega}^{-1}(i)=\{l_k\}>\{l_1,\dots,l_{k-1}\}=\tilde{\omega}^{-1}(i+1)$, which implies that $\omega=t_i\circ\tilde{\omega}<_B\tilde{\omega}$. By a recursive argument, we obtain $\omega<_B\tilde{\omega}\leqslant_B\xi_{n_1,\dots,n_r}$. □

Proposition 25. *Let $\gamma_0,\dots,\gamma_k$ be a family of elements in $\mathcal{P}_\infty$, with $\gamma_i\in\mathcal{P}_{n_i,r_i}$, and let ω and ω' be elements of $SH(r_0,\dots,r_k)$. The inequality $\omega\leqslant_B\omega'$ holds if, and only if $\omega\circ(\gamma_0\times\cdots\times\gamma_k)\leqslant_B\omega'\circ(\gamma_0\times\cdots\times\gamma_k)$.*

Proof. ($\Rightarrow$) Suppose that $\omega\leqslant_B\omega'$, it suffices to consider the cases:

(1) $\omega^{-1}(j)<\omega^{-1}(j+1)$ for some j, and $\omega'=t_j\circ\omega$.
(2) $(\omega')^{-1}(j)>(\omega')^{-1}(j+1)$ for some j, and $\omega=t_j\circ\omega'$.

The proofs of both cases are similar, we give the proof of the second one. Suppose $(\omega')^{-1}(j) > (\omega')^{-1}(j+1)$ for some j, and $\omega = t_j \circ \omega'$.

Since $\omega \in SH(r_0, \dots, r_k)$, there exists $0 \leqslant l \leqslant k-1$ such that

$$(\omega')^{-1}(j+1) \subseteq \{1, \dots, n_0 + \dots + n_l\}, \quad \text{and}$$
$$(\omega')^{-1}(j) \subseteq \{n_0 + \dots + n_l + 1, \dots, n_0 + \dots + n_k\}.$$

Consequently $(\omega' \circ (\gamma_0 \times \dots \times \gamma_k))^{-1}(j) > (\omega' \circ (\gamma_0 \times \dots \times \gamma_k))^{-1}(j+1)$, and $\omega \circ (\gamma_0 \times \dots \times \gamma_k) = t_j \circ \omega' \circ (\gamma_0 \times \dots \times \gamma_k)$. The argument above implies that:

$$\omega \circ (\gamma_0 \times \dots \times \gamma_k) \leqslant_B \omega' \circ (\gamma_0 \times \dots \times \gamma_k).$$

($\Leftarrow$) We need to prove that, if the elements $\omega, \omega' \in SH(r, s)$ are such that $\omega \circ (\gamma \times \delta) \leqslant_B \omega' \circ (\gamma \times \delta)$, then $\omega \leqslant_B \omega'$. It suffices to check this result for two cases:

(a) When there exists i such that $(\omega \circ (\gamma \times \delta))^{-1}(i) < (\omega \circ (\gamma \times \delta))^{-1}(i+1)$ and $\omega' \circ (\gamma \times \delta) = t_i \circ \omega \circ (\gamma \times \delta)$.
(b) When there exists i such that $(\omega \circ (\gamma \times \delta))^{-1}(i) > (\omega \circ (\gamma \times \delta))^{-1}(i+1)$ and $\omega \circ (\gamma \times \delta) = t_i \circ \omega' \circ (\gamma \times \delta)$.

We prove the result for the first case, the proof is similar for the second one.

We have to consider the following situations:

(1) If $|\omega^{-1}(i)| = |\omega^{-1}(i+1)| = 1$, $\omega^{-1}(i) \leqslant r$ and $r+1 \leqslant \omega^{-1}(i+1)$, then $t_i \circ \omega = \omega' \in SH(r, s)$, so $\omega \leqslant_B \omega'$.
(2) If $\omega^{-1}(i) \cap \{1, \dots, r\} = \{j\}$ and $\omega^{-1}(i+1) \cap \{1, \dots, r\} = \{j+1\}$, for $1 \leqslant j < r$, then $\omega' \circ (\gamma \times \delta)(k) = i$ for all $1 \leqslant k \leqslant n$ such that $\gamma(k) = j$ or $\gamma(k) = j+1$. But, since $\omega' \in SH(r, s)$, it is easily seen that $i = \omega' \circ (\gamma \times \delta)(k) < \omega' \circ (\gamma \times \delta)(k') = i$ for $\gamma(k) = j$ and $\gamma(k') = j+1$, which is impossible.
(3) If $|\omega^{-1}(i)| = |\omega^{-1}(i+1)| = 1$, $r+1 \leqslant \omega^{-1}(i)$ and $\omega^{-1}(i+1) \leqslant r$, then $\omega \circ (\gamma \times \delta) \nleqslant_B t_i \circ \omega \circ (\gamma \times \delta)$.
(4) If $\omega^{-1}(i) \cap \{r+1, \dots, r+s\} = \{j\}$ and $\omega^{-1}(i+1) \cap \{r+1, \dots, r+s\} = \{j+1\}$, for $r+1 \leqslant j < r+s$, then $\omega' \circ (\gamma \times \delta)(k) = i$ for all $n+1 \leqslant k \leqslant n+m$ such that $\delta(k) = j-r$ or $\gamma(k) = j-r+1$. Again, $\omega' \in SH(r, s)$, which implies that $i = \omega' \circ (\gamma \times \delta)(k) < \omega' \circ (\gamma \times \delta)(k') = i$ for $\delta(k) = j-r$ and $\gamma(k') = j-r+1$. So, this case cannot occur.

So, the unique possible situation is 1, and in this case $\omega' = t_i \circ \omega \in SH(r, s)$. □

We want to extend some well-known results for the shuffles in S_n to the generalized shuffles in $\mathcal{P}_n$. Let us begin with the associativity of the shuffle.

Proposition 26 *(associativity of SH). Given nonnegative integers n, m, r the following equality holds*:

$$SH(n,m,r) = \bigcup_{1\leqslant j\leqslant n+m} SH(j,r)\circ\big(SH(n,m)\times 1_r\big)$$
$$= \bigcup_{1\leqslant k\leqslant m+r} SH(n,k)\circ\big(1_n\times SH(m,r)\big).$$

Proof. From Lemma 22, we conclude that for any $\gamma \in SH(n,m,r)$ there exist unique integers $1\leqslant j\leqslant n+m$, $1\leqslant l\leqslant r$, $1\leqslant k\leqslant m+r$ and $1\leqslant s\leqslant n$, and unique elements $\gamma_1\in SH(j,l)$, $\gamma_2\in SH(s,k)$, $\delta_1\in\mathcal{P}_{n+m}$, $\delta_2\in\mathcal{P}_r$, $\tau_1\in\mathcal{P}_n$ and $\tau_2\in\mathcal{P}_{m+r}$, such that

$$\gamma=\gamma_1\circ(\delta_1\times\delta_2)=\gamma_2\circ(\tau_1\times\tau_2).$$

The fact that $\gamma\in SH(n,m,r)$ implies that $\gamma_1(\delta_2(1)+n+m)<\cdots<\gamma_1(\delta_2(r)+n+m)$. So, it must be $l=r$. The same argument, applied to $\gamma_2(\tau_1(i))$, for $1\leqslant i\leqslant n$, shows that $s=n$.

To end the proof note that:

- (i) Since $\gamma\in SH(n,m,r)$ and $\gamma_1\in SH(j,r)$, the element δ_1 belongs to $SH(n,m)$ and δ_2 is equal to 1_r.
- (ii) The fact that $\gamma\in SH(n,m,r)$ and that $\gamma_2\in SH(n,k)$ imply that $\tau_1=1_n$ and $\tau_2\in SH(m,r)$. □

For $n,m\geqslant 1$, the set $SH(n,m)$ is the disjoint union of the following three subsets:

$$SH^{\succ}(n,m):=\big\{x\in SH(n,m):\ x(n)<x(n+m)\big\},$$
$$SH^{\bullet}(n,m):=\big\{x\in SH(n,m):\ x(n)=x(n+m)\big\},$$
$$SH^{\prec}(n,m):=\big\{x\in SH(n,m):\ x(n)>x(n+m)\big\}.$$

Lemma 27. *For $n,m\geqslant 1$, the sets $SH^{\succ}(n,m)$, $SH^{\bullet}(n,m)$ and $SH^{\prec}(n,m)$ may be described in terms of the weak Bruhat order as follows*:

(1) $SH^{\succ}(n,m)=\{x\in\mathcal{P}_{n+m}:\ x\leqslant_B \xi_{n,m-1}\times 1_1\}$,
(2) $SH^{\bullet}(n,m)=\{x\in\mathcal{P}_{n+m}:\ z(n-1,m-1,0)\leqslant_B x\leqslant_B (\xi_{n-1,m-1}\times 1_1)\circ z(n-1,m-1,0)\}$,
(3) $SH^{\prec}(n,m)=\{x\in\mathcal{P}_{n+m}:\ z(n-1,m)\leqslant_B x\leqslant_B \xi_{n,m}\}$,

where $z(n-1,m-1,0)=(1,\dots,n-1,n+m-1,n,\dots,n+m-1)$ *and* $z(n-1,m)=(1,\dots,n-1,n+m,n,\dots,n+m-1)$.

Proof. The proof of the formulas follows from Proposition 24. We give the proof of the first equality, the others may be checked in a similar way.

If $\gamma\in SH^{\succ}(n,m)$, then $\gamma=\gamma_1\times 1_1$, with $\gamma_1\in SH(n,m-1)$. By Proposition 24, we obtain that $\gamma_1\leqslant_B\xi_{n,m-1}$. It implies that $\gamma\leqslant_B\xi_{n,m-1}\times 1_1$.

Conversely, let $\gamma\in\mathcal{P}_{n+m}$ be such that $\gamma\leqslant_B\xi_{n,m-1}\times 1_1$. It is easily seen that $\gamma=\gamma_1\times 1_1$, with $\gamma_1\leqslant_B\xi_{n,m-1}$. So, $\gamma\in SH^{\succ}(n,m)$. □

The following result permits us to decompose the associative product given by SH into three different operations.

Proposition 28. *The following identities hold*:

(1) $\bigcup_{i=1}^{m+r} SH^{\succ}(n,i) \circ (1_n \times SH^{\succ}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\succ}(j,r) \circ (SH(n,m) \times 1_r)$,
(2) $\bigcup_{i=1}^{m+r} SH^{\succ}(n,i) \circ (1_n \times SH^{\prec}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\prec}(j,r) \circ (SH^{\succ}(n,m) \times 1_r)$,
(3) $\bigcup_{i=1}^{m+r} SH^{\prec}(n,i) \circ (1_n \times SH(m,r)) = \bigcup_{j=1}^{n+m} SH^{\prec}(j,r) \circ (SH^{\prec}(n,m) \times 1_r)$,
(4) $\bigcup_{i=1}^{m+r} SH^{\bullet}(n,i) \circ (1_n \times SH^{\bullet}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\bullet}(j,r) \circ (SH^{\bullet}(n,m) \times 1_r)$,
(5) $\bigcup_{i=1}^{m+r} SH^{\succ}(n,i) \circ (1_n \times SH^{\bullet}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\bullet}(j,r) \circ (SH^{\succ}(n,m) \times 1_r)$,
(6) $\bigcup_{i=1}^{m+r} SH^{\bullet}(n,i) \circ (1_n \times SH^{\succ}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\bullet}(j,r) \circ (SH^{\prec}(n,m) \times 1_r)$,
(7) $\bigcup_{i=1}^{m+r} SH^{\bullet}(n,i) \circ (1_n \times SH^{\prec}(m,r)) = \bigcup_{j=1}^{n+m} SH^{\prec}(j,r) \circ (SH^{\bullet}(n,m) \times 1_r)$.

Proof. Let us check that the first equality holds.

If $\gamma \in SH(n,m,r)$ is such that $\gamma = \gamma_1 \circ (\delta \times 1_r) = \gamma_2 \circ (1_n \times \tau)$, with $\gamma_1 \in SH(j,r)$, $\delta \in SH(n,m)$, $\gamma_2 \in SH(n,i)$ and $\tau \in SH(m,r)$, for $1 \leqslant j \leqslant n+m$ and $1 \leqslant i \leqslant m+r$, then $\gamma_1(\delta(n)) = \gamma_2(n)$, $\gamma_1(\delta(n+m)) = \gamma_2(\tau(m)+n)$, and $\gamma_1(j+r) = \gamma_2(\tau(m+r)+n)$.

If $\gamma_1 \in SH^{\succ}(j,r)$, then $\gamma_1(j+r) > \gamma_1(k)$, for all $1 \leqslant k \leqslant j$. In particular, it implies that:

(1) $\gamma_2(\tau(m+r)+n) = \gamma_1(j+r) > \gamma_1(\delta(n+m)) = \gamma_2(\tau(m)+n)$. Since $\tau(m+r)$ and $\tau(m)$ belong to $\{1,\dots,i\}$ and $\gamma_2(n+1) < \cdots < \gamma_2(n+i)$, we get the inequality $\tau(m+r) > \tau(m)$. So, $\tau \in SH^{\succ}(m,r)$.
(2) $\gamma_2(\tau(m+r)+n) = \gamma_1(j+r) > \gamma_1(\delta(n)) = \gamma_2(n)$, which implies that $\gamma_2(n+i) > \gamma_2(n)$. So, $\gamma_2 \in SH^{\succ}(n,i)$.
Conversely, suppose that $\gamma_2 \in SH^{\succ}(n,i)$ and $\tau \in SH^{\succ}(m,r)$. We know that $\gamma_1(j) = \gamma_1(\delta(n))$ or $\gamma_1(j) = \gamma_1(\delta(n+m))$. The following statements hold:
  – $\gamma_1(\delta(n)) = \gamma_2(n) < \gamma_2(n+i) = \gamma_2(\tau(m+r)+n) = \gamma_1(j+r)$, and
  – $\gamma_1(\delta(n+m)) = \gamma_2(\tau(m)+n) < \gamma_2(\tau(m+r)+n) = \gamma_1(j+r)$.

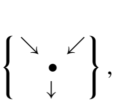
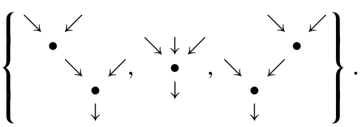
So, $\gamma_1(j) < \gamma_1(j+r)$ and $\gamma_1 \in SH^{\succ}(j,r)$.

The other equalities are proved using similar arguments. □

2.2. Planar rooted trees

In this paper, a *tree* is a finite planar nonempty oriented connected graph t without loops, and such that for any vertex of t there are at least two incoming edges and exactly one outgoing edge.

For $n \geqslant 0$, let $\mathcal{T}_n$ denote the set of planar rooted trees with $n+1$ leaves. For example,

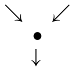
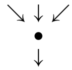
$\mathcal{T}_0 = \{\downarrow\}$, $\mathcal{T}_1 =$ , $\mathcal{T}_2 =$ .

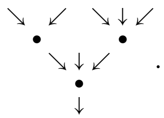
The *degree* of a tree t is n when t belongs to $\mathcal{T}_n$, we denote it by $|t|$.

A tree t is called *binary* if each vertex of t has exactly two incoming edges. We denote by $\mathcal{Y}_n$ the set of binary trees of degree n. Note that the elements of $\mathcal{Y}_n$ are the trees which have exactly $n+1$ leaves and n vertices.

The tree with only one vertex is called a *corolla*. It is denoted by ς_n.

The *grafting* of planar rooted trees $t^0 \in \mathcal{T}_{n_0}, \dots, t^r \in \mathcal{T}_{n_r}$, is the tree $\bigvee(t^0, \dots, t^r)$ of degree $\sum_{i=0}^{r} n_i + r$, obtained by joining the roots of $t^0, \dots, t^r$ to a new vertex and creating a new root. For any tree $t \neq \downarrow$, there exist unique trees $t^0, \dots, t^r$ such that $t = \bigvee(t^0, \dots, t^r)$.

Example 29. Suppose $t^0 =$ , $t^1 = \downarrow$ and $t^2 =$ . The element $\bigvee(t^0, t^1, t^2)$ is the tree:



We have that $\mathcal{T}_n = \bigsqcup_{n_0+\dots+n_r+r=n} \bigvee(\mathcal{T}_{n_0}, \dots, \mathcal{T}_{n_r})$, for $n \geqslant 1$.

Let $\mathcal{T}_\infty$ denote the graded set of all planar trees $\bigcup_{n \geqslant 0} \mathcal{T}_n$.

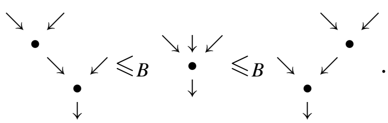
2.2.1. Order on $\mathcal{T}_n$

Definition 30. Let $\leqslant_B$ be the relation on $\mathcal{T}_n$ defined as follows. A tree w covers another tree t if one of the following conditions is satisfied:

(1) If $t = \bigvee(t^0, \dots, t^{i_0}, \dots, t^k)$, $w = \bigvee(t^0, \dots, w^{i_0}, \dots, t^k)$, and w^{i_0} covers t^{i_0}.
(2) If $t = \bigvee(\bigvee(t^0, \dots, t^k), w^0, \dots, w^h)$ and $w = \bigvee(t^0, \dots, t^k, w^0, \dots, w^h)$.
(3) If $t = \bigvee(t^0, \dots, t^k, w^0, \dots, w^h)$ and $w = (t^0, \dots, t^k, \bigvee(w^0, \dots, w^h))$.

We say that $t \leqslant_B t'$ if, and only if, there exist trees $t = t^0, t^1, \dots, t^r = t'$ such that t^{i+1} covers t^i, for $0 \leqslant i \leqslant r-1$.

Example 31. On $\mathcal{T}_2$, we have that



Proposition 32. *The pair $(\mathcal{T}_n, \leqslant_B)$ is a partially ordered set, for all $n \geqslant 1$.*

Proof. We need to check that, given a tree t in $\mathcal{T}_n$, there does not exist a sequence of trees $t = t_0, t_1, \dots, t_m = t$ such that t_{i+1} covers t_i.

Suppose that t_i and t_{i-1} verify one of the three conditions of Definition 30. Note that, if t_i is obtained from t_{i-1} by applying operation of type 2, then $|t_i^0| < |t_{i-1}^0|$. So, the sequence must be obtained applying only operations of types 1 or 3.

If t_i is obtained from t_{i-1} by an operation of type 3, then $|t_j^{r_j}| > |t_{j-1}^{r_{j-1}}|$. We may conclude that the sequence $t = t_0 <_B t_1 <_B \cdots <_B t_m = t$ is obtained by applying only operations of type 1.

In this case, there exists at least one $1 \leqslant k \leqslant m$ and a sequence $t^k = t_0^k <_B \cdots <_B t_m^k = t^k$. But, a recursive argument implies that it is false. So, the relation $\leqslant_B$ is a well defined order on $\mathcal{T}_n$. $\square$

It is immediate to check that $\leqslant_B$ induces an order on the subset $\mathcal{Y}_n$ of $\mathcal{T}_n$. This order is the order defined in [12].

The following result is an immediate consequence of Definition 30 and the proposition above.

Lemma 33. *Let $t = \bigvee(t^0, \ldots, t^k)$ and $z = \bigvee(z^0, \ldots, z^k)$ in $\mathcal{T}_n$, such that $|t^i| = |z^i|$, for $0 \leqslant i \leqslant k$:*

(1) *The inequality $t \leqslant_B z$ holds if, and only if, the inequality $t^i \leqslant_B z^i$ holds for all $0 \leqslant i \leqslant k$.*
(2) *An element $w \in \mathcal{T}_n$ is such that $t \leqslant_B w \leqslant_B z$ if, and only if, $w = \bigvee(w^0, \ldots, w^k)$ with $t^i \leqslant_B w^i \leqslant_B z^i$, for $0 \leqslant i \leqslant k$.*

Define a map $\mathcal{T}_\infty \to \mathcal{T}_\infty$, as follows:

(1) $\bar{\downarrow} := \downarrow$.
(2) $\overline{\bigvee(t^0, \ldots, t^r)} := \bigvee(\overline{t^r}, \ldots, \overline{t^0})$.

Remark 34. The map described above is an anti-isomorphism of the poset $(\mathcal{T}_\infty, \leqslant_B)$ which reflects a tree t in a vertical line passing through its root. Moreover, the order $\leqslant_B$ is the transitive relation spanned by the conditions:

(1) If $t^{i_0} <_B w^{i_0} \in \mathcal{T}_{n_{i_0}}$, then $\bigvee(t^0, \ldots, t^{i_0}, \ldots, t^k) <_B \bigvee(t^0, \ldots, w^{i_0}, \ldots, t^k)$ in $\mathcal{T}_{n_0 + \cdots + n_k + k}$.
(2) For $t^0, \ldots, t^k, w^0, \ldots, w^h \in \mathcal{T}_\infty$,

$$\bigvee\Big(\bigvee(t^0, \ldots, t^k), w^0, \ldots, w^h\Big) <_B \bigvee\big(t^0, \ldots, t^k, w^0, \ldots, w^h\big)$$

(3) The map $t \mapsto \bar{t}$ is an anti-isomorphism of the poset $(\mathcal{T}_\infty, \leqslant_B)$.

2.3. Poset morphism from $\mathcal{P}_\infty$ to $\mathcal{T}_\infty$

2.3.1. Grafting in $\mathcal{P}_\infty$

Let t be a planar rooted tree, we denote by $Vert(t)$ the set of vertices of t.

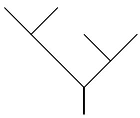
Definition 35. A *leveled planar tree* is a planar tree t together with a surjective map $\phi : Vert(t) \to \{1, \ldots, r\}$, for some $r \leqslant |Vert(t)|$, verifying that $f(v_1) < f(v_2)$, if $v_1, v_2 \in$

$Vert(t)$ are such that there exists a path from v_1 to v_2. The integer r is called the *number of levels* of the leveled tree (t, f).

For example, the pair

$$\left(\begin{array}{ccccccc} \searrow & & \swarrow & & \searrow & & \swarrow \\ & v_1 & & & & v_2 & \\ & & \searrow & & \swarrow & & \\ & & & v_3 & & & \\ & & & \downarrow & & & \end{array}, f\right),$$

where $f(v_i) = i$ for $1 \leqslant i \leqslant 3$, is identified with the leveled tree

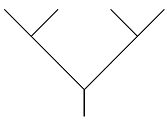


,

while the pair

$$\left(\begin{array}{ccccccc} \searrow & & \swarrow & & \searrow & & \swarrow \\ & v_1 & & & & v_2 & \\ & & \searrow & & \swarrow & & \\ & & & v_3 & & & \\ & & & \downarrow & & & \end{array}, g\right),$$

where $g(v_1) = g(v_2) = 1$ and $g(v_3) = 2$, is identified with the leveled tree



.

Given a family $x^0 = (t^0, f_0), \dots, x^m = (t^m, f_m)$, where $t^i \in T_{n_i}$ and $f_i : Vert(t^i) \to \{1, \dots, r_i\}$ is a surjection, for $0 \leqslant i \leqslant m$, note that the set $Vert(\bigvee(t^0, \dots, t^m)) = \bigcup_{i=0}^m Vert(t^i) \cup \{R\}$, where R denotes the root of $\bigvee(t^0, \dots, t^m)$.

Let $\bar{f} : \bigcup_{i=0}^m Vert(t^i) \to \{1, \dots, r_0 + \dots + r_m\}$ be the map given by $\bar{f}(v) := f_i(v) + r_0 + \dots + r_{i-1}$, for $v \in Vert(t^i)$ and $0 \leqslant i \leqslant m$.

Definition 36. Given a family of leveled planar trees $\{x^i = (t^i, f_i)\}_{0 \leqslant i \leqslant m}$ and an element $\omega \in SH(r_0, \dots, r_m) \cap P_{r_0 + \dots + r_m, p}$, where $\mathrm{Im}(f_i) = \{1, \dots, r_i\}$, the *grafting of* $\{x^i = (t^i, f_i)\}_{0 \leqslant i \leqslant m}$ *over* ω is the leveled tree $(\bigvee(t^0, \dots, t^m), \omega(\bar{f}))$, where

$$\omega(\bar{f})(v) := \begin{cases} \omega \circ \bar{f}(v) & \text{if } v \in \bigcup_{i=0}^m Vert(t^i), \\ p + 1 & \text{if } v = R. \end{cases}$$

The grafting of the family $\{x^i = (t^i, f_i)\}_{0 \leqslant i \leqslant m}$ of leveled planar trees over ω is denoted by $\bigvee_\omega(x^0, \dots, x^m)$.

We want to prove that the set $\mathcal{P}_{n,r}$ may be identified with the set of leveled planar trees with $n+1$ leaves and r levels. In order to do that we introduce the notion of grafting in $\mathcal{P}_\infty$.

Definition 37. Let $(n_0, \dots, n_m)$ be a composition of n and $\gamma_0, \dots, \gamma_m$ a family of elements of $\mathcal{P}_\infty$, with $\gamma_i \in \mathcal{P}_{n_i,r_i}$ or $\gamma_i = (0)$, and let ω be an element of $SH(r_0, \dots, r_m)$. The *grafting* of $\gamma_0, \dots, \gamma_m$ over ω is the element of $\mathcal{P}_{n+m}$ given by:

$$\bigvee_{\omega}(\gamma_0, \dots, \gamma_m) := \big(\big(\omega \circ (\gamma_0 \times \cdots \times \gamma_m)\big) \times 1_1\big) \circ z(n_0, \dots, n_m),$$

where $z(n_0, \dots, n_m) := (1, \dots, n_0, n+1, n_0+1, \dots, n_0+n_1, n+1, \dots, n+1, n_0+\cdots+n_{m-1}+1, \dots, n)$.

Note that: If $\gamma = \bigvee_\omega(\gamma_0, \dots, \gamma_m) \in \mathcal{P}_{n,r}$, with $\gamma_i \in \mathcal{P}_{n_i,r_i}$, for $0 \leqslant i \leqslant m$, then $n = \sum_{i=0}^m n_i + m$ and $r = \sum_{i=0}^m r_i + 1$.

The element $z(n_0, \dots, n_m)$ belongs to $\mathcal{P}_{n+m-1,n+1}$.

Example 38.

(1) The element $1^n := (1, 1, \dots, 1) : \{1, \dots, n\} \to \{1\}$ is the grafting $\bigvee_{(0)}((0), \dots, (0))$.
(2) Let $\omega = (2, 3, 6, 1, 2, 4, 5) \in SH(2, 1, 4)$. If $\gamma_0 = (2, 1, 2)$, $\gamma_1 = (1)$ and $\gamma_2 = (3, 1, 1, 4, 2)$, then $\bigvee_\omega(\gamma_0, \gamma_1, \gamma_2) = (3, 2, 3, 7, 6, 7, 4, 1, 1, 5, 2)$.

Proposition 39. *Let $n, r > 0$. For any $\gamma \in \mathcal{P}_{n,r}$ there exist a unique positive integer m, a unique composition $(r_0, \dots, r_m)$ of $r-1$, and unique elements $\gamma_0, \dots, \gamma_m$ in $\mathcal{P}_\infty$, $\omega \in SH(r_0, \dots, r_m)$ such that $\gamma = \bigvee_\omega(\gamma_0, \dots, \gamma_m)$.*

Proof. Let $\gamma^{-1}(r) = \{l_1 < \cdots < l_m\}$. Define the integers n_i, for $0 \leqslant i \leqslant m$, as follows:

$$n_0 := l_1 - 1, \qquad \dots, \qquad n_i := l_{i+1} - l_i - 1, \qquad \dots, \qquad n_m := n - l_m.$$

It is easy to check that $\gamma = (\alpha \times 1_1) \circ z(n_0, \dots, n_m)$, with $\alpha \in \mathcal{P}_{n-m,r-1}$. From Lemma 14, we conclude that there exist unique elements $\gamma_0 \in \mathcal{P}_{n_0}, \dots, \gamma_m \in \mathcal{P}_{n_m}$ and $\omega \in SH(r_0, \dots, r_m)$ such that $\alpha = \omega \circ (\gamma_0 \times \cdots \times \gamma_m)$.

The argument above proves the existence of the decomposition. Suppose that

$$\gamma = \big(\big(\mu \circ (\zeta_0 \times \cdots \times \zeta_h)\big) \times 1_1\big) \circ z(k_0, \dots, k_h),$$

with $h \geqslant 1$, $\zeta_i \in \mathcal{P}_{k_i,s_i}$, for $1 \leqslant i \leqslant h$, $(s_0, \dots, s_h)$ a composition of $r-1$ and $\mu \in SH(s_0, \dots, s_h)$. It is immediate to see that $\gamma^{-1}(r) = \{n_0+1, \dots, n_0+\cdots+n_{m-1}+m\} = \{k_0+1, \dots, k_0+\cdots+k_{h-1}+h\}$, which implies that $m = h$, $n_i = k_i$, for $1 \leqslant i \leqslant m$, and that $\alpha = \omega \circ (\gamma_0 \times \cdots \times \gamma_m) = \mu \circ (\zeta_0 \times \cdots \times \zeta_m)$.

Lemma 22 implies that $\omega = \mu$ and $\gamma_i = \zeta_i$, for $1 \leqslant i \leqslant m$. $\square$

Remark 40. The proposition above states that there exists a unique bijective map $\tilde{\Gamma}$ from $\mathcal{P}_\infty$ to the set of all planar leveled trees verifying that:

(1) $\tilde{\Gamma}(1^n) := (\varsigma_n, 1)$, for $n \geqslant 1$, where 1 denotes the map which sends the unique vertex of ς_n to 1.
(2) If $\gamma = \bigvee_\omega(\gamma_0, \dots, \gamma_m) \in \mathcal{P}_{n,r}$, with $\gamma_i \in \mathcal{P}_{n_i,r_i}$ for $0 \leqslant i \leqslant m$, and $\omega \in SH(r_0, \dots, r_m)$, then:

$$\tilde{\Gamma}(\gamma) := \bigvee_\omega \bigl(\tilde{\Gamma}(\gamma_0), \dots, \tilde{\Gamma}(\gamma_m)\bigr).$$

The following result is needed in the next section.

Proposition 41. *Let $\gamma \in \mathcal{P}_{n,r}$ be such that $\gamma = \bigvee_\omega(\gamma_0, \dots, \gamma_m)$, for some elements $\omega \in SH(r_0, \dots, r_m)$ and $\gamma_i \in \mathcal{P}_{n_i,r_i}$, $0 \leqslant i \leqslant m$, with $r = \sum_{i=0}^m r_i + 1$.*

For $1 \leqslant j \leqslant r-2$, there exist integers $s_i \in \{r_i, r_i - 1\}$, $1 \leqslant i \leqslant m$, a family $\delta_0, \dots, \delta_m$, with $\delta_i \in \mathcal{P}_{n_i,s_i}$ and $\mu \in SH(s_0, \dots, s_m)$ such that:

$$t_j \circ \gamma = \bigvee_\mu (\delta_0, \dots, \delta_m).$$

Proof.

(1) If $t_j \circ \omega \in SH(r_0, \dots, r_m)$, then $t_j \circ \bigvee_\omega(\gamma_0, \dots, \gamma_m) = \bigvee_{t_j \circ \omega}(\gamma_0, \dots, \gamma_m)$.
(2) If $t_j \circ \omega \notin SH(r_0, \dots, r_m)$, define $\{s_i\}_{0 \leqslant i \leqslant m}$ and $\{\delta_i\}_{0 \leqslant i \leqslant m}$ as follows:

$$s_i := \begin{cases} r_i & \text{if } \{j, j+1\} \nsubseteq \omega(\{r_0 + \cdots + r_{i-1} + 1, \dots, r_0 + \cdots + r_i\}), \\ r_i - 1 & \text{if } \{j, j+1\} \subseteq \omega(\{r_0 + \cdots + r_{i-1} + 1, \dots, r_0 + \cdots + r_i\}), \end{cases}$$

$$\delta_i := \begin{cases} \gamma_i & \text{if } s_i = r_i, \\ t_{j_i} \circ \gamma_i & \text{if } s_i = r_i - 1, \end{cases}$$

where $\omega^{-1}(j) \cap \{r_0 + \cdots + r_{i-1} + 1, \dots, r_0 + \cdots + r_i\} = \{j_i\}$. The element $\mu \in SH(s_0, \dots, s_m)$ is given by

$$\begin{aligned} &\mu(s_0 + \cdots + s_{i-1} + l) \\ &:= \begin{cases} \omega(r_0 + \cdots + r_{i-1} + l) & \text{for } \omega(r_0 + \cdots + r_{i-1} + l) \leqslant j, \\ \omega(r_0 + \cdots + r_{i-1} + l + 1) - 1 & \text{for } \omega(r_0 + \cdots + r_{i-1} + l) > j \text{ and } s_i < r_i, \\ \omega(r_0 + \cdots + r_{i-1} + l) - 1 & \text{for } \omega(r_0 + \cdots + r_{i-1} + l) > j \text{ and } s_i = r_i, \end{cases} \end{aligned}$$

for $1 \leqslant l \leqslant s_i$ and $0 \leqslant i \leqslant m$.
It is not difficult to check that $t_j \circ \gamma = \bigvee_\mu(\delta_0, \dots, \delta_m)$. □

The last results of this section are devoted to give some formulas relating the weak Bruhat order with the grafting product. The corollary below is a consequence of the definition of grafting and of Propositions 20 and 25.

Corollary 42.

(1) *Let* $(n_0, \ldots, n_k)$ *be a composition of* n *and* $\delta_i \in \mathcal{P}_{n_i, s_i}$, $\gamma_i \in \mathcal{P}_{n_1, r_i}$, *for* $1 \leqslant i \leqslant k$, *families of elements in* $\mathcal{P}_\infty$ *such that* $\delta_i \leqslant_B \gamma_i$. *Then*

$$\bigvee_{1_s}(\delta_0, \ldots, \delta_k) \leqslant_B \bigvee_{1_r}(\gamma_0, \ldots, \gamma_k), \quad \text{where } s = \sum_{i=0}^{k} s_i \text{ and } r = \sum_{i=0}^{k} r_i.$$

(2) *Let* $\gamma_0, \ldots, \gamma_k$ *be elements in* $\mathcal{P}_\infty$ *with* $\gamma_i \in \mathcal{P}_{n_i, r_i}$. *The inequality* $\omega \leqslant_B \omega'$ *holds in* $SH(r_0, \ldots, r_k)$ *if, and only if,* $\bigvee_\omega(\gamma_0, \ldots, \gamma_k) \leqslant_B \bigvee_{\omega'}(\gamma_0, \ldots, \gamma_k)$.

Lemma 43. *Let* $\gamma_i \in \mathcal{P}_{n_i, r_i}$, *for* $0 \leqslant i \leqslant k$, *and let* $\omega \in SH(r_0, \ldots, r_k) \cap S_{r-1} = Sh(r_0, \ldots, r_k)$, *where* $r = \sum_{i=0}^{k} r_i$.

(1) *If* $\gamma_l \leqslant_B t_j \circ \gamma_l$ *and* $\omega(r_0 + \cdots + r_{l-1} + j + 1) = \omega(r_0 + \cdots + r_{l-1} + j) + 1$, *for some* $0 \leqslant l \leqslant k$ *and* $1 \leqslant j \leqslant r_l$, *then*:

$$\bigvee_\omega(\gamma_0, \ldots, \gamma_k) \leqslant_B \bigvee_{\omega'}(\gamma_0, \ldots, t_j \circ \gamma_l, \ldots, \gamma_k), \quad \text{where } \omega' := t_{\omega(r_0 + \cdots + r_{l-1} + j)} \circ \omega.$$

(2) *If* $t_j \circ \gamma_l \leqslant_B \gamma_l$ *and* $\omega(r_0 + \cdots + r_{l-1} + j + 1) = \omega(r_0 + \cdots + r_{l-1} + j) + 1$, *for some* $0 \leqslant l \leqslant k$ *and* $1 \leqslant j \leqslant r_l$, *then*:

$$\bigvee_{\omega'}(\gamma_0, \ldots, t_j \circ \gamma_l, \ldots, \gamma_k) \leqslant_B \bigvee_\omega(\gamma_0, \ldots, \gamma_k), \quad \text{where } \omega' := t_{\omega(r_0 + \cdots + r_{l-1} + j)} \circ \omega.$$

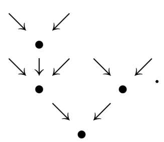
Proof. The proof is immediate, using that $t_{\omega(r_0 + \cdots + r_{l-1} + j)} \circ \omega \in SH(r_0, \ldots, r_k)$. $\square$

2.3.2. The poset morphism Γ

Definition 44. Define $\Gamma : \mathcal{P}_\infty \to \mathcal{T}_\infty$ as the map which sends $\gamma \in \mathcal{P}_\infty \mapsto t$, if $\tilde{\Gamma}(\gamma) = (t, f)$, where $\tilde{\Gamma}$ is the bijection between $\mathcal{P}_\infty$ and the leveled planar trees defined in Remark 40.

Note that $\Gamma(\gamma)$ consists simply in forgetting the levels of $\tilde{\Gamma}(\gamma)$.

Example 45. The tree associated to the map $\gamma = (2, 1, 2, 3, 2) \in \mathcal{P}_5$ is



Remark 46.

(1) It is clear that $\Gamma((1, \ldots, 1)) = \varsigma_n$, for $n \geqslant 1$.

(2) It is immediate from the definition of Γ that $\Gamma(\bar{\gamma}) = \overline{\Gamma(\gamma)}$, for all $\gamma \in \mathcal{P}_\infty$.
(3) Let $\gamma_0, \ldots, \gamma_k$ be a family of elements in $\mathcal{P}_\infty$, with $\gamma_i \in \mathcal{P}_{n_i,r_i}$ and $r = \sum_{i=0}^k r_i$, and let ω be an element in $SH(r_0, \ldots, r_k)$. The images of the elements $\bigvee_\omega(\gamma_0, \ldots, \gamma_k)$ and $\bigvee_{1_r}(\gamma_0, \ldots, \gamma_k)$ under Γ are equal.

Theorem 47. *The map $\Gamma : \mathcal{P}_\infty \to \mathcal{T}_\infty$ verifies that if $\gamma \leqslant_B \delta$ in $\mathcal{P}_n$, then $\Gamma(\gamma) \leqslant_B \Gamma(\delta)$ in $\mathcal{T}_n$.*

Proof. The result is immediate for $n \leqslant 2$. For $n \geqslant 3$, suppose $\gamma = \bigvee_\omega(\gamma_0, \ldots, \gamma_k)$, and $\delta = \bigvee_\mu(\delta_0, \ldots, \delta_h)$, with $\gamma_i \in \mathcal{P}_{n_i,r_i}$ and $\delta_i \in \mathcal{P}_{m_i,s_i}$:

(a) For $\delta = t_j \circ \gamma$, with $\gamma^{-1}(j) < \gamma^{-1}(j+1)$, the following cases must be considered:

(i) If $j < r-1$ and $t_j \circ \omega \in SH(r_0, \ldots, r_k)$, then Proposition 41 asserts that $\Gamma(\delta) = \Gamma(\gamma)$.
(ii) If $j < r-1$ and $t_j \circ \omega \notin SH(r_0, \ldots, r_k)$, then Proposition 41 implies that $k = h$ and

$$\delta_i = \begin{cases} \gamma_i & \text{when } \{j, j+1\} \nsubseteq \omega(\{r_0 + \cdots + r_{i-1} + 1, \ldots, r_0 + \cdots + r_i\}), \\ t_{j_i} \circ \gamma_i & \text{when } \{j, j+1\} \subseteq \omega(\{r_0 + \cdots + r_{i-1} + 1, \ldots, r_0 + \cdots + r_i\}). \end{cases}$$

The fact that $\gamma^{-1}(j) < \gamma^{-1}(j+1)$ states that there exists at most one i_0 such that $\delta_{i_0} \neq \gamma_{i_0}$ and $\gamma_{i_0}^{-1}(j_{i_0}) < \gamma_{i_0}^{-1}(j_{i_0}+1)$. So,

$$\Gamma(\delta) = \bigvee\bigl(\Gamma(\gamma_0), \ldots, \Gamma(t_{j_{i_0}} \circ \gamma_{i_0}), \ldots, \Gamma(\gamma_k)\bigr),$$

and, by recursive hypothesis, $\Gamma(\gamma_{i_0}) \leqslant_B \Gamma(t_{j_{i_0}} \circ \gamma_{i_0})$, which implies that $\Gamma(\gamma) \leqslant_B \Gamma(\delta)$.
(iii) For $j = r-1$, there exists an embedding $\gamma^{-1}(r-1) \subseteq \{1, \ldots, r_0\}$. It is verified that

$$\Gamma(\gamma) = \bigvee\bigl(\Gamma(\gamma_0), \ldots, \Gamma(\gamma_k)\bigr)$$

and that $\Gamma(\delta)$ is the grafting of $\Gamma(\gamma_{0,0}), \ldots, \Gamma(\gamma_{k_0,0}), \Gamma(\gamma_1), \ldots, \Gamma(\gamma_k)$, with $\Gamma(\gamma_0) = \bigvee(\Gamma(\gamma_{0,0}), \ldots, \Gamma(\gamma_{k_0,0}))$. So, $\Gamma(\gamma) \leqslant_B \Gamma(\delta)$.

(b) The proof of the case $t_j \circ \delta = \gamma$, with $\delta^{-1}(j) > \delta^{-1}(j+1)$, is a consequence of (a) and Remarks 19, 34 and 46. □

Definition 48. Given a tree $t \in \mathcal{T}_m$, we define the elements $\mathrm{Min}(t)$ and $\mathrm{Max}(t)$ in $\mathcal{P}_m$ recursively, as follows:

(1) $\mathrm{Min}(\downarrow) := (0) =: \mathrm{Max}(\downarrow)$.
(2) For $t = \bigvee(t^0, \ldots, t^k)$, with $t^i \in \mathcal{T}_{n_i}$ and $n = \sum_{i=0}^k n_i$, the degree of t is $n+k$. Define $\mathrm{Min}(t) := \bigvee_{1_{r-1}}(\mathrm{Min}(t^0), \ldots, \mathrm{Min}(t^k))$, and $\mathrm{Max}(t) := \bigvee_{\xi_{s_0,\ldots,s_k}}(\mathrm{Max}(t^0), \ldots, \mathrm{Max}(t^k))$, where $\mathrm{Min}(t^i) \in \mathcal{P}_{n_i,r_i}$, $\mathrm{Max}(t^i) \in \mathcal{P}_{n_i,s_i}$ and $\sum_{i=0}^k r_i = r-1$.

Proposition 49. *Let $t \in \mathcal{T}_m$ be a tree, the inverse image under Γ of t is an interval for the weak Bruhat order. Moreover, it verifies*

$$\Gamma^{-1}(t) = \{x \in \mathcal{P}_m \text{ such that } \mathrm{Min}(t) \leqslant_B x \leqslant_B \mathrm{Max}(t)\}.$$

Proof. The assertion is obviously true for $m = 0, 1, 2$. For $m \geqslant 3$, let $t \in \mathcal{T}_m$ with $t = \bigvee(t^0, \ldots, t^k)$. By recursive hypothesis, we know that

$$\Gamma^{-1}(t^i) = \{x_i \in \mathcal{P}_{n_i} \text{ such that } \mathrm{Min}(t^i) \leqslant_B x_i \leqslant_B \mathrm{Max}(t^i)\}.$$

Suppose $\gamma = \bigvee_\omega(\gamma_0, \ldots, \gamma_k)$ is such that $\Gamma(\gamma) = t$. Hence $\Gamma(\gamma_i) = t^i$, for $0 \leqslant i \leqslant k$. By Corollary 42 and Lemma 43 we obtain that:

$$\mathrm{Min}(t) = \bigvee_{1_{r-1}} (\mathrm{Min}(t^0), \ldots, \mathrm{Min}(t^k)) \leqslant_B \bigvee_{1_{r'-1}} (\gamma_0, \ldots, \gamma_k) \leqslant_B \gamma,$$

and, there exists $\omega' \leqslant_B \xi_{s_0,\ldots,s_k}$ such that:

$$\gamma \leqslant_B \bigvee_{\omega'} (\mathrm{Max}(t^0), \ldots, \mathrm{Max}(t^k)) \leqslant_B \bigvee_{\xi_{s_0,\ldots,s_k}} (\mathrm{Max}(t^0), \ldots, \mathrm{Max}(t^k)) = \mathrm{Max}(t).$$

Conversely, if $\mathrm{Min}(t) \leqslant_B \gamma \leqslant_B \mathrm{Max}(t)$, then Theorem 47 states that

$$t = \Gamma(\mathrm{Min}(t)) \leqslant_B \Gamma(\gamma) \leqslant_B \Gamma(\mathrm{Max}(t)) = t. \qquad \square$$

So, the weak Bruhat order of $\mathcal{P}_\infty$ induces a partial order $\leqslant$ on $\mathcal{T}_\infty$. Moreover, from Definition 30 and the results above, we obtain that the weak Bruhat order $\leqslant_B$ of $\mathcal{P}_\infty$ induces the weak Bruhat order on $\mathcal{T}_\infty$.

The diagram in Fig. 1 is the Hasse diagram of the order $\leqslant_B$ on $\mathcal{T}_3$.

And the picture in Fig. 2 is a design of the associahedron of dimension 2.

3. Dendriform trialgebras structure on $k[\mathcal{P}_\infty]$ and $k[\mathcal{T}_\infty]$

The following definition is given in [13]. It generalizes the definition of dendriform algebra introduced by J.-L. Loday in [9].

Definition 50. Let k be a field. A *dendriform trialgebra* over k is a vector space V equipped with three k-linear maps $\succ, \cdot, \prec : V \otimes_k V \to V$ verifying the following identities:

(1) $v \succ (w \succ u) = (v * w) \succ u$,
(2) $v \succ (w \prec u) = (v \succ w) \prec u$,
(3) $v \prec (w * u) = (v \prec w) \prec u$,
(4) $v \cdot (w \cdot u) = (v \cdot w) \cdot u$,
(5) $v \succ (w \cdot u) = (v \succ w) \cdot u$,

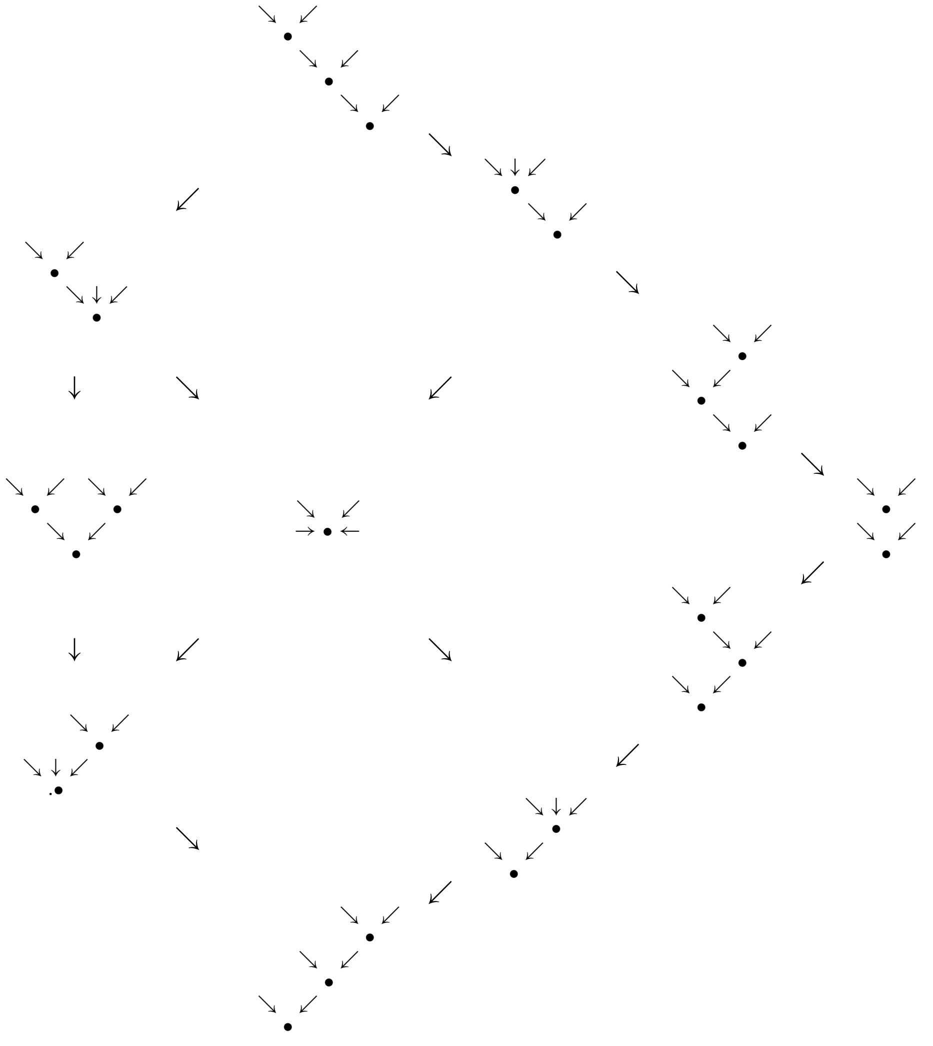


Fig. 1.

(6) $v \cdot (w \succ u) = (v \prec w) \cdot u$,
(7) $v \cdot (w \prec u) = (v \cdot w) \prec u$,

for all v, w and u in V, where $v * w := v \succ w + v \cdot w + v \prec w$.

If $(V, \succ, \cdot, \prec)$ is a dendriform trialgebra, then $(V, *)$ is an associative algebra over k.
Given a set X, we denote by $k[X]$ the k-vector space spanned by X.

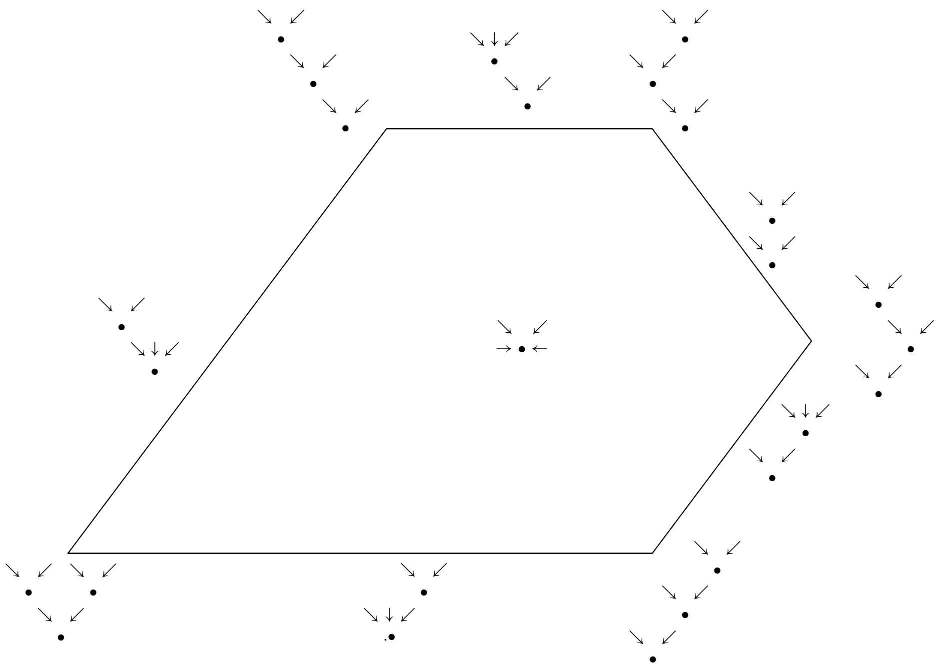


Fig. 2.

3.1. The dendriform trialgebra $k[\mathcal{P}_\infty]$

Let $k[\mathcal{P}_\infty]$ be the graded k-vector space spanned by the set $\mathcal{P}_\infty$. We define operations $\succ$, $\cdot$ and $\prec$ on $k[\mathcal{P}_\infty]$ as follows:

(1) $\gamma \succ \delta := \sum_{\omega \in SH^{\succ}(r,s)} \omega \circ (\gamma \times \delta)$,
(2) $\gamma \cdot \delta := \sum_{\omega \in SH^{\bullet}(r,s)} \omega \circ (\gamma \times \delta)$,
(3) $\gamma \prec \delta := \sum_{\omega \in SH^{\prec}(r,s)} \omega \circ (\gamma \times \delta)$,

for $\gamma \in \mathcal{P}_{n,r}$ and $\delta \in \mathcal{P}_{m,s}$.

Proposition 28 implies that $k[\mathcal{P}_\infty]$ is a dendriform trialgebra over k. For instance, let us show that condition (3) of the definition is fulfilled. Given $\gamma \in \mathcal{P}_{n,r}$, $\delta \in \mathcal{P}_{m,s}$ and $\tau \in \mathcal{P}_{q,l}$, we get the following equality:

$$\gamma \prec (\delta * \tau) = \gamma \prec \left(\sum_{x \in SH(s,l)} x \circ (\delta \times \tau) \right)$$

$$= \sum_{j=1}^{s+l} \sum_{x \in SH^{\prec}(r,j)} \sum_{y \in SH(s,l)} x \circ \big(\gamma \times \big(y \circ (\delta \times \tau)\big)\big)$$

$$= \sum_{j=1}^{s+l} \sum_{x \in SH^{\prec}(r,j)} \sum_{y \in SH(s,l)} x \circ (1_r \times y) \circ (\gamma \times \delta \times \tau)$$

(by point (3) of Proposition 28)

$$= \sum_{i=1}^{r+s} \sum_{x \in SH^{\prec}(i,r)} \sum_{y \in SH^{\prec}(r,s)} x \circ (y \times 1_l) \circ (\gamma \times \delta \times \tau)$$

$$= \left(\sum_{x \in SH^{\prec}(r,s)} x \circ (\gamma \times \delta) \right) \prec \tau = (\gamma \prec \delta) \prec \tau.$$

Moreover, it is easy to see that the associative product $*$ is given by the formula:

$$\gamma * \delta = \sum_{x \in SH(r,s)} x \circ (\gamma \times \delta), \quad \text{for } \gamma \in \mathcal{P}_{n,r} \text{ and } \delta \in \mathcal{P}_{m,s}.$$

Theorem 51. *The operations of the dendriform trialgebra $k[\mathcal{P}_\infty]$ may be described in terms of the weak Bruhat order as follows*:

$$\gamma \succ \delta = \sum_{\gamma \times \delta \leqslant_B x \leqslant_B (\xi_{r,s-1} \times 1_1) \circ (\gamma \times \delta)} x,$$

$$\gamma \cdot \delta = \sum_{z(r-1,s-1,0) \circ (\gamma \times \delta) \leqslant_B x \leqslant_B (\xi_{r-1,s-1} \times 1_1) \circ z(r-1,s-1,0) \circ (\gamma \times \delta)} x,$$

$$\gamma \prec \delta = \sum_{z(r-1,s) \circ (\gamma \times \delta) \leqslant_B x \leqslant_B \xi_{r,s} \circ (\gamma \times \delta)} x,$$

$$\gamma * \delta = \sum_{\gamma \times \delta \leqslant_B x \leqslant_B \xi_{r,s} \circ (\gamma \times \delta)} x,$$

where $\gamma \in \mathcal{P}_{n,r}$, $\delta \in \mathcal{P}_{m,s}$.

Proof. The proof follows from the three points below:

(I) Proposition 25 and Lemma 27 imply that:

- if $\omega \in SH^{\succ}(r,s)$, then $\gamma \times \delta \leqslant_B \omega \circ (\gamma \times \delta) \leqslant_B (\xi_{r,s-1} \times 1_1) \circ (\gamma \times \delta)$,
- if $\omega \in SH^{\bullet}(r,s)$, then $z(r-1,s-1,0) \circ (\gamma \times \delta) \leqslant_B \omega \circ (\gamma \times \delta) \leqslant_B (\xi_{r-1,s-1} \times 1_1) \circ z(r-1,s-1,0) \circ (\gamma \times \delta)$,
- if $\omega \in SH^{\prec}(r,s)$, then $z(r-1,s) \circ (\gamma \times \delta) \leqslant_B \omega \circ (\gamma \times \delta) \leqslant_B \xi_{r,s} \circ (\gamma \times \delta)$.

(II) Conversely, let $\alpha \in \mathcal{P}_{n+m}$, we know by Lemma 8 that $\alpha = \zeta \circ (\tau \times \mu)$, for unique elements $\tau \in \mathcal{P}_{n,p}$, $\mu \in \mathcal{P}_{m,q}$ and $\zeta \in SH(p,q)$ and unique integers p and q.

It is not difficult to see that if α verifies the following conditions:

(1) $\gamma(i) < \gamma(j)$ if, and only if $\alpha(i) < \alpha(j)$, for $1 \leqslant i, j \leqslant n$. Moreover, the equality $\gamma(i) = \gamma(j)$ holds if, and only if $\alpha(i) = \alpha(j)$.
(2) $\delta(i) < \delta(j)$ if, and only if $\alpha(i+n) < \alpha(j+n)$, for $1 \leqslant i, j \leqslant m$. Moreover, the equality $\delta(i) = \delta(j)$ holds if, and only if $\alpha(i+n) = \alpha(j+n)$,

then $\tau = \gamma$ and $\mu = \delta$.

Suppose that $\omega_1 \circ (\gamma \times \delta) \leqslant_B \alpha \leqslant_B \omega_2 \circ (\gamma \times \delta)$, for some $\omega_1, \omega_2 \in SH(r,s)$. Using Lemma 17, we obtain that α verifies conditions (1) and (2) above, so $\alpha = \zeta \circ (\gamma \times \delta)$ for some $\zeta \in SH(r,s)$. □

To end this section we want to show that $(k[\mathcal{P}_\infty], *)$ is free. In order to do that we have to introduce a notion of irreducible element of $\mathcal{P}_\infty$, which generalizes the one given by C. Malvenuto and C. Reutenauer in [15] in the symmetric group case, and to prove an easy result.

Definition 52. Let $\gamma \in \mathcal{P}_n$, we say that γ is an *irreducible element* if

$$\gamma \notin \bigcup_{1 \leqslant p \leqslant n-1} \mathcal{P}_p \times \mathcal{P}_{n-p}.$$

For example, the set of irreducible elements of $\mathcal{P}_3$ is

$$\{(2,3,1), (3,1,2), (3,2,1), (1,2,1), (2,1,1), (2,1,2), (2,2,1), (1,1,1)\}.$$

Lemma 53. *Let $\gamma \in \mathcal{P}_{n,r}$ and $\delta \in \mathcal{P}_{m,s}$ be irreducible elements. If $\omega \circ (\gamma \times \delta)$ belongs to $\bigcup_{1 \leqslant p \leqslant n-1} \mathcal{P}_p \times \mathcal{P}_{n-p}$ for some $\omega \in SH(r,s)$, then $\omega = 1_{r+s}$.*

Proof. Suppose that $\omega \circ (\gamma \times \delta) = \rho \times \tau \in \mathcal{P}_{p,k} \times \mathcal{P}_{n+m-p,l}$, for some $1 \leqslant p \leqslant n+m-1$, $1 \leqslant k \leqslant p$ and $1 \leqslant l \leqslant n+m-p$. Lemma 22 implies that either $\omega = 1_{r+s}$ or $p \neq n$.

Suppose that $p < n$. The following statements hold:

(1) $\{\omega(\gamma(1)), \ldots, \omega(\gamma(p))\} = \{1, \ldots, k\}$ and $\omega \in SH(r,s)$ imply that $\omega(j) = j$, for $1 \leqslant j \leqslant k$, and $\gamma(i) = \rho(i)$ for $1 \leqslant i \leqslant p$.
(2) For $p < i \leqslant n$, the equality $\omega(\gamma(i)) = \tau(i) + k$ holds. Since ω is in $SH(r,s)$, we obtain that $\gamma(i) > k$, for $p < i \leqslant n$.

From (1) and (2) we conclude that there exists $\alpha \in \mathcal{P}_{n-p}$ such that $\gamma = \rho \times \alpha$. So, γ is not irreducible, which contradicts the hypothesis.

If $p > n$, a similar argument shows that $\delta = \beta \times \tau$, which is impossible because δ is irreducible. □

Theorem 54. *The algebra $(k[\mathcal{P}_\infty], *)$ is free over the set of irreducible elements of $\mathcal{P}_\infty$.*

Proof. We denote by Irred the set of irreducible elements of $\mathcal{P}_\infty$. It is immediate to see that given $\gamma \in \mathcal{P}_\infty$ there exists a unique $k \geqslant 1$ and unique irreducible elements $\gamma_1, \ldots, \gamma_k$ such that $\gamma = \gamma_1 \times \cdots \times \gamma_k$. This implies that the tensor space $T(k[\mathrm{Irred}]) = \bigoplus_{n\geqslant 1} k[\mathrm{Irred}]^{\otimes n}$ is isomorphic, as a vector space, to $k[\mathcal{P}_\infty]$. But the k-linear map $T(k[\mathrm{Irred}]) \to k[\mathcal{P}_\infty]$ which sends $\gamma_1 \otimes \cdots \otimes \gamma_k \mapsto \gamma_1 * \cdots * \gamma_k$ is a k-algebra homomorphism, which is surjective by Lemma 53. So, it is an isomorphism. □

3.2. Free dendriform trialgebra structure on $k[\mathcal{T}_\infty]$

Definition 55. Define operations $\succ$, $\cdot$ and $\prec$ on $k[\mathcal{T}_\infty] := \bigoplus_{n\geqslant 1} k[\mathcal{T}_n]$ as follows:

(1) $t * \downarrow = t = \downarrow * t$, for all $t \in \mathcal{T}_\infty$,
(2) $t \succ z = \bigvee(t * z^0, z^1, \ldots, z^h)$,
(3) $t \cdot z = \bigvee(t^0, \ldots, t^{k-1}, t^k * z^0, z^1, \ldots, z^h)$,
(4) $t \prec z = \bigvee(t^0, \ldots, t^{k-1}, t^k * z)$,

for $t = \bigvee(t^0, \ldots, t^k)$ and $z = \bigvee(z^0, \ldots, z^h)$.

From the definition of $\succ$, $\cdot$ and $\prec$, it is easy to check that $(k[T_\infty], \succ, \cdot, \prec)$ is a dendriform trialgebra. The following result is proved in [14].

Theorem 56. *The dendriform trialgebra $(k[\mathcal{T}_\infty], \succ, \cdot, \prec)$ is the free dendriform trialgebra generated by the element of $\mathcal{T}_1$. Moreover, the algebra $(k[\mathcal{T}_\infty], *)$ is the free associative algebra generated by the set of planar rooted trees of type $\bigvee(\,|, t^1, \ldots, t^k)$, with $t^i \in \mathcal{T}_\infty$, for $1 \leqslant i \leqslant k$.*

Let us describe now the operations of the free dendriform algebra $k[\mathcal{T}_\infty]$ in terms of the weak Bruhat order of $\mathcal{T}_\infty$.

Given a tree $t \in \mathcal{T}_n$, we number its leaves from left to right by $0, 1, \ldots, n$.

Definition 57. Let $t \in \mathcal{T}_n$ and $z \in \mathcal{T}_m$ be two planar trees:

(1) For any $0 \leqslant j \leqslant m$, the grafting $t \circ_j z$ of t on the jth leaf of z is the tree obtained by joining the root of t to the leaf j of z.
(2) We denote by t/z the tree $t \circ_0 z$, and by $t\backslash z$ the tree $z \circ_n t$.

Proposition 58. *The operations $\succ$, $\cdot$, $\prec$ and $*$ of the free dendriform trialgebra $k[\mathcal{T}_\infty]$ are defined in terms of the weak Bruhat order and the products over / and under \ by the following conditions*:

(1) $t \succ z = \sum_{t/z \leqslant_B x \leqslant_B \bigvee(t\backslash z^0, z^1, \ldots, z^k)} x$,
(2) $t \cdot z = \sum_{\bigvee(t^0, \ldots, t^{k-1}, t^k/z^0, z^1, \ldots, z^h) \leqslant_B x \leqslant_B \bigvee(t^0, \ldots, t^{k-1}, t^k\backslash z^0, z^1, \ldots, z^h)} x$,
(3) $t \prec w = \sum_{\bigvee(t^0, \ldots, t^{k-1}, t^k/z) \leqslant_B x \leqslant_B t\backslash z} x$,
(4) $t * z = \sum_{t/z \leqslant_B x \leqslant_B t\backslash z} x$.

Proof. For $|t| = 1 = |z|$ it is easy to see that the formulas hold.

For $|t| \geqslant 1$, $|z| \geqslant 1$ and $|t| + |z| \geqslant 2$, a recursive argument states that:

$$t \succ z = \sum_{t/z^0 \leqslant_B x \leqslant_B t \backslash z^0} \bigvee\left(x, z^1, \ldots, z^h\right),$$

$$t \cdot z = \sum_{t^k/z^0 \leqslant_B x \leqslant_B t^k \backslash z^0} \bigvee\left(t^0, \ldots, t^{k-1}, x, s^1, \ldots, z^h\right),$$

$$t \prec z = \sum_{t^k/z \leqslant_B x \leqslant_B t^k \backslash z} \bigvee\left(t^0, \ldots, t^{k-1}, x\right),$$

where $t = \bigvee(t^0, \ldots, t^k)$ and $z = \bigvee(z^0, \ldots, z^h)$.

Now, by Lemma 33, we get the following equalities:

$$t \succ z = \sum_{t/z \leqslant_B x \leqslant_B \bigvee(t \backslash z^0, z^1, \ldots, z^h)} x,$$

$$t \cdot z = \sum_{\bigvee(t^0, \ldots, t^k/z^0, \ldots, z^h) \leqslant_B x \leqslant_B \bigvee(t^0, \ldots, t^k \backslash z^0, \ldots, z^h)} x,$$

$$t \prec z = \sum_{\bigvee(t^0, \ldots, t^k/z) \leqslant_B x \leqslant_B t \backslash z} x.$$

To prove (4), it suffices to note that:

$$\begin{aligned}
&\{x \in \mathcal{T}_\infty \colon t/z \leqslant_B x \leqslant_B t \backslash z\} \\
&\quad = \left\{x \in \mathcal{T}_\infty \colon t/z \leqslant_B x \leqslant_B \bigvee\left(t \backslash z^0, \ldots, z^h\right)\right\} \\
&\qquad \cup \left\{x \in \mathcal{T}_\infty \colon \bigvee\left(t^0, \ldots, t^k/z^0, \ldots, z^h\right) \leqslant_B x \leqslant_B \bigvee\left(t^0, \ldots, t^k \backslash z^0, \ldots, z^h\right)\right\} \\
&\qquad \cup \left\{x \in \mathcal{T}_\infty \colon \bigvee\left(t^0, \ldots, t^k/z\right) \leqslant_B x \leqslant_B \bigvee\left(t^0, \ldots, t^k \backslash z\right)\right\}. \qquad \square
\end{aligned}$$

Consider now the k-linear map $\Gamma^* : k[\mathcal{T}_\infty] \hookrightarrow k[\mathcal{P}_\infty]$ given by $\gamma^*(t) := \sum_{\Gamma(\gamma)=t} \gamma$.

Proposition 59. *Let $t \in \mathcal{T}_n$ and $z \in \mathcal{T}_m$. Given elements $\gamma \in \Gamma^{-1}(t) \cap \mathcal{P}_{n,r}$ and $\delta \in \Gamma^{-1}(z) \cap \mathcal{P}_{m,s}$, the following equalities hold*:

(1) $\Gamma(\gamma \times \delta) = t/z$,
(2) $\Gamma(\xi_{r,s} \circ (\gamma \times \delta)) = t \backslash z$.

Proof.

(1) If $z = \downarrow$, then $\gamma \times (0) = \gamma$ and $\Gamma(\gamma) = t = t/\downarrow$.

If $m \geqslant 1$, then $\delta = \bigvee_{\upsilon}(\delta_0, \ldots, \delta_h)$, for a family $\delta_i \in \Gamma^{-1}(z^i)$, with $0 \leqslant i \leqslant h$, and $\upsilon \in SH(s_0, \ldots, s_h)$. We get the equality: $\gamma \times \delta = \bigvee_{1_r \times \upsilon}(\gamma \times \delta_0, \delta_1, \ldots, \delta_h)$. The recursive hypothesis states that $\Gamma(\gamma \times \delta_0) = t/z^0$, which implies that $\Gamma(\gamma \times \delta) = \bigvee(t/z^0, z^1, \ldots, z^h) = t/z$.

(2) If $t = \downarrow$, then $\xi_{0,r} \circ ((0) \times \delta) = \delta$ and $\Gamma(\delta) = z = \downarrow \backslash z$.
If $n \geqslant 1$, then $\gamma = \bigvee_{\omega}(\gamma_0, \ldots, \gamma_k)$, with $\gamma_i \in \Gamma^{-1}(t^i)$ and $\omega \in SH(r_0, \ldots, r_k)$, with $\sum_{i=0}^{k} r_i = r - 1$. The following equality holds:

$$\xi_{r,s} \circ (\gamma \times \delta) = \bigvee_{\upsilon}\big(\gamma_0, \ldots, \gamma_{k-1}, \big(\xi_{r_k,s} \circ (\gamma_k \times \delta)\big)\big),$$

where $\upsilon := (\omega(1) + s, \ldots, \omega(r_0 + \cdots + r_{k-1}) + s, 1, \ldots, s, \omega(r_0 + \cdots + r_{k-1} + 1) + s, \ldots, \omega(r-1) + s)$, belongs to $SH(r_0, \ldots, r_{k-1}, r_k + s)$.
By recursive hypothesis, $\Gamma(\xi_{r_k,s} \circ (\gamma_k \times \delta)) = t^k \backslash z$.
So, $\Gamma(\xi_{r,s} \circ (\gamma \times \delta)) = \bigvee(t^0, \ldots, t^{k-1}, t^k \backslash z) = t \backslash z$. □

Theorem 60. *The map $\Gamma^* : k[\mathcal{T}_\infty] \hookrightarrow k[\mathcal{P}_\infty]$ is a homomorphism of dendriform trialgebras.*

Proof. Let us show that $\Gamma^*(t \succ z) = \Gamma^*(t) \succ \Gamma^*(z)$, for $t \in \mathcal{T}_n$ and $z \in \mathcal{T}_m$. The equalities $\Gamma^*(t \cdot z) = \Gamma^*(t) \cdot \Gamma^*(z)$, and $\Gamma^*(t \prec z) = \Gamma^*(t) \prec \Gamma^*(z)$, may be obtained using analogous arguments.

We have to prove that:

$$\Big\{\gamma \in \mathcal{P}_{n+m}\colon\ t/z \leqslant_B \Gamma(\gamma) \leqslant_B \bigvee\big(t\backslash z^0, z^1, \ldots, z^h\big)\Big\}$$
$$= \bigcup_{r,s} \ \bigcup_{\omega \in SH^{\succ}(r,s)} \big\{\omega \circ (\tau \times \delta)\colon\ \Gamma(\tau) = t \text{ and } \Gamma(\delta) = z\big\}.$$

($\subseteq$) For $|t| = 1 = |z|$ the result is obvious.

Let $\Gamma(\gamma) = \bigvee_{\mu}(\gamma_0, \ldots, \gamma_l)$ be such that $t/z \leqslant_B \Gamma(\gamma) \leqslant_B \bigvee(t\backslash z^0, z^1, \ldots, z^h)$. We obtain that $t/z^0 \leqslant_B \Gamma(\gamma_0) \leqslant_B t\backslash z^0$ and $\Gamma(\gamma_i) = z^i$, for $1 \leqslant i \leqslant h$. So, $l = h$.

By recursive hypothesis, $\gamma_0 = \alpha \circ (\tau \times \epsilon)$, with $\alpha \in SH(r, s_0)$, $\Gamma(\tau) = t$ and $\Gamma(\epsilon) = z^0$.

Note that $\rho := \mu \circ (\alpha \times 1_{s_1 + \cdots + s_h}) \in SH(r, s_0, \ldots, s_h)$. Moreover, it is easy to check that $\rho = \lambda \circ (1_r \times \nu)$, with $\lambda \in SH(r, s_0 + \cdots + s_h)$ and $\nu \in SH(s_0, \ldots, s_h)$. One has also that $z(r + s_0, s_1, \ldots, s_h) = 1_r \times z(s_0, \ldots, s_h)$. The following equality holds:

$$\begin{aligned}
\gamma &= \big(\big(\lambda \circ \big(\tau \times \big(\nu \circ (\epsilon \times \gamma_1 \times \cdots \times \gamma_h)\big)\big)\big) \times 1_1\big) \circ z(r + s_0, s_1, \ldots, s_h) \\
&= (\lambda \times 1_1) \circ \tau \times \big(\big(\nu \circ (\epsilon \times \gamma_1 \times \cdots \times \gamma_h)\big) \times 1_1\big) \circ \big(1_r \times z(s_0, \ldots, s_h)\big) \\
&= (\lambda \times 1_1) \circ \bigg(\tau \times \bigvee_{\nu}(\epsilon, \gamma_1, \ldots, \gamma_h)\bigg),
\end{aligned}$$

where $\omega := \lambda \times 1_1 \in SH^{\succ}(r, s_0 + \cdots + s_h + 1)$, $\Gamma(\tau) = t$ and $\Gamma(\bigvee_{\nu}(\epsilon, \gamma_1, \ldots, \gamma_h)) = z$.

($\supseteq$) Let $\omega \in SH^{\succ}(r,s)$, $\Gamma(\tau) = t$ and $\Gamma(\delta) = z$. By Proposition 25

$$\tau \times \delta \leqslant_B \omega \circ (\tau \times \delta) \leqslant_B (\xi_{r,s-1} \times 1_1) \circ (\tau \times \delta),$$

for $\tau \in \mathcal{P}_{n,r}$ and $\delta \in \mathcal{P}_{m,s}$.

By Theorem 47 and Proposition 59,

$$\begin{aligned} t/z = \Gamma(\tau \times \delta) \leqslant_B \Gamma\big(\omega \circ (\tau \times \delta)\big) \leqslant_B \Gamma\big((\xi_{r,s-1} \times 1_1) \circ (\tau \times \delta)\big) \\ = \bigvee\big(t \backslash z^0, z^1, \ldots, z^h\big), \end{aligned}$$

which ends the proof. □

Final comments. Given a Coxeter system (W,S) there exists another order on the set $\mathcal{P}^{(W,S)}$ which extends the weak Bruhat order of W. Consider the transitive relation $\leqslant_C$ spanned by:

$$W_J \circ w \leqslant_C W_J \circ \big(\alpha_{J,s}^{-1} \circ w\big), \quad \text{for } J \subseteq S,\ s \in S \setminus J \text{ and } w \in X_{J \cup \{s\}}^{-1},$$

where W_J, X_K and $\alpha_{J,s}$ are the elements defined in Section 1.

Observe that if $x \leqslant_C y$, then $x \leqslant_B y$. So, the weak Bruhat order is weaker than $\leqslant_C$.

This new order preserves the graduation of $\mathcal{P}^{(W,S)}$, that is two elements x and y of $\mathcal{P}^{(W,S)}$ are comparable for $\leqslant_C$ only if $x, y \in \mathcal{P}_r^{(W,S)}$, for some $0 \leqslant r \leqslant |S|$.

The order $\leqslant_C$ gives associative products on the vector spaces $k[\mathcal{P}_\infty]$ and $k[\mathcal{T}_\infty]$, which are the graded associated structures of the ones defined in Section 3. These structures coincide with the associative algebra structures defined by F. Chapoton in [4].

In [16], N. Reading introduced the notion of *translational* families of lattice congruences on the weak Bruhat order of the symmetric groups. He proved that if $\{\Theta_n\}_{n \geqslant 1}$ is a family of such a lattice congruences the subspace $k[Z_\infty^{\Theta}] := \bigoplus_{n \geqslant 0} k[S_n/\Theta_n]$ of $k[S_\infty]$ is a subalgebra of the Malvenuto–Reutenauer algebra, in which case the associative product may be described in terms of the weak Bruhat order. It would be interesting to study the same problem for the family of posets $\{(\mathcal{P}_n, \leqslant_B)\}_{n \geqslant 1}$.

There exist two other infinite series of irreducible Coxeter groups: the family $\{B_n\}_{n \geqslant 1}$ of hyperoctahedral groups, and the family $\{D_n\}_{n \geqslant 3}$. The weak Bruhat order gives structures of modules over the Malvenuto–Reutenauer algebra on the spaces $\bigoplus_{n \geqslant 1} k[B_n]$ and $\bigoplus_{n \geqslant 3} k[D_n]$. In a similar way, the generalized weak Bruhat order of $\mathcal{P}^{B_n}$ and $\mathcal{P}^{D_n}$, induces structures of $k[\mathcal{P}_\infty]$-modules on the spaces $\bigoplus_{n \geqslant 1} k[\mathcal{P}^{B_n}]$ and $\bigoplus_{n \geqslant 3} k[\mathcal{P}^{D_n}]$. These structures will be studied in a future work.

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